New York Journal of Mathematics

New York J. Math. **26** (2020) 1422–1443.

Embedding problems with bounded ramification over function fields of positive characteristic

Moshe Jarden and **Nantsoina Cynthia Ramiharimanana**

Abstract. Let K_0 be an algebraic function field of one variable over a Hilbertian field F of positive characteristic p. Let K be a finite Galois extension of K_0. We prove that every finite embedding problem $\mathbf{1} \to H \to G \to \mathrm{Gal}(K/K_0) \to \mathbf{1}$ whose kernel H is a p-group is properly solvable.

Moreover, the solution can be chosen to locally coincide with finitely many, given in advance, weak local solutions. Finally, and this is the main point of this work, the number of prime divisors of K_0/F that ramify in the solution field is bounded by the number of prime divisors of K_0 that ramify in K plus the length of the maximal G-invariant sequence of subgroups of H.

Contents

Received September 11, 2019.

2010 *Mathematics Subject Classification.* 12F12, 12E30.

Key words and phrases. embedding problems, bounded ramification, function fields of positive characteristic.

ISSN 1076-9803/2020

Introduction

Solving finite embedding problems with solvable kernels and with bounded ramification over global fields is discussed in [JaR18] and [JaR19]. Here is the combination of the main results of those two works.

Theorem A: *Let K/K_0 be a finite Galois extension of global fields of characteristic p, set $\Gamma = \mathrm{Gal}(K/K_0)$, and consider a finite embedding problem*

$$(*) \qquad \begin{tikzcd} & & & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\ 1 \arrow[r] & H \arrow[r] & G \arrow[r, "\alpha"] & \Gamma \arrow[r] & 1, \end{tikzcd}$$

with solvable kernel H. Suppose that

(a1) $\gcd(|H|, |\mu(K)|) = 1$, *and*

(a2) *for each $\mathfrak{p} \in \mathbb{P}(K_0)$ there exists a homomorphism $\psi_\mathfrak{p}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to G$ such that $\alpha \circ \psi_\mathfrak{p} = \rho|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ (we call $\psi_\mathfrak{p}$ a* ***weak local solution****).*

Let T be a finite subset of $\mathbb{P}(K_0)$ that contains $\mathrm{Ram}(K/K_0)$ and for each $\mathfrak{p} \in T$ let $\varphi_\mathfrak{p}$ be a weak local solution.

Then, there exists an epimorphism $\psi\colon \mathrm{Gal}(K_0) \to G$ such that $\alpha \circ \psi = \rho$ (we call ψ a ***proper solution of embedding problem*** *$(*)$), and there exists a set $R \subseteq \mathbb{P}(K_0) \smallsetminus T$ with $|R| = \Omega_p(H, G)$ that satisfies the following conditions:*

(b1) *For each $\mathfrak{p} \in T$ there exists $a \in H$ such that $\psi(\sigma) = a^{-1}\varphi_\mathfrak{p}(\sigma)a$ for all $\sigma \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ (we say that $\psi_\mathfrak{p} := \psi|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ and $\varphi_\mathfrak{p}$ are* ***H-equivalent****).*

(b2) *The fixed field N in $K_{0,\mathrm{sep}}$ of $\mathrm{Ker}(\psi)$ satisfies $\mathrm{Ram}(N/K_0) \subseteq T \cup R$, hence $|\mathrm{Ram}(N/K_0)| \leq |T| + \Omega_p(H, G)$. We call N a* ***solution field of embedding problem*** *$(*)$.*

In this theorem we fix a separable algebraic closure $K_{0,\mathrm{sep}}$ of K_0 and let $\mathrm{Gal}(K_0) = \mathrm{Gal}(K_{0,\mathrm{sep}}/K_0)$ be the absolute Galois group of K_0. We denote the set of primes of K_0 by $\mathbb{P}(K_0)$ and for each $\mathfrak{p} \in \mathbb{P}(K_0)$ we choose a completion $\hat{K}_{0,\mathfrak{p}}$ of K_0 at $\mathfrak{p}$. Then, $\mathrm{Ram}(K/K_0)$ denotes the set of all $\mathfrak{p} \in \mathbb{P}(K_0)$ that ramify in K. Also, $\mu(K)$ is the group of roots of unity in K. Finally, $\Omega_p(H, G)$ is a function that depends on p and the structure of the group H as a normal subgroup of G. In particular, if $p = 0$, then $\Omega_p(H, G)$ is just the number of prime divisors of $|H|$, counted with multiplicity.

Induction on the structure of G acting on H reduces the proof of Theorem A to the case where H is a simple multiplicative G-module C_l^r, where l is a prime number and C_l is the cyclic group of order l.

Then, we use basic tools of algebraic number theory like the strong approximation theorem, the Chebotarev density theorem, and the interplay between global and local fields. In the case where $l \neq p$, we follow Jürgen Neukirch's basic work [Neu79], generalized in [NSW15], and apply class field theory, including Artin's reciprocity theorem and duality theorems for cohomology groups. In the case where $l = p$, we replace class field theory by Hilbert irreducibility theorem for K and Artin-Schreier extensions.

Note that Neukirch's result deals with solving finite embedding problems over number fields with solvable kernels but gives no information about the ramification of the solution field. Our proofs in [JaR18] and [JaR19] add the missing information by using ideas included in [GeJ98].

The present work deals with an algebraic function field K_0 of one variable over a Hilbertian field F of positive characteristic p. A prime divisor of K_0 over F can be considered as an equivalence class of valuations of K_0 which are trivial over F. We denote the set of those prime divisors by $\mathbb{P}(K_0/F)$.

For each $\mathfrak{p} \in \mathbb{P}(K_0/F)$ let $\mathrm{ord}_\mathfrak{p}$ be the normalized discrete valuation of K_0 attached to $\mathfrak{p}$ with $\mathrm{ord}_\mathfrak{p}(a) = 0$ for each $a \in F^\times$. The completion $\hat{K}_{0,\mathfrak{p}}$ of K_0 with respect to $\mathrm{ord}_\mathfrak{p}$ is a field of power series in one variable over the residue field $\bar{K}_{0,\mathfrak{p}}$ of K_0 at $\mathfrak{p}$. We fix an embedding of $K_{0,\mathrm{sep}}$ in $\hat{K}_{0,\mathfrak{p},\mathrm{sep}}$ and use Krasner's lemma to embed $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ into $\mathrm{Gal}(K_0)$.

In the notation of Theorem A, assume that H is a finite p-group. Since there is no root of unity of order p, Condition (a1) is trivially satisfied. Also, since $\mathrm{char}(K_0) = p$, embedding problem $(*)$ has a weak $\mathfrak{p}$-local solution for each $\mathfrak{p} \in \mathbb{P}(K_0/F)$ (Remark 6.1), so we don't have to assume Condition (a2) in this case.

This brings us to our main result, which is restricted to kernels that are finite p-groups but is, in this case, much stronger than Theorem A:

Theorem B: *Let K_0 be a function field of one variable over a Hilbertian field F of positive characteristic p and let K be a finite Galois extension of K_0. Consider the finite embedding problem $(*)$,*

$$\begin{tikzcd} & & & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\ 1 \arrow[r] & H \arrow[r] & G \arrow[r, "\alpha"] & \Gamma \arrow[r] & 1, \end{tikzcd}$$

where the kernel H is a p-group of order p^s. Let T be a finite subset of $\mathbb{P}(K_0/F)$ that contains $\mathrm{Ram}(K/K_0)$. For each $\mathfrak{p} \in T$ let $\varphi_\mathfrak{p} \colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to G$ be a homomorphism such that $\alpha \circ \varphi_\mathfrak{p} = \rho|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$.

Then, there exists an epimorphism $\psi \colon \mathrm{Gal}(K_0) \to G$ such that $\alpha \circ \psi = \rho$ and there exists a set $R \subseteq \mathbb{P}(K_0/F) \smallsetminus T$ with $|R| \le s$ such that:

(a) *For each $\mathfrak{p} \in T$ the homomorphism $\psi|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ is H-equivalent to $\varphi_\mathfrak{p}$.*
(b) *The fixed field N of $\mathrm{Ker}(\psi)$ in $K_{0,\mathrm{sep}}$ satisfies $\mathrm{Ram}(N/K_0) \subseteq T \cup R$, so $|\mathrm{Ram}(N/K_0)| \le |T| + |R| \le |T| + s$.*

The proof of Theorem B follows that part of the proof of Theorem A in [JaR19] in which H is a simple $\mathrm{Gal}(K_0)$-p-module. In particular, we use the Hilbertianity of K_0 to construct linearly disjoint Artin-Schreier extensions of K_0.

Another difference in the proof from that of Theorem A arises from the fact that the Galois group $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p},\mathrm{ur}}/\hat{K}_{0,\mathfrak{p}})$ of the maximal unramified extension of $\hat{K}_{0,\mathfrak{p}}$ for $\mathfrak{p} \in \mathbb{P}(K_0/F)$ is not isomorphic to $\hat{\mathbb{Z}}$ anymore, as is the case where K_0 is a global field. But it is isomorphic to $\mathrm{Gal}(\bar{K}_{0,\mathfrak{p}})$ and $\bar{K}_{0,\mathfrak{p}}$ is a Hilbertian field (because F is). Thus, we are able to apply a result of Ikeda plus a small trick to properly solve the corresponding $\mathfrak{p}$-local embedding problem (see proof of Lemma 6.3).

Another difficulty arises from the fact that a crucial local-global surjectivity theorem for the first cohomology groups (Proposition 4.7) is proved in [NSW15] only for global fields of positive characteristic p, which is not our case. So, we took extra care to prove that result and at points to enhance the proofs of [NSW15], whenever we felt they were too short.

Acknowledgment: *The authors are indebted to Aharon Razon for carefully reading drafts of this work and for his useful comments. Special thanks go to the anonymous referee who suggested an essential simplification of an earlier version of the proof of Proposition 6.8.*

1. Function fields of one variable

Let F be a field of positive characteristic p and let K_0 be an algebraic function field of one variable over F. Thus, K_0 is a finitely generated regular extension of F of transcendence degree 1 [FrJ08, p. 52, Section 3.1]. We fix a separable algebraic closure $K_{0,\mathrm{sep}}$ of K_0 and tacitly assume that each of our separable algebraic extension of K_0 is contained in $K_{0,\mathrm{sep}}$. In particular, this is the case for a finite separable extension K of K_0 that we consider. In this case, K is a finitely generated separable (hence regular) extension of the algebraic closure F' of F in K. Thus, by our convention, K is an algebraic function field of one variable over F'.

We denote the set of all prime divisors of K_0/F by $\mathbb{P}(K_0/F)$ and the normalized discrete valuation of K_0 attached to $\mathfrak{p} \in \mathbb{P}(K_0/F)$ by $\mathrm{ord}_\mathfrak{p}$. Thus, $\mathrm{ord}_\mathfrak{p}(K_0) = \mathbb{Z}$. Similar notation applies also to K, except that we abuse our notation, write $\mathbb{P}(K/F)$ instead of $\mathbb{P}(K/F')$ for the set of prime divisors of K/F', and speak about "primes of K/F" rather than about "primes of K/F'". In addition, we let $\mathrm{Ram}(K/K_0)$ be the set of all $\mathfrak{p} \in \mathbb{P}(K_0/F)$ that ramify in K. It is a finite set.

1.1. Krasner's lemma. A central tool in the study of algebraic function fields of one variable is the **strong approximation theorem** [FrJ08, p. 56, Prop. 3.3.1]:

Proposition 1.1. *Let S be a finite subset of $\mathbb{P}(K_0/F)$, consider $\mathfrak{q} \in \mathbb{P}(K_0/F) \smallsetminus S$, and let $S' = S \cup \{\mathfrak{q}\}$. For each $\mathfrak{p} \in S$ let $x_\mathfrak{p}$ be an element of K_0 and let $m_\mathfrak{p}$ be a positive integer. Then, there exists $x \in K_0$*

with

$$\mathrm{ord}_{\mathfrak{p}}(x - x_{\mathfrak{p}}) = m_{\mathfrak{p}} \text{ for each } \mathfrak{p} \in S \text{ and}$$
$$\mathrm{ord}_{\mathfrak{p}}(x) \geq 0 \text{ for each } \mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus S'.$$

Another tool that we use is **Krasner's lemma** and its consequences. In that lemma, one considers a complete discrete valuation (or, more generally, Henselian) field (E, v) and a complete set of conjugates $x_1, \ldots, x_n$ of an element $x = x_1 \in E_{\mathrm{sep}}$. Then, every $y \in E_{\mathrm{sep}}$ that satisfies

$$v(y - x) > \max_{i \geq 2}(v(x - x_i)),$$

also satisfies $E(x) \subseteq E(y)$ [Lan70, p. 43, Prop. 3].

Krasner's lemma implies the following theorem about the **continuity of the roots**. See [Jar91, Prop. 12.3] or [Efr06, p. 171, Thm. 18.5.2].

Proposition 1.2. *Let (E, v) be a complete discrete valuation field and let $f \in E[X]$ be a monic polynomial of degree n with n distinct roots $x_1, \ldots, x_n$. Then, for each positive integer α there exists a positive integer γ such that the following holds:*

If $g \in E[X]$ is a monic polynomial of degree n with $v(g - f) > \gamma$, then the roots of g are distinct and can be enumerated as $y_1, \ldots, y_n$ such that $v(y_i - x_i) > \alpha$ and $E(x_i) = E(y_i)$.

Here, $v(g - f)$ is the maximal v-value of the differences between the corresponding coefficients of g and f.

1.2. Completions. For each $\mathfrak{p} \in \mathbb{P}(K_0/F)$ we fix a completion $\hat{K}_{0,\mathfrak{p}}$ of K_0 at $\mathfrak{p}$ and a separable algebraic closure $\hat{K}_{0,\mathfrak{p},\mathrm{sep}}$ that contains $K_{0,\mathrm{sep}}$. Then, $K_{0,\mathfrak{p}} = K_{0,\mathrm{sep}} \cap \hat{K}_{0,\mathfrak{p}}$ is a Henselian closure of K_0 at $\mathfrak{p}$.

By the theorem about the continuity of roots, for each $\hat{x} \in \hat{K}_{0,\mathfrak{p},\mathrm{sep}}$ there exists a monic polynomial $f \in K_0[X]$ with $\deg(f) = \deg(\mathrm{irr}(\hat{x}, \hat{K}_{0,\mathfrak{p}}))$[1] and with $\mathrm{ord}_{\mathfrak{p}}(f - \mathrm{irr}(\hat{x}, \hat{K}_{0,\mathfrak{p}}))$ sufficiently large such that there exists a root x of f with $\hat{K}_{0,\mathfrak{p}}(x) = \hat{K}_{0,\mathfrak{p}}(\hat{x})$. In particular, $x \in K_{0,\mathrm{sep}}$. It follows that $K_{0,\mathrm{sep}}\hat{K}_{0,\mathfrak{p}} = \hat{K}_{0,\mathfrak{p},\mathrm{sep}}$, so $\mathrm{Gal}(K_{0,\mathfrak{p}}) \cong \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$. Hence, we may and we will identify $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ with $\mathrm{Gal}(K_{0,\mathfrak{p}})$ via restriction.

2. Cohomological dimension

As in section 1, we consider an algebraic function field K_0 of one variable over a field F of positive characteristic p. Let $\wp$ be the Artin-Schreier operator defined by $\wp(x) = x^p - x$.

For a nonempty proper subset S of $\mathbb{P}(K_0/F)$ and a finite separable extension K of K_0, we define S_K to be the set of prime divisors of K/F that

[1]As usual, we denote the irreducible polynomial of an algebraic element x over a field K by $\mathrm{irr}(x, K)$.

lie over S. Following the convention in algebraic number theory [NSW15, p. 452, Sec. VIII.3], we set

$$\mathcal{O}_{K,S} = \{x \in K \mid \operatorname{ord}_{\mathfrak{P}}(x) \geq 0 \text{ for all } \mathfrak{P} \in \mathbb{P}(K/F) \smallsetminus S_K\}.$$

For a separable algebraic extension L of K_0, we set $\mathcal{O}_{L,S} = \bigcup_K \mathcal{O}_{K,S}$, where K ranges over all finite extensions of K_0 in L. By [Lan58, p. 13, Prop. 4],

(1) $\mathcal{O}_{L,S}$ is the integral closure of $\mathcal{O}_{K_0,S}$ in L.

However, as is customary in commutative algebra, for each $\mathfrak{p} \in \mathbb{P}(K_0/F)$ we write $\mathcal{O}_{\mathfrak{p}} = \{x \in K_0 \mid \operatorname{ord}_{\mathfrak{p}}(x) \geq 0\}$ for the local ring of $\mathfrak{p}$.

Taking into account Footnote 1, the following result is included in Proposition 3.3.2 of [FrJ08].

Lemma 2.1. *The following statements on K_0 and $\mathcal{S}$ hold:*

(a) *$\mathcal{O}_{K_0,S}$ is a Dedekind domain.*

(b) *If $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus S$ and $P = \{x \in \mathcal{O}_{K_0,S} \mid \operatorname{ord}_{\mathfrak{p}}(x) > 0\}$ is the* **center of $\mathfrak{p}$ at** *$\mathcal{O}_{K_0,S}$, then $\mathcal{O}_{\mathfrak{p}} = (\mathcal{O}_{K_0,S})_P$.*

(c) *Every non-zero prime ideal of $\mathcal{O}_{K_0,S}$ is the center of a prime divisor $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus S$.*

(d) *If $\mathfrak{q} \in S$, then $\mathcal{O}_{K_0,S} \not\subseteq \mathcal{O}_{\mathfrak{q}}$.*

Lemma 2.2. *Let (R, v) be a complete discrete valuation domain of positive characteristic p with $v(R) = \mathbb{Z} \cup \{\infty\}$. Then, $P := \{x \in R \mid v(x) \geq 1\} \subseteq \wp(R)$.*

Proof. Let $a \in P$ and consider the polynomial $f(X) = X^p - X - a$ and its derivative $f'(X) = -1$. Then, $v(f(0)) = v(a) \geq 1$ and $v(f'(0)) = v(-1) = 0$. Hence, by Hensel's lemma, there exists $x \in \operatorname{Quot}(R)$ such that $x^p - x - a = 0$ and $v(x) \geq 1$. In particular, $x \in R$. It follows that $a = \wp(x) \in \wp(R)$, as claimed. $\square$

Lemma 2.3. *Let S be a set of prime divisors of K_0/F and let T be a proper finite subset of S. Then, the natural map*

$$\mathcal{O}_{K_0,S} \to \bigoplus_{\mathfrak{p} \in T} \hat{K}_{0,\mathfrak{p}} / \wp \hat{K}_{0,\mathfrak{p}}$$

is surjective.

Proof. Given an element $a_{\mathfrak{p}} \in \hat{K}_{0,\mathfrak{p}}$ for each $\mathfrak{p} \in T$, we use the assumption that $S \smallsetminus T \neq \emptyset$ and the strong approximation theorem (Proposition 1.1) to choose $x \in K_0$ with $\operatorname{ord}_{\mathfrak{p}}(x - a_{\mathfrak{p}}) \geq 1$ for each $\mathfrak{p} \in T$ and $\operatorname{ord}_{\mathfrak{p}}(x) \geq 0$ for every $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus S$. Then, Lemma 2.2 gives for each $\mathfrak{p} \in T$ an element $y_{\mathfrak{p}} \in \hat{K}_{0,\mathfrak{p}}$ with $x - a_{\mathfrak{p}} = \wp(y_{\mathfrak{p}})$. Thus, $x \in \mathcal{O}_{K_0,S}$ and $x \equiv a_{\mathfrak{p}} \mod \wp(\hat{K}_{0,\mathfrak{p}})$, as desired. $\square$

Following [NSW15, p. 171, §3], we denote the category of all p-torsion modules of a profinite group G by $\operatorname{Mod}_p(G)$. As usual, we denote the pth cohomological dimension of G by $\operatorname{cd}_p(G)$.

Lemma 2.4 ([NSW15], p. 172, Prop. 3.3.2)**.** *The following conditions on a profinite group G and a positive integer n are equivalent:*

(a) $\mathrm{cd}_p(G) \leq n$,
(b) $H^q(G, A) = 0$ *for all* $A \in \mathrm{Mod}_p(G)$ *and for all* $q > n$.

Recall that a G-module B is **cohomologically trivial** if $H^i(H, B) = \mathbf{0}$ for every closed subgroup H of G and every positive integer i.

Let R be a Dedekind domain and let N be a Galois extension of $K := \mathrm{Quot}(R)$. Let R_N be the integral closure of R in N. One says that R_N/R is **tamely ramified** if for every finite Galois extension L of K in N and every prime ideal P of R, the prime factorization of P in R_L has the form $PR_L = (Q_1 \cdots Q_g)^e$ with distinct prime ideals $Q_1, \ldots, Q_g$ of R_L and with $\gcd(e, \mathrm{char}(R/P)) = 1$.

Lemma 2.5 ([NSW15], p. 342, Thm. 6.1.10)**.** *Let R be a Dedekind domain with quotient field K, let L be a Galois extension of K, and let R_L be the integral closure of R in L. Then, the following conditions are equivalent:*

(a) *R_L is a cohomologically trivial* $\mathrm{Gal}(L/K)$*-module.*
(b) *R_L/R is tamely ramified.*

We denote the maximal Galois extension of K_0 which is unramified away from S by $K_{0,S}$.

Lemma 2.6 (Generalization of [NSW15], p. 453, Prop. 8.3.1)**.** *Let S be a nonempty proper subset of $\mathbb{P}(K_0/F)$. Let L be a Galois extension of K_0 in $K_{0,S}$. Let K be a Galois extension of K_0 in L. Then, $H^i(\mathrm{Gal}(L/K), \mathcal{O}_{L,S}) = \mathbf{0}$ for every positive integer i.*

Proof. By Lemma 2.1, $\mathcal{O}_{K_0,S}$ is a Dedekind domain and the non-zero prime ideals of $\mathcal{O}_{K_0,S}$ are induced by $\mathbb{P}(K_0/F) \smallsetminus S$. By assumption, each $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus S$ is unramified in L. Thus, the center of $\mathfrak{p}$ at $\mathcal{O}_{K_0,S}$ is unramified in $\mathcal{O}_{L,S}$. It follows from Lemma 2.1(c) that the ring extension $\mathcal{O}_{L,S}/\mathcal{O}_{K_0,S}$ is unramified, hence tamely ramified. Therefore, by Lemma 2.5, $\mathcal{O}_{L,S}$ is a cohomologically trivial $\mathrm{Gal}(L/K_0)$-module. By definition, $H^i(\mathrm{Gal}(L/K), \mathcal{O}_{L,S}) = \mathbf{0}$ for all $i \geq 1$. $\square$

3. *p*-closed extensions

We say that a field L is p-closed if it has no Galois extensions of degree p. Assuming that $p = \mathrm{char}(L) > 0$, the Artin-Schreier map $\wp\colon L \to L$ is in this case surjective. This yields the short exact sequence

$$\mathbf{0} \longrightarrow \mathbb{F}_p \longrightarrow L \xrightarrow{\wp} L \longrightarrow \mathbf{0}. \tag{2}$$

Lemma 3.1 ([NSW15], Cor. 6.1.2)**.** *Let L be a Galois p-closed extension of K_0. Then,*

$$H^n(\mathrm{Gal}(L/K_0), \mathbb{F}_p) = \begin{cases} K_0/\wp K_0 & \text{for } n = 1 \\ \mathbf{0} & \text{for } n \geq 2. \end{cases}$$

As in section 2, let S be a subset of $\mathbb{P}(K_0/F)$. Then, the following result generalizes [NSW15, Cor. 8.3.2].

Lemma 3.2. *Let L be a p-closed Galois extension of K_0 in $K_{0,S}$ and let K be a Galois extension of K_0 in L. Then,*

$$H^n(\mathrm{Gal}(L/K),\mathbb{F}_p)=\begin{cases}\mathcal{O}_{K,S}/\wp\mathcal{O}_{K,S} & \text{for } n=1\\ \mathbf{0} & \text{for } n\geq 2.\end{cases}$$

Proof. First we prove that the map $\wp\colon \mathcal{O}_{L,S}\to\mathcal{O}_{L,S}$ is surjective. To this end, consider $a\in\mathcal{O}_{L,S}$ and choose $x\in K_{0,\mathrm{sep}}$ such that $x^p-x=a$. If $x\notin L$, then by Artin-Schreier, $[L(x):L]=p$, which contradicts the assumption on L. Hence, $x\in L$. Since $\mathcal{O}_{L,S}$ is integrally closed (by (1)), $x\in\mathcal{O}_{L,S}$, as claimed.

The claim yields the following short exact sequence of $\mathrm{Gal}(L/K)$-modules:

$$\mathbf{0}\longrightarrow\mathbb{F}_p\longrightarrow\mathcal{O}_{L,S}\xrightarrow{\wp}\mathcal{O}_{L,S}\longrightarrow\mathbf{0}. \tag{3}$$

The beginning of the corresponding long exact sequence has the form

$$\mathcal{O}_{K,S}\xrightarrow{\wp}\mathcal{O}_{K,S}\longrightarrow H^1(\mathrm{Gal}(L/K),\mathbb{F}_p)\longrightarrow H^1(\mathrm{Gal}(L/K),\mathcal{O}_{L,S}). \tag{4}$$

By Lemma 2.6, the last term of (4) is $\mathbf{0}$. Hence,

$$H^1(\mathrm{Gal}(L/K),\mathbb{F}_p)\cong\mathcal{O}_{K,S}/\wp\mathcal{O}_{K,S}.$$

Similarly, for $i\geq 1$, (3) yields the exact sequence

$$H^i(\mathrm{Gal}(L/K),\mathcal{O}_{L,S})\to H^{i+1}(\mathrm{Gal}(L/K),\mathbb{F}_p)\to H^{i+1}(\mathrm{Gal}(L/K),\mathcal{O}_{L,S}) \tag{5}$$

with trivial first and third terms. Hence, $H^{i+1}(\mathrm{Gal}(L/K),\mathbb{F}_p)=\mathbf{0}$, as claimed. $\square$

We end this section with a generalization of Corollary 8.3.3 on page 453 of [NSW15].

Lemma 3.3. *Assume that F is Hilbertian and let L be a p-closed Galois extension of K_0 in $K_{0,S}$. Then, $\mathrm{cd}_p(\mathrm{Gal}(L/K_0))=1$.*

Proof. Let E be the fixed field in L of a p-Sylow subgroup of $\mathrm{Gal}(L/K_0)$. In particular E contains no p-extension of K_0. Since F is Hilbertian, F has a cyclic extension F' of degree p [FrJ08, p. 304, Prop. 16.4.5]. By Artin-Schreier, there exists $x\in F_{\mathrm{sep}}\smallsetminus F$ such that $a:=x^p-x\in F$ and $F'=F(x)$. Since K_0/F is regular, K_0F' is a cyclic extension of K_0 of degree p. Moreover, K_0F' is unramified over K_0 along $\mathbb{P}(K_0F'/F)\smallsetminus S_{K_0F'}$, so $K_0F'\subseteq K_{0,S}$. Since L is p-closed in $K_{0,S}$, we have $F'\subseteq L$. Finally, by the choice of E, EF' is a cyclic extension of E of order p. Hence, by [NSW15, p. 174, Cor. 3.3.7], $\mathrm{cd}_p(\mathrm{Gal}(L/E))\geq 1$.

By Lemma 3.2, $H^i(\mathrm{Gal}(L/E),\mathbb{F}_p)=\mathbf{0}$ for $i\geq 2$. Since $\mathrm{Gal}(L/E)$ is a pro-p group, it follows from [NSW15, p. 174, Cor. 3.3.6] and the last statement of [NSW15, p. 172, Prop. 3.3.2] that $\mathrm{cd}_p(\mathrm{Gal}(L/E))\leq 1$. It follows that $\mathrm{cd}_p(\mathrm{Gal}(L/E))=1$.

Finally, since $\mathrm{Gal}(L/E)$ is a p-Sylow subgroup of $\mathrm{Gal}(L/K_0)$ it follows from [NSW15, p. 174, Cor. 3.3.6] that $\mathrm{cd}_p(\mathrm{Gal}(L/K_0)) = 1$, as claimed. $\square$

4. Surjectivity of restriction maps

This section establishes a surjectivity theorem for first cohomology groups. As before, we consider an algebraic function field K_0/F of one variable of positive characteristic p.

Notation 4.1. Let S be a subset of $\mathbb{P}(K_0/F)$, let A be a finite $\mathrm{Gal}(K_{0,S}/K_0)$-module, and let x be an element of $H^1(\mathrm{Gal}(K_{0,S}/K_0), A)$. We choose a crossed homomorphism $\chi\colon \mathrm{Gal}(K_{0,S}/K_0) \to A$ that represents x. Then, for $\mathfrak{p} \in \mathbb{P}(K_0/F)$, we set $\chi_\mathfrak{p}$ to be the compositum of the maps

$$\begin{aligned}\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \longrightarrow & \mathrm{Gal}(K_{0,S}\hat{K}_{0,\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}}) \\ & \longrightarrow \mathrm{Gal}(K_{0,S}/K_{0,S} \cap \hat{K}_{0,\mathfrak{p}}) \xrightarrow{\mathrm{incl}} \mathrm{Gal}(K_{0,S}/K_0) \xrightarrow{\chi} A,\end{aligned}$$

where the first two maps are the corresponding restriction maps and the third one is the inclusion map.

The map $\chi \to \chi_\mathfrak{p}$ is compatible with the actions of $\mathrm{Gal}(K_{0,S}/K_0)$ and $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ on A, so $\chi_\mathfrak{p}$ is a crossed homomorphism. We denote the cohomology class of $\chi_\mathfrak{p}$ by $\mathrm{res}_\mathfrak{p}(x)$. Note that the map $\chi \mapsto \chi_\mathfrak{p}$ is multiplicative and maps boundaries onto boundaries. Hence, $\mathrm{res}_\mathfrak{p}\colon H^1(\mathrm{Gal}(K_{0,S}/K_0), A) \to H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is a natural homomorphism.

Our definition implies that if $y \in H^1(\mathrm{Gal}(K_{0,S}/K_0), A)$ and z is the image of y under the map $\inf\colon H^1(\mathrm{Gal}(K_{0,S}/K_0), A) \to H^1(\mathrm{Gal}(K_0), A)$, then $\mathrm{res}_\mathfrak{p}(y) = \mathrm{res}_\mathfrak{p}(z)$.

Remark 4.2. Let S be a subset of $\mathbb{P}(K_0/F)$ and let T be a finite subset of S. By Lemma 3.3, $\mathrm{cd}_p(\mathrm{Gal}(K_{0,S}/K_0)) = 1$. Also, since $\mathrm{char}(\hat{K}_{0,\mathfrak{p}}) = p > 0$, we have $\mathrm{cd}_p(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})) \le 1$ [NSW15, p. 338, Cor. 6.1.3]. Hence, if

$$\mathbf{0} \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow \mathbf{0}$$

is a short exact sequence of p-primary $\mathrm{Gal}(K_{0,S}/K_0)$-modules, then Lemma 2.4 yields the following commutative diagram:

$$\begin{tikzcd} H^1(\mathrm{Gal}(K_{0,S}/K_0), B') \arrow[r] \arrow[d, "\mathrm{res}^1(S,T,B')"] & H^1(\mathrm{Gal}(K_{0,S}/K_0), B) \arrow[r] \arrow[d, "\mathrm{res}^1(S,T,B)"] & H^1(\mathrm{Gal}(K_{0,S}/K_0), B'') \arrow[r] \arrow[d, "\mathrm{res}^1(S,T,B'')"] & \mathbf{0} \\ \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), B') \arrow[r] & \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), B) \arrow[r] & \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), B'') \arrow[r] & \mathbf{0}. \end{tikzcd}$$

The vertical maps in this diagram are the direct sums of the maps $\mathrm{res}_\mathfrak{p}$ for $\mathfrak{p} \in T$ introduced in Notation 4.1.

Observe that surjectivity of the middle vertical arrow implies surjectivity of the right vertical arrow. $\square$

Proposition 4.7 below is an analog of Theorem 9.2.5 on page 539 of [NSW15]. Its proof requires some preparations.

Remark 4.3 (Induced Modules)**.** Consider a short exact sequence

$$\mathbf{1} \longrightarrow H \longrightarrow G \longrightarrow \bar{G} \longrightarrow \mathbf{1}$$

of profinite groups. For a finite H-module A, $\mathrm{Ind}_G^H(A)$ is the G-module that consists of all functions $f\colon G \to A$ such that $f(\eta\sigma) = \eta f(\sigma)$ for all $\eta \in H$ and $\sigma \in G$. The action of G on $\mathrm{Ind}_G^H(A)$ is given for each $\rho \in G$ by $(\rho f)(\sigma) = f(\sigma\rho)$.

The G-module $\mathrm{Ind}_G^H(A)$ is isomorphic to the G-module $\mathrm{Map}(\bar{G}, A)$ of all maps $f\colon \bar{G} \to A$, where an element $\rho \in G$ acts on f by the rule $(\rho f)(\sigma H) = \rho f(\rho^{-1}\sigma H)$. Hence,

$$\mathrm{Ind}_G^H(A) \cong \mathrm{Map}(\bar{G}, A) \cong \mathrm{Ind}_{\bar{G}}^{\mathbf{1}}(A). \tag{6}$$

Shapiro's lemma then gives an isomorphism $H^n(G, \mathrm{Ind}_G^H(A)) \cong H^n(H, A)$ for each $n \geq 0$ [NSW15, p. 62, Prop. 1.6.4].

We are interested in the special case where $A = \mathbb{F}_p$ has a trivial $\bar{G}$-action and $\bar{G}$ is a finite group. In this case $\mathrm{Map}(\bar{G}, \mathbb{F}_p)$ is isomorphic to the $\bar{G}$-module $\mathbb{F}_p[\bar{G}]$ of all formal sums $\sum_{\sigma\in\bar{G}} a_\sigma\sigma$ with $a_\sigma \in \mathbb{F}_p$ for all $\sigma \in \bar{G}$. The action of $\bar{G}$ on $\mathbb{F}_p[\bar{G}]$ is given by $\tau\sum_{\sigma\in\bar{G}} a_\sigma\sigma = \sum_{\sigma\in\bar{G}} a_{\tau^{-1}\sigma}\sigma$. It follows that

$$\mathbb{F}_p[\bar{G}] \cong \mathrm{Map}(\bar{G}, \mathbb{F}_p) \cong \mathrm{Ind}_{\bar{G}}^{\mathbf{1}}(\mathbb{F}_p), \tag{7}$$

as $\bar{G}$-modules. Hence,

$$H^n(G, \mathbb{F}_p[\bar{G}]) \cong H^n(G, \mathrm{Ind}_{\bar{G}}^{\mathbf{1}}(\mathbb{F}_p)) \overset{(6)}{\cong} H^n(G, \mathrm{Ind}_G^H(\mathbb{F}_p)) \cong H^n(H, \mathbb{F}_p). \tag{8}$$

Lemma 4.4. *Let D be an open subgroup of a profinite group G and let Σ be a system of representatives for the quotient set G/D, thus*

$$G = \mathop{\dot{\bigcup}}_{\sigma\in\Sigma} \sigma D = \mathop{\dot{\bigcup}}_{\sigma\in\Sigma} D\sigma^{-1}. \tag{9}$$

For each $\sigma \in \Sigma$ let $D_\sigma = \sigma D\sigma^{-1}$. Then, there exists an isomorphism

$$\Phi\colon \mathrm{Map}(G, \mathbb{F}_p) \to \bigoplus_{\sigma\in\Sigma} \mathrm{Map}(D_\sigma, \mathbb{F}_p). \tag{10}$$

We let G act on the right hand side of (10) *such that Φ becomes an isomorphism of G-modules.*

Proof. Given a continuous map $f\colon G \to \mathbb{F}_p$ and an element $\sigma \in \Sigma$, the continuous map $f_\sigma\colon D_\sigma \to \mathbb{F}_p$ given by $f_\sigma(\sigma d\sigma^{-1}) = f(d\sigma^{-1})$ for each $d \in D$ is well defined. Hence, the map Φ of (10) given by

$$\Phi(f) = (f_\sigma)_{\sigma\in\Sigma} \tag{11}$$

is well defined.

If $f'\colon G \to \mathbb{F}_p$ is another element of $\mathrm{Map}(G, \mathbb{F}_p)$ and $\Phi(f) = \Phi(f')$, then for every $\sigma \in \Sigma$ and $d \in D$ we have $f(d\sigma^{-1}) = f_\sigma(\sigma d\sigma^{-1}) = f'_\sigma(\sigma d\sigma^{-1}) = f'(d\sigma^{-1})$, Hence, $f = f'$. Thus, Φ is injective.

Given $(f_\sigma)_{\sigma\in\Sigma} \in \bigoplus_{\sigma\in\Sigma} \mathrm{Map}(D_\sigma, \mathbb{F}_p)$, we define a map $f\colon G \to \mathbb{F}_p$ by $f(d\sigma^{-1}) = f_\sigma(\sigma d\sigma^{-1})$ for all $d \in D$ and $\sigma \in \Sigma$. By (9), f is well defined and by (11), $\Phi(f) = (f_\sigma)_{\sigma\in\Sigma}$. Hence, Φ is surjective.

It follows that Φ is bijective. Using that bijectivity, the definition $\rho\Phi(f) = \Phi(\rho f)$ for all $\rho \in G$ and $f \in \mathrm{Map}(G, \mathbb{F}_p)$ defines an action of G on the right hand side of (10) such that Φ becomes an isomorphism of G-modules. By definition, it satisfies the final statement of the lemma. □

Lemma 4.5. *Let K_0 be an algebraic function field of one variable over a field F and let $\mathfrak{p}$ be a prime divisor of K_0/F. Let K be a finite Galois extension of K_0 with Galois group G. Then, for each n we have*

$$H^n(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathbb{F}_p[G]) \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^n(\mathrm{Gal}(\hat{K}_\mathfrak{P}), \mathbb{F}_p),$$

where $\mathfrak{P}$ ranges over all prime divisors of K/F that lie above $\mathfrak{p}$.

Proof. By Remark 4.3, $\mathbb{F}_p[G] \cong \mathrm{Map}(G, \mathbb{F}_p)$. By Lemma 4.4, there exists an isomorphism of G-modules

$$\mathrm{Map}(G, \mathbb{F}_p) \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathrm{Map}(D_\mathfrak{P}, \mathbb{F}_p), \tag{12}$$

where for each $\mathfrak{P}$, $D_\mathfrak{P}$ is the decomposition group of $\mathfrak{P}$ over $\mathfrak{p}$. One may lift the action of G on both sides of (12) to an action of $\mathrm{Gal}(K_0)$ and then to restrict it to an action of $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$.

Since $\mathbb{F}_p[G] \cong \mathrm{Map}(G, \mathbb{F}_p)$ (Remark 4.3) and $D_\mathfrak{P} \cong \mathrm{Gal}(\hat{K}_\mathfrak{P}/\hat{K}_{0,\mathfrak{p}})$, it follows from (12) that

$$\begin{aligned}
H^n(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathbb{F}_p[G]) &\cong H^n(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathrm{Map}(G, \mathbb{F}_p)) \\
&\cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^n(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathrm{Map}(D_\mathfrak{P}, \mathbb{F}_p)) \\
&\cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^n(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathrm{Map}(\mathrm{Gal}(\hat{K}_\mathfrak{P}/\hat{K}_{0,\mathfrak{p}}), \mathbb{F}_p)) \\
&\overset{(7)}{\cong} \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^n(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathbb{F}_p[\mathrm{Gal}(\hat{K}_\mathfrak{P}/\hat{K}_{0,\mathfrak{p}})]) \\
&\overset{(8)}{\cong} \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^n(\mathrm{Gal}(\hat{K}_\mathfrak{P}), \mathbb{F}_p),
\end{aligned}$$

as claimed. □

Lemma 4.6 ([Rib70], p. 118, Prop. 4.6)**.** *Let G be a profinite group, for i in a set I let A_i be a G-module, and set $A = \bigoplus_{i\in I} A_i$. Then, $H^q(G,A) = \bigoplus_{i\in I} H^q(G,A_i)$ for each $q \geq 0$.*

Proposition 4.7. *Let F be a field of positive characteristic p and let K_0 be an algebraic function field of one variable over F. Let $T \subseteq S$ be sets of prime divisors of K_0/F such that T is finite and $S \smallsetminus T \neq \emptyset$. Let A be a finite* $\mathrm{Gal}(K_{0,S}/K_0)$*-module such that $pA = \mathbf{0}$. Then, the homomorphism*

$$\mathrm{res}^1(S,T,A)\colon H^1(\mathrm{Gal}(K_{0,S}/K_0),A) \to \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}),A)$$

is surjective.

Proof. We distinguish between four cases.

Case A: $A = \mathbb{F}_p$ By definition, $K_{0,S}$ is p-closed in $K_{0,S}$. Hence, by Lemma 3.2,

$$H^1(\mathrm{Gal}(K_{0,S}/K_0),\mathbb{F}_p) = \mathcal{O}_{K_0,S}/\wp\mathcal{O}_{K_0,S}. \tag{13}$$

By Lemma 2.3, the map $\mathcal{O}_{K_0,S} \to \bigoplus_{\mathfrak{p}\in T} \hat{K}_{0,\mathfrak{p}}/\wp\hat{K}_{0,\mathfrak{p}}$ is surjective. Hence, so is

$$\mathcal{O}_{K_0,S}/\wp\mathcal{O}_{K_0,S} \to \bigoplus_{\mathfrak{p}\in T} \hat{K}_{0,\mathfrak{p}}/\wp\hat{K}_{0,\mathfrak{p}}. \tag{14}$$

In addition, for each $\mathfrak{p} \in \mathbb{P}(K_0/F)$ the field $\hat{K}_{0,\mathfrak{p},\mathrm{sep}}$ is p-closed. Hence, by Lemma 3.1,

$$H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}),\mathbb{F}_p) = \hat{K}_{0,\mathfrak{p}}/\wp\hat{K}_{0,\mathfrak{p}}. \tag{15}$$

It follows from (13), (14), and (15) that the map

$$\mathrm{res}^1(S,T,\mathbb{F}_p)\colon H^1(\mathrm{Gal}(K_{0,S}/K_0),\mathbb{F}_p) \to \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}),\mathbb{F}_p) \tag{16}$$

is surjective.

Case B: $A = \mathbb{F}_p[G]$*, with* $G = \mathrm{Gal}(K/K_0)$*, where K is a finite Galois extension of K_0 in $K_{0,S}$.* By (8),

$$H^1(\mathrm{Gal}(K_{0,S}/K_0),\mathbb{F}_p[G]) \cong H^1(\mathrm{Gal}(K_{0,S}/K),\mathbb{F}_p).$$

By Lemma 4.5, $H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}),\mathbb{F}_p[G]) \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}),\mathbb{F}_p)$. This gives the commutative diagram:

$$\begin{tikzcd}
H^1(\mathrm{Gal}(K_{0,S}/K_0),\mathbb{F}_p[G]) \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}),\mathbb{F}_p[G]) \arrow[d] \\
H^1(\mathrm{Gal}(K_{0,S}/K),\mathbb{F}_p) \arrow[r] & \bigoplus_{\mathfrak{P}\in T_K} H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}),\mathbb{F}_p),
\end{tikzcd}$$

where the vertical arrows are isomorphims. By Case A, applied to K rather than to K_0, the lower horizontal arrow is surjective. It follows that the upper arrow is also surjective.

Case C: *$A = \mathbb{F}_p[G]^n$, with G as in Case B and $n \in \mathbb{N}$.* By Lemma 4.6,

$$H^1(\mathrm{Gal}(K_{0,S}/K_0), \mathbb{F}_p[G]^n) = (H^1(\mathrm{Gal}(K_{0,S}/K_0), \mathbb{F}_p[G])^n.$$

Hence, the statement of our proposition for $A = \mathbb{F}_p[G]^n$ follows from the statement for $A = \mathbb{F}_p[G]$ proven in Case B.

Case D: *A is a finite $\mathrm{Gal}(K_{0,S}/K_0)$-module with $pA = \mathbf{0}$.* Let G be a finite quotient of $\mathrm{Gal}(K_{0,S}/K_0)$ such that the kernel of the map $\mathrm{Gal}(K_{0,S}/K_0) \to G$ acts trivially on A. Then, $A = \sum_{i=1}^n \mathbb{F}_p[G]a_i$ for some $a_1, \dots, a_n \in A$. We write $\mathbb{F}_p[G]^n = \bigoplus_{i=1}^n \mathbb{F}_p[G]e_i$ with e_i being an n-tuple whose coordinates are 0 except 1 in the ith place. Then, the map $(e_1, \dots, e_n) \mapsto (a_1, \dots, a_n)$ extends to an epimorphism $\mathbb{F}_p[G]^n \to A$. Since by Case C, the map $H^1(\mathrm{Gal}(K_{0,S}/K_0), \mathbb{F}_p[G]^n) \to \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \mathbb{F}_p[G]^n)$ is surjective, it follows from Remark 4.2 that the map

$$H^1(\mathrm{Gal}(K_{0,S}/K_0), A) \to \bigoplus_{\mathfrak{p}\in T} H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$$

is also surjective. $\square$

5. Choosing an element in the first cohomology group

We prove an analogue of the strong approximation theorem for the first cohomology group (Lemma 5.2).

Let K_0 be either a global field or an algebraic function field of one variable over a Hilbertian field of positive characteristic p. Let K be a finite Galois extension of K_0 and let A be a multiplicative $\mathrm{Gal}(K_0)$-module with an action from the right.

Definition 5.1. Consider $\mathfrak{p} \in \mathbb{P}(K_0/F)$, let A be a finite $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$-module, and let h be a homomorphism of $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ into another group. We say that h is **unramified** if $h(\hat{I}_\mathfrak{p}) = \mathbf{1}$.

Likewise, an element $x \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is **unramified** if $\chi(\hat{I}_\mathfrak{p}) = \mathbf{1}$ for each (alternatively, for one) crossed homomorphism χ: $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to A$ that represents x.

Now we generalize Lemma 2.5 of [JaR19] (removing the unnecessary assumption about the simplicity of the module A).

Lemma 5.2. *Let K_0 be an algebraic function field of one variable over a field F of positive characteristic p, let K be a finite Galois extension of K_0, let $C_{p,1}, \dots, C_{p,r}$ be isomorphic copies of C_p, and let $A = C_{p,1} \times \cdots \times C_{p,r}$ be a $\mathrm{Gal}(K/K_0)$-module. Let T be a finite subset of $\mathbb{P}(K_0/F)$ that contains $\mathrm{Ram}(K/K_0)$. For each $\mathfrak{p} \in T$ consider an element $y_\mathfrak{p} \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$. Finally, let $\mathfrak{q} \in \mathbb{P}(K_0/F) \smallsetminus T$.*

Then, there exists $z \in H^1(\mathrm{Gal}(K_0), A)$ such that

(a) $\mathrm{res}_\mathfrak{p}(z) = y_\mathfrak{p}$ *for each* $\mathfrak{p} \in T$ *and*
(b) $\mathrm{res}_\mathfrak{p}(z)$ *is unramified for each* $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T \cup \{\mathfrak{q}\})$.

Proof. We set $T' = T \cup \{\mathfrak{q}\}$ and let $K_{0,T'}$ be the maximal Galois extension of K_0 which is unramified away of T'. Since $\mathrm{Ram}(K/K_0) \subseteq T \subset T'$, we have $K \subseteq K_{0,T'}$. Hence, A can be considered as a $\mathrm{Gal}(K_{0,T'}/K_0)$-module with a trivial action of $\mathrm{Gal}(K_{0,T'}/K)$ on A. By Proposition 4.7 applied to T and T' rather than to T and S, there exists $y \in H^1(\mathrm{Gal}(K_{0,T'}/K_0), A)$ such that $\mathrm{res}_{\mathfrak{p}}(y) = y_{\mathfrak{p}}$ for each $\mathfrak{p} \in T$. Let $\inf\colon H^1(\mathrm{Gal}(K_{0,T'}/K_0), A) \to H^1(\mathrm{Gal}(K_0), A)$ be the inflation map and set $z = \inf(y) \in H^1(\mathrm{Gal}(K_0), A)$. Then, $\mathrm{res}_{\mathfrak{p}}(z) = \mathrm{res}_{\mathfrak{p}}(y) = y_{\mathfrak{p}}$ for each $\mathfrak{p} \in T$ (Notation 4.1). If $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus T'$, then $\mathfrak{p}$ is unramified in $K_{0,T'}$, so the inertia subgroup $\hat{I}_{\mathfrak{p}}$ of $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ is contained in $\mathrm{Gal}(K_{0,T'})$. Let $\chi\colon \mathrm{Gal}(K_{0,T'}/K_0) \to A$ be a crossed homomorphism that represents y. Then, $\psi = \chi \circ \mathrm{res}_{K_{0,\mathrm{sep}}/K_{0,T'}}$ is a crossed homomorphism that represents z. Hence, for each $\sigma \in \hat{I}_{\mathfrak{p}}$ and $\bar{\sigma}$ being the restriction of σ to $K_{0,T'}$ we have $\psi(\sigma) = \chi(\bar{\sigma}) = \chi(1) = 1$. Thus, z is unramified at $\mathfrak{p}$, as desired. $\square$

6. Embedding problems with simple kernels

Let F be a Hilbertian field of positive characteristic p, let K_0 be an algebraic function field of one variable over F, and let K be a finite Galois extension of K_0. We set $\Gamma = \mathrm{Gal}(K/K_0)$ and consider the embedding problem

$$\begin{tikzcd} & & & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\ \mathbf{1} \arrow[r] & A \arrow[r] & \bar{G} \arrow[r, "\bar{\alpha}"] & \Gamma \arrow[r] & \mathbf{1}, \end{tikzcd} \tag{17}$$

where $A = C_p^r$ with $r \geq 1$ and $\rho = \mathrm{res}_{K_{0,\mathrm{sep}}/K}$. Suppose that the action of Γ on A defined by $a^{\bar{\alpha}(\bar{g})} = \bar{g}^{-1} a \bar{g}$ for $\bar{g} \in \bar{G}$ makes A a simple (multiplicative) Γ-module. Then, lifting the action of Γ on A via ρ to an action of $\mathrm{Gal}(K_0)$ on A, the group A becomes a simple $\mathrm{Gal}(K_0)$-module on which $\mathrm{Gal}(K)$ acts trivially.

Remark 6.1. Our goal in this section is to properly solve Embedding Problem (17) with bounded ramification. As a first step toward that goal we note that in any case, the problem is weakly solvable. Indeed, since $\mathrm{char}(K_0) = p > 0$, we have $\mathrm{cd}_p(\mathrm{Gal}(K_0)) \leq 1$ [NSW15, p. 338, Cor. 6.1.3]. Since A is a finite p-group, [NSW15, p. 192, Thm. 3.5.6] yields a homomorphism $\psi_0\colon \mathrm{Gal}(K_0) \to \bar{G}$ such that $\bar{\alpha} \circ \psi_0 = \rho$.

Notation 6.2. Given a homomorphism h of $\mathrm{Gal}(K_0)$ into a group G and a prime divisor $\mathfrak{p} \in \mathbb{P}(K_0/F)$, we set $h_{\mathfrak{p}} = h|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$. Note that a homomorphism $h_{\mathfrak{p}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to G$ may appear in our text without being the restriction of a homomorphism h as above.

A local version of Remark 6.1 makes the existence of local weak solutions in our case redundant and also replaces the crucial lemma 8.1 of [JaR18].

Lemma 6.3. *For every $\mathfrak{p} \in \mathbb{P}(K_0/F)$ there exists a homomorphism*

$$\varphi_{\mathfrak{p}} \colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \bar{G}$$

such that, in the notation of Diagram (17), $\bar{\alpha} \circ \varphi_{\mathfrak{p}} = \rho_{\mathfrak{p}}$, where $\rho_{\mathfrak{p}} = \rho|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$.

Moreover, if a prime divisor $\mathfrak{p} \in \mathbb{P}(K_0/F)$ is unramified in K, then

(a) *one can choose the local solution $\varphi_{\mathfrak{p}}$ to be unramified and*
(b) *if in addition, the short exact sequence in embedding problem (17) splits, then $\varphi_{\mathfrak{p}}$ can be chosen to be surjective.*

Proof. The existence of a weak solution $\varphi_{\mathfrak{p}}$ of the local embedding problem at $\mathfrak{p}$ follows as in Remark 6.1 from the fact that $\mathrm{char}(\hat{K}_{0,\mathfrak{p}}) = p$ and that A is a p-group.

Next, we assume that $\mathfrak{p}$ is unramified in K. Hence, $K \subseteq \hat{K}_{0,\mathfrak{p},\mathrm{ur}}$. Since ρ is trivial on $\mathrm{Gal}(K)$, it is also trivial on $\hat{I}_{\mathfrak{p}} = \mathrm{Gal}(\hat{K}_{0,\mathfrak{p},\mathrm{ur}})$. Hence, there exists an epimorphism $\bar{\rho}_{\mathfrak{p}} \colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p},\mathrm{ur}}/\hat{K}_{0,\mathfrak{p}}) \to \Gamma$ such that $\bar{\rho}_{\mathfrak{p}} \circ \mathrm{res}_{\hat{K}_{0,\mathfrak{p},\mathrm{sep}}/\hat{K}_{0,\mathfrak{p},\mathrm{ur}}} = \rho_{\mathfrak{p}}$, where $\hat{K}_{0,\mathfrak{p},\mathrm{ur}}$ is the maximal unramified extension of $\hat{K}_{0,\mathfrak{p}}$.

By Section 1, $\mathrm{ord}_{\mathfrak{p}}$ is a discrete valuation of K_0 and $\hat{K}_{0,\mathfrak{p}}$ is the completion of K_0 at $\mathrm{ord}_{\mathfrak{p}}$. Hence, by [CaF67, p. 28, Cor. 2], there is an isomorphism

$$\mathrm{Gal}(\hat{K}_{0,\mathfrak{p},\mathrm{ur}}/\hat{K}_{0,\mathfrak{p}}) \cong \mathrm{Gal}(\bar{K}_{0,\mathfrak{p}}), \tag{18}$$

where $\bar{K}_{0,\mathfrak{p}}$ is the residue field of K_0 at $\mathfrak{p}$. We identify the groups $\mathrm{Gal}(\bar{K}_{0,\mathfrak{p}})$ and $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p},\mathrm{ur}}/\hat{K}_{0,\mathfrak{p}})$ under the isomorphism (18).

As in Remark 6.1, the relation $\mathrm{char}(\bar{K}_{0,\mathfrak{p}}) = p > 0$ yields a homomorphism $\bar{\varphi}_{\mathfrak{p}} \colon \mathrm{Gal}(\bar{K}_{0,\mathfrak{p}}) \to \bar{G}$ such that $\bar{\alpha} \circ \bar{\varphi}_{\mathfrak{p}} = \bar{\rho}_{\mathfrak{p}}$. Let $\varphi_{\mathfrak{p}} = \bar{\varphi}_{\mathfrak{p}} \circ \mathrm{res}_{\hat{K}_{0,\mathfrak{p},\mathrm{sep}}/\hat{K}_{0,\mathfrak{p},\mathrm{ur}}}$. Then, $\bar{\alpha} \circ \varphi_{\mathfrak{p}} = \rho_{\mathfrak{p}}$. Moreover, $\varphi_{\mathfrak{p}}(\hat{I}_{\mathfrak{p}}) = \mathbf{1}$, so $\varphi_{\mathfrak{p}}$ is unramified.

$$\begin{tikzcd}
& & & \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \arrow[d, "\mathrm{res}"] \arrow[ddl, bend right, "\varphi_{\mathfrak{p}}"'] \arrow[dd, bend left=60, "\rho_{\mathfrak{p}}"] & \\
& & & \mathrm{Gal}(\hat{K}_{0,\mathfrak{p},\mathrm{ur}}/\hat{K}_{0,\mathfrak{p}}) \arrow[d, "\bar{\rho}_{\mathfrak{p}}"] \arrow[dl, "\bar{\varphi}_{\mathfrak{p}}"'] & \\
\mathbf{1} \arrow[r] & A \arrow[r] & \bar{G} \arrow[r, "\bar{\alpha}"] & \Gamma \arrow[r] & \mathbf{1}
\end{tikzcd} \tag{19}$$

This proves (a).

Next assume that the short exact sequence in (17) (hence, in (19)) splits. Note that the residue field $\bar{K}_{0,\mathfrak{p}}$ is a finite extension of the Hilbertian field F. Hence, by [FrJ08, p. 227, Prop. 12.3.3], $\bar{K}_{0,\mathfrak{p}}$ is also Hilbertian. By Ikeda [FrJ08, p. 304, Prop. 16.4.5], we may choose the homomorphism $\bar{\varphi}_{\mathfrak{p}}$ in this case to be surjective. Under that choice, $\varphi_{\mathfrak{p}}$ will also be surjective, as stated in (b). $\square$

Definition 6.4. Two weak solutions $\psi, \psi'\colon \mathrm{Gal}(K_0) \to \bar{G}$ of (17) are **A-equivalent** if there exists $a \in A$ such that $\psi'(\sigma) = a^{-1}\psi(\sigma)a$ for all $\sigma \in \mathrm{Gal}(K_0)$. We denote the A-equivalence class of ψ by $[\psi]$ and let

$$\mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$$

be the set of all A-equivalence classes.

Remark 6.5. (a) For A-equivalent weak solutions ψ and ψ' of (17), ψ is surjective (respectively, unramified, or trival) if and only if ψ' is. Thus, we say that $[\psi]$ is **surjective**, **unramified**, **trivial** if one (alternatively, every) representative of the class has the corresponding property.

We denote the set of all surjective classes $[\psi]$ by $\mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})_{\mathrm{sur}}$.

(b) The cohomolgy group $H^1(\mathrm{Gal}(K_0), A)$ acts freely and transitively on $\mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$: If $x \in H^1(\mathrm{Gal}(K_0), A)$, $\chi\colon \mathrm{Gal}(K_0) \to A$ is a crossed homomorphism that represents x, and $[\psi] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$, then $[\psi]^x := [\psi \cdot \chi]$. This action makes $\mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$ a principal homogeneous space over $H^1(\mathrm{Gal}(K_0), A)$ ([JaR18, Lemma 10.4]). In particular, the action of $H^1(\mathrm{Gal}(K_0), A)$ is transitive.

(c) Similarly, for each $\mathfrak{p} \in \mathbb{P}(K_0/F)$, the set of all equivalence classes

$$\mathcal{H}\mathrm{om}_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G})$$

of the weak solutions of the corresponding local embedding problem at the prime divisor $\mathfrak{p}$ of K_0/F is a principal homogeneous space over $H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$.

The proof of the following result is a verbatim repetition of Lemma 10.5 of [JaR18].

Lemma 6.6. *Consider $\mathfrak{p} \in \mathbb{P}(K_0/F)$, let $[\psi] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$, and let $x \in H^1(\mathrm{Gal}(K_0), A)$. Suppose that $[\psi]$ is unramified at $\mathfrak{p}$ and $\mathrm{res}_{\mathfrak{p}}(x) \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is unramified. Then, $[\psi]^x$ is unramified at $\mathfrak{p}$.*

Lemma 6.7. *Let $\mathfrak{p}$ be a prime divisor in $\mathbb{P}(K_0/F)$ which is unramified in K. Then, in the notation of Embedding Problem (17), there exists an element $[\varphi_{\mathfrak{p}}] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G})$ such that if for $[\bar{\psi}] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$ we have $[\bar{\psi}_{\mathfrak{p}}] = [\varphi_{\mathfrak{p}}]$, then*

(a) *$[\bar{\psi}]$ is unramified at $\mathfrak{p}$ and*

(b) *$[\bar{\psi}]$ is surjective.*

Proof. By Lemma 6.3, there exists an unramified homomorphism

$$\varphi_{\mathfrak{p}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \bar{G}$$

such that

(20a) $\bar{\alpha} \circ \varphi_{\mathfrak{p}} = \rho_{\mathfrak{p}}$ and

(20b) if the short exact sequence in Embedding Problem (17) splits, then $\varphi_{\mathfrak{p}}$ is surjective.

Claim: *For every weak solution $\bar\psi$ of Embedding Problem* (17)*, the intersection $A\cap \mathrm{Im}(\bar\psi)$ is a* $\mathrm{Gal}(K_0)$*-module.* Indeed, if $\sigma\in \mathrm{Gal}(K_0)$ and $a=\bar\psi(\sigma)\in A\cap\mathrm{Im}(\bar\psi)$ and $\tau\in\mathrm{Gal}(K_0)$, then $\bar\alpha(\bar\psi(\tau))=\rho(\tau)$. Hence, by definition, $a^\tau=\bar\psi(\sigma)^\tau=\bar\psi(\sigma)^{\bar\psi(\tau)}=\bar\psi(\sigma^\tau)$. Therefore, $a^\tau\in A\cap\mathrm{Im}(\bar\psi)$.

If $[\bar\psi_{\mathfrak p}]=[\varphi_{\mathfrak p}]$, then by Remark 6.5, $[\bar\psi_{\mathfrak p}]$ is unramified, so $[\bar\psi]$ is unramified at $\mathfrak p$.

We distinguish between two cases.

Case A: *The short exact sequence in Diagram* (17) *does not split.* This implies that $A\cap\mathrm{Im}(\bar\psi)\neq\mathbf 1$. Since A is a simple $\mathrm{Gal}(K_0)$-module, we conclude from the claim that $A=A\cap\mathrm{Im}(\bar\psi)\le\mathrm{Im}(\bar\psi)$. Since $\bar\alpha(\mathrm{Im}(\bar\psi))=\rho(\mathrm{Gal}(K_0))=\Gamma$, this implies that $\mathrm{Im}(\bar\psi)=\bar G$. This means that $\bar\psi$ is surjective.

Case B: *The short exact sequence in Diagram* (17) *splits.* Then, by (20b), $[\bar\psi]$ is surjective, as asserted. □

Proposition 6.8. *Let T be a finite subset of $\mathbb P(K_0/F)$ that contains $\mathrm{Ram}(K/K_0)$. For each $\mathfrak p\in T$ let $[\varphi_{\mathfrak p}]\in\mathcal{H}om_{\Gamma,\rho_{\mathfrak p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak p}),\bar G)$.*

Then, there exist $\mathfrak q\in\mathbb P(K_0/F)\smallsetminus T$ and a proper solution $\bar\psi\colon\mathrm{Gal}(K_0)\to\bar G$ of Embedding Problem (17) *such that*

(a) $[\bar\psi_{\mathfrak p}]=[\varphi_{\mathfrak p}]$ *in $\mathcal{H}om_{\Gamma,\rho_{\mathfrak p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak p}),\bar G)$ for each $\mathfrak p\in T$ and*
(b) $[\bar\psi]$ *is unramified at $\mathfrak p\in\mathbb P(K_0/F)\smallsetminus(T\cup\{\mathfrak q\})$, so if $\bar N$ is the solution field corresponding to $\bar\psi$, then $\mathrm{Ram}(\bar N/K_0)\subseteq T\cup\{\mathfrak q\}$.*

Proof. We break the proof into four parts.

Part A: *Surjectivity.* Use Lemma 6.7 to choose $\mathfrak q_0\in\mathbb P(K_0/F)\smallsetminus T$ and an element $[\varphi_{\mathfrak q_0}]\in\mathcal{H}om_{\Gamma,\rho_{\mathfrak q_0},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak q_0}),\bar G)$ such that if

$$[\bar\psi]\in\mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0),\bar G)$$

satisfies $[\bar\psi_{\mathfrak q_0}]=[\varphi_{\mathfrak q_0}]$, then

(21a) $[\bar\psi]$ is unramified at $\mathfrak q_0$ and

(21b) $[\bar\psi]$ is surjective.

Part B: *A weak solution.* By Remark 6.1, there exists an element $[\psi_0]\in\mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0),\bar G)$. Our goal is to adjust $[\psi_0]$ by the action of an element $x\in H^1(\mathrm{Gal}(K_0),A)$ such that $[\psi_0]^x$ satisfies Conditions (a) and (b) of the proposition.

Part C: *Constructing an element $x\in H^1(\mathrm{Gal}(K_0),A)$.* Let $T^*=T\cup\{\mathfrak q_0\}$ and let $\mathfrak r_1,\dots,\mathfrak r_s$ be the prime divisors in $\mathbb P(K_0/F)\smallsetminus T^*$ where ψ_0 ramifies. Set $T^{**}=T^*\cup\{\mathfrak r_1,\dots,\mathfrak r_s\}$. Then,

(22) ψ_0 is unramified along $\mathbb P(K_0/F)\smallsetminus T^{**}$.

Since $\mathrm{Ram}(K/K_0)\subseteq T$, each $\mathfrak p\in\{\mathfrak r_1,\dots,\mathfrak r_s\}$ is unramified in K, so

$$\rho_{\mathfrak p}\colon\mathrm{Gal}(\hat K_{0,\mathfrak p})\to\Gamma$$

is unramified (Definition 5.1). Hence, by Lemma 6.3(a),

(23) there exists an unramified element $[\varphi_\mathfrak{p}] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), \bar G)$.

Consider the system $([\varphi_\mathfrak{p}] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), \bar G))_{\mathfrak{p}\in T^{**}}$. By Remark 6.5(c), for each $\mathfrak{p} \in T^{**}$ there exists a (unique) element $y_\mathfrak{p} \in H^1(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), A)$ such that

$$[\psi_{0,\mathfrak{p}}]^{y_\mathfrak{p}} = [\varphi_\mathfrak{p}]. \tag{24}$$

By Lemma 5.2, there exist $x \in H^1(\mathrm{Gal}(K_0), A)$ and a prime divisor $\mathfrak{q} \in \mathbb{P}(K_0/F) \smallsetminus T^{**}$ such that

(25a) $\mathrm{res}_\mathfrak{p}(x) = y_\mathfrak{p}$ for each $\mathfrak{p} \in T^{**}$ and
(25b) $\mathrm{res}_\mathfrak{p}(x)$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T^{**} \cup \{\mathfrak{q}\})$.

Part D: *The proper solution $\bar\psi$.* Let $[\bar\psi] = [\psi_0]^x \in \mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0), \bar G)$.

Proof of (a). For each $\mathfrak{p} \in T^{**}$ we have that

$$[\bar\psi_\mathfrak{p}] = [\psi_{0,\mathfrak{p}}]^{\mathrm{res}_\mathfrak{p}(x)} \overset{(25a)}{=} [\psi_{0,\mathfrak{p}}]^{y_\mathfrak{p}} \overset{(24)}{=} [\varphi_\mathfrak{p}] \text{ in } \mathcal{H}\mathrm{om}_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), \bar G). \tag{26}$$

In particular, (26) holds for each $\mathfrak{p} \in T$, so conclusion (a) of the proposition holds.

Proof of (b). Let $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T \cup \{\mathfrak{q}\})$. If $\mathfrak{p} = \mathfrak{q}_0$, then by (26), $[\bar\psi_\mathfrak{p}] = [\varphi_\mathfrak{p}]$. Hence, by (21a), $[\bar\psi_\mathfrak{p}]$ is unramified. If $\mathfrak{p} \in \{\mathfrak{r}_1, \dots, \mathfrak{r}_s\}$, then by 26, $[\bar\psi_\mathfrak{p}] = [\varphi_\mathfrak{p}]$, hence by (23), $[\bar\psi_\mathfrak{p}]$ is unramified. Now if

$$\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T^{**} \cup \{\mathfrak{q}\}),$$

then by (22) and (25b), both $[\psi_{0,\mathfrak{p}}]$ and $\mathrm{res}_\mathfrak{p}(x)$ are unramified. Hence, $[\bar\psi_\mathfrak{p}] = [\psi_{0,\mathfrak{p}}]^{\mathrm{res}_\mathfrak{p}(x)}$ is unramified (Lemma 6.6). It follows that conclusion (b) holds in all cases. $\square$

7. Finite embedding problems with a p-group kernel

Using induction, we prove in this section our main result: Every finite embedding problem over K_0 whose kernel is a p-group has a proper solution with bounded ramification that satisfies finitely many local conditions.

Definition 7.1. Suppose that a finite group G acts on a finite group H with $|H| = p^s$. Let

$$\mathbf{1} = H_m < \cdots < H_2 < H_1 < H_0 = H \tag{27}$$

be a maximal G-series in H. In other words, for each $1 \le i \le m$, the group H_i is a proper normal subgroups of H_{i-1} which is maximal among all proper normal subgroups of H_{i-1} that are G-invariant. Since H_{i-1} is a p-group, $H_{i-1}/H_i \cong C_p^{r_i}$ is a simple G-module, where r_i is a positive integer (see e.g. [JaR19, Lemma 5.1]). If $\mathbf{1} = H'_{m'} < \cdots < H'_2 < H'_1 < H'_0 = H$ is another maximal G-series in H, then by Jordan-Hölder, $m = m'$ and there is a permutation κ of $\{0, 1, \dots, m\}$ such that $\kappa(0) = 0$, $\kappa(m) = m'$, and H_{i-1}/H_i is G-isomorphic to $H'_{\kappa(i-1)}/H'_{\kappa(i)}$ for $i = 1, \dots, m$ [Rob82, p. 66, Thm. 3.14].

We set $\Omega_p(H,G)=m$, in particular

$$\Omega_p(H_{i-1}/H_i,G)=1 \tag{28}$$

and $\Omega_p(H,G)\le s$. Note that if H' is a G-invariant normal subgroup of H, then G acts on H/H' and

$$\Omega_p(H,G)=\Omega_p(H/H',G/H')+\Omega_p(H',G). \tag{29}$$

Setup 7.2. Let F be a Hilbertian field of characteristic p, K_0 an algebraic function field of one variable over F, and K a finite Galois extension of K_0. Consider the embedding problem

$$\begin{tikzcd} & & & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\ \mathbf{1} \arrow[r] & H \arrow[r] & G \arrow[r, "\alpha"] & \Gamma \arrow[r] & \mathbf{1}, \end{tikzcd} \tag{30}$$

where $\Gamma=\mathrm{Gal}(K/K_0)$, G is a finite group, α is an epimorphism, $\rho=\mathrm{res}_{K_{0,\mathrm{sep}}/K}$, and $H=\mathrm{Ker}(\alpha)$ is a p-group. In particular, H is normal in G, so G acts on H by conjugation. Hence, each G-invariant subgroup of H is normal in H.

Let H_1 be a maximal G-invariant subgroup of H. Then, by Definition 7.1, there exists a positive integer r_1 such that $H/H_1\cong C_p^{r_1}$. In particular, H/H_1 is a simple G-module. Hence, H/H_1 is also a simple Γ-module, so also a simple $\mathrm{Gal}(K_0)$-module on which $\mathrm{Gal}(K)$ acts trivially. Moreover, $p^{r_1}|H_1|=|H|$ and we have the following commutative diagram:

$$\begin{tikzcd} & H_1 \arrow[r, equal] \arrow[d] & H_1 \arrow[d] & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\ \mathbf{1} \arrow[r] & H \arrow[r] \arrow[d, "\lambda"'] & G \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & \Gamma \arrow[r] \arrow[d, equal] & \mathbf{1} \\ \mathbf{1} \arrow[r] & H/H_1 \arrow[r] & G/H_1 \arrow[r, "\bar{\alpha}"] & \Gamma \arrow[r] & \mathbf{1} \end{tikzcd} \tag{31}$$

with exact short horizontal sequences such that both maps λ are quotient maps.

With this we arrive at our main result:

Theorem 7.3. *Let K_0 be an algebraic function field of one variable over a Hilbertian field F of positive characteristic p. Consider the finite Embedding Problem (30) whose kernel H is a p-group of order p^s. Let T be a finite subset of $\mathbb{P}(K_0/F)$ that contains $\mathrm{Ram}(K/K_0)$. For each $\mathfrak{p}\in T$ let $[\varphi_\mathfrak{p}]\in \mathcal{H}\mathrm{om}_{\Gamma,\rho_\mathfrak{p},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}),G)$.*

Then, there exists an element $[\psi]\in\mathcal{H}\mathrm{om}_{\Gamma,\rho,\alpha}(\mathrm{Gal}(K),G)_{\mathrm{sur}}$ and there exists a set $R\subseteq\mathbb{P}(K_0/F)\smallsetminus T$ with $|R|=\Omega_p(H,G)$ such that

(a) $[\psi_{\mathfrak{p}}] = [\varphi_{\mathfrak{p}}]$ *in* $\mathcal{H}\mathrm{om}_{\Gamma,\rho_{\mathfrak{p}},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G)$ *for each* $\mathfrak{p} \in T$ *and*
(b) $[\psi]$ *is unramified at each* $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T \cup R)$. *Thus, the fixed field* N *of* $\mathrm{Ker}(\psi)$ *in* $K_{0,\mathrm{sep}}$ *satisfies* $\mathrm{Ram}(N/K_0) \subseteq T \cup R$.

Proof. As in Setup 7.2, let H_1 be a maximal G-invariant subgroup of H. Thus, H/H_1 is a simple Γ-module and $H/H_1 \cong C_p^{r_1}$ for some positive integer r_1. Let $\lambda\colon G \to G/H_1$ be a homomorphism as in the commutative diagram (31). Then, for each $\mathfrak{p} \in T$ the homomorphism $\bar{\varphi}_{\mathfrak{p}} := \lambda \circ \varphi_{\mathfrak{p}}$ satisfies $\bar{\alpha} \circ \bar{\varphi}_{\mathfrak{p}} = \rho_{\mathfrak{p}}$. The rest of the proof breaks up into three parts.

Part A: *An embedding problem whose kernel is a simple* $\mathrm{Gal}(K_0)$*-module.* Since $H/H_1 \cong C_p^{r_1}$ is a simple Γ-module, Proposition 6.8 yields a prime divisor $\mathfrak{q}_1 \in \mathbb{P}(K_0/F) \smallsetminus T$ and an element

$$[\psi_1] \in \mathcal{H}\mathrm{om}_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), G/H_1)_{\mathrm{sur}} \tag{32}$$

such that

(33a) $[\psi_{1,\mathfrak{p}}] = [\bar{\varphi}_{\mathfrak{p}}]$ in $\mathcal{H}\mathrm{om}_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G/H_1)$ for each $\mathfrak{p} \in T$ and
(33b) $[\psi_1]$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T \cup \{\mathfrak{q}_1\})$.

Part B: *The induction step.* Part A gives rise to an embedding problem

$$\begin{tikzcd} & & & \mathrm{Gal}(K_0) \arrow[d, "\psi_1"'] & \\ \mathbf{1} \arrow[r] & H_1 \arrow[r] & G \arrow[r, "\lambda"] & G/H_1 \arrow[r] & \mathbf{1} \end{tikzcd} \tag{34}$$

with a p-group kernel H_1. Set $s_1 = s - r_1$. Then, $|H_1| = |H|/|H/H_1| = p^{s_1}$.

By (33a), there exists for each $\mathfrak{p} \in T$ an element $a_{\mathfrak{p}} \in H$ such that

$$\begin{aligned} \psi_{1,\mathfrak{p}}(\sigma) = \lambda(a_{\mathfrak{p}})^{-1}\bar{\varphi}_{\mathfrak{p}}(\sigma)\lambda(a_{\mathfrak{p}}) &= \lambda(a_{\mathfrak{p}}^{-1})\lambda(\varphi_{\mathfrak{p}}(\sigma))\lambda(a_{\mathfrak{p}}) \\ &= \lambda(a_{\mathfrak{p}}^{-1}\varphi_{\mathfrak{p}}(\sigma)a_{\mathfrak{p}}) = \lambda \circ \varphi_{\mathfrak{p}}^{a_{\mathfrak{p}}}(\sigma) \end{aligned}$$

for each $\sigma \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$. Hence,

$$[\varphi_{\mathfrak{p}}^{a_{\mathfrak{p}}}] \in \mathcal{H}\mathrm{om}_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G) \text{ for every } \mathfrak{p} \in T. \tag{35}$$

Given $\mathfrak{p} \in T$, we set

$$\varphi_{1,\mathfrak{p}} = \varphi_{\mathfrak{p}}^{a_{\mathfrak{p}}} \tag{36}$$

and for $\mathfrak{q}_1$ we use Lemma 6.3 to choose

$$[\varphi_{1,\mathfrak{q}_1}] \in \mathcal{H}\mathrm{om}_{G/H_1,\psi_{1,\mathfrak{q}_1},\lambda}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G).$$

Since H_1 is a p-group with $|H_1| < |H|$, an induction hypothesis on the order of the kernel of the embedding problem gives a set

$$R_1 \subseteq \mathbb{P}(K_0/F) \smallsetminus (T \cup \{\mathfrak{q}_1\})$$

with $|R_1| = \Omega_p(H_1, G)$ and an element

$$[\psi] \in \mathcal{H}\mathrm{om}_{G/H_1,\psi_1,\lambda}(\mathrm{Gal}(K_0), G)_{\mathrm{sur}} \tag{37}$$

such that

(38a) $[\psi_{\mathfrak{p}}] = [\varphi_{1,\mathfrak{p}}] \in \mathcal{H}\mathrm{om}_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G)$ for each $\mathfrak{p} \in T \cup \{\mathfrak{q}_1\}$, and

(38b) $[\psi]$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0/F) \smallsetminus (T \cup \{\mathfrak{q}_1\} \cup R_1)$.

Let $R = \{\mathfrak{q}_1\} \cup R_1$. Then,

$$|R| = 1 + |R_1| \overset{(28)}{=} \Omega_p(H/H_1, G/H_1) + \Omega_p(H_1, G) \overset{(29)}{=} \Omega_p(H, G).$$

Part C: *Conclusion of the proof.* We prove that the class $[\psi]$ introduced in Part B satisfies the conclusion of the theorem. Indeed,

$$\alpha \circ \psi \overset{(31)}{=} \bar{\alpha} \circ \lambda \circ \psi \overset{(37)}{=} \bar{\alpha} \circ \psi_1 \overset{(32)}{=} \rho.$$

In addition, by (37), ψ is surjective, so $\psi \in \mathcal{H}\mathrm{om}_{\Gamma,\rho,\alpha}(\mathrm{Gal}(K_0), G)_{\mathrm{sur}}$.

$$\begin{tikzcd}
 & & \mathbf{1} \arrow[d] & & \\
 & & H_1 \arrow[d] & \mathrm{Gal}(K_0) \arrow[dl, "\psi"'] \arrow[ddl, "\psi_1"] \arrow[dd, "\rho"] & \\
\mathbf{1} \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[d, "\lambda"'] \arrow[dr, "\alpha"] & & \\
\mathbf{1} \arrow[r] & H/H_1 \arrow[r] & G/H_1 \arrow[r, "\bar{\alpha}"] \arrow[d] & \Gamma \arrow[r] & \mathbf{1} \\
 & & \mathbf{1} & &
\end{tikzcd}$$

Moreover, by (38a), for each $\mathfrak{p} \in T$ there exists $b_{\mathfrak{p}} \in H_1$ such that for each $\sigma \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ we have

$$\psi_{\mathfrak{p}}(\sigma) = b_{\mathfrak{p}}^{-1}\varphi_{1,\mathfrak{p}}(\sigma)b_{\mathfrak{p}} \overset{(36)}{=} b_{\mathfrak{p}}^{-1}a_{\mathfrak{p}}^{-1}\varphi_{\mathfrak{p}}(\sigma)a_{\mathfrak{p}}b_{\mathfrak{p}} = (a_{\mathfrak{p}}b_{\mathfrak{p}})^{-1}\varphi_{\mathfrak{p}}(\sigma)(a_{\mathfrak{p}}b_{\mathfrak{p}}).$$

Since $a_{\mathfrak{p}} \in H$ and $b_{\mathfrak{p}} \in H_1$, we have $a_{\mathfrak{p}}b_{\mathfrak{p}} \in H$. Therefore, $[\psi_{\mathfrak{p}}] = [\varphi_{\mathfrak{p}}]$ in $\mathcal{H}\mathrm{om}_{\Gamma,\rho_{\mathfrak{p}},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G)$ for each $\mathfrak{p} \in T$, as desired. □

References

[CaF67] Cassels, John W.S.; Fröhlich, Albrecht, dirs. Algebraic number theory. Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union. *Academic Press, London; Thompson Book Co., Inc., Washington, D.C.*, 1967. xviii+366 pp. MR0215665, Zbl 0153.07403. 1436

[Efr06] Efrat, Ido. Valuations, orderings, and Milnor K-theory. Mathematical Surveys and Monographs, 124. *American Mathematical Society, Providence, RI*, 2006. xiv+288 pp. ISBN: 0-8218-4041-X. MR2215492, Zbl 1103.12002, doi: 10.1090/surv/124. 1426

[FrJ08] Fried, Michael D.; Jarden, Moshe. Field Arithmetic. Third edition. Revised by Jarden. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 11. *Springer-Verlag, Berlin*, 2008. xxiv+792 pp. ISBN: 978-3-540-77269-9. MR2445111, Zbl 1145.12001, doi: 10.1007/978-3-540-77270-5. 1425, 1427, 1429, 1436

[GeJ98] Geyer, Wulf-Dieter; Jarden, Moshe. Bounded realization of l-groups over global fields. The method of Scholz and Reichardt. *Nagoya Math. J.* **150** (1998), 13–62. MR1633151, Zbl 0906.12002, doi: 10.1017/S0027763000025046. 1424

[Jar91] Jarden, Moshe. Intersections of local algebraic extensions of a Hilbertian field. *Generators and relations in groups and geometries* (Lucca, 1990), 343–405, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci, 333, *Kluwer Acad. Publ., Dordrecht*, 1991. MR1206921, Zbl 0737.12001, doi: 10.1007/978-94-011-3382-1_13. 1426

[JaR18] Jarden, Moshe; Ramiharimanana, Nantsoina Cynthia. Solving embedding problems with bounded ramification. *Proc. Lond. Math. Soc. (3)* **117** (2018), no. 1, 149–191. MR3830893, Zbl 06928215, doi: 10.1112/plms.12126. 1423, 1424, 1435, 1437

[JaR19] Jarden, Moshe; Ramiharimanana, Nantsoina Cynthia. Embedding problems with bounded ramifications over global fields of positive characteristic. *J. Lond. Math. Soc. (2)* **100** (2019), no. 1, 323–340. MR3999693, Zbl 07118851, doi: 10.1112/jlms.12215. 1423, 1424, 1434, 1439

[Lan58] Lange, Serge. Introduction to algebraic geometry. *Interscience Publishers, Inc., New York-London* 1958. xi+260 pp. MR0100591, Zbl 0095.15301. 1427

[Lan70] Lange, Serge. Algebraic number theory. *Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Don Mills, Ont.*, 1970. xi+354 pp. MR0282947, Zbl 0211.38404. 1426

[Neu79] Neukirch, Jürgen. On solvable number fields. *Invent. Math.* **53** (1979), no. 2, 135–164. MR0560411, Zbl 0447.12008 doi: 10.1007/BF01390030 1423

[NSW15] Neukirch, Jürgen; Schmidt, Alexander; Wingberg, Kay. Cohomology of number fields. Second edition. Grundlehren der Mathematischen Wissenschaften, 323. *Springer-Verlag, Berlin*, 2008. xvi+825 pp. ISBN: 978-3-540-37888-4. Second Edition, corrected version 2.2, July 2015. Electronic edition obtained from www.mathi.uni-heidelberg.de/∼schmidt/NSW2e. MR2392026, Zbl 1136.11001, doi: 10.1112/blms/33.2.252. 1423, 1425, 1427, 1428, 1429, 1430, 1431, 1435

[Rib70] Ribes, Luis. Introduction to profinite groups and Galois cohomology. Reprint of the 1970 original, with errata. Queen's Papers in Pure and Applied Mathematics, 24. *Queen's University, Kingston, Ont.*, 1999. vi+316 pp. ISBN: 0-88911-834-5. MR1705274, Zbl 0948.11043. 1433

[Rob82] Robinson, Derek John Scott. A course in the theory of groups. Graduate Texts in Mathematics, 80. *Springer-Verlag, New York-Berlin*, 1982. xvii+481 pp. ISBN: 0-387-90600-2. MR0648604 Zbl 0483.20001 doi: 10.1007/978-1-4419-8594-1 1439

(M. Jarden) School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel
jarden@tauex.tau.ac.il

(N. C. Ramiharimanana) Department of Mathematics, Clemson University, Clemson, SC 29634, USA
nantsoina@aims.ac.za

This paper is available via http://nyjm.albany.edu/j/2020/26-56.html.