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Primary decompositions of knot concordance

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Abstract. For all $n > 0$ there exists a homomorphism from the smooth concordance group of knots in dimension $2n+1$ to an algebraically defined group $\mathcal{G}^{\mathbb{Q}}$. This algebraic concordance group splits as a direct sum of groups indexed by polynomials. For $n > 1$ the homomorphism is injective, and this leads to what is called a primary decomposition theorem. In the classical dimension, the kernel of this homomorphism includes the smooth concordance group of topologically slice knots, $\mathcal{T}$, which has become an important focus of research about smooth knot concordance. Here we will show that primary decompositions of $\mathcal{T}$ of a strong type cannot exist.

In more detail, it is shown that there exists a topologically slice knot K for which there is a factorization of its Alexander polynomial, $\Delta_K(t) = f_1(t)f_2(t)$, where f_1 and f_2 are relatively prime and each is the Alexander polynomial of a topologically slice knot, but K is not smoothly concordant to any connected sum $K_1 \# K_2$ for which $\Delta_{K_i}(t) = f_i(t)^{n_i}$ for any nonnegative integers n_i.

Contents

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1. Introduction

A central problem in three-dimensional knot concordance theory in the smooth category consists of understanding $\mathcal{T}$, the concordance group of topologically slice knots. Freedman [7, 8] proved that the subgroup $\mathcal{T}^1$ generated by knots with Alexander polynomial one satisfies $\mathcal{T}^1 \subset \mathcal{T}$. Early work proving that $\mathcal{T}^1$ is nontrivial includes that of Akbulut, Casson, and Cochran-Gompf [4]; in [5] it was shown that $\mathcal{T}^1$ contains an infinitely generated free subgroup. Using Heegaard Floer theory, in [9, 10] it was shown that $\mathcal{T}/\mathcal{T}^1$ contains an infinitely generated free subgroup and infinite two-torsion.

Recently, Jae Choon Cha [3] has undertaken an in-depth investigation of *primary decompositions* of $\mathcal{T}$. One motivation for studying primary decompositions arises from Levine's work [12] in which it was shown that for all integers $n > 0$ there is a homomorphism from the smooth concordance group of knotted $2n-1$ spheres in S^{2n+1} to a group called the *rational algebraic concordance group*: $\psi_{2n-1}\colon \mathcal{C}_{2n-1} \to \mathcal{G}^{\mathbb{Q}}$. It was also proved that there is a decomposition $\mathcal{G}^{\mathbb{Q}} \cong \oplus_{p\in\mathcal{A}} \mathcal{G}^{\mathbb{Q}}_p$, where $\mathcal{A}$ is the set of irreducible Alexander polynomials. Such a decomposition does not exist using integer rather than rational coefficients, but the failure was completely analyzed by Stoltzfus [16]. In all odd higher dimensions ψ_{2n-1} is injective, leading to decomposition theorems for knot concordance groups. In the classical dimension, $n = 1$, the map ψ_1 is not injective [2]; the kernel is infinitely generated and contains the subgroup $\mathcal{T}$.

To briefly summarize the perspective of Cha's work, we let $\mathcal{Q} \subset \mathbb{Z}[t]$ be the set of irreducible polynomials $q(t)$ satisfying $q(1) = 1$. Let

$$\mathcal{P} = \{q(t)q(t^{-1}) \in \mathbb{Z}[t, t^{-1}] \bigm| q(t) \in \mathcal{Q}\}.$$

According to Fox and Milnor [6], if a knot K is smoothly slice, then its Alexander polynomial is a product of elements in $\mathcal{P}$. The same result holds for topologically locally flat slice knots, as proved using the existence of normal bundles for locally flat disks (see Freedman-Quinn [7, Section 9.3]). According to Terasaka [17], every product of elements in $\mathcal{P}$ is the Alexander polynomial of some slice knot.

Given any subset $\mathcal{P}_0 \subset \mathcal{P}$, let $\mathcal{T}^{\mathcal{P}_0} \subset \mathcal{T}$ denote the subgroup generated by topologically slice knots with Alexander polynomial a product of polynomials p for $p \in \mathcal{P}_0$. In the case that $\mathcal{P}_0$ is a singleton $\{p\}$, we write $\mathcal{T}^p$. Hence, as above, $\mathcal{T}^1$ denotes the subgroup generated by knots with Alexander polynomial one. Notice that for any pair of elements $p, q \in \mathcal{P}$, we have $\mathcal{T}^1 \subset \mathcal{T}^p \cap \mathcal{T}^q$. Thus, in the following question (which is closely related to a series of problems and conjectures made in [3]) it is necessary to consider the quotients $\mathcal{T}^p_\Delta = \mathcal{T}^p/\mathcal{T}^1$ and $\mathcal{T}_\Delta = \mathcal{T}/\mathcal{T}^1$.

Question 1. Do the canonical homomorphisms $\mathcal{T}_\Delta^p \to \mathcal{T}_\Delta$ induce an isomorphism

$$\Phi\colon \bigoplus_{p\in\mathcal{P}} \mathcal{T}_\Delta^p \to \mathcal{T}_\Delta?$$

In [3], Cha identifies and studies a specific infinite set $\mathcal{P}_0 \subset \mathcal{P}$ with two properties: first, for all $p \in \mathcal{P}_0$, he proves that $\mathcal{T}_\Delta^p$ contains an infinitely generated free subgroup $\mathcal{S}_\Delta^p$; second, he proves that the restriction of Φ is injective on $\oplus_{p\in\mathcal{P}^0}\mathcal{S}_\Delta^p$.

The main goal of this paper is to provide a counterexample to a splitting property related to the surjectivity of Φ, considered by Cha under the name *strong existence* (see [3, Appendix A]). Although this does not provide a complete answer to the Question 1, it adds strong evidence that the answer is "no." More specifically, it indicates that Φ is probably not surjective.

Theorem 1.1. *There exists a set of three polynomials, $\mathcal{P}_0 = \{f_1, f_2, f_3\} \subset \mathcal{P}$, such that the natural homomorphism*

$$\mathcal{T}_\Delta^{f_1} \oplus \mathcal{T}_\Delta^{f_2} \oplus \mathcal{T}_\Delta^{f_3} \to \mathcal{T}_\Delta^{\mathcal{P}_0}$$

is not surjective.

The use of three factors is an artifact of the proof. It will be clear that without the restriction of irreducibility for elements in $\mathcal{Q}$, we could have used two factors, as was stated in the abstract. To be more precise, there is the following statement.

Theorem 1.2. *There exist Alexander polynomials $f_1(t)$ and $f_2(t)$ having no common factors and a topologically slice knot K with $\Delta_K(t) = f_1(t)^2 f_2(t)^2$ such that K is not concordant to any connected sum of knots $K_1 \# K_2$ where $\Delta_{K_i}(t) = f_i(t)^{n_i}$ and $n_1, n_2 \in \mathbb{Z}$.*

Notice that it follows that K is a topologically slice knot that is not smoothly concordant to a knot with Alexander polynomial one. The first examples of such knots were described in [10]. The example and proof here are closely related to that earlier work.

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2. Rational homology cobordism and an example

The proof of Theorem 1.1 can be reduced to a result concerning rational homology cobordism, as we now describe. Recall first that the Alexander polynomial of a knot determines the order of the first homology of $M(K)$, the 2–fold cyclic branched cover of S^3 branched over K: $\left|H_1(M(K))\right| =$

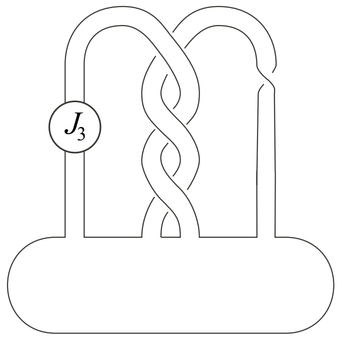

Figure 1. Knot

$|\Delta_K(-1)|$. Also, 2–fold branched covers of concordant knots are rationally homology cobordant.

Figure 1 is a schematic diagram of a topologically slice knot K_n, illustrated in the case of $n = 3$. The value of n will be chosen to satisfy $n \equiv 3 \mod 4$ and the knot J_n will be the connected sum of $(3n-1)/4$ copies of the positive clasped untwisted double of the trefoil knot, $D(T(2,3))$. As illustrated, K_n is the boundary of a punctured Klein bottle, built by adding two bands to a disk. In the figure, the right band has one half-twist; the left band has the knot J_n tied in it and is untwisted, meaning that a simple closed curve on the surface that passes over the band once, and does not go over the band on the right, links its push-off 0 times. There are n half-twist between the two bands. Formally, K_n is a satellite of J_n, with the satellite operation depending on the parameter n. Similar knots were used in [10] to prove the nontriviality of $\mathcal{T}_\Delta = \mathcal{T}/\mathcal{T}^1$. A quick calculation in Section 4 will show that if $n = pq$ for odd primes p and q, then

$$\Delta_{K_n}(t) = \big(\phi_{2p}(t)\phi_{2q}(t)\phi_{2pq}(t)\big)^2,$$

where $\phi_k(t)$ denotes the k–cyclotomic polynomial.

Theorem 2.1. *If $n = pq$, then the homology of the 2–fold branched cover of K_n satisfies $|H_1(M(K_n))| = n^2$. If K_n is concordant to a connected sum $L_1 \# L_2 \# L_3$, with $\Delta_{L_1}(t) = \phi_{2p}(t)^{m_1}$, $\Delta_{L_2}(t) = \phi_{2q}(t)^{m_2}$, and $\Delta_{L_3}(t) = \phi_{2pq}(t)^{m_3}$, then $M(K_n)$ is rationally homology cobordant to a connected sum $M_1 \# M_2 \# M_3$, where $|H_1(M_1)| = p^{m_1}$, $|H_1(M_2)| = q^{m_2}$ and $|H_1(M_3)| = 1$.*

Proof. These all follow immediately from the facts that $\phi_{2p}(-1) = p$, $\phi_{2q}(-1) = q$, and $\phi_{2pq}(-1) = 1$. See Lemma 4.2 for details. □

The topic of [11] was the general problem of finding a primary splitting of the rational homology cobordism group. We will show that the techniques used there can be applied to prove that for $n = 15$, a rational homology cobordism from $M(K_n)$ to such a connected sum, $M_1 \# M_2 \# M_3$, does not exist. Notice that if it did exist, we could let $N_2 = M_2 \# M_3$, and reduce our work to obstructing the existence of a rational homology cobordism to a connected sum of two manifolds, $M_1 \# N_2$. Our goal will be to prove that $M(K_n)$ is not rational homology cobordant to any connected sum $M_1 \# M_2$, where $\big|H_1(M_1)\big| = 3^{m_1}$ and $\big|H_1(M_2)\big| = 5^{m_2}$ for some integers m_1 and m_2.

3. Obstructions from d–invariants

We use the following notation: for any abelian group G and prime integer p, let $G_{(p)}$ denote the subgroup consisting of all elements of order p^n for some n.

All the three-manifolds M we will be working with are $\mathbb{Z}_2$–homology three-spheres. Since $H_1(M)$ is of odd order, there are natural identifications: $H_1(M) \cong H^2(M) \cong \mathrm{Spin}^c(M)$. For $g \in H_1(M)$ or $g \in H^2(M)$ we denote by $d(K, g)$ the Heegaard Floer correction term associated to g viewed as a Spin^c–structure. Basic results concerning $\mathrm{Spin}^c(M)$, the d–invariant, and its basic properties are in [13]. Further details and examples are provided in [11].

If M bounds a rational homology four-ball, then there is a subgroup $\mathcal{M} \subset H^2(M)$ such that: $\big|\mathcal{M}\big|^2 = \big|H^2(M)\big|$; the nonsingular linking form vanishes on $\mathcal{M}$; and $d(M, g) = 0$ for all $g \in \mathcal{M}$. (The linking form is usually defined on the first homology; it can be viewed as a form on the second cohomology via the natural isomorphism $H_1(M) \cong H^2(M)$. In general, $\mathcal{M}$ can be viewed as a subgroup of either $H_1(M)$ or $H^2(M)$.)

Theorem 3.1. *Let p and q be distinct odd primes and let M be a three-manifold satisfying $H_1(M) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{q^2}$, generated by elements a and b of order p^2 and q^2, respectively. If M is rationally homology cobordant to a connected sum $M_1 \# M_2$ where $H_1(M_1)_{(q)} = 0$ and $H_1(M_2)_{(p)} = 0$, then the value of*

$$d(M, ipa + jqb) - d(M, ipa) - d(M, jqb)$$

is independent of i and j.

Proof. Since M and $M_1 \# M_2$ are rationally homology concordant, $M \# -M_1 \# -M_2$ bounds a rational homology 4–ball W. The image of $H^2(W)$ in $H^2(M \# -M_1 \# -M_2)$ is the desired subgroup $\mathcal{M}$ satisfying $\big|\mathcal{M}\big|^2 = \big|H^2(M \# -M_1 \# -M_2)\big|$. Furthermore, $\mathcal{M}$ is self-annihilating with respect to the nonsingular linking form. The order of a self-annihilating subgroup of a group G is of order at most $\sqrt{|G|}$; it follows that the subgroup $\mathcal{M}_{(p)} \subset H^2(M)_{(p)} \oplus H^2(-M_1)_{(p)}$ cannot be contained in $H^2(-M_1)_{(p)}$. In particular, some element of the form $(x_p, y_p) \in H^2(M)_{(p)} \oplus H^2(-M_1)_{(p)}$ with $x_p \neq 0 \in \mathbb{Z}_{p^2}$ is contained in $\mathcal{M}$. By taking a multiple, we can assume $x_p = pa$. Similarly, there is an element $(qb, y_q) \in H^2(M) \oplus H^2(-M_2)$ in $\mathcal{M}_{(q)}$.

Notice that we are viewing $H^2(-M_1) \subset H^2(-M_1) \oplus H^2(-M_2)$, so in this sense y_p can be interpreted as an ordered pair $(y_p, 0) \in H^2(-M_1) \oplus H^2(-M_2)$; similarly, y_q represents an ordered pair $(0, y_q) \in H^2(-M_1) \oplus$

$H^2(-M_2)$. The correction term is additive under connected sum and vanishes for elements in $\mathcal{M}$. Thus for any i and j:

$$d(M, ipa) - d(M_1, iy_p) - d(M_2, 0) = 0,$$

$$d(M, jqb) - d(M_1, 0) - d(M_2, jy_q) = 0,$$

and

$$d(M, ipa + jqb) - d(M_1, iy_p) - d(M_2, jy_q) = 0.$$

Subtracting the first two equations from the third shows that for all i and j,

$$d(M, ipa + jqb) - d(M, ipa) - d(M, jqb) = -d(M_1, 0) - d(M_2, 0).$$

The right hand side is independent of i and j. □

4. The knots K_n and their Alexander polynomials

As described in Section 2, we are considering the knots K_n illustrated schematically in Figure 1.

Theorem 4.1. *For n odd, the Alexander polynomial of K_n is given by*

$$\Delta_{K_n}(t) = \left(\frac{t^n + 1}{t + 1}\right)^2.$$

Proof. This knot is a winding number two satellite of J_n. Since J_n has Alexander polynomial one, a standard formula for the Alexander polynomial of a satellite knot [15] implies that the Alexander polynomial of K_n is the same as what it would be if the knot J_n were replaced with the unknot in constructing K_n. In this case, a simple manipulation shows that $K_n = P(n, -n, n-1)$, a three-stranded pretzel knot, illustrated in the case of $n = 3$ in Figure 2. To compute its Alexander polynomial, we consider instead the Conway polynomial $\nabla_{K_n}(z)$. (Recall that for an arbitrary oriented knot K, $\Delta_K(t) = \nabla_K(t^{1/2} - t^{-1/2})$.)

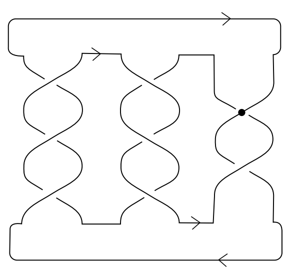

FIGURE 2. The pretzel knot $P(3, -3, 2)$

The standard crossing change formula for the Conway polynomial is

$$\nabla_+(z) - \nabla_-(z) = -z\nabla_s(z),$$

where $\nabla_{\pm}$ denotes the Conway polynomial of an oriented link with a specified crossing made positive or negative and ∇_s is the Conway polynomial of the link formed by smoothing that same crossing. This can be applied to a crossing on the right-most band of the pretzel knot, indicated by the dot in Figure 2. Smoothing the crossing yields an unlink, which has 0 Conway polynomial. Changing the crossing removes two half-twists. Thus, $\nabla_{P(n,-n,n-1)}(z) = \nabla_{P(n,-n,n-3)}(z)$. Since n is odd, continuing in this way removes the right-most crossings, ultimately yielding the connected sum $-T(2,n)\#T(2,n)$. The Alexander polynomials of torus knots is well-known; in this case it is

$$\Delta_{T(2,n)}(t) = \frac{(t^{2n}-1)(t-1)}{(t^2-1)(t^n-1)} = \frac{t^n+1}{t+1}.$$

□

Lemma 4.2. *For distinct odd primes p and q, there is the following identity, where the ϕ_i are cyclotomic polynomials.*

$$\frac{t^{pq}+1}{t+1} = \phi_{2p}(t)\phi_{2q}(t)\phi_{2pq}(t).$$

Furthermore, $\phi_{2p}(-1) = p$, $\phi_{2q}(-1) = q$, and $\phi_{2pq}(-1) = 1$.

Proof. The polynomial $t^n - 1$ has factors $\phi_d(t)$ for all divisors d of n. Thus

$$t^{2pq} - 1 = \phi_{2pq}(t)\phi_{2p}(t)\phi_{2q}(t)\phi_p(t)\phi_q(t)\phi_2(t)\phi_1(t) \tag{1}$$

and

$$t^{pq} - 1 = \phi_{pq}(t)\phi_p(t)\phi_q(t)\phi_1(t).$$

Dividing the first equation by the second, and then dividing by $\phi_2(t) = t+1$ yields

$$\frac{t^{pq}+1}{t+1} = \phi_{2pq}(t)\phi_{2p}(t)\phi_{2q}(t).$$

L'Hopital's rule can be used to determine that the left hand side evaluated at -1 is pq. Thus, if we show $\phi_{2p}(-1) = p$, and, similarly, $\phi_q(-1) = q$, we are done. Proceeding as before,

$$t^{2p} - 1 = \phi_{2p}(t)\phi_p(t)\phi_2(t)\phi_1(t)$$

and

$$t^p - 1 = \phi_p(t)\phi_1(t).$$

Dividing yields

$$\frac{t^p+1}{t+1} = \phi_{2p}(t).$$

In this case, L'Hopital's rule shows that $\phi_{2p}(-1) = p$. □

Corollary 4.3. *If $n = pq$, where p and q are distinct odd primes, then $\Delta_{K_n}(t) = \big(\phi_{2p}(t)\phi_{2q}(t)\phi_{2pq}(t)\big)^2$, where the $\phi_k(t)$ are cyclotomic polynomials. For r an odd prime, $\phi_{2r}(-1) = r$ and for a product of two distinct odd primes, $\phi_{2pq}(-1) = 1$.*

5. Computing $d(M(K), i)$; the completion of the proof of Theorem 1.1

We will now restrict to the case of $n = 15$. The methods of [1] apply to show that the 2–fold branched cover of S^3 branched over K_n is given by 15^2–surgery on $T_{14,15} \# 22D(T_{2,3})$. We denote its 2–fold cover of S^3 branched over K_{15} simply by M. Note that $H^2(M) \cong H_1(M) \cong \mathbb{Z}_{15^2}$.

Suppose now that K_{15} were concordant to a connected sum of three knots, one with Alexander polynomial $\phi_6(t)^{m_1}$, one with Alexander polynomial $\phi_{10}(t)^{m_2}$, and one with Alexander polynomial $\phi_{30}(t)^{m_3}$. Then, as described in the introduction, M would by rationally homology cobordant to $M_1 \# M_2$, where $H_1(M_1)_{(5)} = 0$ and $H_1(M_2)_{(3)} = 0$. Thus, Theorem 3.1 would apply.

In [10, Section 6], an algorithm is presented for computing the values of the d–invariants of $M(K_n)$. Here is the results of the computation; readers are referred to [10] for general background. In Appendix A we provide a summary of the details of this specific computation. Since $H^2(M) \cong \mathbb{Z}_{225}$, the order 9 subgroup is generated by $a = 25$ and the order 25 subgroup is generated by $b = 9$. The values that result from the computation of $d(M, ipa + jqb)$ are as shown in Table 1.

	$i=0$	$i=1$	$i=2$
$j=0$	16	6	6
$j=1$	10	2	12
$j=2$	4	8	14
$j=3$	4	14	8
$j=4$	10	12	2

Table 1. Values of $d(M, ipa + jqb)$

The values of $d(M, ipa+jqb) - d(M, ipa) - d(M, jqb)$ are as shown (with sign reversed for readability) in Table 2; they are not all equal.

	$i=0$	$i=1$	$i=2$
$j=0$	16	16	16
$j=1$	16	14	4
$j=2$	16	2	-4
$j=3$	16	-4	2
$j=4$	16	4	14

Table 2. Values of $-\big(d(M, ipa + jqb) - d(M, ipa) - d(M, jqb)\big)$

5.1. Infinite families. Let $\{p_i\}$ be an infinite increasing sequence of primes for which $p_i \equiv 3 \mod 4$ if and only if i is odd. Let

$$\mathcal{P}_0 = \cup_{i=1}^{\infty}\{\phi_{2p_i-1}(t), \phi_{2p_i}(t), \phi_{2p_{2i-1}p_{2i}}(t)\}.$$

Most of the previous argument is easily generalized. The only step that we have not been able to complete in general is the computation of the d–invariants. Our expectation is that this would lead to the conclusion that

$$\bigoplus_{p\in\mathcal{P}_0} \mathcal{T}_{\Delta}^{p} \to \mathcal{T}_{\Delta}^{\mathcal{P}_0}$$

is not surjective.

Appendix A. Computation of d–invariants

Here we will describe the computation of the d–invariants for the 2–fold branched cover of S^3 branched over K_{15}. A related example was presented in [10, Section 6] with further background material but lacking a few of the details that we provide here.

For this knot, recall that J_{15} is the connected sum of 11 copies of the untwisted Whitehead double of the trefoil knot, $Wh(T(2,3))$. The 2–fold branched cover, which we denote by M_{15}, can be described as 15^2 surgery on the knot $L = T(14,15)\,\#_{22}\,Wh(T(2,3))$. That is, $M_{15} = S^3_{15^2}(L)$.

Our goal is to compute the d–invariants $d(M_{15}, m)$, which in [10] were denoted $d(M_{15}, \mathfrak{s}_m)$. The first step is to determine the Heegaard Floer knot complex $\mathrm{CFK}(L)$. This is a chain complex with coefficients in $\mathbb{F}$, the field with two elements. It is $\mathbb{Z}$–graded, supports two increasing filtrations, and is a free $\mathbb{F}[U, U^{-1}]$–module. The action of U lowers gradings by 2 and filtration levels by 1.

According to [14], complexes of connected sums of knots are the tensor products of the corresponding complexes for the individual knots, so we first need to describe $\mathrm{CFK}(T(14,15))$ and $\mathrm{CFK}(\#_{22}Wh(T(2,3)))$. In [10] it is shown that these complexes are of the form $\big(C_1 \otimes \mathbb{F}[U,U^{-1}]\big) \oplus A_1$ and $\big(C_2 \otimes \mathbb{F}[U,U^{-1}]\big) \oplus A_2$, where A_1 and A_2 are acyclic. The acyclic summands do not affect the value of the d–invariant of surgery on the knots, so can be ignored. Both C_1 and C_2 are *stairway* complexes; in particular, they are freely generated by elements of grading 0 and of grading 1. Each has one dimensional homology, and that homology is at grading 0. All the grading 0 generators are homologous cycles.

For the complex C_1, the grading 0 generators have bifiltration levels given by the following set, along with the symmetric values; for example, since $(0,91)$ is listed, there is also a generator at bifiltration level $(91,0)$. There are 14 generators; the following seven and their reflections:

$$\{(0,91),(1,78),(3,66),(6,55),(10,45),(15,36),(21,28)\}.$$

The corresponding list of the 23 generators of C_2 are given in the following list, where we present one element from each symmetric pair:

$$\{(0,22),(1,21),(2,20),(3,19),(4,18),(5,17),(6,16),(7,15),\\ (8,14),(9,13),(10,12),(11,11)\}.$$

The tensor product of the two complexes has $14 \times 23 = 322$ generators of grading 0, all of which are cycles representing the generator of homology. The set of all bifiltration levels of these generators is formed by taking all possible sums of the bifiltration levels from each set. Call the set of these bifiltration levels $\mathcal{S}$.

In [10, Theorem 5.3, Section 6], it is described how the value of d–invariant $d(M,m)$ is computed using these generators. Here is a concise summary. For any m satisfying $|m| \leq 112$, for each generator at filtration level (α,β), one computes the value of the function $\Psi(\alpha,\beta)$ defined by

$$\Psi(\alpha,\beta) = \begin{cases} \beta - m, & \text{if } \beta - \alpha \geq m, \\ \alpha, & \text{if } \beta - \alpha < m. \end{cases}$$

Next, one lets

$$\delta_m(\mathcal{S}) = \min\{\Psi(s) \bigm| s \in \mathcal{S}\}.$$

The next result presents the final result that is needed to complete the computation.

Theorem A.1. *For m satisfying $|m| \leq 112$,*

$$d(M,m) = -2\delta_m(\mathcal{S}) - \frac{-(2m-225)^2 + 225}{(4)(225)}.$$

With these results, the computer computations of the values in Table 1 and Table 2 are straightforward.

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