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On local automorphisms of some quantum mechanical structures of Hilbert space operators

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Abstract. In this paper we substantially strengthen several formerly obtained results stating that all 2-local automorphisms of certain quantum structures consisting of Hilbert space operators are necessarily automorphisms.

Contents

1. The structures under consideration and their automorphism groups

In this paper we present results on the automorphism groups of various quantum mechanics related structures which consist of bounded linear operators acting on a complex Hilbert space. Our results have a quite common content stating that the majority of those automorphism groups are very rigid in a certain sense, their elements are very strongly determined by their local actions. The precise meaning of this will be given below in the second section.

Let us first introduce the structures and their automorphism groups which we consider in the paper. Let H be a complex Hilbert space with $\dim H > 1$.

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We denote by $\mathbb{B}(H)$ the C^*-algebra of all bounded linear operators acting on H. An operator $A \in \mathbb{B}(H)$ is called positive semidefinite if $\langle Ax, x\rangle \geq 0$ holds for all $x \in H$, where $\langle ., .\rangle$ stands for the inner product in H. The collection of all positive semidefinite operators is denoted by $\mathbb{B}(H)^+$. This concept of positivity induces the natural (Löwner) partial order: if A, B are self-adjoint elements of $\mathbb{B}(H)$, then we write $A \leq B$ if and only if $B - A$ is a positive semidefinite operator.

The sets of operators which we will deal with are the following:

- the set $\mathbb{S}(H)$ of all self-adjoint elements of $\mathbb{B}(H)$;
- the set $\mathbb{P}(H)$ of all orthogonal projections on H;
- the set $\mathbb{D}(H)$ of all density operators on H, i.e., the set of all positive semidefinite operators A with $\operatorname{Tr} A = 1$, where Tr is the usual trace functional;
- the set $\mathbb{E}(H)$ of all Hilbert space effects which consists of all positive semidefinite operators A on H which are bounded by the identity, $0 \leq A \leq I$.
- the set $\mathbb{B}(H)^{++}$ of all positive definite operators (i.e., invertible positive semidefinite operators) on H.

According to the mathematical formalism of quantum mechanics introduced by von Neumann, the elements of these sets have physical contents. If a quantum system is represented by the Hilbert space H, then the operators in $\mathbb{S}(H)$ correspond to (bounded) quantum observables. The elements of $\mathbb{P}(H)$ can be viewed as propositions about those observables. The density operators, the elements of $\mathbb{D}(H)$ describe the (mixed) states of the quantum system, and the elements of $\mathbb{E}(H)$ correspond to yes-no measurements on the system which can be unsharp.

There are important relevant algebraic operations on those collections of operators and the objects of our present investigations are the corresponding automorphism groups. Concerning the content of the next few paragraphs we refer the reader to the following sources: the paper [5], Chapter 5 in the book [15], Section 0.3 in the Introduction of the monograph [23], and the seminal paper [30].

Let us recall that a conjugate linear surjective isometry on H is called an antiunitary operator. On any domain $\mathcal{D} \subset \mathbb{B}(H)$, transformations of the form $A \mapsto UAU^*$, where U is either a unitary or an antiunitary operator on H, are called unitary-antiunitary conjugations.

After these, the operations in question on the above introduced sets and the corresponding automorphism groups are the following. The most important and natural products on $\mathbb{S}(H)$ are the ring theoretical and the algebraic Jordan products which are the operations $(A, B) \mapsto AB + BA$ and $(A, B) \mapsto (1/2)(AB + BA)$. The corresponding automorphisms of the Jordan ring or Jordan algebra $\mathbb{S}(H)$ are usually called Jordan-Segal automorphisms (Chapter 5 in [15], [30]). The structure of all linear bijections of $\mathbb{S}(H)$ which preserve any one of those two Jordan operations are well-known to be unitary-antiunitary congruence transformations ([5], [15], [30]). Actually more is true, we can omit the linearity condition and still get the same conclusion. Indeed, Theorem 2.2 in [1] tells the following.

For any complex Hilbert space H with $\dim H > 1$, the bijective transformations $\phi : \mathbb{S}(H) \to \mathbb{S}(H)$ satisfying either

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A), \quad A, B \in \mathbb{S}(H)$$

or

$$\phi((1/2)(AB + BA)) = (1/2)(\phi(A)\phi(B) + \phi(B)\phi(A)), \quad A, B \in \mathbb{S}(H)$$

are exactly the maps of the form

$$\phi(A) = UAU^*, \quad A \in \mathbb{S}(H),$$

where U is either a unitary or an antiunitary operator on H, i.e., those maps are exactly the unitary-antiunitary congruence transformations on $\mathbb{S}(H)$.

It is well-known that the linear Jordan-Segal automorphisms automatically preserve the so-called Jordan triple product which is the operation $(A, B) \mapsto ABA$ (see, e.g., the argument given in the proof of (c) in 6.3.2 Lemma in [27]). It is an interesting fact that the linearity can be dropped also in relation with the transformations respecting this triple operation. From Theorem 2.1 in [1] we learn the following.

For any complex Hilbert space H with $\dim H > 1$, the bijective transformation $\phi : \mathbb{S}(H) \to \mathbb{S}(H)$ satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \quad A, B \in \mathbb{S}(H)$$

if and only if it is of the form

$$\phi(A) = cUAU^*, \quad A \in \mathbb{S}(H)$$

where $c \in \{-1, 1\}$ and U is either a unitary or an antiunitary operator on H.

The second set we consider is the collection $\mathbb{P}(H)$ of all projections on H equipped with the relation $\leq$ of order and the operation $P \mapsto P^\perp = I - P$ of orthocomplementation. We call the corresponding automorphisms of $\mathbb{P}(H)$ von Neumann automorphisms (Chapter 5 in [15]). Let $\wedge$ stand for the infimum in the lattice of projections ($P \wedge Q$ is the projection projecting onto the intersection of the ranges of P and Q). Assume that $\phi : \mathbb{P}(H) \to \mathbb{P}(H)$ is a bijective map. It is easy to see that we have that ϕ is a von Neumann automorphism meaning that

$$P \leq Q \iff \phi(P) \leq \phi(Q) \quad \text{and} \quad \phi(P)^\perp = \phi(P^\perp)$$

hold for any $P, Q \in \mathbb{P}(H)$ if and only if it satisfies

$$\phi(P \wedge Q^\perp) = \phi(P) \wedge \phi(Q)^\perp, \quad P, Q \in \mathbb{P}(H).$$

Consequently, although the order and the orthocomplementation are two different objects (the first one is a relation and the second one is an operation), the corresponding automorphisms can be expressed using only one operation which is $(P,Q) \mapsto P \wedge Q^\perp$. (Let us point out that the operation $(P,Q) \mapsto P \vee Q^\perp$ could similarly be used to describe the von Neumann automorphisms of $\mathbb{P}(H)$. Indeed, this is apparent from the identity $P \vee Q^\perp = (Q \wedge P^\perp)^\perp$, $P,Q \in \mathbb{P}(H)$. It is just a question of taste which one of the two operations one prefers.) The following statement is well-known ([5], [15]).

For a Hilbert space H with $\dim H \geq 3$, the von Neumann isomorphisms of $\mathbb{P}(H)$ are exactly the unitary-antiunitary congruence transformations on $\mathbb{P}(H)$.

The next set is $\mathbb{D}(H)$, the collection of all density operators on H. It is a convex set and the affine bijections of $\mathbb{D}(H)$ (bijections preserving all convex combinations) are called Kadison automorphisms. Their structure is again the same ([15], [30]).

The Kadison automorphisms are exactly the unitary-antiunitary congruence transformations on $\mathbb{D}(H)$.

In fact, even more is true. Namely, we need not to assume that all convex combinations are respected by our transformation, the preservation of the arithmetic mean alone is sufficient as we show this in the next proposition.

Proposition 1.1. *Let $\phi : \mathbb{D}(H) \to \mathbb{D}(H)$ be a bijective map which preserves the arithmetic mean, i.e., assume that ϕ satisfies*

$$\phi\left(\frac{A+B}{2}\right) = \frac{\phi(A)+\phi(B)}{2}, \quad A,B \in \mathbb{D}(H). \tag{1}$$

Then ϕ is necessarily of the form

$$\phi(A) = UAU^*, \quad A \in \mathbb{D}(H)$$

with a unitary or antiunitary operator U on H.

Proof. Let us recall that functional equations of the form (1) are called Jensen equations. It is shown in the paper [9] that every function on a $\mathbb{Q}$-convex subset of a $\mathbb{Q}$-linear space X into a $\mathbb{Q}$-linear space Y that satisfies the Jensen equation is necessarily of the form $x \mapsto L(x)+C$, with some fixed element $C \in Y$ and additive function $L : X \to Y$. As $\mathbb{D}(H)$ is an $\mathbb{R}$-convex subset of the $\mathbb{R}$-linear space $\mathbb{S}(H)$, we conclude that there is an element $C \in \mathbb{S}(H)$ and an additive map $L : \mathbb{S}(H) \to \mathbb{S}(H)$ such that $\phi(A) = L(A)+C$ holds for every $A \in \mathbb{D}(H)$.

We show that L has a certain homogeneity property. To see this, fix an arbitrary element $B_0 \in \mathbb{D}(H)$ and set $\mathcal{D} = \mathbb{D}(H) - B_0$. We assert that L is bounded from below on $\mathcal{D}$. Indeed, we compute

$$L(\mathcal{D}) = L(\mathbb{D}(H) - B_0) = L(\mathbb{D}(H)) - L(B_0) = \phi(\mathbb{D}(H)) - C - L(B_0)$$

which set is bounded by $-C - L(B_0)$ from below (with respect to the partial order $\leq$). For every $X \in \mathcal{D}$ and $t \in [0,1]$ we have $tX \in \mathcal{D}$. Indeed, if X is of the form $X = A - B_0$ for some $A \in \mathbb{D}(H)$, then

$$tX = t(A - B_0) = tA + (1-t)B_0 - B_0 \in \mathcal{D}.$$

It follows that for any vector $x \in H$, the additive function $t \mapsto \langle L(tX)x, x\rangle$ is bounded from below on the unit interval. A famous result of Ostrowski says that any additive function on the reals which is bounded from below on a set of positive Lebesgue measure is continuous and hence a constant multiple of the identity (see, e.g., Theorem 9.3.1 in [14]). Therefore, we have $\langle L(tX)x, x\rangle = t\langle L(X)x, x\rangle$ for any real number t. Since this holds for every $x \in H$, we have $L(tX) = tL(X)$ for all $X \in \mathcal{D}$ and real number t.

Finally, we can show that ϕ is an affine bijection of $\mathbb{D}(H)$, i.e., a Kadison automorphism, which then implies that it is of the desired form. For any $A, B \in \mathbb{D}(H)$ and $t \in [0, 1]$ we can compute as follows

$$\begin{aligned}\phi(tA + (1-t)B) &= L(tA + (1-t)B) + C\\ &= L(t(A - B_0) + (1-t)(B - B_0)) + L(B_0) + C\\ &= tL(A - B_0) + (1-t)L(B - B_0) + L(B_0) + C\\ &= tL(A) + (1-t)L(B) + C = t\phi(A) + (1-t)\phi(B).\end{aligned}$$

□

Again, the content of the result above is that the automorphisms of $\mathbb{D}(H)$ with respect to one operation, that is one single convex combination, coincide with the automorphisms of $\mathbb{D}(H)$ with respect to a parametrized family of operations, the family of all convex combinations.

The next set is $\mathbb{E}(H)$, the set of all Hilbert space effects. It is again a convex set and its corresponding affine automorphisms are called Ludwig automorphisms ([15]). Their structure was determined in [18]. By Corollary 2 in that paper, for every Ludwig automorphism $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ we have either a unitary or an antiunitary operator U on H such that either

$$\phi(A) = UAU^*, \quad A \in \mathbb{E}(H) \tag{2}$$

or

$$\phi(A) = U(I - A)U^*, \quad A \in \mathbb{E}(H). \tag{3}$$

We observe that also in the case of these types of automorphisms, there is no need to assume the preservation of all convex combinations, that of the arithmetic mean alone is sufficient. Namely, we have the following proposition.

Proposition 1.2. *Let $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ be a bijective map satisfying*

$$\phi\left(\frac{A+B}{2}\right) = \frac{\phi(A) + \phi(B)}{2}, \quad A, B \in \mathbb{E}(H).$$

Then ϕ is necessarily a Ludwig automorphism of $\mathbb{E}(H)$ and hence it is of one of the two forms (2), (3).

Proof. One can follow an argument similar to the proof of Proposition 1.1 but the situation here is simpler: we do not need to translate the convex set $\mathbb{E}(H)$ what we did concerning $\mathbb{D}(H)$. □

In the literature, they also consider the operation of partial addition on $\mathbb{E}(H)$ (i.e., the usual addition restricted for pairs of elements of $\mathbb{E}(H)$ whose sums belong to $\mathbb{E}(H)$) and the corresponding concept of so-called **E**-automorphisms ([5]). The bijective map $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ is said to be an **E**-automorphism if for any $A, B \in \mathbb{E}(H)$ we have

$$A + B \leq I \Longleftrightarrow \phi(A) + \phi(B) \leq I$$

and for any such pair $A, B \in \mathbb{E}(H)$, the following holds

$$\phi(A + B) = \phi(A) + \phi(B).$$

According to the section 6. Conclusion in [5] we have the following.

*For any Hilbert space H with $\dim H > 1$, the **E**-automorphisms are exactly the unitary-antiunitary congruence transformations on $\mathbb{E}(H)$.*

There is another operation on $\mathbb{E}(H)$, the so-called the sequential product $(A, B) \mapsto \sqrt{A}B\sqrt{A}$ which is closely related to the Jordan triple product on $\mathbb{S}(H)$. This operation was introduced by Gudder and Greechie in [10] (also see [11] and [12]). In Theorem 2.7 in [10] they showed that all sequential automorphisms of $\mathbb{E}(H)$ are unitary-antiunitary congruence transformations provided that $\dim H \geq 3$. In Corollary 7 in [21] it was shown that the condition $\dim H \geq 3$ can in fact be dropped. Therefore, we have the following.

For any Hilbert space H with $\dim H > 1$, the bijective map $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ is a sequential automorphism if and only there is either a unitary or an antiunitary operator U on H such that

$$\phi(A) = UAU^*, \quad A \in \mathbb{E}(H).$$

In closing this section, let us point out that the operation $(A, B) \mapsto \sqrt{A}B\sqrt{A}$ provides the most natural K-loop structure on the positive definite cone of a C^*-algebra, see [4]. As it was mentioned in that paper, this structure has important applications among others in connection with the Einstein velocity addition, the operation which plays so fundamental role in the special theory of relativity. As for the corresponding continuous automorphisms of $\mathbb{B}(H)^{++}$, by Theorem 1 in [22] we have the following.

Assume that H is an infinite dimensional Hilbert space. The continuous bijective map $\phi : \mathbb{B}(H)^{++} \to \mathbb{B}(H)^{++}$ satisfies

$$\phi(\sqrt{A}B\sqrt{A}) = \sqrt{\phi(A)}\phi(B)\sqrt{\phi(A)}, \quad A, B \in \mathbb{B}(H)^{++}$$

if and only if there is a unitary or antiunitary operator U on H such that either

$$\phi(A) = UAU^*, \quad A \in \mathbb{B}(H)^{++} \tag{4}$$

or

$$\phi(A) = UA^{-1}U^*, \quad A \in \mathbb{B}(H)^{++}. \tag{5}$$

The case of a finite dimensional Hilbert space is different, then multiplication by a fixed power of the determinant functional can show up, see [22].

2. A new look at 2-local automorphisms

In this section we present our new results. Before doing that, let us recall the following. In the paper [28], Šemrl introduced the fruitful notion of 2-local automorphisms as follows. If $\mathcal{A}$ is any algebra, the map $\phi : \mathcal{A} \to \mathcal{A}$ is called a 2-local automorphism if for any $A, B \in \mathcal{A}$ there is an algebra automorphism $\phi_{A,B}$ of $\mathcal{A}$ such that

$$\phi(A) = \phi_{A,B}(A) \quad \text{and} \quad \phi(B) = \phi_{A,B}(B). \tag{6}$$

It is important to emphasize here (and this concerns the material below, too) that ϕ is not assumed to be linear, surjective or continuous, it is only a simple map having the property above. Observe that in a very similar way we can define the concept of 2-local automorphisms of any other algebraic structures. Also, one can easily introduce concepts of other types of 2-local maps related to given collections of transformations (derivation algebra, isometry group, etc.). It is a remarkable fact if every 2-local automorphism of the algebra $\mathcal{A}$ is in fact an automorphism of $\mathcal{A}$ (implying that we get the bijectivity, linearity, multiplicativity of those 2-local maps "for free"). This reflects a kind of rigidity of the automorphism group, that the automorphisms are completely determined by their actions on the two element subsets of $\mathcal{A}$. If this is the case, then one could say that the automorphism group is 2-reflexive (although this denomination may be somewhat confusing since it is used also in different contexts in the literature). Theorem 1 in [28] is a fundamental result which says that for the infinite dimensional separable Hilbert space H, the group of all algebra automorphisms of $\mathbb{B}(H)$ has that property.

To mention some older and some recent papers concerning 2-local automorphisms, we list the articles [6, 13, 16, 31], [20, 26], and [2, 7, 8, 17].

Regarding the results to be presented below, we especially refer to two papers. Firstly, in [19] it was shown that for the infinite dimensional separable Hilbert space H,

- the 2-local von Neumann automorphisms of $\mathbb{P}(H)$ are von Neumann automorphisms (Proposition in [19]),
- the 2-local Jordan-Segal automorphisms of $\mathbb{S}(H)$ are Jordan-Segal automorphisms (Corollary in [19]).

(As for the finite dimensional cases, we remark that the former result holds true whenever $\dim H \geq 3$, while the second one remains valid even without that restriction.) Secondly, in [3] it was shown that for the infinite dimensional separable Hilbert space H,

- every 2-local Kadison automorphism of $\mathbb{D}(H)$ is a Kadison automorphism (Theorem 2 in [3]),
- any 2-local **E**-automorphism of $\mathbb{E}(H)$ is an **E**-automorphism (Theorem 3 in [3]),

- every 2-local sequential automorphism of $\mathbb{E}(H)$ is the same kind of an automorphism (Theorem 3 in [3] again),
- any 2-local Ludwig automorphism of $\mathbb{E}(H)$ is a Ludwig automorphism (Theorem 4 in [3]).

(Also concerning the above given four results we mention that they are valid in the finite dimensional cases as well.)

The previous six statements could be summed up saying that they demonstrate that the 2-reflexivity property holds for various automorphism groups of quantum structures of Hilbert space operators.

And now about the new results that we have promised in the abstract and which concern a much stronger 2-reflexivity property of several of the automorphism groups in question. As a matter of fact, the starting point of our present investigations is the current paper [25], where the second author has observed that the group of *-automorphisms of the C^*-algebra $\mathbb{B}(H)$ has a much stronger property than the "usual" 2-reflexivity. Namely, it has turned out in [25] that we can add or multiply the two equations in (6) (used to define 2-local automorphisms) hence squeezing them into one equation and we still have the same or almost the same conclusion as above. More precisely, in Theorem 1 in [25] we have proved that assuming H is a separable Hilbert space with $\dim H \geq 3$, if a map $\phi : \mathbb{B}(H) \to \mathbb{B}(H)$ has the property that for any $A, B \in \mathbb{B}(H)$ there is a unitary operator $U_{A,B}$ on H such that

$$\phi(A) + \phi(B) = U_{A,B}(A + B)U_{A,B}^*,$$

then there is a unitary operator $U \in \mathbb{B}(H)$ such that

$$\phi(A) = UAU^*, \quad A \in \mathbb{B}(H).$$

Theorem 2 in [25] tells that for any separable Hilbert space H, if a map $\phi : \mathbb{B}(H) \to \mathbb{B}(H)$ has the property that for any $A, B \in \mathbb{B}(H)$ we have a unitary operator $U_{A,B}$ on H such that

$$\phi(A)\phi(B) = U_{A,B}(AB)U_{A,B}^*,$$

then there is a unitary operator $U \in \mathbb{B}(H)$ such that either

$$\phi(A) = UAU^*, \quad A \in \mathbb{B}(H)$$

or

$$\phi(A) = -UAU^*, \quad A \in \mathbb{B}(H)$$

holds true.

In this section we present results of similar spirit concerning the automorphism groups of the quantum structures of Hilbert space operators which we have listed in the first section. Namely, we will squeeze the two equations defining 2-local automorphisms of any such structure through the corresponding operation to obtain one single equation and investigate whether the transformations satisfying that much weaker assumption are necessarily automorphisms.

We begin with the case of Jordan-Segal automorphisms, more precisely with the automorphisms of $\mathbb{S}(H)$ with respect to any of the two Jordan products $(A, B) \mapsto AB + BA$ and $(A, B) \mapsto (1/2)(AB + BA)$. We show that a transformation of $\mathbb{S}(H)$ which satisfies the equation obtained by taking any of the two Jordan-type products (clearly, the consideration only one of them is sufficient) of the two equations defining 2-local Jordan-Segal automorphisms, is either a Jordan-Segal isomorphism or the negative of a Jordan-Segal isomorphism. More precisely, the result reads as follows.

Theorem 2.1. *Let H be a separable Hilbert space and let $\phi : \mathbb{S}(H) \to \mathbb{S}(H)$ be a transformation with the property that for any $A, B \in \mathbb{S}(H)$ there exists either a unitary or an antiunitary operator U_{AB} on H such that*

$$\phi(A)\phi(B) + \phi(B)\phi(A) = U_{AB}(AB + BA)U_{AB}^*. \tag{7}$$

Then there is a unitary or antiunitary operator U on H such that we have either

$$\phi(A) = UAU^*, \quad A \in \mathbb{S}(H)$$

or

$$\phi(A) = -UAU^*, \quad A \in \mathbb{S}(H).$$

Before presenting the proof, let us recall a famous theorem by Wigner concerning the structure of quantum mechanical symmetry transformations which will play an essential role in the next and also in the other proofs in the paper. Denote by $\mathbb{P}_1(H)$ the set of all rank-one projections on H (its elements represent the pure states of a quantum system). For any pair $P, Q \in \mathbb{P}_1(H)$, the quantity $\operatorname{Tr} PQ$ is called transition probability. Maps $\phi : \mathbb{P}_1(H) \to \mathbb{P}_1(H)$ which preserve the transition probability, i.e., which satisfy

$$\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ, \quad P, Q \in \mathbb{P}_1(H)$$

are called quantum mechanical symmetry transformations or Wigner transformations. Wigner's famous theorem, which plays a very important role in the mathematical foundations of quantum theory, asserts that for any such map ϕ, there is a linear or conjugate linear isometry J on H such that

$$\phi(P) = JPJ^*, \quad P \in \mathbb{P}_1(H), \tag{8}$$

see, e.g., Section 2.1 in [23]. In fact, originally, Wigner considered bijective such maps ϕ (and in that case J is necessarily a unitary or antiunitary operator on H) but here we definitely need this non-bijective version of his celebrated result.

Proof of Theorem 2.1. In the beginning we note that some of the ideas of the proof are borrowed from the proof of Theorem 2 in our recent paper [25].

Clearly, by (7), for any rank-one projection $P \in \mathbb{P}_1(H)$ considering $A = B = P$, we have that $\phi(P)^2$ is a rank-one projection. Since $\phi(P)$ is also self-adjoint, we have that it is either a rank-one projection or the negative of a rank-one projection. We proceed by showing that the sign does not depend on the particular choice of P. Indeed, let P, Q, P', Q' be rank-one projections on H with $PQ \neq 0$ such that $\phi(P) = P'$ and $\phi(Q) = -Q'$. Then using (7), we infer that for some unitary or antiunitary operator U on H

$$\begin{aligned} 0 < 2 \operatorname{Tr} PQ &= \operatorname{Tr} U(QP + PQ)U^* = \operatorname{Tr}(\phi(P)\phi(Q) + \phi(Q)\phi(P)) \\ &= \operatorname{Tr}(-P'Q' - Q'P') = -2 \operatorname{Tr} P'Q' \leq 0, \end{aligned} \tag{9}$$

which is an obvious contradiction. If $PQ = 0$, we can pick a rank-one projection R on H such that $PR \neq 0$ and $QR \neq 0$ and proceed as above. Thus, considering the map $-\phi$ if necessary, we can and do assume that $\phi(P) \in \mathbb{P}_1(H)$ holds for every $P \in \mathbb{P}_1(H)$. By the computation in (9), for any pair $P, Q \in \mathbb{P}_1(H)$ we obtain that $\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ$. Therefore, Wigner's theorem applies and hence there is a linear or conjugate linear isometry J on H such that

$$\phi(P) = JPJ^* \tag{10}$$

holds for every $P \in \mathbb{P}_1(H)$. What we do in the remaining part of the proof is that we prove that J is unitary or antiunitary and that the formula (10) holds for all operators $A \in \mathbb{S}(H)$. (Indeed, as we will see, this is the basic and general strategy of most of the other proofs in the paper, too.) We argue as follows.

Since the square of a self-adjoint operator belongs to the trace class exactly when it is a Hilbert-Schmidt operator, it follows easily from the property (7) that ϕ maps self-adjoint Hilbert-Schmidt operators to self-adjoint Hilbert-Schmidt operators and ϕ preserves the Hilbert-Schmidt inner product of such operators. Then, for any self-adjoint Hilbert-Schmidt operators A, B, C on H and any real number λ we compute

$$\begin{aligned} &\langle \phi(A + \lambda B) - (\phi(A) + \lambda\phi(B)), \phi(C)\rangle \\ &= \langle \phi(A + \lambda B), \phi(C)\rangle - \langle \phi(A), \phi(C)\rangle - \lambda\langle \phi(B), \phi(C)\rangle \\ &= \langle A + \lambda B, C\rangle - \langle A, C\rangle - \lambda\langle B, C\rangle = 0. \end{aligned}$$

By the real linearity of the inner product, the equality

$$\langle \phi(A + \lambda B) - (\phi(A) + \lambda\phi(B)), \phi(A + \lambda B) - (\phi(A) + \lambda\phi(B))\rangle = 0$$

follows and we obtain

$$\phi(A + \lambda B) = \phi(A) + \lambda\phi(B). \tag{11}$$

This gives the real linearity of ϕ on the space of all self-adjoint Hilbert-Schmidt operators on H.

From (7) we have that $\phi(I)^2 = I$, and since $\phi(I)$ is self-adjoint, it follows that

$$\phi(I) = P_1 + (-P_2),$$

where P_1, P_2 are orthogonal projections on H, $P_2 = P_1^\perp$.

We show that for the range $\operatorname{rng} J$ of the linear or conjugate linear isometry J in (10) the inclusion $\operatorname{rng} J \subseteq \operatorname{rng} P_1$ holds. To verify this, observe that by (7), for an arbitrary unit vector $x \in H$ we have

$$\phi(I)\phi(x \otimes x) + \phi(x \otimes x)\phi(I) = 2Q, \tag{12}$$

where Q is a rank-one projection. We know that $\phi(x \otimes x)$ is also a rank-one projection, in fact, by (8), with the unit vector $p = Jx$ we have $\phi(x \otimes x) = p \otimes p$. Since $\operatorname{rng} P_1$ and $\operatorname{rng} P_2$ give an orthogonal decomposition of H, thus p can be written as

$$p = p_1 + p_2, \quad p_1 \in \operatorname{rng} P_1, p_2 \in \operatorname{rng} P_2.$$

Computing the inner product of the value of left hand side of (12) at p_2 with p_2, we obtain $-2\|p_2\|^4$ and, since this should equal $\langle 2Qp_2, p_2 \rangle$ which is non-negative, we deduce that $p_2 = 0$. This means that $Jx = p \in \operatorname{rng} P_1$ verifying the inclusion $\operatorname{rng} J \subseteq \operatorname{rng} P_1$.

We next show that J is surjective. In the finite dimensional case it is obvious. In the infinite dimensional case, let $(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal sequence in H and $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers which is square summable. By the real linearity of ϕ on the space of all self-adjoint Hilbert-Schmidt operators (see (11)), for every $N \in \mathbb{N}$ we have

$$\phi\left(\sum_n \lambda_n e_n \otimes e_n\right) = \sum_{n=0}^{N} \lambda_n \phi(e_n \otimes e_n) + \phi\left(\sum_{n=N+1}^{\infty} \lambda_n e_n \otimes e_n\right).$$

Since ϕ preserves the Hilbert-Schmidt norm, the second summand converges to 0 as $N \to \infty$, and thus we obtain

$$\begin{aligned}\phi\left(\sum_n \lambda_n e_n \otimes e_n\right) &= \sum_n \lambda_n \phi(e_n \otimes e_n)\\ = \sum_n \lambda_n J e_n \otimes e_n J^* &= J\left(\sum_n \lambda_n e_n \otimes e_n\right) J^*.\end{aligned} \tag{13}$$

Using the local form (7), we have

$$\begin{aligned}\phi(I)\phi\left(\sum_n \lambda_n e_n \otimes e_n\right) + \phi\left(\sum_n \lambda_n e_n \otimes e_n\right)\phi(I)\\ = 2V\left(\sum_n \lambda_n e_n \otimes e_n\right) V^*\end{aligned} \tag{14}$$

for some unitary or antiunitary operator V on H. Since $\phi(I)$ acts as the identity on the range of J (recall that $\operatorname{rng} J \subseteq \operatorname{rng} P_1$), we obtain $\phi(I)J = J$, $J^*\phi(I) = J^*$ and then using (13) we deduce

$$\phi(I)\phi\left(\sum_n \lambda_n e_n \otimes e_n\right) = \phi\left(\sum_n \lambda_n e_n \otimes e_n\right)\phi(I) = \phi\left(\sum_n \lambda_n e_n \otimes e_n\right).$$

By (14), it follows that

$$J\left(\sum_n \lambda_n e_n \otimes e_n\right) J^* = V\left(\sum_n \lambda_n e_n \otimes e_n\right) V^*.$$

The operator on the right hand side has dense range from which we infer that J has dense range too, that is, it is either a unitary or an antiunitary operator on H.

Finally, we show that $\phi(A) = JAJ^*$ holds for every $A \in \mathbb{S}(H)$. Indeed, let x be an arbitrary unit vector in H and set $P = x \otimes x$. We compute

$$2\operatorname{Tr} J^*\phi(A)JP = 2\operatorname{Tr}\phi(A)JPJ^*$$
$$= \operatorname{Tr}(\phi(A)\phi(P) + \phi(P)\phi(A)) = \operatorname{Tr}(AP + PA) = 2\operatorname{Tr} AP,$$

which can also be written as

$$\langle J^*\phi(A)Jx, x\rangle = \langle Ax, x\rangle.$$

Since x was arbitrary, we obtain the desired identity $\phi(A) = JAJ^*$. This completes the proof of the theorem. □

We continue with the case of the automorphism group of $\mathbb{S}(H)$ corresponding to the Jordan triple product $(A, B) \mapsto ABA$. As we have pointed out in the first section of the paper, the elements of that group are exactly the unitary-antiunitary congruence transformations and their negatives. Therefore, in view of the previous result, it is natural to investigate the following question.

Assuming $\phi : \mathbb{S}(H) \to \mathbb{S}(H)$ is a transformation with the property that for any $A, B \in \mathbb{S}(H)$ there exists either a unitary or an antiunitary operator U_{AB} on H and a number $\varepsilon_{AB} \in \{-1, 1\}$ such that

$$\phi(A)\phi(B)\phi(A) = \varepsilon_{AB}U_{AB}ABAU_{AB}^*, \tag{15}$$

is it true that we "globally" have a unitary or antiunitary operator U on H and a number $\varepsilon \in \{-1, 1\}$ such that

$$\phi(A) = \varepsilon UAU^*, \quad A \in \mathbb{S}(H)?$$

The following trivial example shows that the answer to this question is negative. Consider the transformation $\phi : \mathbb{S}(H) \to \mathbb{S}(H)$ defined by

$$\phi(A) = \begin{cases} A & \text{if } A \neq I; \\ -I & \text{if } A = I. \end{cases} \tag{16}$$

It is easy to check that ϕ satisfies the condition given in (15). However, for any nontrivial projection $P \in \mathbb{S}(H)$ we have $\phi(PIP) = \phi(P) = P \neq -P = \phi(P)\phi(I)\phi(P)$. Consequently, the transformation ϕ defined in (16) is not a Jordan triple automorphism of $\mathbb{S}(H)$.

However, if we restrict our attention to the subgroup of the group of all Jordan triple automorphisms of $\mathbb{S}(H)$ whose elements send the identity to the indentity (which is the "half" of the full automorphism group, more precisely a subgroup with index 2), we then have the following positive result.

Theorem 2.2. *Let H be a separable Hilbert space and let $\phi : \mathbb{S}(H) \to \mathbb{S}(H)$ be a transformation with the property that for any $A, B \in \mathbb{S}(H)$ there exists either a unitary or an antiunitary operator U_{AB} on H such that*

$$\phi(A)\phi(B)\phi(A) = U_{AB}ABAU_{AB}^*. \tag{17}$$

Then we have a unitary or antiunitary operator U on H for which

$$\phi(A) = UAU^*, \quad A \in \mathbb{S}(H).$$

Proof. Inserting $A = B = I$ into (17), we have $\phi(I)^3 = I$, and by the self-adjointness of $\phi(I)$, $\phi(I) = I$ follows. This implies that for all $A \in \mathbb{S}(H)$, A and $\phi(A)$ are unitarily or antiunitarily congruent. In particular, ϕ maps projections to projections and preserves their rank. For any pair $P, Q \in \mathbb{P}_1(H)$, by (17) we have

$$\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} \phi(P)\phi(Q)\phi(P) = \operatorname{Tr} PQP = \operatorname{Tr} PQ.$$

Referring to Wigner's theorem, there is a linear or conjugate linear isometry J on H such that $\phi(P) = JPJ^*$ holds for every $P \in \mathbb{P}_1(H)$.

We proceed by showing that J is surjective, hence it is either a unitary or an antiunitary operator. In the finite dimensional case, it is apparent. To verify it in the infinite dimensional case, let $(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal sequence in H and $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers. Set $P_n = e_n \otimes e_n$, $n \in \mathbb{N}$. By (17), we have unitary or antiunitary operators U, V_1 on H such that

$$(JP_1J^*)U\left(\sum_n \lambda_n P_n\right)U^*(JP_1J^*) = \phi(P_1)\phi\left(\sum_n \lambda_n P_n\right)\phi(P_1)$$
$$= V_1P_1\left(\sum_n \lambda_n P_n\right)P_1V_1^* = \lambda_1 V_1P_1V_1^*.$$

From this we obtain

$$V_1^*(JP_1J^*)U\left(\sum_n \lambda_n P_n\right)U^*(JP_1J^*)V_1 = \lambda_1 P_1$$

and then

$$\lambda_1 = \left\langle \left(\sum_n \lambda_n P_n\right)U^*JP_1J^*V_1e_1, U^*JP_1J^*V_1e_1 \right\rangle.$$

Since λ_1 is the largest eigenvalue of the compact operator $\sum_n \lambda_n P_n$, for some complex number ε with $|\varepsilon| = 1$ we necessarily have

$$\varepsilon e_1 = U^*JP_1J^*V_1e_1 = (U^*Je_1 \otimes V^*Je_1)e_1 = \langle e_1, V^*Je_1\rangle U^*Je_1.$$

It follows easily that the unit vectors e_1, U^*Je_1, V^*Je_1 are all pairwise linearly dependent (one needs to use also the criterion for equality in the

Cauchy-Schwarz inequality to verify this). We can continue in a similar fashion and obtain that for some unitary or antiunitary operator V_2 on H we have

$$(JP_2J^*)U\left(\sum_n \lambda_n P_n\right)U^*(JP_2J^*) = V_2P_2\left(\sum_n \lambda_n P_n\right)P_2V_2^* = \lambda_2 V_2P_2V_2^*$$

from which we deduce

$$\lambda_2 = \left\langle \left(\sum_n \lambda_n P_n\right)U^*JP_2J^*V_2e_2, U^*JP_2J^*V_2e_2\right\rangle. \tag{18}$$

The unit vector U^*Je_2 is orthogonal to U^*Je_1 and hence to e_1, too. Consequently, we have that

$$U^*JP_2J^*V_2e_2 = \langle e_2, V_2^*Je_2\rangle U^*Je_2$$

is also orthogonal to e_1. It then follows from (18) that $U^*JP_2J^*V_2e_2$ needs to be a scalar multiple of e_2, the scalar being of modulus one, and then we obtain that the unit vectors $e_2, U^*Je_2, V_2^*Je_2$ are all pairwise linearly dependent. We can continue in a similar way and get that every e_n is in the range of U^*J, which implies that J is surjective.

For the final step of the proof, let x be an arbitrary unit vector in H and set $P = x \otimes x$. Using (17), for any $A \in \mathbb{S}(H)$ we can compute

$$\begin{aligned}\operatorname{Tr} J^*\phi(A)JP &= \operatorname{Tr}\phi(A)JPJ^* = \operatorname{Tr}(JPJ^*)\phi(A)(JPJ^*)\\ &= \operatorname{Tr}\phi(P)\phi(A)\phi(P) = \operatorname{Tr} PAP = \operatorname{Tr} AP,\end{aligned}$$

from which we obtain $\langle J^*\phi(A)Jx, x\rangle = \langle Ax, x\rangle$. Since x was arbitrary, we have that $\phi(A) = JAJ^*$. This completes the proof of the statement. □

We next turn to the von Neumann automorphisms of $\mathbb{P}(H)$. The question what we are going to investigate reads as follows. Assume that $\phi : \mathbb{P}(H) \to \mathbb{P}(H)$ is a transformation with the property that for any $P, Q \in \mathbb{P}(H)$ we have either a unitary or an antiunitary operator U_{PQ} on H such that

$$\phi(P) \wedge \phi(Q)^\perp = U_{PQ}(P \wedge Q^\perp)U_{PQ}^*. \tag{19}$$

Does it follow that ϕ is a unitary-antiunitary congruence transformation on $\mathbb{P}(H)$? The next example shows that the answer is negative in the infinite dimensional case.

Let $W \in \mathbb{B}(H)$ be a non-surjective isometry and let R denote the orthogonal projection onto the orthogonal complement of $\operatorname{rng} W$. Define $\phi : \mathbb{P}(H) \to \mathbb{P}(H)$ by

$$\phi(P) = \begin{cases} WPW^* & \text{if } P \text{ has finite rank;}\\ WPW^* + R & \text{otherwise.}\end{cases}$$

Then ϕ is obviously not a unitary-antiunitary congruence transformation on $\mathbb{P}(H)$ but we can show that for any $P, Q \in \mathbb{P}(H)$ we have either a unitary or an antiunitary operator U_{PQ} on H such that (19) holds.

To see this, we only need to check that for arbitrary $P, Q \in \mathbb{P}(H)$, the projections $P \wedge Q^\perp$ and $\phi(P) \wedge \phi(Q)^\perp$ have the same rank and the same corank.

If P has finite rank, then in the case where Q has finite as well as in the case where Q has infinite rank, we have $\phi(P) \wedge \phi(Q)^\perp = W(P \wedge Q^\perp)W^*$ which projection has the same rank and corank as $P \wedge Q^\perp$.

If P has infinite rank and Q has finite rank, then we have

$$\phi(P) \wedge \phi(Q)^\perp = (WPW^* + R) \wedge (WQW^*)^\perp = W(P \wedge Q^\perp)W^* + R.$$

Clearly, the kernels of $P \wedge Q^\perp$ and $W(P \wedge Q^\perp)W^* + R$ have the same dimension and, as $P \wedge Q^\perp$ has infinite rank, the same is true for their ranges.

Finally, if both P, Q have infinite rank, then

$$\phi(P) \wedge \phi(Q)^\perp = (WPW^* + R) \wedge (WQW^* + R)^\perp = W(P \wedge Q^\perp)W^*.$$

It is clear that the ranges of $P \wedge Q^\perp$ and $W(P \wedge Q^\perp)W^*$ have the same dimension. As for the kernels, since $(P \wedge Q^\perp)^\perp = P^\perp \vee Q$ is infinite dimensional, the same is true.

Therefore, the desired sort of strong 2-reflexivity property does not hold for the group of all von Neumann automorphisms of $\mathbb{P}(H)$. However, if we pose one extra condition on the transformation under consideration, namely, that the rank-one projections are all included in its range, then we get the next positive result.

Theorem 2.3. *Assume that H is a separable Hilbert space with dimension at least 3. Let $\phi : \mathbb{P}(H) \to \mathbb{P}(H)$ be a transformation with the property that for any $P, Q \in \mathbb{P}(H)$ there exists either a unitary or an antiunitary operator U_{PQ} on H such that*

$$\phi(P) \wedge \phi(Q)^\perp = U_{PQ}(P \wedge Q^\perp)U_{PQ}^*. \tag{20}$$

If we also have $\mathbb{P}_1(H) \subseteq \operatorname{rng} \phi$, then there is a unitary or antiunitary operator U on H for which

$$\phi(P) = UPU^*, \quad P \in \mathbb{P}(H).$$

Proof. By choosing $P = I$ and $Q = 0$ in (20), we see that $\phi(I) = \phi(0)^\perp = I$ must hold. It immediately follows that for any $P \in \mathbb{P}(H)$, we have $\phi(P) = \phi(P) \wedge \phi(0)^\perp = V(P \wedge 0^\perp)V^* = VPV^*$ with some unitary or antiunitary operator V on H, hence ϕ preserves the rank and the corank of any projection on H.

We proceed by showing that $\phi|_{\mathbb{P}_1(H)}$ is injective and preserves orthogonality in both directions. Assume that there are $P, Q \in \mathbb{P}_1(H)$ such that $\phi(P) = \phi(Q)$, $P \neq Q$. Then the coranks of $P^\perp, Q^\perp$ are both 1 and the corank $P^\perp \wedge Q^\perp$ is 2. Hence

$$VP^\perp V^* = \phi(P^\perp) \wedge \phi(P)^\perp = \phi(P^\perp) \wedge \phi(Q)^\perp = V'(P^\perp \wedge Q^\perp)V'^*$$

holds for some unitaries or antiunitaries V, V' on H, a contradiction. This proves the injectivity of $\phi|_{\mathbb{P}_1(H)}$. Since the range of ϕ contains $\mathbb{P}_1(H)$ and

ϕ preserves the rank, we obtain that $\phi|_{\mathbb{P}_1(H)}$ is a bijection of $\mathbb{P}_1(H)$. To see the orthogonality preserving property, we need to show that for any $P, Q \in \mathbb{P}_1(H)$, we have

$$PQ = 0 \iff \phi(P)\phi(Q) = 0,$$

or equivalently,

$$P \wedge Q^\perp = P \iff \phi(P) \wedge \phi(Q)^\perp = \phi(P).$$

This follows easily from the property (20). Indeed, if $P \wedge Q^\perp = P$, then $\phi(P) \wedge \phi(Q)^\perp = VPV^*$ for some unitary or antiunitary V on H. It follows that $VPV^* \leq \phi(P)$ and, since the projections on both sides are of rank 1, we infer that they coincide, i.e., $\phi(P) \wedge \phi(Q)^\perp = VPV^* = \phi(P)$. Conversely, if $\phi(P) \wedge \phi(Q)^\perp = \phi(P)$, then $\phi(P) = V(P \wedge Q^\perp)V^*$ holds for some unitary or antiunitary V on H. Thus $P \wedge Q^\perp$ has rank 1, which implies that $P \wedge Q^\perp = P$.

We have proved that $\phi|_{\mathbb{P}_1(H)} : \mathbb{P}_1(H) \to \mathbb{P}_1(H)$ is a bijection which preserves orthogonality in both directions. By the famous Uhlhorn's version of Wigner's theorem (see, e.g., Section 0.3 in [23]), there exists either a unitary or an antiunitary operator U on H such that $\phi(P) = UPU^*$ for all $P \in \mathbb{P}_1(H)$.

Considering the transformation $U^*\phi(.)U$ on $\mathbb{P}(H)$, we can and do assume that $\phi|_{\mathbb{P}_1(H)}$ is the identity. Now, to complete the proof, we need to show that this implies that ϕ is the identity on the entire set $\mathbb{P}(H)$. For an arbitrary $P \in \mathbb{P}(H)$ and $Q \in \mathbb{P}_1(H)$ with $Q \leq P$, we have $Q \wedge \phi(P^\perp)^\perp = V(Q \wedge P)V^* = VQV^*$ for some unitary or antiunitary V on H. From this, we deduce $Q = VQV^*$ and then $Q \leq \phi(P^\perp)^\perp$. Since this holds for any rank-one projection Q with $Q \leq P$, we obtain $P \leq \phi(P^\perp)^\perp$, which means that $\phi(P^\perp) \leq P^\perp$. By the arbitrariness of $P \in \mathbb{P}(H)$, we have $\phi(P) \leq P$, $P \in \mathbb{P}(H)$. Assuming that for one $P \in \mathbb{P}(H)$ we have $\phi(P) \neq P$, it follows that there is a $Q \in \mathbb{P}_1(H)$ such that $\phi(P), \phi(Q) = Q$ are orthogonal and $Q \leq P$. But the argument given two paragraph above (see the last two sentences there and interchange the roles of P, Q) shows that the orthogonality of $\phi(P), \phi(Q)$ implies the orthogonality of P, Q which contradicts $Q \leq P$. Therefore, we have that ϕ is the identity on the whole set $\mathbb{P}(H)$ and it completes the proof of the statement. □

Let us make an important remark here. In Proposition 2.6 in [29], Šerml proved the following strengthening of Uhlhorn's theorem.

If $\dim H \geq 3$ *and* $\phi : \mathbb{P}_1(H) \to \mathbb{P}_1(H)$ *is an injective map which sends maximal orthogonal collections in* $\mathbb{P}_1(H)$ *to maximal orthogonal collections, then it is a unitary-antiunitary congruence transformation.*

Checking our proof above, one can see that even if we do not assume the extra condition $\mathbb{P}_1(H) \subset \operatorname{rng} \phi$, we have that our transformation ϕ restricted to $\mathbb{P}_1(H)$ is injective, its range is included in $\mathbb{P}_1(H)$, and it preserves orthogonality in both directions. In the finite dimensional case we then obtain easily that ϕ satisfies the requirements of Šemrl's former proposition and hence we obtain that it is a unitary-antiunitary congruence transformation on $\mathbb{P}_1(H)$.

As a consequence, we can see that, in the finite dimensional case, the conclusion in Theorem 2.3 remains valid even without the mentioned extra condition and hence in that case we do have the strong 2-reflexivity property for the group of von Neumann automorphisms of $\mathbb{P}(H)$.

Concerning the automorphism group of the set $\mathbb{D}(H)$ of density operators with respect to the operation of the arithmetic mean, we have the following fully positive result.

Theorem 2.4. *Let H be a separable Hilbert space and let $\phi : \mathbb{D}(H) \to \mathbb{D}(H)$ be a transformation with the property that for any $A, B \in \mathbb{D}(H)$ there exists either a unitary or an antiunitary operator U_{AB} on H such that*

$$\frac{\phi(A) + \phi(B)}{2} = U_{AB} \frac{A + B}{2} U_{AB}^*. \tag{21}$$

Then we have a unitary or antiunitary operator U on H for which

$$\phi(A) = UAU^*, \quad A \in \mathbb{D}(H).$$

Proof. Clearly, for any $P \in \mathbb{P}_1(H)$, choosing $A = B = P$ in (21), we see that $\phi(P)$ is a rank-one projection. For any two rank-one projections P, Q on H, by our condition on ϕ, we have that $\operatorname{Tr}(\phi(P)+\phi(Q))^2 = \operatorname{Tr}(P+Q)^2$, and from this, $\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ$ follows. Thus, Wigner's theorem applies and there exists a linear or conjugate linear isometry J on H such that $\phi(P) = JPJ^*$ holds for every $P \in \mathbb{P}_1(H)$.

We prove that J is in fact surjective. This requires verification only in the infinite dimensional case. So, let $(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal sequence in H and set $P_n = e_n \otimes e_n$, $n \in \mathbb{N}$. Select a strictly decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive numbers such that $\sum_n \lambda_n = 1$ and define $A = \sum_n \lambda_n P_n$. Let W be a unitary or antiunitary operator on H such that $\phi(A) = WAW^*$.

By (21), we have that $\phi(A) + \phi(P_1) = U_1(A + P_1)U_1^*$ holds for some unitary or antiunitary operator U_1 on H and hence

$$\|WAW^* + JP_1J^*\| = \|A + P_1\| = \lambda_1 + 1.$$

($\|.\|$ stands for the operator norm.) It is not difficult to see that we necessarily have $WP_1W^* = JP_1J^*$. Next, consider the operator

$$\phi(A)+\phi(P_2) = WAW^*+JP_2J^* = \lambda_1 WP_1W^*+\left(\sum_{n=2}^{\infty} \lambda_n WP_nW^* + JP_2J^*\right).$$

Since all of the rank-one projections WP_nW^*, $n \geq 2$ and JP_2J^* are orthogonal to $WP_1W^* = JP_1J^*$, we have that

$$\|WAW^* + JP_2J^*\| = \max\left\{\lambda_1, \left\|\sum_{n=2}^{\infty} \lambda_n WP_nW^* + JP_2J^*\right\|\right\}. \tag{22}$$

On the other hand, by the property (21), $WAW^*+JP_2J^* = \phi(P)+\phi(P_1) = U_2(A+P_2)U_2^*$ holds for some unitary or antiunitary operator U_2 on H. Therefore, the left hand side of (22) is equal to λ_2+1. It follows that

$$\left\| \sum_{n=2}^{\infty} \lambda_n W P_n W^* + J P_2 J^* \right\| = \lambda_2 + 1.$$

As above, this implies that $WP_2W^* = JP_2J^*$. We can continue in this way and obtain that $WP_nW^* = JP_nJ^*$ holds for every $n \in \mathbb{N}$. Since W is a unitary or antiunitary operator on H, it then follows that the Je_n's form a complete orthonormal sequence in H which gives us that J is also a unitary or antiunitary operator on H.

Considering the transformation $J^*\phi(.)J$, we can and do assume that our original map ϕ is the identity on the set $\mathbb{P}_1(H)$. We prove that this implies that ϕ is the identity on the whole set $\mathbb{D}(H)$. Let $A \in \mathbb{D}(H)$ now be arbitrary and choose a unitary or antiunitary operator W on H such that $\phi(A) = WAW^*$. By (21) again, the equality $\|A+P\| = \|\phi(A)+P\|$ holds for any rank-one projection P on H. We assert that $A = \phi(A)$. Indeed, let the decreasing sequence of the different nonzero eigenvalues of A be $\mu_1, \mu_2, \ldots$. Denote the corresponding spectral projections by $Q_1, Q_2, \ldots$. Clearly, we have $\|A+P\| = 1+\mu_1$ if and only if $P \leq Q_1$, while $\|\phi(A)+P\| = 1+\mu_1$ holds if and only if $P \leq WQ_1W^*$. It follows that $Q_1 = WQ_1W^*$. Next, for any rank-one projection P on H which is orthogonal to $Q_1 = WQ_1W^*$, we have $\|A+P\| = 1+\mu_2$ if and only if $P \leq Q_2$ and $\|\phi(A)+P\| = 1+\mu_2$ if and only if $P \leq WQ_2W^*$. Therefore, $Q_2 = WQ_2W^*$ follows. Continuing in this way, we deduce that $\phi(A) = A$ really holds which completes the proof of the statement. □

We next turn to the automorphisms of the set $\mathbb{E}(H)$ of Hilbert space effects. The cases of Ludwig automorphisms and **E**-automorphisms are quite similar. Since the latter one is simpler, we present the corresponding result first.

Theorem 2.5. *Let H be a separable Hilbert space and let $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ be a transformation with the property that for any $A, B \in \mathbb{E}(H)$ with $A+B \in \mathbb{E}(H)$ there exists either a unitary or an antiunitary operator U_{AB} on H such that*

$$\phi(A)+\phi(B) = U_{AB}(A+B)U_{AB}^*. \tag{23}$$

Then we have a unitary or antiunitary operator U on H for which

$$\phi(A) = UAU^*, \quad A \in \mathbb{E}(H)$$

Proof. It is clear from the property (23) that $\phi(0) = 0$. Next, for every $A \in \mathbb{E}(H)$, we have that $\phi(A)$ is unitarily or antiunitarily congruent to A. Specifically, ϕ sends projections to projections, preserves the rank and corank of projections, and preserves the spectrum of any element of $\mathbb{E}(H)$. One can also deduce from (23) that $\phi(I-A) = I-\phi(A)$ for any $A \in \mathbb{E}(H)$.

We show that $\phi(tP) = t\phi(P)$ holds for every rank-one projection P on H and real number $t \in]0,1[$. Let p be a unit vector in H such that $\phi(P) = p \otimes p$. We have $\phi(tP) = tq \otimes q$ for some unit vector $q \in H$. From (23) we obtain that

$$tq \otimes q + I - p \otimes p = \phi(tP) + \phi(P^\perp) = V(tP + P^\perp)V^*$$

holds with some unitary or antiunitary operator V on H. We can compute in turn as follows

$$\begin{aligned} tI \leq V(tP + P^\perp)V^* = tq \otimes q + I - p \otimes p, \\ p \otimes p \leq tq \otimes q + (1-t)I, \\ 1 \leq t|\langle q, p\rangle|^2 + 1 - t, \\ 1 \leq |\langle q, p\rangle|^2. \end{aligned}$$

By the equality case in Cauchy-Schwarz inequality, this implies that $q \otimes q = p \otimes p$ and, consequently, we obtain $\phi(tP) = t\phi(P)$ what was asserted.

Pick any two rank-one projections P, Q on H. Since $\phi(P/2) = \phi(P)/2$ and $\phi(Q/2) = \phi(Q)/2$, it follows from (23) that $\phi(P) + \phi(Q)$ is unitarily or antiunitarily congruent to $P + Q$, from which we obtain that $\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ$ as in the proof of Theorem 2.4. It follows that ϕ is a Wigner transformation on $\mathbb{P}_1(H)$ and hence there is a linear or conjugate linear isometry J on H such that $\phi(P) = JPJ^*$ holds for every $P \in \mathbb{P}_1(H)$.

If H is finite dimensional, then we immediately obtain that J is unitary or antiunitary. Assume that H is infinite dimensional. We apply an argument similar to the corresponding part of the proof of Theorem 2.4. Let $(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal sequence in H. Choose any strictly decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real numbers in $]0,1[$. Set $A = \sum_n \lambda_n e_n \otimes e_n$. Then $\phi(A) = \sum_n \lambda_n f_n \otimes f_n$ holds with a possibly different complete orthonormal sequence $(f_n)_{n\in\mathbb{N}}$ in H. Then we have that

$$\begin{aligned} \sum_n \lambda_n f_n \otimes f_n + (1-\lambda_1)J(e_1 \otimes e_1)J^* &= \phi(A) + (1-\lambda_1)\phi(e_1 \otimes e_1) \\ &= \phi(A) + \phi((1-\lambda_1)e_1 \otimes e_1) = V(A + (1-\lambda_1)e_1 \otimes e_1)V^* \end{aligned} \tag{24}$$

holds with some unitary or antiunitary V on H. This gives us that

$$\left\| \sum_n \lambda_n f_n \otimes f_n + (1-\lambda_1)J(e_1 \otimes e_1)J^* \right\| = 1, \tag{25}$$

which implies that $f_1 \otimes f_1 = Je_1 \otimes Je_1$, i.e., f_1 is a scalar multiple of Je_1 (the scalar being of modulus one). Similarly,

$$\begin{aligned} \sum_n \lambda_n f_n \otimes f_n + (1-\lambda_2)J(e_2 \otimes e_2)J^* &= \phi(A) + (1-\lambda_2)\phi(e_2 \otimes e_2) \\ &= \phi(A) + \phi((1-\lambda_2)e_2 \otimes e_2) = V(A + (1-\lambda_2)e_2 \otimes e_2)V^* \end{aligned} \tag{26}$$

holds with some unitary or antiunitary V on H. This gives us that

$$\left\| \sum_n \lambda_n f_n \otimes f_n + (1-\lambda_2) J(e_2 \otimes e_2) J^* \right\| = 1 \tag{27}$$

and, taking into account that Je_2 is orthogonal to Je_1 and hence to f_1 as well, we can infer that $f_2 \otimes f_2 = Je_2 \otimes Je_2$. We can continue in the same way and obtain that the vectors Je_n form a complete orthonormal sequence in H and this implies that J is unitary or antiunitary. Observe that we also have $\phi(A) = JAJ^*$. Considering the transformation $J^*\phi(.)J$, we can and do assume that our original map ϕ is the identity on $\mathbb{P}_1(H)$ and also on the set of all operators of the form $\sum_n \lambda_n e_n \otimes e_n$, where $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal sequence in H and $(\lambda_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of real numbers in $]0,1[$.

Let now $A \in \mathbb{E}(H)$ be any invertible operator with spectrum $\sigma(A) = \{\mu_1, \ldots, \mu_n\}$, where the elements in $\{\mu_1, \ldots, \mu_n\}$ appear in the strictly decreasing order. Denote the corresponding spectral projections of A by $P_1, \ldots, P_n$ and for any $i = 1, ..., n$, let $(e_{i,j})_j$ be an orthonormal basis in $\operatorname{rng} P_i$. For an arbitrary positive real number $\varepsilon < \min\{\mu_1 - \mu_2, \mu_2 - \mu_3, \ldots, \mu_n\}$, let $(t_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers such that $t_n \to \varepsilon$. Define a perturbation A_ε of A by $A_\varepsilon = \sum_{i=1}^n \sum_j (\mu_i - t_j) e_{i,j} \otimes e_{i,j}$. By the property of ϕ, we have $\phi(A_\varepsilon) = VA_\varepsilon V^*$ with some unitary or antiunitary V on H. Following the method presented in the previous paragraph, we can verify that

$$V \left(\sum_j (\mu_1 - t_j) e_{1,j} \otimes e_{1,j} \right) V^* = \sum_j (\mu_1 - t_j) e_{1,j} \otimes e_{1,j},$$

and then that

$$V \left(\sum_j (\mu_2 - t_j) e_{2,j} \otimes e_{2,j} \right) V^* = \sum_j (\mu_2 - t_j) e_{2,j} \otimes e_{2,j},$$

and so forth. Consequently, we can infer that $\phi(A_\varepsilon) = A_\varepsilon$.

Clearly, as $\varepsilon \to 0$, we have $A_\varepsilon \to A$ in the operator norm. Using the well-known fact that the spectrum is continuous at the normal elements of $\mathbb{B}(H)$, we can compute

$$\begin{aligned} \sigma(\phi(I-A)+A) &= \sigma(\phi(I-A) + \lim_{\varepsilon\to 0} A_\varepsilon) = \sigma(\phi(I-A) + \lim_{\varepsilon\to 0} \phi(A_\varepsilon)) = \\ &= \lim_{\varepsilon\to 0} \sigma(\phi(I-A) + \phi(A_\varepsilon)) = \lim_{\varepsilon\to 0} \sigma(I - A + A_\varepsilon) = \\ &= \sigma(I - A + \lim_{\varepsilon\to 0} A_\varepsilon) = \sigma(I-A+A) = \{1\}, \end{aligned} \tag{28}$$

which implies that $\phi(I-A) = I-A$. Therefore, we obtain that ϕ is the identity on the set of all elements of $\mathbb{E}(H)$ which have finite spectra that do not contain the point 1. Since every element of $\mathbb{E}(H)$ can be approximated from below by such elements in the operator norm, the reasoning followed in (28) can be applied again to obtain that $\phi(I-A) = I - A$ holds for all $A \in \mathbb{E}(H)$. This completes the proof both in the finite and infinite dimensional cases. □

Concerning the automorphism group of $\mathbb{E}(H)$ with respect to the operation of the arithmetic mean (as a particular convex combination) we have the following positive result. Recall that by Proposition 1.2, that automorphism group consists of all unitary-antiunitary congruence transformations as well as their compositions with the reflection $A \mapsto I - A$.

Theorem 2.6. *Let H be a separable Hilbert space and let $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ be a transformation with the property that for any $A, B \in \mathbb{E}(H)$ there exists a unitary or antiunitary operator U_{AB} on H such that either*

$$\frac{\phi(A)+\phi(B)}{2} = U_{AB}\frac{A+B}{2}U_{AB}^* \tag{29}$$

or

$$\frac{\phi(A)+\phi(B)}{2} = U_{AB}\left(I - \frac{A+B}{2}\right)U_{AB}^* \tag{30}$$

holds. Then we have a unitary or antiunitary operator U on H for which either

$$\phi(A) = UAU^*, \quad A \in \mathbb{E}(H) \tag{31}$$

or

$$\phi(A) = U(I-A)U^*, \quad A \in \mathbb{E}(H). \tag{32}$$

Proof. In what follows, usually we will consider not the equations (29) and (30) but their multiplications by 2. Clearly, we have either $\phi(I) = I$ or $\phi(I) = 0$. In the latter case, considering the transformation $A \mapsto I - \phi(A)$ we can still assume that our map satisfies $\phi(I) = I$.

After this, we will follow the proof of Theorem 2.5 closely. Indeed, we will show that as in the occurrences of the equation (23) there, in the majority of the cases here we neccessarily have that, out of (29) and (30), the possibility (29) must hold true while in the other cases the situation can easily be handled.

To begin, first notice that $\phi(I-A) = I - \phi(A)$ holds for every $A \in \mathbb{E}(H)$. Next, for an arbitrary $A \in \mathbb{E}(H)$, we have a unitary or antiunitary V on H such that

$$\phi(A) = \phi(A) + \phi(I) - I = \begin{cases} V(A+I)V^* - I = VAV^* \\ \text{or} \\ 2I - V(A+I)V^* - I = -VAV^*. \end{cases}$$

As the latter case is clearly impossible for a nonzero A, we condclude that A and $\phi(A)$ are unitarily or antiunitarily congruent for every $A \in \mathbb{E}(H)$. Consequently, ϕ sends projections to projections, it preserves the norm, the rank and the spectrum of the elements of $\mathbb{E}(H)$.

We next show that $\phi(tP) = t\phi(P)$ holds for any $P \in \mathbb{P}_1(H)$ and real number $t \in]0,1[$. Indeed, let $\phi(P) = p \otimes p$, $\phi(tP) = tq \otimes q$, with some unit vectors $p, q \in H$, and suppose that

$$tq \otimes q + I - p \otimes p = \phi(tP) + \phi(P^\perp) = 2I - V(tP + P^\perp)V^*$$

holds with some unitary or antiunitary V on H (i.e., out of (29) and (30), we in fact have the latter possibility). This would imply that

$$tq \otimes q - p \otimes p = (1-t)VPV^*$$

and then that

$$0 > t|\langle p, q\rangle|^2 - |\langle p, p\rangle|^2 = (1-t)\langle VPV^* p, p\rangle$$

which is a clear contradiction. Therefore, it follows that the other possibility,

$$tq \otimes q + I - p \otimes p = \phi(tP) + \phi(P^\perp) = V(tP + P^\perp)V^*$$

i.e., (29) holds with some unitary or antiunitary V on H. From this we can deduce $\phi(tP) = t\phi(P)$ just as in the corresponding part of the proof of Theorem 2.5.

Our next claim is that the restriction of ϕ to $\mathbb{P}_1(H)$ is a Wigner transformation. Let $\dim H \geq 3$ and select $P, Q \in \mathbb{P}_1(H)$. Then, by the properties of ϕ, we necessarily have $\phi(P) + \phi(Q) = V(P+Q)V^*$ for some unitary or antiunitary V on H. Indeed, if $\phi(P) + \phi(Q) = 2I - V(P+Q)V^*$, then, for example, by taking trace we immediately arrive at a contradiction. So, $\phi(P) + \phi(Q) = V(P+Q)V^*$ holds and we have $\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ$ as in the first paragraph of the proof of Theorem 2.4. Assume now that $\dim H = 2$. Then the possibility $\phi(P) + \phi(Q) = 2I - V(P+Q)V^*$ can in principal occur. But taking squares on both sides in this equality and then taking the trace, we can again easily arrive at $\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ$. Therefore, by Wigner's theorem, in all cases we have a linear or conjugate linear isometry J on H such that $\phi(P) = JPJ^*$ for all $P \in \mathbb{P}_1(H)$.

In the remaining part of the proof, to verify that J is unitary or antiunitary and that $\phi(A) = JAJ^*$ holds for all $A \in \mathbb{E}(H)$, we can essentially follow the argument given in the proof of Theorem 2.5. In fact, as for the reasoning presented in the fourth paragraph there, we need to apply a little bit of adjustment. Namely, in order to avoid the appearance of the possibility (30), we consider not $1 - \lambda_1$ in (24), (25) but $\lambda - \lambda_1$, not $1 - \lambda_2$ in (26), (27) but $\lambda - \lambda_2$, etc., where λ is any fixed number in $]\lambda_1, 1[$. Indeed, for such λ, the equality

$$\sum_n \lambda_n f_n \otimes f_n + (\lambda - \lambda_1)J(e_1 \otimes e_1)J^* = V(2I - (A + (\lambda - \lambda_1)e_1 \otimes e_1))V^* \tag{33}$$

cannot happen since, as one can easily check, we have the inequality

$$V(2I - (A + (\lambda - \lambda_1)e_1 \otimes e_1))V^* \geq (2 - \lambda)I$$

and then taking the values in (33) at f_n, their inner product with f_n, and letting $n \to \infty$, we would obtain $\lim_n \lambda_n \geq 2 - \lambda > 1$, a contradiction. The same reasoning applies for the indices $2, 3, \ldots$ in the place of 1 above.

The argument in the fifth paragraph of the proof of Theorem 2.5 can be closely followed. Concerning the sixth one, we observe that the spectra $\sigma(I - A + A_\varepsilon)$ and $\sigma(2I - (I - A + A_\varepsilon))$ both converge to $\{1\}$ as $\varepsilon \to 0$. Hence the reasoning presented there can be used to complete the proof of the statement. □

Concerning the sequential automorphisms of $\mathbb{E}(H)$ we again have a positive result.

Theorem 2.7. *Let H be a separable Hilbert space and let $\phi : \mathbb{E}(H) \to \mathbb{E}(H)$ be a transformation with the property that for any $A, B \in \mathbb{E}(H)$ there exists either a unitary or an antiunitary operator U_{AB} on H such that*

$$\sqrt{\phi(A)}\phi(B)\sqrt{\phi(A)} = U_{AB}\sqrt{A}B\sqrt{A}U_{AB}^*. \tag{34}$$

Then we have a unitary or antiunitary operator U on H for which

$$\phi(A) = UAU^*, \quad A \in \mathbb{E}(H).$$

Proof. The equality $\phi(I) = I$ is evident from (34). The rest of proof is essentially identical with the proof of Theorem 2.2. □

We remark that one can formulate and prove the same statement as in Theorem 2.7 for the set $\mathbb{B}(H)^+$ of all positive semidefinite operators in the place of the set $\mathbb{E}(H)$ of Hilbert space effects.

We finish the paper with our result on the group of all (continuous) automorphisms of the standard K-loop operation on $\mathbb{B}(H)^{++}$. To be precise, let us recall that those automorphisms are exactly the transformations of one of the two forms (4), (5) only in the case where H is an infinite dimensional Hilbert space (we have already mentioned that in the finite dimensional case multiplication by a fixed power of the determinant functional also shows up, see Theorem 1 in [22]).

Theorem 2.8. *Assume that H is a separable Hilbert space and let $\phi : \mathbb{B}(H)^{++} \to \mathbb{B}(H)^{++}$ be a transformation with the property that for any $A, B \in \mathbb{B}(H)^{++}$ we have a unitary or antiunitary operator $U_{A,B}$ on H such that either*

$$\sqrt{\phi(A)}\phi(B)\sqrt{\phi(A)} = U_{A,B}\sqrt{A}B\sqrt{A}U_{A,B}^*$$

or

$$\sqrt{\phi(A)}\phi(B)\sqrt{\phi(A)} = U_{A,B}\left(\sqrt{A}B\sqrt{A}\right)^{-1}U_{A,B}^*.$$

Then we have a unitary or antiunitary operator U on H for which either

$$\phi(A) = UAU^*, \quad A \in \mathbb{B}(H)^{++}$$

or

$$\phi(A) = UA^{-1}U^*, \quad A \in \mathbb{B}(H)^{++}.$$

Before presenting the proof we recall the concept of the so-called Thompson metric defined on the positive definite cone of $\mathbb{B}(H)$ (or, more generally, of any C^*-algebra). For arbitrary $A, B \in \mathbb{B}(H)^{++}$, we set

$$d_T(A, B) = \| \log \sqrt{A}^{-1} B \sqrt{A}^{-1} \|.$$

It is well-known that d_T is a metric, called Thompson metric, on $\mathbb{B}(H)^{++}$ and it is not difficult to see that the convergence with respect to d_T coincides with the convergence with respect to the operator norm. The probably most important property of the Thompson metric is that it makes the positive definite cone (which is open in the topology of the operator norm) a complete metric space. For the description of the isometry group of $\mathbb{B}(H)^{++}$ equipped with the Thompson metric and for some more information we refer to [24]. Interestingly, we can utilize the metric d_T in the proof of our statement above.

Proof of Theorem 2.8. Choosing $A = B = I$, we easily see that $\phi(I)^2 = I$ which implies $\phi(I) = I$. Next, choosing $B = A^{-1}$, we deduce $\phi(A^{-1}) = \phi(A)^{-1}$, $A \in \mathbb{B}(H)^{++}$.

In what follows, for an arbitrary $A \in \mathbb{B}(H)^{++}$, we write $A^{[-1]}$ to denote either the operator A or its inverse A^{-1}.

For any $A, B \in \mathbb{B}(H)^{++}$, we have that

$$\sqrt{\phi(A)}^{-1} \phi(B) \sqrt{\phi(A)}^{-1} = \sqrt{\phi(A^{-1})} \phi(B) \sqrt{\phi(A^{-1})}$$

is unitarily or antiunitarily congruent to $(\sqrt{A}^{-1} B \sqrt{A}^{-1})^{[-1]}$ which clearly implies that

$$d_T(\phi(A), \phi(B)) = d_T(A, B), \qquad A, B \in \mathbb{B}(H)^{++}$$

i.e., ϕ is a Thompson isometry. Since, as we have written above, the topology of d_T on $\mathbb{B}(H)^{++}$ coincides with the topology of the operator norm, we obtain that our map ϕ is continuous in that latter topology, too.

From the condition in the theorem we know that for any $A \in \mathbb{B}(H)^{++}$, the operator $\phi(A)$ is unitarily or antiunitarily congruent to $A^{[-1]}$. Pick any positive number ε and consider the convex (and hence connected) set of all elements A of $\mathbb{B}(H)^{++}$ for which $(1+\varepsilon)I \leq A$ holds. Concerning the image $\phi(A)$ of any such element A we have either $(1+\varepsilon)I \leq \phi(A)$ or $\phi(A) \leq (1/(1+\varepsilon))I$. By the continuity of ϕ and the connectedness of the above set, it follows that either we have the former possibility for all considered A, or we have the latter possibility for all considered A. Obviously, when we let ε decrease and tend to 0, the above property does not change and using the continuity of ϕ, we finally obtain that either we have $\phi(A) \geq I$ for all $A \geq I$ or we have $\phi(A) \leq I$ for all $A \geq I$.

Now, without loss of generality, we can assume that the former case takes place, i.e., for all $A \geq I$ we have $\phi(A) \geq I$ (indeed, otherwise, we consider the transformation $A \mapsto \phi(A)^{-1}$). That means that for any $A \geq I$ we have that $\phi(A)$ is unitarily or antiunitarily congruent to A.

Picking any rank-one projection P on H, it follows that we have $\phi(I+P) = I+\psi(P)$ for some rank-one projection $\psi(P)$. Pick a pair $P, Q \in \mathbb{P}_1(H)$. Denote $\alpha = \sqrt{2}-1$. We know that

$$\begin{aligned} &(I+\alpha\psi(P))(I+\psi(Q))(I+\alpha\psi(P)) \\ &= \sqrt{I+\psi(P)}(I+\psi(Q))\sqrt{I+\phi(P)} \\ &= \sqrt{\phi(I+P)}\phi(I+Q)\sqrt{\phi(I+P)} \\ &= W(\sqrt{I+P}(I+Q)\sqrt{(I+P)}W^* \\ &= W((I+\alpha P)(I+Q)(I+\alpha P))W^* \end{aligned} \tag{35}$$

holds for some unitary or antiunitary operator W on H. Subtracting the identity from both sides of the equality

$$(I+\alpha\psi(P))(I+\psi(Q))(I+\alpha\psi(P)) = W((I+\alpha P)(I+Q)(I+\alpha P))W^*$$

and taking trace, we easily obtain that

$$\operatorname{Tr}\psi(P)\psi(Q) = \operatorname{Tr} PQ.$$

This shows that $\psi : \mathbb{P}_1(H) \to \mathbb{P}_1(H)$ is a Wigner transformation. Therefore, there is an either linear or conjugate linear isometry $J : H \to H$ such that $\psi(P) = JPJ^*$ holds for all rank-one projections P on H.

Our next aim is, as before, to show that J is unitary or antiunitary. To prove it in the nontrivial case, where H is infinite dimensional, take a complete orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in H and let $P_n = e_n \otimes e_n$ for every $n \in \mathbb{N}$. Furthermore, let $(\lambda_n)_{n\in\mathbb{N}}$ be a stricly decreasing sequence of positive numbers and define $K = \sum_n \lambda_n P_n$ and consider the operator $I+K$. Since $\phi(I+K)$ is unitarily or antiunitarily congruent to $I+K$, there is a complete orthonormal sequence $(f_n)_{n\in\mathbb{N}}$ in H such that for $P'_n = f_n \otimes f_n$, $n \in \mathbb{N}$ and $K' = \sum_n \lambda_n P'_n$ we have $\phi(I+K) = \sum_n (1+\lambda_n)P'_n = I+K'$. We compute

$$\begin{aligned} (I+\alpha\psi(P_1))(I+K')(I+\alpha\psi(P_1)) &= \sqrt{\phi(I+P_1)}\phi(I+K)\sqrt{\phi(I+P_1)} \\ &= W((I+\alpha P_1)(I+K)(I+\alpha P_1))W^* \end{aligned}$$

for some unitary or antiunitary operator W on H. Again, subtracting the identity and then taking trace, we get

$$\operatorname{Tr} K'\psi(P_1) = \operatorname{Tr} KP_1.$$

It follows easily that we necessarily have $\psi(P_1) = P'_1$. In a similar manner we get

$$\operatorname{Tr}(K'\psi(P_2)) = \operatorname{Tr}(KP_2).$$

We know that $\psi(P_2)$ is orthogonal to $\psi(P_1)$. Therefore, from the above equality we obtain that $\psi(P_2) = P'_2$. We can go on in the same way and conclude that $\psi(P_n) = P'_n$ holds for every positive integer n. It follows that all f_n's are in the range of J implying that J is unitary or antiunitary.

Now, considering the transformation $J^*\phi(.)J$, we can assume that $\phi(I+P)=I+P$ holds for every rank-one projection P on H. What remains to prove is that ϕ is the identity on the whole set $\mathbb{B}(H)^{++}$.

First we prove that $\phi(I+tP)=I+tP$ holds for any rank-one projection P on H and positive number t. To see this, observe that for any given such t, P we have that $\phi(I+tP)=I+tP'$ holds for some rank-one projection P' on H. As in (35), we obtain

$$\begin{gathered}(I+\alpha P)(I+tP')(I+\alpha P)=\sqrt{\phi(I+P)}\phi(I+tP)\sqrt{\phi(I+P)}\\ =W(\sqrt{I+P}(I+tP)\sqrt{I+P})W^*=W((I+\alpha P)(I+tP)(I+\alpha P))W^*\end{gathered}$$

for some unitary or antiunitary operator W on H. As before, subtracting the identity and taking trace, we obtain $\operatorname{Tr} PP'=1$ which implies that $P'=P$ yielding $\phi(I+tP)=I+tP$.

Finally, pick an arbitrary $A\in\mathbb{B}(H)^{++}$. Let P be an arbitrary rank-one projection on H and t be any positive real number. We have

$$\begin{gathered}(\sqrt{1+t}P+(I-P))\phi(A)(\sqrt{1+t}P+(I-P))\\ =(\sqrt{I+tP})\phi(A)(\sqrt{I+tP})=(\sqrt{\phi(I+tP)})\phi(A)(\sqrt{\phi(I+tP)})\\ =W((\sqrt{1+t}P+(I-P))A(\sqrt{1+t}P+(I-P)))^{[-1]}W^*\end{gathered}\tag{36}$$

holds with some unitary or antiunitary operator W on H. Assume now, that for any positive real number M there exists $t>M$ such that the inverse shows up in the last term in (36). Then we have a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$\begin{gathered}\sigma((\sqrt{1+t_n}P+(I-P))\phi(A)(\sqrt{1+t_n}P+(I-P)))\\ =\sigma\left(\left(\frac{1}{\sqrt{1+t_n}}P+(I-P)\right)A^{-1}\left(\frac{1}{\sqrt{1+t_n}}P+(I-P)\right)\right)\end{gathered}$$

holds for all $n\in\mathbb{N}$. Dividing by $(1+t_n)$ and letting n tend to infinity, by the continuity of the spectrum, we see that the right hand side tends to $\{0\}$ while the left hand side tends to $\sigma(P\phi(A)P)$. But this yields $\sigma(P\phi(A)P)=\{0\}$, a contradiction.

Consequently, it follows that for large enough t, we necessarily have

$$\begin{gathered}\sigma((\sqrt{1+t}P+(I-P))\phi(A)(\sqrt{1+t}P+(I-P)))\\ =\sigma((\sqrt{1+t}P+(I-P))A(\sqrt{1+t}P+(I-P))).\end{gathered}$$

Again, dividing by $(1+t)$ and letting t tend to infinity we obtain $\sigma(P\phi(A)P)=\sigma(PAP)$. Choosing $P=x\otimes x$ for any unit vector $x\in H$, this gives that $\langle\phi(A)x,x\rangle=\langle Ax,x\rangle$. That clearly implies $\phi(A)=A$ and the proof is complete. □

Let us close the paper with an apparent question. In all of the statements above we have assumed that the underlying Hilbert space is separable. It is very natural to ask what happens if we drop that assumption. We leave this as an open problem.

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