

# On $BT_1$ group schemes and Fermat curves

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ABSTRACT. Let  $p$  be a prime number and let  $k$  be an algebraically closed field of characteristic  $p$ . A  $BT_1$  group scheme over  $k$  is a finite commutative group scheme which arises as the kernel of  $p$  on a  $p$ -divisible (Barsotti–Tate) group. We compare three classifications of  $BT_1$  group schemes, due in large part to Kraft, Ekedahl, and Oort, and defined using words, canonical filtrations, and permutations. Using this comparison, we determine the Ekedahl–Oort types of Fermat quotient curves and we compute four invariants of the  $p$ -torsion group schemes of these curves.

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## 1. Introduction

Fix a prime number  $p$  and let  $k$  be an algebraically closed field of characteristic  $p$ . Suppose  $C$  is a smooth irreducible projective curve of genus  $g$  over  $k$ . Its Jacobian  $\text{Jac}(C)$  is a principally polarized abelian variety of dimension  $g$ . The  $p$ -torsion group scheme  $G = \text{Jac}(C)[p]$  is a polarized  $BT_1$  group

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scheme of rank  $p^{2g}$ . The isomorphism class of  $G$  is uniquely determined by the de Rham cohomology of  $C$  [Oda69].

An important way to describe the isomorphism class of  $G$  is via a combinatorial invariant called the *Ekedahl–Oort type*, or E–O type. The E–O type is a sequence  $[\psi_1, \dots, \psi_g]$  of integers such that  $\psi_i - \psi_{i-1} \in \{0, 1\}$  for  $1 \leq i \leq g$  (letting  $\psi_0 = 0$ ). The  $p$ -rank and  $a$ -number of  $G$  can be quickly computed from the E–O type. The E–O type gives key information about the stratification of the moduli space of principally polarized abelian varieties of dimension  $g$ .

Recently, there has been a lot of interest in studying  $p$ -torsion group schemes for curves, for example [Moo04], [EP13], [PW15], [DH], and [Moo20]. Despite this, there are very few examples of curves for which the Ekedahl–Oort type has been computed. In this paper, our main result is Theorem 7.1, in which we determine the Ekedahl–Oort type for the Jacobian  $J_d$  of the smooth projective curve  $\mathcal{C}_d$  with affine equation  $y^d = x(1-x)$ , for all positive integers  $d$  that are relatively prime to  $p$ .

Here is our motivation for studying the curve  $\mathcal{C}_d$ . First, it is a quotient of the Fermat curve  $F_d$  of degree  $d$ . There has been a lot of work on  $p$ -divisible groups of Fermat curves. For example, Yui determined their Newton polygons [Yui80, Thm. 4.2]. Several authors studied the  $p$ -ranks and  $a$ -numbers of Fermat curves, e.g., [KoW88], [Gon97], and [MonS18]. It turns out that most of the interesting features of the  $p$ -torsion group scheme for  $F_d$  are already present for that of  $\mathcal{C}_d$ .

Second, the de Rham cohomology of  $\mathcal{C}_d$ , with its Frobenius and Verschiebung operators, has a streamlined description. Typically it is complicated to discern the Ekedahl–Oort type from a computation of these operators. In the case of  $\mathcal{C}_d$ , the actions of  $F$  and  $V$  are given by permutation data that is easy to analyze. For this reason, we are able to describe the mod  $p$  Dieudonné module of  $J_d[p]$  in several different ways and give a closed form formula for its Ekedahl–Oort type.

Third, in our companion paper [PU], we prove that every polarized  $BT_1$ -group scheme over  $k$  occurs as a direct factor of the  $p$ -torsion group scheme of  $J_d$  for infinitely many  $d$  as long as  $p > 3$ . In other words, the class of curves  $\mathcal{C}_d$  for  $p \nmid d$  includes all possible structures of  $p$ -torsion group schemes of principally polarized abelian varieties.

To prove Theorem 7.1, we rely on a detailed comparison of three classifications of  $BT_1$  group schemes, essentially due to Kraft, Ekedahl, and Oort. We develop this comparison in Sections 2 through 4, working in general, not restricting to Jacobians of curves. The Kraft classification uses words on a two-letter alphabet  $\{f, v\}$ ; it interacts well with direct sums and identifies the indecomposable objects in the category. The Ekedahl–Oort classification uses the interplay between  $F$  and  $V$  to build a “canonical filtration” and is well suited to moduli-theoretic questions. The third classification is given in terms of a finite set  $S$  with a permutation and a partition into two subsets,

and it is particularly well suited to studying Fermat curves. The material in these sections is fundamental to the proof of the theorem and does not appear in a self-contained way elsewhere in the literature.

In Section 5, we study homomorphisms between  $BT_1$  group schemes to analyze two well-known invariants, the  $p$ -rank and  $a$ -number, and two newer invariants related to the  $p$ -torsion group scheme of a supersingular elliptic curve, called the  $s_{1,1}$ -multiplicity and  $u_{1,1}$ -number.

In Section 6, we recall two results about the  $BT_1$  modules of Fermat curves and their quotients from [PU]. In Section 7, we prove the main result about the Ekedahl–Oort type of  $\mathcal{C}_d$  for all positive integers  $d$  relatively prime to  $p$ .

In the rest of the paper, we provide explicit examples of the Ekedahl–Oort structure of  $\mathcal{C}_d$  and the four associated invariants under various conditions on  $d$  and  $p$ . In Section 8, we separate out the case  $p = 2$ . In Section 9, we determine the  $a$ -number of  $\mathcal{C}_d$  for all  $d$  relatively prime to  $p$ . In Sections 10 and 11, we analyze the cases  $d = p^\ell - 1$  and  $d = p^\ell + 1$ , for any natural number  $\ell$ . We call  $d = p^\ell - 1$  the “encompassing case”; it is in some sense the general case because for every  $d'$ , the group scheme  $\text{Jac}(\mathcal{C}_{d'})[p]$  is a direct factor of  $\text{Jac}(\mathcal{C}_d)[p]$  where  $d = p^\ell - 1$  for some  $\ell$ .

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## 2. Groups and modules

In this section, we review certain categories of group schemes and their Dieudonné modules.

### 2.1. Group schemes of $p$ -power order and their Dieudonné modules.

Our general reference for the assertions in this section is [Fon77].

Let  $W(k)$  denote the Witt vectors over  $k$ . Write  $\sigma$  for the absolute Frobenius of  $k$ , and extend it to  $W(k)$  by  $\sigma(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$ . Define the *Dieudonné ring*  $\mathbb{D} = W(k)\{F, V\}$  as the  $W(k)$ -algebra generated by symbols  $F$  and  $V$  with relations

$$FV = VF = p, \quad F\alpha = \sigma(\alpha)F, \quad \text{and} \quad \alpha V = V\sigma(\alpha) \quad \text{for } \alpha \in W(k). \quad (2.1)$$

Let  $\mathbb{D}_k = \mathbb{D}/p\mathbb{D} \cong k\{F, V\}$ .

Let  $G$  be a finite commutative group scheme of order  $p^\ell$  over  $k$ . Let  $M(G)$  be the (contravariant) Dieudonné module of  $G$ . This is a  $W(k)$ -module of length  $\ell$  with semi-linear operators  $F$  and  $V$ .

Let  $G^D$  be the Cartier dual of  $G$ . For  $M$  a  $\mathbb{D}$ -module of finite length over  $W(k)$ , let  $M^*$  be its dual module. A basic result of Dieudonné theory is that  $M(G^D) \cong M(G)^*$ .

**2.2.  $BT_1$  group schemes and  $BT_1$  modules.** By definition, a  $BT_1$  group scheme over  $k$  is a finite commutative group scheme  $G$  that is killed by  $p$  and that has the properties

$$\text{Ker}(F : G \rightarrow G^{(p)}) = \text{Im}(V : G^{(p)} \rightarrow G)$$

and

$$\text{Im}(F : G \rightarrow G^{(p)}) = \text{Ker}(V : G^{(p)} \rightarrow G).$$

The notation  $BT_1$  is an abbreviation of “Barsotti–Tate of level 1” reflecting the fact [Ill85, Prop. 1.7] that  $BT_1$  group schemes are precisely those which occur as the kernel of  $p$  on a Barsotti–Tate ( $=p$ -divisible) group.

By definition, a  $BT_1$  module over  $k$  is a  $\mathbb{D}_k$ -module  $M$  of finite dimension over  $k$  such that

$$\text{Ker}(F : M \rightarrow M) = \text{Im}(V : M \rightarrow M)$$

and

$$\text{Im}(F : M \rightarrow M) = \text{Ker}(V : M \rightarrow M).$$

(Oort also calls these  $DM_1$  modules.) Clearly, a  $\mathbb{D}_k$ -module  $M$  is a  $BT_1$  module if and only if it is the Dieudonné module of a  $BT_1$  group scheme over  $k$ .

The group schemes  $\mathbb{Z}/p\mathbb{Z}$ ,  $\mu_p$ , and  $G_{1,1}^1$  are  $BT_1$  group schemes. On the other hand,  $\alpha_p$  is not, since  $\text{Ker } F = M(\alpha_p) \neq 0 = \text{Im } V$ .

A  $BT_1$  group scheme  $G$  is *self-dual* if there exists an isomorphism  $G \cong G^D$ . Similarly, a  $BT_1$  module  $M$  is *self-dual* if  $M \cong M^*$ . Clearly,  $G$  is self-dual if and only if  $M(G)$  is self-dual.

One may ask that a duality  $\phi : G \rightarrow G^D$  be skew, meaning that  $\phi^D : G \cong (G^D)^D \rightarrow G^D$  satisfies  $\phi^D = -\phi$ . This is equivalent to any of the following three conditions:

- the bilinear pairing  $G \times G \rightarrow \mathbb{G}_m$  induced by  $\phi$  is skewsymmetric;
- the duality  $M(\phi) : M(G)^* \rightarrow M(G)$  is skew, meaning that  $M(\phi)^* = -M(\phi)$ ;
- the induced pairing  $M(G)^* \times M(G)^* \rightarrow k$  is skewsymmetric.

Interestingly, when  $p = 2$ , there exist  $BT_1$  group schemes  $G$  with an *alternating* pairing ( $\langle x, x \rangle = 0$  for all  $x$ ) such that the induced pairing on  $M(G)$  is skewsymmetric but not alternating.

For this reason, one defines a *polarized  $BT_1$  module* over  $k$  as a  $BT_1$  module over  $k$  with a non-degenerate, alternating pairing, and one defines a *polarized  $BT_1$  group scheme* over  $k$  as a  $BT_1$  group scheme  $G$  over  $k$  with a pairing that induces a non-degenerate, alternating pairing on  $M(G)$ . Corollary 4.2 says that every self-dual  $BT_1$  module admits a polarization which is unique up to isomorphism.

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<sup>1</sup>the kernel of  $p$  on a supersingular elliptic curve

If  $A$  is a principally polarized abelian variety of dimension  $g$  over  $k$ , its  $p$ -torsion subscheme  $A[p]$  is naturally a polarized  $BT_1$  group scheme of order  $p^{2g}$ .

### 3. Review of classifications of $BT_1$ group schemes

In this section, we review bijections between isomorphism classes of  $BT_1$  modules over  $k$  and three other classes of objects of combinatorial nature. More precisely, following Kraft [Kra]<sup>2</sup> Ekedahl, and Oort [Oor01], we will construct a diagram of isomorphism classes:

$$\begin{array}{ccc}
 BT_1 \text{ modules} & \xrightarrow{\sim} & \text{canonical types} & (3.1) \\
 \uparrow \sim & & \downarrow \sim & \\
 \text{multisets of (primitive)} & \longleftrightarrow & \text{(admissible) permutations} & \\
 \text{cyclic words on } \{f, v\} & & \text{of } S = S_f \cup S_v. & 
 \end{array}$$

The top horizontal map is the Ekedahl–Oort classification of  $BT_1$  modules and the left vertical map is the Kraft classification. While some of the material in this section is known, there are significant reasons to cover it, and we take the opportunity to correct a few minor imprecisions in the literature. First, it is helpful to have a self-contained short description of this material. Second, we need a precise dictionary between the classification on the lower right of the diagram, given in terms of permutations, and the others. The permutation classification is not as well known and is particularly well suited to studying the  $p$ -torsion group schemes of quotients of Fermat curves. Third, the work of Oort uses covariant Dieudonné theory, but the contravariant theory is more convenient for studying Fermat curves.

There are other classifications of  $BT_1$  modules, one due to Moonen [Moo01], involving cosets of Weyl groups; and another due to van der Geer [vdG99, §6] in terms of Young diagrams and partitions. We will not need this material, so we omit any further discussion.

**3.1. Words and permutations.** In this section, we describe the bijection on the bottom row of Diagram (3.1). Let  $\mathcal{W}$  be the monoid of words  $w$  on the two-letter alphabet  $\{f, v\}$ , and write  $1$  for the empty word. By convention, the first (resp. last) letter of  $w$  is its leftmost (resp. rightmost) letter. The complement  $w^c$  of  $w$  is the word obtained by exchanging  $f$  and  $v$  at every letter.

For a positive integer  $\lambda$ , write  $\mathcal{W}_\lambda$  for the words of length  $\lambda$ . Endow  $\mathcal{W}_\lambda$  with the lexicographic ordering with  $f < v$ . If  $w \in \mathcal{W}_\lambda$ , we write  $w = u_{\lambda-1} \cdots u_0$  where  $u_i \in \{f, v\}$  for  $0 \leq i \leq \lambda - 1$ . Define an action of the group  $\mathbb{Z}$  on  $\mathcal{W}$  by requiring that  $1 \in \mathbb{Z}$  map  $w = u_{\lambda-1} \cdots u_0$  to  $u_0 u_{\lambda-1} \cdots u_1$ . If  $w$  and  $w'$  are in the same orbit of this action, we say  $w'$  is a *rotation* of  $w$ .

<sup>2</sup>Kraft’s manuscript is unpublished and, as far as we know, not available online. However, the results from Kraft that we use are reestablished in [Oor01].

The orbit  $\bar{w}$  of  $w$  under the action of  $\mathbb{Z}$  is called a *cyclic word*. Write  $\bar{\mathcal{W}}$  for the set of cyclic words.

A word  $w$  is *primitive* if  $w$  is not of the form  $(w')^e$  for some word  $w'$  and some integer  $e > 1$ . If  $w$  has length  $\lambda > 0$ , it is primitive if and only if the subgroup of  $\mathbb{Z}$  fixing  $w$  is exactly  $\lambda\mathbb{Z}$ . Write  $\mathcal{W}'$  for the set of primitive words and  $\bar{\mathcal{W}}'$  for the set of primitive cyclic words. At the level of multisets, one can define a retraction of  $\bar{\mathcal{W}}' \subset \bar{\mathcal{W}}$  by sending the class of a word  $w = (w')^e$  where  $w'$  is primitive to the class of  $w'$  with multiplicity  $e$ .

Consider a finite set  $S$  written as the disjoint union  $S = S_f \cup S_v$  of two subsets and a permutation  $\pi : S \rightarrow S$ . Two such collections of data  $(S = S_f \cup S_v, \pi)$  and  $(S' = S'_f \cup S'_v, \pi')$  are isomorphic if there is a bijection  $\iota : S \rightarrow S'$  such that  $\iota(S_f) = S'_f$ ,  $\iota(S_v) = S'_v$ , and  $\iota\pi = \pi'\iota$ .

Given  $(S = S_f \cup S_v, \pi)$ , there is an associated multiset of cyclic words on  $\{f, v\}$  defined as follows. For  $a \in S$  with orbit of size  $\lambda$ , define the word  $w_a = u_{\lambda-1} \cdots u_0$  where

$$u_j = f \text{ if } \pi^j(a) \in S_f, \text{ and } u_j = v \text{ if } \pi^j(a) \in S_v.$$

Then  $\bar{w}_a$  depends only on the orbit of  $a$ . This gives a well-defined map from orbits of  $\pi$  to cyclic words, i.e., elements of  $\bar{\mathcal{W}}$ . Taking the union over orbits, we can associate to  $(S = S_f \cup S_v, \pi)$  a multiset of cyclic words. If  $S$  and  $S'$  are isomorphic, then they yield the same multiset.

Conversely, given a multiset of cyclic words, let  $S$  be the set of all words representing them (repeated to account for multiplicities), let  $S_f$  be the subset of those words ending with  $f$ , let  $S_v$  be the subset of those words ending with  $v$ , and let  $\pi$  be defined by the action of  $1 \in \mathbb{Z}$  as above.

The data  $(S = S_f \cup S_v, \pi)$  is *admissible* if the associated words  $w_a$  for  $a \in S$  are primitive.

**3.2. Cyclic words to  $BT_1$  modules.** Following Kraft [Kra], we attach a  $BT_1$  module to a multiset of primitive cyclic words. This defines the left vertical arrow of Diagram (3.1).

**3.2.1. Construction.** Suppose that  $w \in \mathcal{W}'$  is a primitive word, say  $w = u_{\lambda-1} \cdots u_0$  with  $u_j \in \{f, v\}$ . Let  $M(w)$  be the  $k$ -vector space with basis  $e_j$  with  $j \in \mathbb{Z}/\lambda\mathbb{Z}$  and define a  $p$ -linear map  $F : M(w) \rightarrow M(w)$  and a  $p^{-1}$ -linear map  $V : M(w) \rightarrow M(w)$  by setting

$$F(e_j) = \begin{cases} e_{j+1} & \text{if } u_j = f, \\ 0 & \text{if } u_j = v, \end{cases} \quad \text{and} \quad V(e_{j+1}) = \begin{cases} e_j & \text{if } u_j = v, \\ 0 & \text{if } u_j = f. \end{cases}$$

(Note that  $F(e_j) \neq 0$  exactly when the last letter of the  $j$ -th rotation of  $w$  is  $f$ , and  $V(e_{j+1}) \neq 0$  exactly when the last letter of the  $j$ -th rotation of  $w$  is  $v$ .) This construction yields a  $BT_1$  module which up to isomorphism only depends on the primitive cyclic word  $\bar{w}$  associated to  $w$ .

Kraft proves that  $M(w)$  is indecomposable and that every indecomposable  $BT_1$  module is isomorphic to one of the form  $M(w)$  for a unique primitive

cyclic word  $\bar{w}$ . Thus every  $BT_1$  module  $M$  is isomorphic to a direct sum  $\oplus M(w_i)$  where  $\bar{w}_i$  runs through a uniquely determined multiset of primitive cyclic words.

If  $w$  is a word that is not necessarily primitive, the formulas above define a  $BT_1$  module. If  $w = (w')^e$ , Kraft also proves that  $M(w) \cong M(w')^e$ .

It is clear that  $M(f) = M(\mathbb{Z}/p\mathbb{Z})$ ,  $M(v) = M(\mu_p)$ , and  $M(fv)$  is the Dieudonné module of the kernel of  $p$  on a supersingular elliptic curve. More generally, if  $w$  has length  $> 1$  and is primitive, then  $M(w)$  is the Dieudonné module of a unipotent, connected  $BT_1$  group scheme.

**3.2.2. Generators and relations.** Let  $w$  be a primitive word with associated  $BT_1$  module  $M(w)$ . It will be convenient to have a presentation of  $M(w)$  by generators and relations. Clearly,  $M(f) = \mathbb{D}_k/(F - 1, V)$  and  $M(v) = \mathbb{D}_k/(F, V - 1)$ .

Now suppose  $w$  has length  $> 1$ . Then, after rotating  $w$  if necessary, we may assume its last letter is  $f$  and its first letter is  $v$ . (Both letters appear because  $w$  is primitive, so in particular is not  $f^m$  nor  $v^n$ .) We then write  $w$  in exponential notation as

$$w = v^{n_r} f^{m_r} \dots v^{n_1} f^{m_1},$$

for some positive integers  $r, m_1, \dots, m_r, n_1, \dots, n_r$ .

**Lemma 3.1.** *The  $BT_1$  module  $M(w)$  admits generators  $E_i$  and relations  $F^{m_i} E_{i-1} = V^{n_i} E_i$  for  $i \in \mathbb{Z}/r\mathbb{Z}$ .*

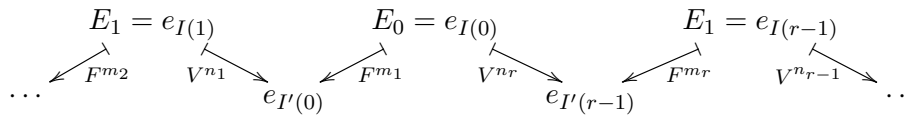
**Proof.** Indeed, for  $i = 0, \dots, r - 1$ , let  $I(i) = \sum_{j=1}^i (m_j + n_j)$ , let  $I'(i) = I(i) + m_{i+1}$ , and let  $E_i = e_{I(i)}$ . The  $E_i$  generate  $M(w)$  as a  $\mathbb{D}_k$ -module because

$$\text{if } I(i) \leq j \leq I'(i), \text{ then } e_j = F^{j-I(i)} E_i,$$

$$\text{and if } I'(i) \leq j \leq I(i+1), \text{ then } e_j = V^{j-I'(i)} E_{i+1},$$

and there are relations  $F^{m_i} E_{i-1} = e_{I'(i-1)} = V^{n_i} E_i$ . □

The following diagram illustrates this presentation of  $M(w)$ :



**3.3.  $BT_1$  modules to canonical types.** Following Oort [Oor01], we explain how to describe the isomorphism class of a  $BT_1$  module in terms of certain combinatorial data. This defines the top horizontal arrow in Diagram (3.1). Readers are invited to work through the example in Section 3.3.2 while reading this section.

Warning: Many of our formulas differ from those in [Oor01] for two reasons: first, we use the contravariant Dieudonné theory, whereas Oort uses the covariant theory; second, Oort studies a filtration defined by  $F^{-1}$  and  $V$ , whereas we use  $F$  and  $V^{-1}$ . The two approaches are equivalent (and

exchanged under duality), but the latter is more convenient for studying Fermat curves.

**3.3.1. The canonical filtration.** Recall that  $\mathcal{W}$  denotes the monoid of words on  $\{f, v\}$ . Let  $M$  be a  $BT_1$  module, and define a left action of  $\mathcal{W}$  on the set of  $k$ -subspaces of  $M$  by requiring that

$$fN := F(N) \text{ and } vN := V^{-1}(N).$$

In other words,  $f$  sends a subspace  $N$  to its image under  $F$  and  $v$  sends  $N$  to its inverse image under  $V$ . Note that if  $N_1 \subset N_2$ , then  $fN_1 \subset fN_2$  and  $vN_1 \subset vN_2$ . If  $N$  is a  $\mathbb{D}_k$ -module, so are  $fN$  and  $vN$ , and  $fN \subset N \subset vN$ . Also note that  $fM = \text{Im } F = \text{Ker } V = v0$ .

Let  $M$  be a  $BT_1$  module. An *admissible filtration* on  $M$  is a filtration by  $\mathbb{D}_k$ -modules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M, \quad (3.1)$$

such that for all  $i$ , there exist indices  $\phi(i)$  and  $\nu(i)$  such that  $fM_i = M_{\phi(i)}$  and  $vM_i = M_{\nu(i)}$ .

**Definition 3.2.** The *canonical filtration* on  $M$  is the coarsest admissible filtration on  $M$ . If Equation (3.1) is the canonical filtration, the *blocks* of  $M$  are  $B_i = M_{i+1}/M_i$  for  $0 \leq i \leq s-1$ .

The canonical filtration of  $M$  is constructed by enumerating all subspaces of  $M$  of the form  $wM$  and indexing them in order of containment. Define  $s$  to be the number of steps in the filtration and  $r$  to be the integer such that  $M_r = fM = v0$ . Define functions  $\phi$ ,  $\nu$ , and  $\rho$  by

$$\begin{aligned} \phi : \{0, \dots, s\} &\rightarrow \{0, \dots, r\}, & fM_i &= M_{\phi(i)}, \\ \nu : \{0, \dots, s\} &\rightarrow \{r, \dots, s\}, & vM_i &= M_{\nu(i)}, \end{aligned}$$

and

$$\rho : \{0, \dots, s\} \rightarrow \mathbb{Z}, \quad \rho(i) = \dim_k M_i.$$

This data has the following properties.

**Proposition-Definition 3.3.** *The data  $(r, s, \phi, \nu, \rho)$  associated to the canonical filtration of  $M$  is a canonical type, i.e.,  $s > 0$ ,  $0 \leq r \leq s$ , and the functions  $\phi$ ,  $\nu$ , and  $\rho$  have the following properties:*

- (1)  $\phi$  and  $\nu$  are monotone nondecreasing and surjective;
- (2)  $\rho$  is strictly increasing with  $\rho(0) = 0$ ;
- (3)  $\nu(i+1) > \nu(i)$  if and only if  $\phi(i+1) = \phi(i)$ ;
- (4) if the equivalent conditions in (3) are true, then  $\rho(i+1) - \rho(i) = \rho(\nu(i)+1) - \rho(\nu(i))$ , while if not, then  $\rho(i+1) - \rho(i) = \rho(\phi(i)+1) - \rho(\phi(i))$ ;
- (5) and every integer in  $\{1, \dots, s\}$  can be obtained by repeatedly applying  $\phi$  and  $\nu$  to  $s$ .



If the data  $(r, s, \phi, \nu, \rho)$  comes from the canonical filtration of a  $BT_1$  module, then it is clear from the definitions that  $0 \leq r \leq s$ , with  $s > 0$ , and that properties (1), (2), and (5) hold. Oort proves that properties (3) and (4) hold in [Oor01, §2].

The properties imply that  $\nu(i + 1) - \nu(i)$  and  $\phi(i + 1) - \phi(i)$  are either 0 or 1, that exactly one of them is 1, and that  $\nu(i) + \phi(i) = r + i$ . The next lemma will be used in later sections.

**Lemma 3.4.** [Oor01, Lemma 2.4] *Let (3.1) denote the canonical filtration of  $M$  and let  $B_i = M_{i+1}/M_i$  for  $0 \leq i \leq s - 1$ . If  $\phi(i + 1) > \phi(i)$  then  $F$  induces a  $p$ -linear isomorphism  $B_i \xrightarrow{\sim} B_{\phi(i)}$ , and if  $\nu(i + 1) > \nu(i)$ , then  $V^{-1}$  induces a  $p$ -linear isomorphism  $B_i \xrightarrow{\sim} B_{\nu(i)}$ .*

The key assertion is that the canonical type of  $M$  determines  $M$  up to isomorphism:

**Proposition 3.5.** *If the canonical types of two  $BT_1$  modules  $M, M'$  are equal, then  $M \cong M'$ .*

Oort proves a related result [Oor01, Thm. 9.4] with quasi-polarizations (pairings) which is more involved and only applies to self-dual  $BT_1$  modules. Moonen proves the result stated here [Moo01, §4] in the more general context where the module  $M$  also has endomorphisms by a semi-simple  $\mathbb{F}_p$ -algebra  $D$ ; taking  $D = \mathbb{F}_p$  yields Proposition 3.5.

*Remark 3.6.* Let  $\mu : \{0, \dots, s - 1\} \rightarrow \mathbb{Z}$  be defined by  $\mu(i) = \rho(i + 1) - \rho(i)$ . Equivalently,  $\mu(i) = \dim_k(B_i)$ . Property (2) says  $\mu$  takes positive values, and property (4) says if  $\nu(i + 1) > \nu(i)$ , then  $\mu(i) = \mu(\nu(i))$  and if  $\phi(i + 1) > \phi(i)$ , then  $\mu(i) = \mu(\phi(i))$ .

*Remark 3.7.* Oort defines a canonical type to be data as above satisfying properties (1)–(4), i.e., he omits (5), and he states [Oor01, Remark 2.8] that every canonical type comes from a  $BT_1$  module. With this definition, it is true that every canonical type comes from an admissible filtration on a  $BT_1$  module, but not necessarily from the *canonical* filtration. Here is a counterexample: Let  $r = 0, s = 2$  and  $\phi(i) = 0, \nu(i) = i$  and  $\rho(i) = i$  for  $i = 0, 1, 2$ . This data comes from a filtration on  $N = M((\mu_2)^2)$  and it satisfies properties (1)–(4), but not (5). The canonical type of  $N$  has  $r = 0, s = 1, \phi(i) = 0, \nu(i) = i$  for  $i = 0, 1$ , and  $\rho(0) = 0, \rho(1) = 2$ .

**3.3.2. An example.** Let  $M$  be the  $k$ -vector space with basis  $e_1, \dots, e_7$  and action of  $F, V$  given by

$e$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$F(e)$	0	0	0	$e_1$	$e_2$	0	$e_3$
$V(e)$	0	0	0	$e_1$	$e_2$	$e_3$	$e_6$

Using  $\langle \dots \rangle$  to denote the span of a set of vectors, the canonical filtration of  $M$  is

$$\begin{aligned} M_0 = 0 \subset M_1 = \langle e_1, e_2 \rangle \subset M_2 = \langle e_1, e_2, e_3 \rangle \subset M_3 = \langle e_1, e_2, e_3, e_4, e_5 \rangle \\ \subset M_4 = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle \subset M_5 = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle = M. \end{aligned}$$

The canonical type is given by  $s = 5$ ,  $r = 2$ , and the functions  $\phi, \nu, \rho$  below:

$i$	0	1	2	3	4	5
$\phi(i)$	0	0	0	1	1	2
$\nu(i)$	2	3	4	4	5	5
$\rho(i)$	0	2	3	5	6	7

**3.4. Canonical types to permutations.** Following [Oor01, Section 2], we explain how to use a canonical type to define a partitioned set  $S = S_f \cup S_v$  with permutation  $\pi : S \rightarrow S$ , thus defining the right vertical arrow of Diagram (3.1). These results cast considerable light on the structure of  $BT_1$  modules, but they will not be used explicitly in the rest of the paper.

Let  $(r, s, \phi, \nu, \rho)$  be as in Definition 3.3, and let  $\Gamma = \{0, \dots, s-1\}$ . Define  $\Pi : \Gamma \rightarrow \Gamma$  by:

$$\Pi(i) = \begin{cases} \phi(i) & \text{if } \phi(i+1) > \phi(i), \\ \nu(i) & \text{if } \nu(i+1) > \nu(i). \end{cases}$$

Property (3) of Definition 3.3 shows that  $\Pi$  is well defined. Property (1) implies that  $\Pi$  is injective and thus bijective. By Property (4),  $\mu(i) = \rho(i+1) - \rho(i)$  is constant on the orbits of  $\Pi$ .

We partition  $\Gamma$  as a disjoint union  $\Gamma_f \cup \Gamma_v$  where  $i \in \Gamma_f$  if and only if  $\pi(i) = \phi(i)$ . Equivalently,

$$\Gamma_f = \{i \in \Gamma \mid \phi(i+1) > \phi(i)\} \quad \text{and} \quad \Gamma_v = \{i \in \Gamma \mid \nu(i+1) > \nu(i)\}.$$

Then  $(r, s, \phi, \nu, \rho)$  is determined by the data  $\Gamma = \Gamma_f \cup \Gamma_v$ ,  $\Pi : \Gamma \rightarrow \Gamma$ , and  $\mu : \Gamma \rightarrow \mathbb{Z}$ .

**3.4.1. Example 3.3.2 continued.** In this case, the permutation  $\Pi$  of  $\Gamma = \{0, 1, 2, 3, 4\}$  is  $(0, 2)(1, 3, 4)$ , the partition is given by  $\Gamma_f = \{2, 4\}$  and  $\Gamma_v = \{0, 1, 3\}$ , and the associated words are

$$w_0 = fv, \quad w_1 = fvv, \quad w_2 = vf, \quad w_3 = vfv, \quad \text{and} \quad w_4 = vvf.$$

Note that  $\mu(0) = \mu(2) = 2$  and  $\mu(1) = \mu(3) = \mu(4)$ , so  $\mu$  is constant on the orbits of  $\Pi$ .

To complete the definition of the right vertical arrow of Diagram (3.1), we use  $\mu$  as a set of “multiplicities” to expand  $\Gamma$  into  $S$ . More precisely, define

$$S := \{e_{i,j} \mid i \in \Gamma, 1 \leq j \leq \mu(i)\}$$

with partition

$$S_f = \{e_{i,j} \in S \mid i \in \Gamma_f\} \quad \text{and} \quad S_v = \{e_{i,j} \in S \mid i \in \Gamma_v\}$$

and permutation  $\pi : S \rightarrow S$  with  $\pi(e_{i,j}) := e_{\Pi(i),j}$ .

**Lemma 3.8.** *The data  $(S = S_f \cup S_v, \pi)$  is an admissible permutation.*

**Proof.** The set of cyclic words associated to  $(S = S_f \cup S_v, \pi)$  is the same as the set of cyclic words associated to  $(\Gamma = \Gamma_f \cup \Gamma_v, \Pi)$ . We need to show that these words are primitive. If  $w = w_i$  for  $i \in \Gamma$ , then  $w_{\Pi^j(i)}$  is the  $j$ -th cyclic rotation of  $w_i$ . Thus to show that the  $w_i$  are all primitive, it suffices to show that they are distinct.

To that end, define a left action of the monoid  $\mathcal{W}$  on the set  $\{0, \dots, s\}$  by requiring that  $f(i) = \phi(i)$  and  $v(i) = \nu(i)$ . If  $i \in \Gamma$ , and if  $w_i$  is the word associated to  $i$ , then  $w_i$  fixes  $i$  and  $i + 1$ . This is a manifestation in the canonical type of the isomorphisms from Lemma 3.4:

$$B_i \xrightarrow{\sim} B_{\Pi(i)} \xrightarrow{\sim} B_{\Pi^2(i)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} B_i.$$

Now assume that  $i, j \in \Gamma$ ,  $i < j$ , and  $w_i = w_j = w$ . We will deduce a contradiction of property (5) in Definition 3.3. Since  $\phi$  and  $\nu$  are nondecreasing, for all  $n > 0$  we have

$$w^n(s) \geq i + 1 > i \geq w^n(0) \quad \text{and} \quad w^n(s) \geq j + 1 > j \geq w^n(0),$$

so  $w^n(s) \geq j + 1$  and  $i \geq w^n(0)$  for all  $n > 0$ .

Choose some  $i'$  with  $i < i' \leq j$ . By property (5) of Definition 3.3, there is a word  $w'$  with  $w'(s) = i'$ . Choose  $n > 0$  large enough that  $w^n$  is at least as long as  $w'$ , and then replace  $w'$  with  $w'v^m$  where  $m$  is chosen so  $w^n$  and  $w'$  have the same length. Since  $v(s) = \nu(s) = s$ , we still have  $w'(s) = i'$ . If  $w' \geq w^n$  (in the lexicographic order from Section 3.1), then  $i' = w'(s) \geq w^n(s) \geq j + 1$ , a contradiction; and if  $w' < w^n$ , then  $i' = w'(s) \leq w^n(0) \leq i$ , again a contradiction. We conclude that there can be no  $i < j$  with  $w_i = w_j$ , and thus  $\pi$  is admissible.  $\square$

*Remark 3.9.* A more thorough analysis along these lines shows that if  $M$  is a  $BT_1$  module, then there are finitely many primitive words  $w_i$  such that  $w_i^n M \supsetneq w_i^{n-1} M$  for all  $n > 1$ . Enumerate these as  $w_0, \dots, w_{s-1}$  and choose integers  $n_i$  so that  $w_i^{n_i} M = w_i^{n_i-1} M$  and  $w_i^{n_i} M \supsetneq w_i^{n_i-1} M$  for all  $n \geq n_i$ , and so that the lengths of the  $w_i^{n_i}$  are all the same. Define  $\tilde{w}_i = w_i^{n_i}$ . Reorder the  $w_i$  so that

$$\tilde{w}_0 < \tilde{w}_1 < \cdots < \tilde{w}_{s-1}.$$

(Numbering the  $\tilde{w}_i$  from  $i = 0$  turns out to be most convenient; see the proof of Lemma 9.6.) Let  $\tilde{w}_{-1} M = 0$ . Then the  $w_i$  are distinct, the canonical filtration of  $M$  is

$$0 \subsetneq \tilde{w}_0 M \subsetneq \cdots \subsetneq \tilde{w}_{s-1} M = M,$$

and the primitive words associated to  $M$  are precisely the  $w_i$ , with the multiplicity of  $w_i$  being

$$\mu(i) = \dim_k(\tilde{w}_i M / \tilde{w}_{i-1} M).$$

**3.5. Words to canonical types.** In this section, we describe the map from multisets of (not necessarily primitive) cyclic words to canonical type. This will be used in Sections 5 and 7.

For  $1 \leq i \leq n$ , let  $\bar{w}_i$  be cyclic words with multiplicities  $m_i$ . Let  $M(\bar{w}_i)$  be the Kraft module discussed in Section 3.2. Our goal is to describe the canonical type of the  $BT_1$  module

$$M = \bigoplus_{i=1}^n M(\bar{w}_i)^{m_i}.$$

Let  $\lambda_i$  be the length of  $\bar{w}_i$  and choose a representative  $w_i = u_{i,\lambda_i-1} \cdots u_{i,0}$  of  $\bar{w}_i$  with  $u_{i,j} \in \{f, v\}$ . The  $k$ -vector space underlying  $M$  has basis  $e_{i,j,k}$  where  $1 \leq i \leq n$ ,  $j \in \mathbb{Z}/\lambda_i\mathbb{Z}$ , and  $1 \leq k \leq m_i$ . Its  $\mathbb{D}_k$ -module structure is given by

$$F(e_{i,j,k}) = \begin{cases} e_{i,j+1,k} & \text{if } u_{i,j} = f, \\ 0 & \text{if } u_{i,j} = v, \end{cases}$$

and

$$V(e_{i,j+1,k}) = \begin{cases} e_{i,j,k} & \text{if } u_{i,j} = v, \\ 0 & \text{if } u_{i,j} = f. \end{cases}$$

Let  $w_{i,j} = u_{i,j-1} \cdots u_{i,0} u_{i,\lambda_i} \cdots u_{i,j}$  be the  $j$ -th rotation of  $w_i$ . Let  $\ell = \text{LCM}(\lambda_i)_{i=1..n}$  and set  $\tilde{w}_{i,j} = w_{i,j}^{\ell/\lambda_i}$ . Thus each  $\tilde{w}_{i,j}$  has length  $\ell$  and we may compare them in lexicographic order.

Let  $\Sigma$  be the multiset obtained by including each  $\tilde{w}_{i,j}$  with multiplicity  $m_i$ . (The  $\tilde{w}_{i,j}$  need not be distinct, and if there are repetitions, one should add multiplicities.) Now relabel the distinct elements of  $\Sigma$  as  $\omega_t$  for  $0 \leq t \leq s-1$  and ordered so that  $\omega_0 < \omega_1 < \cdots < \omega_{s-1}$ . Let  $\tau$  be the function such that  $\tau(i, j) = t$  if and only if  $\tilde{w}_{i,j} = \omega_t$ . Let  $\mu(t)$  be the multiplicity of  $\omega_t$  in  $\Sigma$ .

For  $1 \leq t \leq s$ , let  $M_t$  be the  $k$ -subspace of  $M$  spanned by those  $e_{i,j,k}$  with  $\tau(i, j) \leq t-1$ . We claim that the canonical filtration of  $M$  is

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M.$$

Indeed, it is easy to see that if  $\omega_t$  ends with  $f$  and  $\omega_{t'}$  is the first rotation of  $\omega_t$ , then  $F$  induces a  $p$ -linear isomorphism  $M_{t+1}/M_t \xrightarrow{\sim} M_{t'+1}/M_{t'}$ . On the other hand, if  $\omega_t$  ends with  $v$ , then  $V^{-1}$  induces a  $p$ -linear isomorphism  $M_{t+1}/M_t \xrightarrow{\sim} M_{t'+1}/M_{t'}$ . Thus the displayed filtration is an admissible filtration. Via the action of  $\mathcal{W}$  on  $M$ , the word  $\omega_t$  induces a semi-linear automorphism of  $M_{t+1}/M_t$ , while a power of  $\omega_t$  induces the zero map of  $M_{t'+1}/M_{t'}$  if  $t \neq t'$ . It follows that  $\omega_t^n M = M_{t+1}$  for large enough  $n$ , so this is the coarsest filtration, thus the canonical filtration.

The dimension of the block  $B_t = M_{t+1}/M_t$  equals  $\mu(\omega_t)$ , the multiplicity of  $\omega_t$  in  $\Sigma$ .

It remains to record the values of  $r$  and the functions  $\phi$ ,  $\nu$ , and  $\rho$  associated to  $M$ :

- $r = \#\{t \mid 0 \leq t < s, \omega_t \text{ ends with } f\}$ ;
- $\phi(i) = \#\{t \mid 0 \leq t < i, \omega_t \text{ ends with } f\}$ ;
- $\nu(i) = r + \#\{t \mid 1 \leq t < i, \omega_t \text{ ends with } v\}$ ; and
- $\rho(i) = \sum_{t=0}^{i-1} \mu(t)$ .

**3.5.1. Example.** Let  $\bar{w}_1 = \overline{fv}$ ,  $w_2 = \overline{fvfv}$ , and  $m_1 = m_2 = 1$ . Taking  $n_1 = 2$  and  $n_2 = 1$ , one finds that  $\Sigma$  contains  $\omega_0 = fvf$  and  $\omega_1 = vfv$ , each with multiplicity 3. The function  $\tau$  is

$$\tau(1, 0) = \tau(2, 0) = \tau(2, 2) = 0 \quad \text{and} \quad \tau(1, 1) = \tau(2, 1) = \tau(2, 1) = 1,$$

and  $\mu(0) = \mu(1) = 3$ . Thus  $s = 2$ ,  $r = 1$ , and the functions  $\phi$ ,  $\nu$ , and  $\rho$  are given by

$i$	0	1	2
$\phi(i)$	0	0	1
$\nu(i)$	1	2	2
$\rho(i)$	0	3	6

## 4. Duality and E–O structures

**4.1. Duality of  $BT_1$  modules.** We record how duality of  $BT_1$  modules interacts with the objects in Diagram (3.1). All the assertions in this section will be left to the reader.

For a  $BT_1$  module  $M$ , let  $M^*$  is its dual. If  $N \subset M$  is a  $k$ -subspace, then

$$F(N^\perp) = (V^{-1}N)^\perp \quad \text{and} \quad V^{-1}(N^\perp) = (FN)^\perp.$$

Let

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_s = M$$

be the canonical filtration of  $M$ ; setting  $M_i^* = (M_{s-i})^\perp$ , then the canonical filtration of  $M^*$  is

$$0 = M_0^* \subsetneq M_1^* \subsetneq \dots \subsetneq M_s^* = M^*.$$

If the canonical data attached to  $M$  is  $(r, s, \phi, \nu, \rho)$ , and the canonical data attached to  $M^*$  is  $(r^*, s^*, \phi^*, \nu^*, \rho^*)$ , then  $s^* = s$ ,  $r^* = s - r$ , and for  $0 \leq i \leq s$ ,

$$\phi^*(i) = s - \nu(s - i), \quad \nu^*(i) = s - \phi(s - i), \quad \text{and} \quad \rho^*(i) = \rho(s) - \rho(s - i).$$

It follows that  $M$  is self-dual if and only if the associated canonical data satisfies  $s = 2r$ ,

$$\phi(i) + \nu(s - i) = s, \quad \text{and} \quad \rho(i) + \rho(s - i) = \rho(s). \tag{4.1}$$

The relationship between the partitioned set with permutation associated to  $M$  and to  $M^*$  is  $S^* = S$ ,  $S_f^* = S_v$ ,  $S_v^* = S_f$ , and  $\pi^* = \pi$ . It follows that  $M$  is self-dual if and only if there exists a bijection  $\iota : S \rightarrow S$  which satisfies  $\iota(S_f) = S_v$  and  $\pi \circ \iota = \iota \circ \pi$ .

If  $w$  is a primitive word on  $\{f, v\}$ , define  $w^c$  to be the word obtained by exchanging  $f$  and  $v$ . This operation descends to a well-defined involution on

cyclic words and  $M(w)^* \cong M(w^c)$ . It follows that  $M$  is self-dual if and only if the associated multiset of cyclic primitive words consists of self-dual words ( $\overline{w^c} = \overline{w}$ ) and pairs of dual words ( $\{\overline{w}, \overline{w^c}\}$ ).

**4.2. Ekedahl–Oort classification of polarized  $BT_1$  modules.** Clearly, a polarized  $BT_1$  module is self-dual. Conversely, as we will see below (Corollary 4.2), any self-dual  $BT_1$  module can be given a polarization. In this section, we review the Ekedahl–Oort classification [Oor01] of polarized  $BT_1$  modules.

**4.2.1. Elementary sequences.** Elementary sequences are a convenient repackaging of the data of a self-dual canonical type  $(r, s, \phi, \nu, \rho)$ . Using Equation (4.1), the restrictions of  $\phi$  and  $\rho$  to  $\{0, \dots, r\}$  determine the rest of the data. An *elementary sequence* of length  $g$  is a sequence  $\Psi = [\psi_1, \dots, \psi_g]$  of integers with  $\psi_{i-1} \leq \psi_i \leq \psi_{i-1} + 1$  for  $i = 1, \dots, g$  and  $\psi_0 = 0$ . The set of elementary sequences of length  $g$  has cardinality  $2^g$ .

Given  $(r, s, \phi, \nu, \rho)$ , define an elementary sequence as follows. Let  $g = \rho(r)$ . Set  $\psi_0 = 0$ . For each  $1 \leq j \leq g$ , let  $i$  be the unique integer  $0 < i \leq r$  such that  $\rho(i-1) < j \leq \rho(i)$ . Define

$$\psi_j = \begin{cases} \psi_{j-1} & \text{if } \phi(i) = \phi(i-1), \\ \psi_{j-1} + 1 & \text{if } \phi(i) > \phi(i-1). \end{cases}$$

(Put more vividly, the sequence  $\psi_j$  increases for  $\mu(i-1)$  steps if  $\phi(i) > \phi(i-1)$  and it stays constant for  $\mu(i-1)$  steps if  $\phi(i) = \phi(i-1)$ .)

We leave it as an exercise for the reader to check that, given an elementary sequence, there is a unique self-dual canonical type giving rise to it by this construction.

Elementary sequences can be obtained directly from a self-dual  $BT_1$  module as follows: The canonical filtration can be refined into a “final filtration,” i.e., a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_{2g} = M$$

respected by  $F$  and  $V^{-1}$  and such that  $\dim_k(M_i) = i$ . Then  $\Psi$  is defined by  $\psi_i = \dim_k(FM_i)$ .

**Theorem 4.1.** [Oor01, Thm. 9.4] *Every elementary sequence of length  $g$  arises from a polarized  $BT_1$  module of dimension  $2g$ , and two polarized  $BT_1$  modules over  $k$  with the same elementary sequences are isomorphic. More precisely, there is an isomorphism of  $BT_1$  modules which respects the alternating pairings. (This isomorphism is not unique in general.)*

The elementary sequence attached to a  $BT_1$  module is also called its *Ekedahl–Oort structure*.

**Corollary 4.2.** *Every self-dual  $BT_1$  module admits a polarization, i.e., a non-degenerate alternating pairing, and this pairing is unique up to (non-unique) isomorphism.*

**Proof.** If  $M$  is a self-dual  $BT_1$  module, construct its canonical type, and its elementary sequence  $\Psi$  as in Section 4.2.1. Theorem 4.1 furnishes a polarized  $BT_1$  module with the same underlying  $BT_1$  module, and this proves the existence of a polarization. For uniqueness, note that the construction of  $\Psi$  does not depend on the pairing. So, given two alternating pairings on  $M$ , Theorem 4.1 shows there is a (not necessarily unique) automorphism intertwining the pairings.  $\square$

## 5. Homomorphisms

The Kraft description of  $BT_1$  modules is well adapted to computing homomorphisms. We work out three important examples in this section.

### 5.1. Homs from $\mathbb{Z}/p\mathbb{Z}$ or $\mu_p$ .

**Definition 5.1.** If  $G$  is a  $BT_1$  group scheme, the  $p$ -rank of  $G$  is the largest integer  $f$  such that there is an injection  $(\mathbb{Z}/p\mathbb{Z})^f \hookrightarrow G$ . Alternatively,  $f$  is the dimension of the largest quotient space of  $M(G)$  on which Frobenius acts bijectively.

**Lemma 5.2.** *If  $G$  is a  $BT_1$  group scheme, the  $p$ -rank of  $G$  is equal to the multiplicity of the word  $f$  in the multiset of primitive words corresponding to  $M(G)$ . Similarly, the largest  $f$  such that  $\mu_p^f$  embeds in  $M(G)$  is the multiplicity of the word  $v$ .*

**Proof.** From the presentation in terms of generators and relations, we see that if  $w$  is a primitive word other than  $f$ , then there is no non-zero homomorphism from  $\mathbb{Z}/p\mathbb{Z} = M(f)$  to  $M(w)$ . It follows that if  $M(G)$  is given in the Kraft classification by a multiset of primitive cyclic words, the  $p$ -rank is the multiplicity of the word  $f$ . The assertion for  $\mu_p$  is proved analogously.  $\square$

**5.2. Homs from  $\alpha_p$ .** Let  $G$  be a finite group scheme over  $k$  killed by  $p$ . Define the *foot* (or *socle*) of  $G$  to be the largest semisimple subgroup of  $G$ . The simple objects in the category of finite group schemes over  $k$  killed by  $p$  are  $\mathbb{Z}/p\mathbb{Z}$ ,  $\mu_p$ , and  $\alpha_p$ . Thus the foot of  $G$  is a direct sum of these (with multiplicity). If  $G$  is connected and unipotent, then its foot is of the form  $\alpha_p^\ell$  for some positive integer  $\ell$ . Note that  $M(\alpha_p) = \mathbb{D}_k/(F, V)$ .

Similarly, if  $M$  is a  $\mathbb{D}_k$ -module of finite length, the *head* (or *co-socle*) of  $M$  is its largest semisimple quotient. We write  $\mathcal{H}(M)$  for the head of  $M$ . If  $M$  is a  $\mathbb{D}_k$ -module on which  $F$  and  $V$  are nilpotent, then  $\mathcal{H}(M)$  is a  $k$ -vector space on which  $F = V = 0$ . The presentation of  $M(w)$  by generators and relations makes it clear that if  $w = v^{n_\ell} f^{m_\ell} \dots v^{n_1} f^{m_1}$ , then  $\mathcal{H}(M(w))$  has dimension  $\ell$ . More precisely, the images of the generators  $E_0, \dots, E_{\ell-1}$  in  $\mathcal{H}(M(w))$  are a basis.

**Definition 5.3.** If  $G$  is a finite group scheme over  $k$  killed by  $p$ , the  $a$ -number of  $G$ , denoted  $a(G)$ , is the largest integer  $a$  such that there is an injection  $\alpha_p^a \hookrightarrow G$  of group schemes. If  $G$  is connected and unipotent,  $p^{a(G)}$  equals the

order of the foot of  $G$ . Similarly, if  $M$  is a  $\mathbb{D}_k$ -module of finite length, the  $a$ -number of  $M$ , denoted  $a(M)$ , is the largest integer  $a$  such that there is a surjection  $M \rightarrow M(\alpha_p) = \mathbb{D}_k/(F, V)$  of  $\mathbb{D}_k$ -modules.

It is clear that the  $a$ -number is additive in direct sums and that  $a(M(f)) = a(M(v)) = 0$ .

**Lemma 5.4.** *The  $a$ -numbers of  $BT_1$  modules have the following properties:*

- (1) *If  $\ell, m_1, \dots, m_\ell$ , and  $n_1, \dots, n_\ell$  are positive integers, then  $a(M(v^{n_\ell} f^{m_\ell} \dots v^{n_1} f^{m_1})) = \ell$ .*
- (2) *Also  $a(M(w))$  is the number of rotations of  $w$  which start with  $v$  and end with  $f$ .*

**Proof.** For part (1),  $a(M) = \dim_k(\mathcal{H}(M))$  and, by the discussion above, if  $w = v^{n_\ell} f^{m_\ell} \dots v^{n_1} f^{m_1}$ , then  $a(M(w)) = \ell$ . Part (2) is immediate from part (1).  $\square$

**5.3. Homs from  $G_{1,1}$ .** Write  $M_{1,1} := \mathbb{D}_k/(F - V) \cong M(fv)$ . Let  $G_{1,1}$  be the  $BT_1$  group scheme over  $k$  such that  $M(G_{1,1}) \cong M_{1,1}$ . The group scheme  $G_{1,1}$  appears “in nature” as the kernel of multiplication by  $p$  on a supersingular elliptic curve over  $k$  (e.g., see [Ulm91, Prop. 4.1]).

**Definitions 5.5.** Let  $G$  be a  $BT_1$  group scheme over  $k$  and let  $M$  be a  $BT_1$  module.

- (1) Define the  $s_{1,1}$ -multiplicity of  $G$  as the largest integer  $\mathfrak{s}$  such that there is an isomorphism of group schemes

$$G \cong G_{1,1}^{\mathfrak{s}} \oplus G'.$$

Define the  $s_{1,1}$ -multiplicity of  $M$  as the largest integer  $\mathfrak{s}$  such that there is an isomorphism of  $\mathbb{D}_k$ -modules  $M \cong M_{1,1}^{\mathfrak{s}} \oplus M'$ .

- (2) Define the  $u_{1,1}$ -number of  $G$  as the largest integer  $\mathfrak{u}$  such that there exists an injection

$$G_{1,1}^{\mathfrak{u}} \hookrightarrow G$$

of group schemes. Define the  $u_{1,1}$ -number of  $M$  as the largest integer  $\mathfrak{u}$  such that there is a surjection  $M \rightarrow M_{1,1}^{\mathfrak{u}}$  of  $\mathbb{D}_k$ -modules.

The notation  $s_{1,1}$  (resp.  $u_{1,1}$ ) is motivated by the word superspecial (resp. unpolarized), see Section 5.5. Note the equalities  $s_{1,1}(M(G)) = s_{1,1}(G)$  and  $u_{1,1}(M(G)) = u_{1,1}(G)$ . It is clear that  $u_{1,1}(G) \geq s_{1,1}(G)$ . The  $s_{1,1}$ -multiplicity and  $u_{1,1}$ -number are additive in direct sums.

By the Kraft classification, if  $M(G)$  is described by a multiset of primitive cyclic words, then  $s_{1,1}(G)$  equals the multiplicity of the cyclic word  $fv$  in the multiset.

We want to compute the  $u_{1,1}$ -number of the standard  $BT_1$  modules  $M(w)$ . Trivially,

$$u_{1,1}(M(f)) = u_{1,1}(M(v)) = 0.$$



A straightforward exercise shows that  $u_{1,1}(M(fv)) = 1$ , and more precisely that

$$\text{Hom}_{\mathbb{D}_k}(M(fv), M(fv)) \cong \mathbb{F}_{p^2} \times k,$$

with  $(c, d) \in \mathbb{F}_{p^2} \times k$  identified with the homomorphism that sends the class of  $1 \in \mathbb{D}_k/(F - V)$  to the class of  $c + dF$ . The surjective homomorphisms are those where  $c \neq 0$ .

We may thus assume that  $w$  has length  $\lambda > 2$ . We will evaluate the  $u_{1,1}$ -number of  $M(w)$  by computing  $\text{Hom}_{\mathbb{D}_k}(M(w), M_{1,1})$  explicitly. To that end, write  $w = v^{n_\ell} f^{m_\ell} \dots v^{n_1} f^{m_1}$ . Since  $\lambda > 2$  and  $w$  is non-periodic, we may replace  $w$  with a rotated word so that  $m_1 > 1$  or  $n_\ell > 1$  (or both). Roughly speaking, the following proposition says that the  $u_{1,1}$ -number of  $M(w)$  is the number of appearances in  $w$  of subwords of the form  $v^{>1}(fv)^e f^{>1}$  where  $e \geq 0$ . For example, if  $w = v^2 f^2 v^3 f v f^4$ , then the  $u_{1,1}$ -number is 2.

**Proposition 5.6.** *Suppose  $w = v^{n_\ell} f^{m_\ell} \dots v^{n_1} f^{m_1}$  where  $m_1 > 1$  or  $n_\ell > 1$ . Define  $u$  by:*

$$u := \# \{1 \leq i \leq \ell \mid m_i > 1 \text{ and } n_i > 1\} \\ + \# \{1 \leq i < j \leq \ell \mid n_j > 1, m_j = n_{j-1} = \dots = n_i = 1, \text{ and } m_i > 1\}.$$

Then

- (1) the  $s_{1,1}$  number of  $M(w)$  is  $\ell$  if  $m_i = n_i = 1$  for all  $i$ , and 0 otherwise,
- (2)  $\text{Hom}_{\mathbb{D}_k}(M(w), M_{1,1})$  is in bijection with  $k^{u+\ell}$ , and
- (3) the  $u_{1,1}$ -number of  $M(w)$  is  $u$ .

**Proof.** Part (1) follows from the fact that for a primitive word  $w$ , the  $s_{1,1}$  number of  $M(w)$  is 1 if  $\bar{w} = \overline{fv}$  and is zero otherwise.

For part (2), we use Lemma 3.1 to present  $M(w)$  with generators  $E_0, \dots, E_{\ell-1}$  (with indices taken modulo  $\ell$ ) and relations  $F^{m_i} E_{i-1} = V^{n_i} E_i$ . Let  $z_0, z_1$  be a  $k$ -basis of  $M_{1,1}$  with  $Fz_0 = Vz_0 = z_1$  and  $Fz_1 = Vz_1 = 0$ . Then a homomorphism  $\psi : M(w) \rightarrow M_{1,1}$  is determined by its values on the generators  $E_i$ . Write

$$\psi(E_i) = a_{i,0}z_0 + a_{i,1}z_1.$$

Then  $\psi$  is a  $\mathbb{D}_k$ -module homomorphism if and only if  $F^{m_i}\psi(E_{i-1}) = V^{n_i}\psi(E_i)$  for  $i = 1, \dots, \ell$ . By (2.1),

$$F^{m_i}\psi(E_{i-1}) = \begin{cases} a_{i-1,0}^p z_1 & \text{if } m_i = 1 \\ 0 & \text{if } m_i > 1 \end{cases}$$

and

$$V^{n_i}\psi(E_i) = \begin{cases} a_{i,0}^{1/p} z_1 & \text{if } n_i = 1 \\ 0 & \text{if } n_i > 1. \end{cases}$$

This system of equations places no constraints on  $a_{i,1}$  for  $i = 1, \dots, \ell$ , because  $Vz_1 = Fz_1 = 0$ . The constraints on  $a_{i,0}$  for  $i = 1, \dots, \ell$  are: if  $m_i = n_i = 1$ , then  $a_{i-1,0}^p = a_{i,0}^{1/p}$ ; if  $m_i = 1$  and  $n_i > 1$ , then  $a_{i-1,0} = 0$ ; if  $m_i > 1$  and

$n_i = 1$ , then  $a_{i,0} = 0$ ; if  $m_i > 1$  and  $n_i > 1$ , then no constraint. Using that  $m_1 > 1$  or  $n_\ell > 1$ , we find a triangular system of equations for  $a_{i,0}$ , and it is a straightforward exercise to show that the solutions are in bijection with  $k^u$ . Combining with the  $k^\ell$  unconstrained values of  $\{a_{i,1} \mid 1 \leq i \leq \ell\}$  yields part (2).

For part (3), the homomorphism  $\psi : M(w) \rightarrow M_{1,1}$  is surjective if and only if at least one of the  $a_{i,0}$  is not zero, which is equivalent to  $\psi$  inducing a surjection  $\mathcal{H}(M(w)) \rightarrow \mathcal{H}(M_{1,1}) = k$ . Part (1) implies that there are  $u$  independent such  $\psi$  (and no more). This shows that  $u$  is the largest integer such that there is a surjection  $M(w) \rightarrow M_{1,1}^u$ , completing the proof of part (3).  $\square$

**5.4. Motivation.** The next proposition motivates the  $s_{1,1}$ -multiplicity and the  $u_{1,1}$ -number.

**Proposition 5.7.** *Let  $A/k$  be an abelian variety and let  $E/k$  be a supersingular elliptic curve.*

- (1) *If there is an abelian variety  $B$  and an isogeny  $E^s \times B \rightarrow A$  of degree prime to  $p$ , then  $s$  is less than or equal to the  $s_{1,1}$ -multiplicity of  $A[p]$ .*
- (2) *If there is a morphism of abelian varieties  $E^u \rightarrow A$  with finite kernel of order prime to  $p$ , then  $u$  is less than or equal to the  $u_{1,1}$ -number of  $A[p]$ .*

**Proof.** An isogeny as in (1) shows that  $E[p] \cong G_{1,1}$  is a direct factor of  $A[p]$  with multiplicity  $s$ , so  $s \leq s_{1,1}(A[p])$ . A morphism as in (2) shows that  $E[p] \cong G_{1,1}$  appears in  $A[p]$  with multiplicity at least  $u$ , so  $u \leq u_{1,1}(A[p])$ .  $\square$

**5.5. Connection to the superspecial rank.** Suppose  $G$  is a polarized  $BT_1$  group scheme. In [AP15, Def. 3.3], Achter and Pries define the *superspecial rank* of  $G$  as the largest integer  $s$  such that there is an injection  $G_{1,1}^s \hookrightarrow G$  and such that the polarization on  $G$  restricts to a non-degenerate pairing on  $G_{1,1}^s$ . In this situation, the pairing allows them to define a complement (see [AP15, Lemma 3.4]), so that  $G \cong G_{1,1}^s \oplus G'$  (a direct sum of polarized  $BT_1$  group schemes). A self-dual  $BT_1$  group scheme  $G$  equipped with a decomposition  $G \cong G_{1,1} \oplus G'$  (just of  $BT_1$  group schemes) automatically admits a polarization compatible with the direct sum decomposition. Therefore, the superspecial rank of  $G$  equals its  $s_{1,1}$ -multiplicity.

They also define an *unpolarized superspecial rank*, which is the same as our  $u_{1,1}$ -number, and prove a result [AP15, Lemma 3.8] which is closely related to and implied by Proposition 5.6.

Next, we consider a correction to [AP15, Thm. 3.14]. Let  $K$  be the function field of an irreducible, smooth, proper curve  $X$  over  $k$ , let  $J_X$  be the Jacobian of  $X$ , and let  $E$  be a supersingular elliptic curve over  $k$  which we regard as a curve over  $K$  by base change. Let  $\text{Sel}(K, p)$  denote the Selmer group for the multiplication-by- $p$  isogeny of  $E/K$ . See [Ulm91] for details.

**Proposition 5.8.** *With notation as above, let  $a$ ,  $u_{1,1}$ , and  $s_{1,1}$  be the  $a$ -number,  $u_{1,1}$ -number, and  $s_{1,1}$ -multiplicity of  $J_X[p]$  respectively. Then the group  $\text{Sel}(K, p)$  is isomorphic to the product of a finite group and a  $k$ -vector space of dimension  $a + u_{1,1} - s_{1,1}$ .*

**Proof.** Applying [Ulm19, Props. 6.2 and 6.4] with  $\mathcal{C} = X$ ,  $\mathcal{D} = E$ ,  $\Delta = 1$ , and  $n = 1$  shows that  $\text{Sel}(K, p)$  and  $\text{Hom}_{\mathbb{D}}(H_{dR}^1(X), M_{1,1})$  differ by a finite group. By [Oda69, Cor. 5.11],  $H_{dR}^1(X) \cong M(J_X[p])$ . Write  $M(J_X[p])$  as a sum of indecomposable  $BT_1$  modules  $M(w)$  for suitable cyclic words  $w$ . In Section 4.3, we computed  $H_w = \text{Hom}_{\mathbb{D}}(M(w), M_{1,1})$  for a primitive cyclic word  $w$ . Recall that  $H_w = 0$  if  $w = f$  or  $w = v$  and  $H_w \cong \mathbb{F}_{p^2} \times k$  if  $w = fv$ . Also  $H_w \cong k^{u+\ell}$ , if  $w \notin \{1, f, v, fv\}$  where  $u = u_{1,1}(M(w))$  and  $\ell$  is the  $a$ -number of  $M(w)$ . The result follows from the additivity of  $a$ -numbers,  $u_{1,1}$ -numbers, and  $s_{1,1}$ -multiplicities.  $\square$

Next, we compute the numerical invariants of a module  $M = \oplus M(\bar{w}_i)^{m_i}$  in terms of multiplicities of words. In Section 3.5, we defined multiplicities  $\mu$  by considering all lifts of  $\bar{w}_i$  to words and taking powers of those words so they all have the same length  $\ell = \text{lcm}(\text{lengths of } \bar{w}_i)$ . Let  $\mu(f-v)$  be the sum of the multiplicities of all these words of length  $\ell$  starting with  $f$  and ending with  $v$ ; and let  $\mu(-vf)$  be the sum of the multiplicities of all words of length  $\ell$  ending with  $vf$ .

**Proposition 5.9.** *The  $BT_1$  module  $M = \oplus M(\bar{w}_i)^{m_i}$  has*

- (1)  $p$ -rank equal to  $\mu(f^\ell)$ ,
- (2)  $a$ -number equal to  $\mu(f-v) = \mu(-vf) = \mu(-fv)$ ,
- (3)  $s_{1,1}$ -multiplicity equal to  $\mu((fv)^{\ell/2})$  if  $\ell$  is even and to 0 if  $\ell$  is odd,
- (4) and  $u_{1,1}$ -number equal to the sum of the  $s_{1,1}$  multiplicity and

$$\mu(-v^2f^2) + \mu(-v^2fvf^2) + \dots + \mu(-v^2(fv)^{\lfloor (\ell-4)/2 \rfloor} f^2).$$

**Proof.** Parts (1) and (2) follow from Lemma 5.2 and Lemma 5.4 respectively. Parts (3) and (4) follow from Proposition 5.6.  $\square$

**5.6. Examples.** For small genus, we give tables of elementary sequences (“E–O”), matched with the self-dual multisets of primitive cyclic words (“K”), together with their  $p$ -ranks,  $a$ -numbers,  $s_{1,1}$ -multiplicities, and  $u_{1,1}$ -numbers.

From Section 4.2.1, for the  $BT_1$  module of the elementary sequence  $\Psi = [\psi_1, \dots, \psi_g]$ , the  $p$ -rank is the largest  $i$  such that  $\psi_i = i$  and the  $a$ -number is  $g - \psi_g$ . We do not know how to compute the  $s_{1,1}$ -multiplicity or  $u_{1,1}$ -number directly from  $\Psi$ .

For the  $BT_1$  module of a multiset of cyclic words, the  $p$ -rank is the multiplicity of the word  $f$  by Lemma 5.2 and the  $a$ -number can be computed using Lemma 5.4. The  $s_{1,1}$ -multiplicity is the multiplicity of the cyclic word  $fv$  and the  $u_{1,1}$ -number can be computed using Proposition 5.6.

$g = 1$					
E-O	K	$p$ -rank	$a$ -number	$s_{1,1}$ -mult.	$u_{1,1}$ -number
[0]	$\{fv\}$	0	1	1	1
[1]	$\{f, v\}$	1	0	0	0

$g = 2$					
E-O	K	$p$ -rank	$a$ -number	$s_{1,1}$ -mult.	$u_{1,1}$ -number
[0, 0]	$\{(fv)^2\}$	0	2	2	2
[0, 1]	$\{ffvv\}$	0	1	0	1
[1, 1]	$\{f, v, fv\}$	1	1	1	1
[1, 2]	$\{(f)^2, (v)^2\}$	2	0	0	0

If  $G$  is a polarized  $BT_1$  group scheme of order  $p^{2g}$  with positive  $p$ -rank, then  $G \cong G' \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$  where  $G'$  is a polarized  $BT_1$  group scheme of order  $p^{2g-2}$ . Thus the rows of the table for  $g$  for  $G$  with positive  $p$ -rank can be deduced from the table for  $g-1$ . In passing from genus  $g-1$  to genus  $g$ : the elementary sequence changes from  $[\psi_1, \dots, \psi_{g-1}]$  to  $[1, \psi_1 + 1, \dots, \psi_{g-1} + 1]$ ; the multiplicity of the words  $f$  and  $v$  increases by 1; the  $p$ -rank increases by 1; and the  $a$ -number,  $s_{1,1}$ -multiplicity, and  $u_{1,1}$ -number stay the same. In light of this, when  $g = 3$  and  $g = 4$ , we only include the  $BT_1$  group schemes with  $p$ -rank 0 in the table.

$g = 3$					
E-O	K	$p$ -rank	$a$ -number	$s_{1,1}$ -mult.	$u_{1,1}$ -number
[0, 0, 0]	$\{(fv)^3\}$	0	3	3	3
[0, 0, 1]	$\{fv, fvv\}$	0	2	1	2
[0, 1, 1]	$\{fvv, vff\}$	0	2	0	0
[0, 1, 2]	$\{ffvfv\}$	0	1	0	1

The E-O structure  $[0, 1, 1]$  is the first which is decomposable as a  $BT_1$  group scheme but indecomposable as a polarized  $BT_1$  group scheme.

$g = 4$					
E-O	K	$p$ -rank	$a$ -number	$s_{1,1}$ -mult.	$u_{1,1}$ -number
[0, 0, 0, 0]	$\{(fv)^4\}$	0	4	4	4
[0, 0, 0, 1]	$\{(fv)^2, fvv\}$	0	3	2	3
[0, 0, 1, 1]	$\{ffvfvfv\}$	0	3	0	1
[0, 0, 1, 2]	$\{(ffvv)^2\}$	0	2	0	2
[0, 1, 1, 1]	$\{fv, ffv, vvf\}$	0	3	1	1
[0, 1, 1, 2]	$\{fv, fffvv\}$	0	2	1	2
[0, 1, 2, 2]	$\{fffv, fvv\}$	0	2	0	0
[0, 1, 2, 3]	$\{ffffvvvv\}$	0	1	0	1

### 6. Fermat Jacobians

In this section, we recall two results on the  $BT_1$  modules of Fermat curves from [PU].

For each positive integer  $d$  not divisible by  $p$ , let  $F_d$  be the Fermat curve of degree  $d$ , i.e., the smooth, projective curve over  $k$  with affine model

$$F_d : \quad X^d + Y^d = 1,$$

and let  $J_{F_d}$  be its Jacobian. Let  $\mathcal{C}_d$  be the smooth, projective curve over  $k$  with affine model

$$\mathcal{C}_d : \quad y^d = x(1 - x). \tag{6.1}$$

The substitution  $X^d$  for  $x$  and  $XY$  for  $y$  in the equation for  $\mathcal{C}_d$  shows that  $\mathcal{C}_d$  is a quotient of  $F_d$ . The map  $F_d \rightarrow \mathcal{C}_d$  is the quotient of  $F_d$  by the subgroup

$$\{(\zeta, \zeta^{-1}) \mid \zeta \in \mu_d\} \subset (\mu_d)^2 \subset \text{Aut}(F_d)$$

of index  $d$ . The Riemann-Hurwitz formula shows that the genus of  $\mathcal{C}_d$  is

$$g(\mathcal{C}_d) = \lfloor (d - 1)/2 \rfloor = \begin{cases} (d - 1)/2 & \text{if } d \text{ is odd,} \\ (d - 2)/2 & \text{if } d \text{ is even.} \end{cases}$$

**Theorem 6.1** (= [PU, Thm. 5.5]). *The Dieudonné module  $M(J_d[p])$  is the  $BT_1$  module with data*

$$S = \mathbb{Z}/d\mathbb{Z} \setminus \{0, d/2\} \text{ if } d \text{ is even, and } S = \mathbb{Z}/d\mathbb{Z} \setminus \{0\} \text{ if } d \text{ is odd,}$$

$$S_f = \{a \in S \mid d/2 < a < d\}, \quad S_v = \{a \in S \mid 0 < a < d/2\},$$

and the permutation  $\pi : S \rightarrow S$  given by  $\pi(i) = pa$ .

**Theorem 6.2** (= [PU, Thm. 5.9]). *The Dieudonné module  $M(J_{F_d}[p])$  is the  $BT_1$  module with data*

$$T = \{(a, b) \in (\mathbb{Z}/d\mathbb{Z})^2 \mid a \neq 0, b \neq 0, a + b \neq 0\},$$

$$T_f = \{(a, b) \in S \mid a + b > d\}, \quad T_v = \{(a, b) \in S \mid a + b < d\},$$

and the permutation  $\sigma(a, b) = (pa, pb)$ .

*Remark 6.3.* Let  $\mu_d$  be the group of  $d$ -th roots of unity in  $k$ , and note that  $\mu_d$  acts on  $\mathcal{C}_d$  by multiplication on the  $y$  coordinate. Although this action does not appear explicitly in Theorem 6.1, it plays a key role in the proof. Indeed, the  $k$ -valued characters of  $\mu_d$  can be identified with  $\mathbb{Z}/d\mathbb{Z}$ , and the set  $S$  is precisely the set of characters of  $\mu_d$  which appear in  $H_{dR}^1(\mathcal{C}_d)$ . Each character that appears does so with multiplicity one. The subset  $S_v$  consists of those characters appearing in the subspace  $H^0(\mathcal{C}_d, \Omega^1) \subset H_{dR}^1(\mathcal{C}_d)$ . Moreover, the indecomposable submodules of  $M(J_d[p])$  as a  $BT_1$  module with  $\mu_d$  action correspond to the orbits of  $p$  on  $S$ . As noted in [PU, Rem. 5.6], such a submodule may become decomposable if we ignore the  $\mu_d$  action.

*Remark 6.4.* The action of  $\mu_d$  also allows one to decompose the Jacobian  $J_d$  (i.e., the motive of  $\mathcal{C}_d$ ) into parts indexed by divisors of  $d$ . If  $d'$  divides  $d$ , there is a projection  $\mathcal{C}_d \rightarrow \mathcal{C}_{d'}$  and an induced inclusion  $J_{d'} \rightarrow J_d$ . Defining  $J_d^{new}$  as the quotient of  $J_d$  by the sum of the images of  $J_{d'} \rightarrow J_d$  where  $d'$  runs through proper divisors of  $d$ , we obtain an isogeny

$$J_d \sim \bigoplus_{d'|d} J_{d'}^{new}$$

of degree prime to  $p$ . In particular, the  $p$ -divisible group of  $J_d$  is isomorphic to the direct sum of the  $p$ -divisible groups of the  $J_{d'}^{new}$ . For  $d = 1$  and  $d = 2$ ,  $J_d^{new}$  is trivial; if  $d > 2$ , then  $J_d^{new}$  is an abelian variety with complex multiplication by the cyclotomic field  $\mathbb{Q}(\mu_d)$ . A slight generalization of the proof of Theorem 6.1 allows one to compute the CM type of  $J_d^{new}$ . Using this, one finds that the  $p$ -divisible group of  $J_d^{new}$  is a “standard ordinary” in the sense of [Moo04, 1.2.3]. See also [PU, Rem. 5.10].

*Remark 6.5.* The quotient  $\mathcal{C}_d \rightarrow F_d$  induces an inclusion  $S \hookrightarrow T$  sending  $a$  to  $(a, a)$ , and the partition and permutation of  $T$  are compatible with those of  $S$ . There are other quotients  $\mathcal{C}$  of  $F_d$  by subgroups of  $\mu_d^2$ . (For example, the curves  $v^e = u^r(1-u)^s$  where  $e \mid d$ ,  $\gcd(r, s, e) = 1$ , and  $r + s < e$ .) Each gives rise to a set  $S_{\mathcal{C}}$  with partition and permutation and the projection  $F_d \rightarrow \mathcal{C}$  induces an inclusion  $S_{\mathcal{C}} \hookrightarrow T$  compatible with the partitions and permutations. The interesting features of  $J_{F_d}[p]$  are already present in  $J_{\mathcal{C}_d}[p]$ , so we restrict to studying this case for simplicity.

## 7. Ekedahl–Oort structures of Fermat quotients: general case

In this section, we determine the Ekedahl–Oort type of the Jacobian  $J_d$  of the Fermat quotient curve  $\mathcal{C}_d$  with affine model  $y^d = x(1-x)$  for  $d$  relatively prime to  $p$ .

**7.1. Words, patterns, and multiplicities.** By Theorem 6.1, the  $BT_1$  module of  $J_d[p]$  is the one obtained from diagram (3.1) using the data  $S \subset \mathbb{Z}/d\mathbb{Z}$  with its usual partition and permutation. This data gives rise to a multiset of cyclic words. To compute the E–O structure, as in Section 3.5, we consider all representative words of the cyclic words, take powers so they all have the same length, and compute their multiplicities to find the dimensions of the blocks  $B_i = M_{i+1}/M_i$  in the canonical filtration.

We want to reformulate this to go directly from the set  $S$  to the multiplicities  $\mu$ . To do so, we introduce the *pattern* of an element of  $S$ . This is a variant of the word which takes into account the powers mentioned in the previous paragraph.

Let  $\ell = |\langle p \rangle|$  be the multiplicative order of  $p$  modulo  $d$ . Let  $\mathcal{W}_{\ell}$  be the set of words of length  $\ell$  on  $\{f, v\}$ . Define a map  $\text{Pat} : S \rightarrow \mathcal{W}_{\ell}$  as follows: for

$a \in S$ , define  $\text{Pat}(a) = u_{\ell-1} \cdots u_0$  where

$$u_j = f \text{ if } p^j a \in S_f, \text{ and } u_j = v \text{ if } p^j a \in S_v.$$

If the word  $w_a$  for  $a$  has length  $\ell$  (which happens when  $\gcd(a, d) = 1$ ), then  $\text{Pat}(a) = w_a$  whereas for  $a$  with a shorter word,  $\text{Pat}(a)$  is a power of  $w_a$ .

For example, take  $d = 9$  and  $p = 2$ , so that  $\ell = 6$ . The orbits of  $\langle p \rangle$  are  $1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 7 \rightarrow 5 \rightarrow 1$  and  $3 \rightarrow 6 \rightarrow 3$ . For  $a = 3$ , the word is  $fv$  and  $\text{Pat}(3) = fvf vfv$ . For  $a = 6$ , the word is  $vf$  and  $\text{Pat}(6) = vfvfvf$ . For  $a \neq 3, 6$ , then  $\text{Pat}(a) = w_a$ ; for example  $\text{Pat}(1) = fffvfv$ .

For  $w \in \mathcal{W}_\ell$ , define the *multiplicity* of  $w$  to be the cardinality of its inverse image under  $\text{Pat}$ :

$$\mu(w) := |\text{Pat}^{-1}(w)|.$$

This is the same multiplicity as defined in Section 3.5 for the module determined by  $S$ .

**7.2. Ekedahl–Oort structure of  $J_d[p]$ .** Recall from Section 4.2.1 that the E–O structure associated to a self-dual  $BT_1$  module is the sequence of integers  $[\psi_1, \dots, \psi_g]$  which starts from  $\psi_0 = 0$  and has sections of length equal to the block sizes  $\dim(B_i)$  which are increasing (resp. constant) if the word attached to the block ends with  $f$  (resp.  $v$ ).

To simplify this description, we introduce some notation:  $\nearrow^m$  (resp.  $\rightarrow^m$ ) stands for an increasing (resp. constant) sequence of integers of length  $m$ . Thus,

$$[\nearrow^3 \rightarrow^2] = [1, 2, 3, 3, 3] \quad \text{and} \quad [\rightarrow^2 \nearrow^3] = [0, 0, 1, 2, 3].$$

Now enumerate the elements of  $\mathcal{W}_\ell$  that start with  $f$  in lexicographic order:

$$w_0 = f^\ell, w_1 = f^{\ell-1}v, w_2 = f^{\ell-2}vf, w_3 = f^{\ell-2}vv, \dots, w_{2^{\ell-1}-1} = fv^{\ell-1}. \tag{7.1}$$

Let  $\mu_j = \mu(w_j)$ .

Using Sections 3.5 and 4.2.1, we get the following description of the E–O structure of  $J_d[p]$ :

**Theorem 7.1.** *Let  $\ell$  be the multiplicative order of  $p$  modulo  $d$ , and let  $\mu_0, \dots, \mu_{2^{\ell-1}-1}$  be the multiplicities of the words above. Let  $J_d$  be the Jacobian of the Fermat quotient curve  $C_d$  with affine model  $y^d = x(1-x)$ . The Ekedahl–Oort structure of  $J_d[p]$  is given by*

$$[\nearrow^{\mu_0} \rightarrow^{\mu_1} \nearrow^{\mu_2} \dots \rightarrow^{\mu_{2^{\ell-1}-1}}].$$

**Proof.** Indeed, for the  $BT_1$  module  $M = M(J_d[p])$  and for  $0 \leq j < 2^{\ell-1}$ , the subspace  $M_{j+1}$  in the canonical filtration is the span of the basis vectors  $e_a$  indexed by  $a \in S$  such that  $\text{Pat}(a) \leq w_j$ . In particular,  $\dim(M_{j+1}) = \sum_{i=0}^j \mu_i$ . By Definition 3.2,  $B_j = M_{j+1}/M_j$ , so  $\dim(B_j) = \mu_j$ . If  $j$  is even, then  $w_j$  ends with  $f$  so  $B_j$  is mapped isomorphically onto its image by  $F$ . If  $j$  is odd, then  $w_j$  ends with  $v$  so  $B_j$  is killed by  $F$ . Thus the  $\mu_j$  are the values of  $\rho(j+1) - \rho(j)$  in the canonical type of  $J_d[p]$ , and they give the

lengths of the runs where the elementary sequence is increasing ( $j$  even) or constant ( $j$  odd).  $\square$

## 8. Ekedahl–Oort structures of Fermat quotients: the case $p = 2$

In this section, suppose  $p = 2$ . In this case, the curve  $\mathcal{C}_d$  is an Artin-Schreier cover of the projective line with equation  $x^2 - x = y^d$ , and the formulas when  $p = 2$  are different from the case when  $p$  is odd. We do not include proofs in this section because most of the results already appear in the literature.

The genus of  $\mathcal{C}_d$  is  $g_d = (d - 1)/2$ .

**Corollary 8.1.** *Let  $p = 2$  and  $d > 1$  be odd. Let  $J_d$  be the Jacobian of  $\mathcal{C}_d : y^d = x(1 - x)$ . Then:*

- (1) [EP13, special case of Thm. 1.3] *the Ekedahl–Oort type of  $J_d[2]$  has the form*

$$[0, 1, 1, 2, 2, \dots, \lfloor g_d/2 \rfloor].$$

- (2) [Sub75, Theorem 4.2] (*Deuring–Shafarevich formula*) *the 2-rank of  $J_d[2]$  is 0;*
- (3) [EP13, Prop. 3.4] *the  $a$ -number of  $J_d[2]$  is  $\frac{d-1}{4}$  if  $d \equiv 1 \pmod{4}$  and  $\frac{d+1}{4}$  if  $d \equiv 3 \pmod{4}$ ;*
- (4) [AP15, Application 5.3] *the  $s_{1,1}$ -multiplicity of  $J_d[2]$  is 1 if  $d \equiv 0 \pmod{3}$  and is 0 otherwise.*

## 9. The $a$ -number of the Fermat quotient curve

Suppose  $p$  is odd. Let  $J_d$  be the Jacobian of the curve  $\mathcal{C}_d$  with affine model  $y^d = x(1 - x)$ . We find a closed-form formula for the  $a$ -number of  $J_d$  and some information about its  $p$ -rank.

If  $d = 1, 2$ , then  $\mathcal{C}_d$  is rational; we exclude this trivial case in the next results.

**Proposition 9.1.** *Suppose  $p$  is odd,  $d > 2$ , and  $p$  does not divide  $d$ . Then:*

- (1) *The  $a$ -number of  $J_d[p]$  is*

$$\begin{aligned} \sum_{j=1}^{(p-1)/2} \left( \left\lfloor \frac{2jd}{2p} \right\rfloor - \left\lfloor \frac{(2j-1)d}{2p} \right\rfloor \right) \\ = \frac{(p-1)d}{p} \frac{1}{4} - \sum_{j=1}^{(p-1)/2} \left( \left\langle \frac{2jd}{2p} \right\rangle - \left\langle \frac{(2j-1)d}{2p} \right\rangle \right). \end{aligned}$$

Here  $\langle \cdot \rangle$  denotes the fractional part.

- (2) *If  $d \equiv \pm 1 \pmod{2p}$ , then the  $a$ -number of  $J_d$  is*

$$(p-1)(d \mp 1)/4p.$$



(3) If  $d \equiv p \pm 1 \pmod{2p}$ , then the  $a$ -number of  $J_d$  is

$$(p - 1)(d \pm (p - 1))/4p.$$

*Remark 9.2.* An analogue of parts (2) and (3) for the Fermat curve  $F_d$  was proven by Montanucci and Speziali [MonS18] using the Cartier operator.

**Proof.** By Proposition 5.9, the  $a$ -number of  $J_d[p]$  is  $\mu(-fv)$  which equals  $\mu(-vf)$  since  $J_p[d]$  is self-dual. This is the number of elements  $a \in S$  such that the pattern of  $a$  ends with  $fv$ . These are precisely the elements with  $a \in S_v$  and  $pa \in S_f$ , and we may count them using archimedean considerations. More precisely,  $a \in S_v$  means  $0 < a < d/2$  and  $pa \in S_f$  means that (the least positive residue of)  $pa$  satisfies  $d/2 < pa < d$ .

We prove part (1) and leave the other parts as exercises. The elements of  $S$  which contribute to the  $a$ -number are represented by integers satisfying one of the inequalities

$$\frac{d}{2p} < a < \frac{2d}{2p}, \quad \frac{3d}{2p} < a < \frac{4d}{2p}, \quad \dots, \quad \frac{(p-2)d}{2p} < a < \frac{(p-1)d}{2p},$$

and the number of such integers is the left hand side of the displayed equation in part (1). The equality in the displayed equation in part (1) is immediate from the definitions of  $\lfloor \cdot \rfloor$  and  $\langle \cdot \rangle$ .  $\square$

*Remark 9.3.* If  $p$  is large and  $d$  is large with respect to  $p$ , by Proposition 9.1, the  $a$ -number of  $J_d$  is close to  $\frac{(p-1)d}{4p}$  which is close to  $g/2$ . The second expression in part (1) shows that the difference is less than  $(p-1)/2$  in absolute value. Numerical experiments suggest that the difference is bounded by  $\frac{(p-1)^2}{4p}$  and part (3) shows that the difference equals this when  $d \equiv p \pm 1 \pmod{2p}$ .

An abelian variety  $A$  is *superspecial* if its  $a$ -number is equal to its dimension. This is equivalent to  $A$  being isomorphic to a product of supersingular elliptic curves. The next result shows that  $J_d$  is superspecial if and only if  $\mathcal{C}_d$  is a quotient of  $\mathcal{C}_{p+1}$  by a subgroup of  $\mu_{p+1}$ .

**Proposition 9.4.** *Suppose  $d > 2$  and  $p \nmid d$ . Then  $J_d$  is superspecial if and only if  $d$  divides  $p + 1$ .*

An analogue of Proposition 9.4 for  $F_d$  was proven by Kodama and Washio [KoW88, Cor. 1, p. 192]. The “if” direction of our proposition follows from their result.

**Proof.** By Proposition 5.9(2),  $J_d$  is superspecial if and only if  $\mu((fv)^{g/2}) = g$ . This is the case if and only if  $p$  exchanges  $S_f$  and  $S_v$ , i.e.,  $pS_f = S_v$  and  $pS_v = S_f$ . Note also that if this statement holds for  $(p, d)$  then it holds for  $(p, d')$  for any divisor  $d' > 2$  of  $d$ .

We claim that if  $H \subset (\mathbb{Z}/d\mathbb{Z})^\times$  is a subgroup such that  $hS_v = S_v$  (equivalently  $hS_f = S_f$ ) for all  $h \in H$ , then  $H = \{1\}$ . Suppose that  $H$  is such a subgroup,  $1 < a < d$ , and the class of  $a$  lies in  $H$ . If  $a > d/2$ , then  $a$  sends

$1 \in S_v$  to  $a \notin S_v$ , a contradiction. If  $a < d/2$ , then there is an integer  $b$  in the interval  $(d/2a, d/a)$ , so  $b \in S_v$ , and  $d/2 < ab < d$ , so the class of  $ab$  lies in  $S_f$ , again a contradiction. Thus  $H = \{1\}$ .

Suppose that  $p$  exchanges  $S_f$  and  $S_v$ . By applying the claim to  $H = \langle p^2 \rangle$ , we see that  $p$  has order 2 modulo  $d$ , and the same holds for  $p$  modulo  $d'$  for any divisor  $d'$  of  $d$ . Since 1 does not exchange  $S_f$  and  $S_v$ , the order of  $p$  is exactly 2 modulo any  $d' > 2$  dividing  $d$ . If  $d'$  is an odd prime power, this implies  $p \equiv -1 \pmod{d'}$ .

More generally, let  $d' = 2^e$  be the largest power of 2 dividing  $d$ . If  $d' = 1$  or 2, then  $p \equiv -1 \pmod{d'}$ . If  $d' = 4$ , there is a unique class of order exactly 2 modulo  $d'$ , namely  $-1$ , and again  $p \equiv -1 \pmod{d'}$ . Finally, if  $e > 2$ , then there are three elements of order exactly 2 modulo  $d'$ , but only one of them reduces to an element of order 2 modulo  $2^{e-1}$ , namely  $-1$ . Again, we find  $p \equiv -1 \pmod{d'}$ . In all three cases,  $p \equiv -1 \pmod{d'}$  and so  $p \equiv -1 \pmod{d}$ , as required.  $\square$

**9.1. Observations about the  $p$ -rank.** The behavior of the  $p$ -rank,  $s_{1,1}$ -multiplicity, and  $u_{1,1}$ -number of  $J_d$  for arbitrary  $p$  and  $d$  seems rather erratic. In later sections, we give closed form formulas for these invariants under restrictions on  $d$ . Here we include some observations about when the  $p$ -rank is as large or small as possible.

An abelian variety is *ordinary* if its  $p$ -rank is equal to its dimension. The next result shows that  $J_d$  is ordinary if and only if  $\mathcal{C}_d$  is a quotient of  $\mathcal{C}_{p-1}$  by a subgroup of  $\mu_{p-1}$ .

**Proposition 9.5.** *Suppose  $d > 2$  and  $p \nmid d$ . Then  $J_d$  is ordinary if and only if  $d$  divides  $p - 1$ .*

An analogue of Proposition 9.5 for the Fermat curve  $F_d$  was proven by Yui [Yui80, Thm. 4.2] using exponential sums; (see also [Gon97, Prop 5.1] for a proof using the Cartier operator when  $d$  is prime). The “if” direction of our proposition can be deduced from Yui’s result.

**Proof.** By Proposition 5.9(1),  $J_d$  is ordinary if and only if  $\mu(f^\ell) = g$ . If  $d$  divides  $p - 1$ , then  $\ell = 1$ , the orbits of  $\langle p \rangle$  on  $S \subset \mathbb{Z}/d\mathbb{Z}$  are singletons, and  $\mu(f) = |S_f| = g$ , so  $J_d$  is ordinary.

The converse follows from the claim in the proof of Proposition 9.4.  $\square$

At the opposite extreme, the  $p$ -rank of  $J_d$  is 0 when  $d$  divides  $p + 1$ , by Proposition 9.4. If  $J_d$  is supersingular, then the  $p$ -rank of  $J_d$  is zero (the converse is not necessarily true for  $g \geq 3$ ); by [Gon97, Prop. 5.1],  $J_d$  is supersingular when  $d$  is prime and the order of  $p$  in  $(\mathbb{Z}/d\mathbb{Z})^\times$  is even.

**9.2. Breaks in words.** We end this section with an elementary combinatorial result used later.

Fix  $d$  and let  $\ell$  be the order of  $\langle p \rangle \subset (\mathbb{Z}/d\mathbb{Z})^\times$ . Given a pattern of length  $\ell$ , say  $w = u_{\ell-1} \cdots u_0$ , we say that  $0 \leq j < \ell - 1$  is a *break* of  $w$  if  $u_{j+1} \neq u_j$ ,

and we say  $j = \ell - 1$  is a break of  $w$  if  $u_0 \neq u_{\ell-1}$ . If  $k$  is the number of breaks of  $w$ , then  $k$  is even and  $0 \leq k \leq \ell$ . Moreover, a pattern  $w$  is determined by its set of breaks and by its last letter  $u_0$ , and there are  $2^{\binom{\ell}{k}}$  words of length  $\ell$  with  $k$  breaks. The sum of these numbers for  $0 \leq k \leq \ell$  equals  $2^\ell$ .

We also consider “self-dual” words of length  $\ell = 2\lambda$ , i.e., words of the form  $w^c \cdot w$  where  $w$  has length  $\lambda$ . Such a word is determined by its last half  $w$ , and we may encode  $w$  by specifying its last letter and its “breaks” as above, ignoring the last potential break: If  $w = u_{\lambda-1} \cdots u_0$  we say that  $0 \leq j < \lambda - 1$  is a break if  $u_{j+1} \neq u_j$ . (We ignore  $j = \lambda - 1$  because whether or not  $u_{\lambda-1}$  is a break of  $w^c \cdot w$  is already determined by the other data.) There are  $2^{\binom{\lambda-1}{k}}$  words  $w$  with  $k$  breaks. The sum of these numbers for  $0 \leq k \leq \lambda - 1$  is  $2^\lambda$ .

Fix  $\ell \geq 1$ . As in (7.1), list all words of length  $\ell$  which begin with  $f$  in lexicographic order:

$$w_0 = f^\ell, w_1 = f^{\ell-1}v, w_2 = f^{\ell-2}vf, \dots, w_{2^{\ell-1}-1} = fv^{\ell-1}.$$

Let  $k(i)$  be the number of breaks of  $w_i$  (not looking at possible wrap-around breaks).

**Lemma 9.6.** *The function  $k(i)$  has the properties: (i)  $k(0) = 0$ ; and (ii) if  $2^j \leq i < 2^{j+1}$ , then  $k(i) = k(2^{j-1} - 1 - i) + 1$ , and it is characterized by these properties.*

**Proof.** Clearly  $k(0) = 0$ . In the list of words,  $w_i$  is the binary representation of the integer  $i$  where  $f$  stands for 0,  $v$  stands for 1, and the leftmost letters are the most significant digits. Thus, if  $2^j \leq i < 2^{j+1}$ , then  $w_i = f^{\ell-j-1}vt$  and  $w_{2^{j-1}-1-i} = f^{\ell-j}(t^c)$  for some  $t$ , and it is visible that  $w_i$  has one more break than  $w_{2^{j-1}-1-i}$  does. This proves the second property of  $k$ . The two properties clearly characterize  $k$ .  $\square$

The function  $i \mapsto k(i)$  is independent of  $\ell$  if  $2^{\ell-1} > i$ . Its first few values are:

$i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$k(i)$	0	1	2	1	2	3	2	1	2	3	4	3	2	3	2	1

### 10. $J_d[p]$ in the “encompassing” case

Suppose  $p$  is odd. In this section, we fix  $d = p^\ell - 1$ , which we call the “encompassing” case. The reason is that if  $p \nmid d'$ , then  $d'$  divides  $p^\ell - 1$  for some  $\ell$ , with the quotient  $(p^\ell - 1)/d'$  being prime-to- $p$ , and so  $J_{d'}[p]$  is a direct factor of  $J_{p^\ell-1}[p]$ . Let  $S = S_f \cup S_v$  and  $\pi$  be defined as before.

**10.1.  $p$ -adic digits.** Elements  $a \in S$  correspond to  $p$ -adic expansions

$$a = a_0 + a_1p + \cdots + a_{\ell-1}p^{\ell-1}. \tag{10.1}$$

where  $a_i \in \{0, \dots, p - 1\}$  and we exclude the following cases: all  $a_i = 0$  (when  $a = 0$ ); all  $a_i = p - 1$  (when  $a = d$ ); and the case all  $a_i = (p - 1)/2$

(when  $a = d/2$ ). Multiplication by  $p$  corresponds to permuting the digits cyclically.

**10.2. Multiplicities.** Given  $a \in S$ , let  $\text{Pat}(a) = u_{\ell-1} \cdots u_0$  be the pattern of  $a$ . Let (10.1) be the  $p$ -adic expansion of  $a$ . Then  $a \in S_v$  (meaning  $u_0 = v$ ) if and only if  $a < d/2$ . In other words, the condition is that the first  $p$ -adic digit to the left of  $a_{\ell-1}$  (inclusive) which is not  $(p-1)/2$  is in fact less than  $(p-1)/2$ .

Similarly  $a \in S_f$  (meaning  $u_0 = f$ ) if and only if  $a > d/2$ . This is true if and only if the first  $p$ -adic digit to the left of  $a_{\ell-1}$  (inclusive) which is not  $(p-1)/2$  is in fact greater than  $(p-1)/2$ . The other letters  $u_j$  of  $\text{Pat}(a)$  are determined similarly by looking at the  $p$ -adic digits of  $a$  to the left of  $a_{\ell-1-j}$ . (Finding the first digit  $\neq (p-1)/2$  may require wrapping around.)

For example, if  $\ell = 4$  and  $p > 3$ , then  $\text{Pat}(a) = f f v f$  when

$$a = (p-1)/2 + (p-2)p + 0p^2 + (p-2)p^3.$$

The following proposition records the ‘‘multiplicities’’ of each pattern.

**Proposition 10.1.** *Let  $p$  be odd and  $d = p^\ell - 1$  and define  $S = S_f \cup S_v$  as usual.*

- (1) *For  $w \in \mathcal{W}_\ell$ , write  $\mu(w)$  for the number of elements  $a \in S$  with  $\text{Pat}(a) = w$ . Then*

$$\mu(f^\ell) = \mu(v^\ell) = \left(\frac{p+1}{2}\right)^\ell - 2.$$

*If  $w$  has  $k > 0$  breaks, then*

$$\mu(w) = \left(\frac{p-1}{2}\right)^k \left(\frac{p+1}{2}\right)^{\ell-k}.$$

- (2) *More generally,*

$$\mu(-f^e) = \mu(-v^e) = \left(\frac{p+1}{2}\right) p^{\ell-e} - 2,$$

$$\begin{aligned} \mu(-v^{e_k} \cdots f^{e_1}) &= \mu(-f^{e_k} \cdots v^{e_1}) \\ &= \left(\frac{p+1}{2}\right)^{\sum e_j - k} \left(\frac{p-1}{2}\right)^{k-1} \left(\frac{p^{\ell+1} - \sum e_j - 1}{2}\right), \end{aligned}$$

*and*

$$\begin{aligned} \mu(-f^{e_{k+1}} v^{e_k} \cdots f^{e_1}) &= \mu(-v^{e_{k+1}} f^{e_k} \cdots v^{e_1}) \\ &= \left(\frac{p+1}{2}\right) p^{\sum e_j - k - 1} \left(\frac{p-1}{2}\right)^k \left(\frac{p^{\ell+1} - \sum e_j + 1}{2}\right). \end{aligned}$$

**Proof.** We prove part (1) and leave part (2) for the reader. The number of elements  $a \in S$  whose pattern is  $f^\ell$  or  $v^\ell$  is  $(\frac{p+1}{2})^\ell - 2$ . Indeed, for  $f^\ell$  (resp.  $v^\ell$ ), we may choose each  $a_j$  freely with  $(p-1)/2 \leq a_j \leq p-1$  (resp.  $0 \leq a_j \leq (p-1)/2$ ) except that we may not take them all to be  $(p-1)/2$  nor all  $p-1$  (resp. 0). This proves the first claim in (1).

Suppose  $w$  is a pattern with breaks. Then the inequalities at the beginning of this subsection show that an element  $a$  with pattern  $w$  should have digits  $a_j$  satisfying:

$$\begin{cases} a_j \leq (p-1)/2 & \text{if } u_{\ell-1-j} = v \text{ and } j \text{ is not a break of } w, \\ a_j < (p-1)/2 & \text{if } u_{\ell-1-j} = v \text{ and } j \text{ is a break of } w, \\ a_j \geq (p-1)/2 & \text{if } u_{\ell-1-j} = f \text{ and } j \text{ is not a break of } w, \\ a_j > (p-1)/2 & \text{if } u_{\ell-1-j} = f \text{ and } j \text{ is a break of } w. \end{cases}$$

The count displayed in the second claim in (1) is then immediate. □

**Theorem 10.2.** *Let  $p$  be odd and  $d = p^\ell - 1$ . Let  $\mathcal{C}_d$  be the curve  $y^d = x(1-x)$  and let  $J_d$  be its Jacobian. Then the Ekedahl-Oort type of  $J_d$  has the form*

$$[\nearrow^{\mu_0} \rightarrow^{\mu_1} \dots \rightarrow^{\mu_{2^{\ell-1}-1}}]$$

where  $\mu_0 = (\frac{p+1}{2})^\ell - 2$  and for  $1 \leq i \leq 2^{\ell-1} - 1$ , letting  $k(i)$  be the function in Lemma 9.6,

$$\mu_i = \begin{cases} \left(\frac{p-1}{2}\right)^{k(i)} \left(\frac{p+1}{2}\right)^{\ell-k(i)} & \text{if } i \text{ is even,} \\ \left(\frac{p-1}{2}\right)^{k(i)+1} \left(\frac{p+1}{2}\right)^{\ell-k(i)-1} & \text{if } i \text{ is odd.} \end{cases}$$

**Proof.** This follows immediately from Theorem 7.1, the calculation of multiplicities in Proposition 10.1, and the evaluation of the number of breaks in Lemma 9.6. (One should note that the  $k$  appearing in Proposition 10.1 for  $w_i$  is  $k(i)$  if  $i$  is even, and it is  $k(i) + 1$  if  $i$  is odd.) □

**10.3. Examples.** Suppose  $p$  is odd.

- (1) If  $\ell = 1$ ,  $\mathcal{C}_d$  has genus  $(p-3)/2$  and is ordinary. The list of words as at (7.1) is just  $f$ . The elementary sequence is

$$[\nearrow^{(p-3)/2}] = [1, 2, \dots, (p-3)/2].$$

- (2) If  $\ell = 2$ , the list of words is  $ff, fv$ . The elementary sequence has an increasing section of length  $(p+1)^2/4 - 2$  and a constant section of length  $(p-1)^2/4$ , i.e., it is:

$$\begin{aligned} &[\nearrow^{(p+1)^2/4-2} \rightarrow^{(p-1)^2/4}] \\ &= [1, 2, \dots, (p+1)^2/4 - 2, \dots, (p+1)^2/4 - 2]. \end{aligned}$$

- (3) If  $\ell = 3$ , the list of words is  $f^3, f^2v, fvf, fvv$ . The elementary sequence is:

$$[\nearrow^m \rightarrow^n \nearrow^n \rightarrow^n],$$

where  $m = (p + 1)^3/8 - 2$  and  $n = (p + 1)(p - 1)^2/8$ .

- (4) If  $\ell = 4$ , the word  $f^3v$  occurs; this is the smallest example whose  $BT_1$  group scheme was not previously known to occur as a factor of the  $p$ -torsion of a Jacobian for all primes  $p$ . If  $p \equiv 2, 3 \pmod 5$ , then this group scheme occurs generically for the family of genus 4 curves that are degree 5 cyclic covers of  $\mathbb{P}^1$  given by an equation of the form  $y^5 = x(x - 1)(x - \lambda)$ , see [LMPT19, Notation 6.3, Theorem 7.3].

**Proposition 10.3.** *Let  $p$  be odd and let  $d = p^\ell - 1 > 2$ . Then:*

- (1) *the  $p$ -rank of  $J_d[p]$  is  $(\frac{p+1}{2})^\ell - 2$ ;*
- (2) *the  $a$ -number of  $J_d[p]$  is  $\frac{p-1}{2} \frac{p^{\ell-1}-1}{2}$ ;*
- (3) *the  $s_{1,1}$ -multiplicity of  $J_d[p]$  is 0 if  $\ell$  is odd and is  $(\frac{p^2-1}{4})^{\ell/2}$  if  $\ell$  is even;*
- (4) *and the  $u_{1,1}$ -number of  $J_d$  is the sum of the  $s_{1,1}$ -multiplicity and*

$$\sum_{j=0}^{\lfloor (\ell-4)/2 \rfloor} \left(\frac{p+1}{2}\right)^2 \left(\frac{p-1}{2}\right)^{2j+1} \binom{p^{\ell-3-2j}-1}{2}.$$

**Proof.** This follows from Proposition 5.9 using the multiplicities in Proposition 10.1. □

### 11. $J_d[p]$ in the Hermitian case

Suppose  $p$  is odd. Fix an integer  $\lambda \geq 1$  and let  $d = p^\lambda + 1$ . In this case, the Fermat curve of degree  $d$  is isomorphic (over  $\overline{\mathbb{F}}_p$ ) to the Hermitian curve  $H_q$  with equation  $y_1^{q+1} = x_1^q + x_1$  where  $q = p^\lambda$ . It is well known that  $H_q$  is supersingular and its Ekedahl–Oort type was studied in [PW15]. Since  $C_d$  is a quotient of  $H_q$ , it is also supersingular in this case.

**11.1.  $p$ -adic digits.** Let  $S = \mathbb{Z}/d\mathbb{Z} \setminus \{0, d/2\}$ . Let  $\pi : S \rightarrow S$  be induced by multiplication by  $p$ . Let  $S_v = \{b \in S \mid 0 < b < d/2\}$  and  $S_f = \{b \in S \mid d/2 < b < d\}$ . Let

$$S' = \{(b_1, \dots, b_\lambda) \mid 0 \leq b_j \leq p - 1 \text{ and not all } b_j = (p - 1)/2\}.$$

There is a bijection  $S' \rightarrow S$  given by

$$(b_1, \dots, b_\lambda) \mapsto b = 1 + \sum_{j=1}^{\lambda} b_j p^{j-1}.$$

Under this bijection, the permutation  $\pi$  is given by

$$(b_1, \dots, b_\lambda) \mapsto (p - 1 - b_\lambda, b_1, \dots, b_{\lambda-1}).$$

An element  $b$  belongs to  $S_v$  if and only if  $b < (p^\lambda + 1)/2$ . This is true if and only if, in the tuple, the entry  $b_j$  with largest  $j$  such that  $b_j \neq (p - 1)/2$  has the property that  $b_j < (p - 1)/2$ .

**11.2. Multiplicities.** The multiplicative order of  $p$  modulo  $d$  is  $\ell = 2\lambda$ . We define a map  $\text{Pat}' : S \rightarrow \mathcal{W}_\lambda$  as follows: Given  $b \in S$ , the *pattern*  $\text{Pat}'(b)$  of  $b$  is the word  $w = u_{\lambda-1} \cdots u_0$  given by

$$u_j = f \text{ if } p^j b \in S_f \text{ and } u_j = v \text{ if } p^j b \in S_v.$$

(The notation  $\text{Pat}'$  is used to distinguish this from the pattern in the encompassing case.) If the word for  $b$  has length  $\ell$  (the maximum length), then it is  $\text{Pat}'(b)^c \cdot \text{Pat}'(b)$  (where the  $c$  stands for the complementary word). For any  $b \in S$ , the word for  $b$  has a power with length  $\ell$  and this power equals  $\text{Pat}'(b)^c \cdot \text{Pat}'(b)$ . Note that since  $p^\lambda = -1 \pmod{d}$ , the word of  $b$  is “self-dual”, i.e., of the form  $t^c t$ , for every  $b \in S$ .

For a word  $w$  of length  $\lambda$ , let  $\mu'(w)$  be the number of elements  $b \in S$  with  $\text{Pat}'(b) = w$ . For a word  $t$  of length  $\leq \lambda$ , let  $\mu'(-t)$  be the number of elements  $b \in S$  with  $\text{Pat}'(b) = t' \cdot t$  for some  $t'$ , in other words, the number of  $b$  with pattern ending in  $t$ .

**Proposition 11.1.** *Let  $p$  be odd.*

- (1) *Suppose  $e_1, \dots, e_k$  are positive integers with  $\sum e_i = \lambda$ . If  $k$  is odd, then*

$$\mu'(f^{e_k} v^{e_{k-1}} \cdots f^{e_1}) = \mu'(v^{e_k} f^{e_{k-1}} \cdots v^{e_1}) = \left(\frac{p+1}{2}\right)^{\lambda-k} \left(\frac{p-1}{2}\right)^k,$$

*and if  $k$  is even, then*

$$\mu'(v^{e_k} \cdots f^{e_1}) = \mu'(f^{e_k} \cdots v^{e_1}) = \left(\frac{p+1}{2}\right)^{\lambda+1-k} \left(\frac{p-1}{2}\right)^{k-1}.$$

- (2) *More generally, given integers  $e_1, \dots, e_k > 0$ , let  $\lambda' = \sum e_i$  and suppose  $\lambda' \leq \lambda$ . If  $k$  is odd, and  $t$  has the form  $t = f^{e_k} v^{e_{k-1}} \cdots v^{e_2} f^{e_1}$ , then*

$$\mu'(-t) = \mu'(-t^c) = \left(\frac{p+1}{2}\right)^{\lambda'-k} \left(\frac{p-1}{2}\right)^{k-1} \left(\frac{p^{\lambda+1-\lambda'} - 1}{2}\right),$$

*and if  $k$  is even, and  $t$  has the form  $t = v^{e_k} \cdots f^{e_1}$ , then*

$$\mu'(-t) = \mu'(-t^c) = \left(\frac{p+1}{2}\right)^{\lambda'-k} \left(\frac{p-1}{2}\right)^{k-1} \left(\frac{p^{\lambda+1-\lambda'} + 1}{2}\right).$$

Part (1) of Proposition 11.1 contradicts [PW15, Lemma 4.3], which we believe is in error.

**Proof.** Part (1) follows from part (2), so we will prove the latter. It is clear that  $\mu'(-t) = \mu'(-t^c)$ .

Suppose  $k$  is odd and  $t = f^{e_k} v^{e_{k-1}} \cdots v^{e_2} f^{e_1}$ . Write  $f^{e_k} \cdots f^{e_1} = u_{\lambda'-1} \cdots u_0$  with  $u_j \in \{f, v\}$ . Then  $b \in S$  has pattern  $-t$  if and only if the  $p$ -adic digits

$(b_1, \dots, b_\lambda)$  satisfy, for  $\lambda + 1 - \lambda' < j \leq \lambda$ ,

$$\begin{cases} b_j \leq (p - 1)/2 & \text{if } u_{\lambda-j} = v \text{ and } j \text{ is not a break of } t, \\ b_j < (p - 1)/2 & \text{if } u_{\lambda-j} = v \text{ and } j \text{ is a break of } t, \\ b_j \geq (p - 1)/2 & \text{if } u_{\lambda-j} = f \text{ and } j \text{ is not a break of } t, \\ b_j > (p - 1)/2 & \text{if } u_{\lambda-j} = f \text{ and } j \text{ is a break of } t, \end{cases} \tag{11.1}$$

and the number corresponding to the tuple  $\beta = (b_1, \dots, b_{\lambda+1-\lambda'})$  is large, namely

$$p^{\lambda+1-\lambda'} + 1 > 1 + \sum_{j=1}^{\lambda+1-\lambda'} b_j p^{j-1} > (p^{\lambda+1-\lambda'} + 1)/2.$$

So there are  $(p^{\lambda+1-\lambda'} - 1)/2$  choices for  $\beta$ . Taking the product with the number of possibilities for  $b_j$  for  $\lambda + 1 - \lambda' < j \leq \lambda$  yields the quantity in the statement.

Similarly, if  $k$  is even and  $t = v^{e_k} \cdots f^{e_1} = u_{\lambda'-1} \cdots u_0$  with  $u_j \in \{f, v\}$ , then  $b \in S$  has pattern  $-t$  if and only if the  $p$ -adic digits  $(b_1, \dots, b_\lambda)$  satisfy (11.1), for  $\lambda + 1 - \lambda' < j \leq \lambda$  and the number corresponding to the tuple  $\beta$  is small, namely

$$0 < 1 + \sum_{j=1}^{\lambda+1-\lambda'} b_j p^{j-1} \leq (p^{\lambda+1-\lambda'} + 1)/2.$$

So there are  $(p^{\lambda+1-\lambda'} + 1)/2$  choices for  $\beta$ . Again, taking the product with the number of possibilities for  $b_j$  for  $\lambda + 1 - \lambda' < j \leq \lambda$  yields the quantity in the statement.  $\square$

**Theorem 11.2.** *Let  $p$  be odd and  $d = p^\lambda + 1$ . Let  $C_d$  be the curve  $y^d = x(1 - x)$  and let  $J_d$  be its Jacobian. Then the Ekedahl–Oort type of  $J_d$  has the form  $[\rightarrow \mu'_0]$  if  $\lambda = 1$  and*

$$[\rightarrow \mu'_0 \nearrow \mu'_1 \dots \nearrow \mu'_{2^\lambda-1-1}]$$

if  $\lambda > 1$ , where, letting  $k(i)$  be the function described in Lemma 9.6,

$$\mu'_i = \begin{cases} \left(\frac{p+1}{2}\right)^{\lambda-k(i)-1} \left(\frac{p-1}{2}\right)^{k(i)+1} & \text{if } i \text{ is even,} \\ \left(\frac{p+1}{2}\right)^{\lambda-k(i)} \left(\frac{p-1}{2}\right)^{k(i)} & \text{if } i \text{ is odd.} \end{cases}$$

**Proof.** This follows immediately from Theorem 7.1, the multiplicities in Proposition 11.1, and the number of breaks in Lemma 9.6. (Note that the  $k$  in Proposition 11.1 for  $w_i$  is  $k(i) + 1$ .)  $\square$



**11.3. Examples.** Let  $p$  be odd.

- (1) If  $\lambda = 1$ , the curve  $\mathcal{C}_d$  has genus  $(p-1)/2$  and is superspecial: the list of words starting with  $f$  and with positive multiplicity is  $fv$ , and the elementary sequence is

$$[\rightarrow^{(p-1)/2}] = [0, \dots, 0].$$

- (2) If  $\lambda = 2$ , the list of words is  $ffvv, fvvf$ , and the elementary sequence has a constant section of length  $(p^2-1)/4$  and an increasing section of length  $(p^2-1)/4$ :

$$[\rightarrow^{(p^2-1)/4} \nearrow^{(p^2-1)/4}] = [0, \dots, 0, 1, 2, \dots, (p^2-1)/4].$$

- (3) If  $\lambda = 3$ , the list of words is  $f^3v^3, f^2v^3f, (fv)^3, fv^3f^2$ , and the elementary sequence has four segments and has the form

$$[\rightarrow^m \nearrow^m \rightarrow^n \nearrow^m],$$

where  $m = (p+1)^2(p-1)/8$  and  $n = (p-1)^3/8$ .

**Proposition 11.3.** *Let  $p$  be odd, let  $\lambda$  be a positive integer, and let  $d = p^\lambda + 1$ . Then:*

- (1) *the  $p$ -rank of  $J_d[p]$  is 0;*
- (2) *the  $a$ -number of  $J_d[p]$  is  $(p-1)(p^{\lambda-1} + 1)/4$ ;*
- (3) *the  $s_{1,1}$ -multiplicity of  $J_d[p]$  is 0 if  $\lambda$  is even and  $(\frac{p-1}{2})^\lambda$  if  $\lambda$  is odd;*
- (4) *and the  $u_{1,1}$ -number of  $J_d[p]$  is the sum of the  $s_{1,1}$ -multiplicity and*

$$\sum_{j=0}^{\lfloor (\lambda-4)/2 \rfloor} \binom{p+1}{2}^2 \binom{p-1}{2}^{2j+1} \binom{p^{\lambda-3-2j} + 1}{2} + \begin{cases} 0 & \text{if } \lambda = 1, \\ \binom{p+1}{2}^2 \binom{p-1}{2}^{\lambda-2} & \text{if } \lambda > 1 \text{ and odd,} \\ \binom{p+1}{2} \binom{p-1}{2}^{\lambda-1} & \text{if } \lambda \text{ even.} \end{cases}$$

The analogue of the  $a$ -number calculation in part (2) for the Fermat curve of degree  $d = p^\lambda + 1$  is given in [Gro90, Prop. 14.10].

**Proof.** This follows from Proposition 5.9 using the multiplicities in Proposition 11.1. □

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