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Toeplitz operators on the space of all entire functions

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Abstract. We introduce and characterize the class of Toeplitz operators on the Fréchet space of all entire functions. We completely describe Fredholm, semi-Fredholm, invertible and one-sided invertible operators in this class.

Contents

1. Introduction

The theory of Toeplitz operators on the Hardy, Bergman or Fock space is well-established. The latter space is a subspace of the space of all entire functions. In this paper we show that there also exists an interesting theory of Toeplitz operators on the space of all entire functions $H(\mathbb{C})$. This is a Fréchet space and importantly the fundamental tools of functional analysis such as the Hahn-Banach theorem, the uniform boundedness principle and the Open mapping/Closed graph theorem are available – this will play an important role in the proofs. In our study we are primarily guided by the known results in the Hardy space case and our recent results for Toeplitz operators on the space of real analytic functions ([9], [23], [24], [25]). There

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are excellent references for the classical Hardy space theory of Toeplitz operators ([7], [29]). For the Fock space we refer the reader to [42]. In [37] the author studies the algebras generated by Toeplitz operators on the Bergman spaces. Below we first present our results, underline the differences and similarities comparing to the Hardy space case and the real analytic case. Then we describe the motivation which led us to our study.

A continuous linear operator on $H(\mathbb{C})$ is a Toeplitz operator if its matrix is a Toeplitz matrix. The matrix of an operator is the one defined with respect to the Schauder basis $(z^n)_{n\in\mathbb{N}_0}$. We shall write down the details in Section 2. Our first result says that such an operator is necessarily of the form $\mathcal{C}M_F$, where M_F is the operator of multiplication by the symbol F and $\mathcal{C}$ is the (appropriately defined) Cauchy transform. This is an analog of the classical result of Brown and Halmos [6] which characterizes Toeplitz operators on the Hardy space as operators of the form PM_ϕ, where P is the Riesz projection and M_ϕ is multiplication by a bounded function ϕ, [7, Theorem 2.7]. In the case of the space of all entire functions $H(\mathbb{C})$ the symbol space $\mathfrak{S}(\mathbb{C})$ turns out to be

$$H(\mathbb{C}) \oplus H_0(\infty).$$

The symbol $H_0(\infty)$ stands for the space of all germs at ∞ of holomorphic functions which vanish at ∞. This is our Theorem 2.2. Hence any function F which is holomorphic in a punctured neighborhood $U \setminus \{\infty\}$ of ∞ (the neighborhood U of ∞ may be assumed simply connected in the Riemann sphere $\mathbb{C}_\infty$, since we work with germs) defines the Toeplitz operator T_F by the formula

$$(T_F f)(z) := \frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta.$$

Here, f is entire and $\gamma \subset \mathbb{C}$ is a C^∞ smooth Jordan curve such that both $\mathbb{C}\setminus U$ and the point $z \in \mathbb{C}$ are contained in the interior $I(\gamma)$ of the curve γ (recall Jordan's theorem) – we emphasize that the interior is relative to $\mathbb{C}$. The characterization of Toeplitz operators on $H(\mathbb{C})$ is a consequence of the Köthe-Grothendieck-da Silva characterization of the spaces dual to $H(G)$, G a domain in $\mathbb{C}$. We work down the details in Section 2. The space of all entire function is isomorphic as a Fréchet space with the power series space $\Lambda_\infty(n)$, which is a sequence space. We therefore formulate our characterization also in terms of this space. This is just as in the Hardy space $H^2(\mathbb{T})$, which is isometrically isomorphic with the sequence space $l^2(\mathbb{N})$. This will also be presented in Section 2.

Our next main result characterizes semi-Fredholm and Fredholm Toeplitz operators.

Theorem 1. *Assume that $T_F\colon H(\mathbb{C}) \to H(\mathbb{C})$ with $F \in \mathfrak{S}(\mathbb{C})$ is a Toeplitz operator.*

(i) *The operator T_F is always a Φ_+-operator.*

(ii) *The operator* T_F *is a* Φ_-*-operator if and only if* T_F *is a Fredholm operator.*

(iii) *The operator* T_F *is a Fredholm operator if and only if* F *does not vanish. In this case*

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

The statement of (iii) in Theorem 1 is deceptively similar to the Hardy space case. A symbol $F \in \mathfrak{S}(\mathbb{C})$ is essentially a germ. We say that F *does not vanish* if F is the equivalence class of a function $\tilde{F}$ holomorphic in a punctured neighborhood of ∞, which does not vanish. Observe that in general F which does not vanish may be the equivalence class in $\mathfrak{S}(\mathbb{C})$ of a function $\tilde{G} \in H(U \setminus \{\infty\})$, U an open neighborhood of ∞ in $\mathbb{C}_\infty$, such that $\tilde{G}(z) = 0$ for some $z \in U \setminus \{\infty\}$. However, the infinity cannot be an accumulation point of the zeros of any $\tilde{F}$ which represents F. If ∞ is an accumulation point of zeros of any $\tilde{F}$, which represents F, then we shall say that F *vanishes*. Thus our theory is essentially asymptotic in nature. One should therefore be careful as far as the definition of the winding number of $F \in \mathfrak{S}(\mathbb{C})$ is concerned. In order to prove (iii) we invoke our results concerning Fredholm Toeplitz operators on the spaces of germs $H(K)$, where K is a finite closed interval [24, Theorem 5.3].

Let us recall that an operator $T\colon H(\mathbb{C}) \to H(\mathbb{C})$ is a Φ_+-operator if the range of T is closed and the kernel is finite dimensional. It is a Φ_--operator if the range of T is of finite codimension. Just as in the Banach space case, a continuous linear operator, the range of which is of finite codimension in $H(\mathbb{C})$ has necessarily closed range. The proof of this fact relies on the Open mapping theorem [28, Theorem 24.30], which holds true for Fréchet spaces. Since we shall use this profound theorem, we recall it below as Theorem 3.4. A Fréchet space has a web [28, Corollary 24.29], and is ultrabornological [28, Proposition 24.13, Remark 24.15 (b)]. We shall refrain from writing down the details of the proof that the range of a continuous linear operator which is of finite codimension is closed. Instead we refer the reader to [9, Proposition 5.1] for essentially the same argument.

We shall show that if F vanishes, that is, if ∞ is an accumulation point of the zeros of any $\tilde{F}$, which represents F, then the operator T_F is injective and has closed range (Theorem 3.5 and Corollary 3.6). In order to prove that the range is closed we shall represent the space $H(\mathbb{C})$ as

$$\operatorname{lim\,proj} H^2(r\mathbb{T}),$$

where $H^2(r\mathbb{T})$ is the Hardy space on the disk of radius $r > 0$ and use the Fredholm index theory of Toeplitz operators on the Hardy spaces. Injectivity is in turn a consequence of density of the image of the adjoint operator T_F', which follows from our previous results [24, Theorem 5.6]. We also show that if F vanishes and (z_n) are the zeros of $\tilde{F}$, which represents F, then for any g in the range of T_F it holds that

$$\lim_{n\to\infty} g(z_n) = 0.$$

Hence, the range $R(T_F)$ is of infinite codimension. This suffices to prove (ii) of Theorem 1.

We remark that in the case of the Hardy spaces there are Toeplitz operators of class Φ_-, which are not Fredholm. Also, not every Toeplitz operator is a Φ_+-operator. A characterization of these classes of operators was given by Douglas and Sarason [17], see also [7, Theorem 2.75]. For instance, for a unimodular symbol $\varphi \in L^\infty$ it holds that the Toeplitz operator with symbol φ is a Φ_+-operator if and only if $\operatorname{dist}_{L^\infty}(\varphi, C + H^\infty) < 1$. This is a metric characterization. One can therefore rather not expect such a result in the case of real analytic functions, the topology of which is not metrizable. Instead we proved in [23, Main Theorem 1.2] that a Toeplitz operator is a Φ_+-operator if and only if the symbol has no non-real zeros accumulating at a real point. We also proved that a Toeplitz operator is Φ_--operator if and only if the symbol has no real zeros going to infinity [24, Main Theorem 1.3]. The methods worked out in that case can be applied to prove Theorem 1. We emphasize however that the symbol spaces in the $H(\mathbb{C})$ case and the $\mathcal{A}(\mathbb{R})$ case are different. Roughly speaking there is 'only one infinity' in the $H(\mathbb{C})$ space case. This is the key difference comparing with the real analytic case which is responsible for the difference between Theorem 1 above and [24, Main Theorem 1.2 and Main Theorem 1.3].

Next we characterize invertible Toeplitz operators.

Theorem 2. *Assume that $T_F\colon H(\mathbb{C}) \to H(\mathbb{C})$ with $F \in \mathfrak{S}(\mathbb{C})$ is a Toeplitz operator.*

(i) *Either* $\ker T_F = \{0\}$ *or* $\ker T_F' = \{0\}$.

(ii) *The operator T_F is invertible if and only if it is a Fredholm operator of index* 0.

Theorem 2 is in perfect analogy with the Hardy space case. In particular, statement (i) is the classical Coburn-Simonenko theorem [8], [34], also [7, Theorem 2.38]. Theorem 2 is essentially a consequence of Liouville's theorem. We proved such results also in the real analytic case [23, Theorem 1.2 and Theorem 1.3].

It is a consequence of standard functional analytic arguments that an injective Fredholm operator is left invertible and a surjective Fredholm operator is right invertible. In other words, in view of Theorem 2, if $F \in \mathfrak{S}(\mathbb{C})$ does not vanish and $\operatorname{winding} F \geq 0$, then T_F is left invertible. If $\operatorname{winding} F \leq 0$, then T_F is right invertible. Our next main result concerns symbols which vanish. We shall show that if $F \in \mathfrak{S}(\mathbb{C})$ vanishes then there exists a sequence of functionals $\xi_n \in H(\mathbb{C})'$ such that

$$R(T_F) = \bigcap_{n=1}^{\infty} \ker \xi_n. \tag{1.1}$$

This is Theorem 5.2 below. Then for these functionals we consider a generalized interpolation (Toeplitz) problem and study the map

$$\Xi\colon H(\mathbb{C}) \ni f \mapsto (\xi_n(f))_{n\in\mathbb{N}} \in \omega.$$

Here, the symbol ω stands for the Fréchet space of all sequences. We show that the map Ξ is surjective by means of Eidelheit's theorem, which we recall as Theorem 5.4 below. Then we show that $\ker \Xi$, which by (1.1) is equal to $R(T_F)$, is not a complemented subspace of $H(\mathbb{C})$. This part of the argument is essentially that of [32, p. 162]. This establishes our third main result.

Theorem 3. *If $F \in \mathfrak{S}(\mathbb{C})$ vanishes then the range of the operator*

$$T_F\colon H(\mathbb{C}) \to H(\mathbb{C})$$

is not a complemented subspace of $H(\mathbb{C})$.

As a result, the operator T_F is left invertible if and only if F does not vanish and winding $F \geq 0$*. The operator T_F is right invertible if and only if F does not vanish and* winding $F \leq 0$*.*

We initiated the research on operators on the space of real analytic functions on the real line $\mathcal{A}(\mathbb{R})$ defined by Toeplitz matrices in [9]. This was continued in [23], [24] and [25]. As we mentioned above, we characterized Fredholm, semi-Fredholm, invertible and one sided invertible Toeplitz operators on the space $\mathcal{A}(\mathbb{R})$. We worked out *ibid.* some methods to investigate this class of operators. We apply them, with necessary modifications, in the case of the space of all entire functions in this paper. Actually the case of entire functions is essentially easier. The reader who knows our previous research notices that some arguments could be made shorter. We could just indicate the difference between the current case and the case of real analytic functions. This would be at the cost of clarity and completeness. We however strive to make the paper self-contained. We emphasize again that the most important difference lies in the symbol space.

It seems natural to study operators defined by Toeplitz matrices also on the other locally convex spaces of holomorphic functions, not only on $\mathcal{A}(\mathbb{R})$. These operators played important roles in mathematics before they were called Toeplitz operators – below we give some interesting examples. In some sense $\mathcal{A}(\mathbb{R})$ is a large space, the other extreme case is the space of all entire functions $H(\mathbb{C})$. This is arguably rather a small space. We emphasize that there is nowadays growing interest in operators on different locally convex spaces, especially spaces of holomorphic and differentiable functions, which are not Banach spaces. The literature is really vast, we mention therefore here only these papers which have some influence on our research [1], [3], [5] and [4]. In [25] we presented the motivation which led us to the study of Toeplitz operators on $\mathcal{A}(\mathbb{R})$. The arguments for the current research are essentially the same. We feel however that we should at least sketch them here to place our study in the correct perspective.

Perhaps the most important example of operators which we investigate is given by the following Cauchy's integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\prod_{k=0}^n(\zeta - z_k)}\frac{d\zeta}{\zeta - z}. \tag{1.2}$$

Here, $z_0, \ldots, z_n \in \mathbb{C}$ and γ is a C^∞-smooth Jordan curve such that the point z and the points $z_0, \ldots, z_n$ are contained in the interior $I(\gamma)$ of the curve γ. For any $z \in \mathbb{C}$ such a curve can be chosen, by Cauchy's theorem the value of (1.2) does not depend on γ. Hence, the integral defines an entire function, when f is entire.

An important situation occurs when $z_k = k, k = 0, 1, \ldots, n$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\prod_{k=0}^n(\zeta - k)}\frac{d\zeta}{\zeta - z} = \frac{(-1)^n}{n!}\sum_{k=0}^n \binom{n}{k}(-1)^k f(k) = \frac{(-1)^n}{n!}D_n[f].$$

This integral is usually called Nörlund-Rice integral. This is an important object in theoretical computer science and discrete mathematics. The method of estimating it is considered 'one of the basic asymptotic techniques of the analysis of algorithms' [18], [30]. We refer the reader especially to [18] for the fascinating presentation of the problem of estimating finite differences. The methods used *ibid.* show the unity of the whole mathematics, its discrete and continuous faces.

Observe that for (1.2) to make sense, it suffices that f is holomorphic in some neighborhood of the real line $\mathbb{R}$. This was our starting point in [9], [23], [24] and [25]. However in many applications one can assume that f is entire. This is our perspective here.

For a general choice of the points $z_0, \ldots, z_n \in \mathbb{C}$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\prod_{k=0}^n(\zeta - z_k)}\frac{d\zeta}{\zeta - z} = [z, z_0, \ldots, z_n].$$

The symbol $[z, z_0, \ldots, z_n]$ is the divided difference

$$[z, z_0] := \frac{[z] - [z_0]}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0}$$
$$\cdots$$
$$[z, z_0, \ldots, z_n] := \frac{[z, z_0, \ldots, z_{n-1}] - [z_0, \ldots, z_n]}{z - z_n}.$$

This is an object of fundamental importance in interpolation theory. We refer the reader to the beautiful books [19] and [27] and our previous research [25] for further information. Here we only say that the divided differences are discrete analogs of the derivatives and they are used in Newton interpolation formula. Some holomorphic functions develop into Newton series

$$a_0 + \sum_{n=1}^\infty \frac{a_n}{n!}(z - z_0)\cdot\cdots\cdot(z - z_{n-1}). \tag{1.3}$$

Then

$$\frac{a_n}{n!} = [z_0, \dots, z_n]$$

and the convergence of (1.3) is governed by the Toeplitz operator (1.2). In fact, for an entire function f

$$f(z) = [z_0] + [z_0, z_1](z - z_0) + [z_0, z_1, z_2](z - z_0)(z - z_1) + \dots + \\ + [z_0, \dots, z_n](z - z_0)(z - z_1) \dots (z - z_{n-1}) + R_n(z),$$

where the rest

$$R_n(z) = \frac{(z - z_0) \dots (z - z_n)}{2\pi i} \int_\gamma \frac{f(\zeta)}{\prod_{k=0}^{n} (\zeta - z_k)} \frac{d\zeta}{\zeta - z}$$

is the product of Toeplitz operators, which we study in the current paper [19, p. 12 and p. 34], see also [27, II.2.11]. This is the well-known Newton interpolation formula.

We mention here that some answers to the question of the convergence of Newton series are known exactly for entire functions [19, Satz 1, p. 43, Satz 1, p. 47] and also [19, Chapter II, §2 and §3]. The fact that a holomorphic function develops into Newton series may have profound consequences. Pólya characterized functions holomorphic in some half-space $\Re z > \alpha$, satisfying certain growth condition which take integer values at integers as polynomials [1] (see [19, p. 113, Satz 3]. The estimate of integrals of the type (1.2) is also one of the key elements in the theorem which says that α and e^α cannot be simultaneously algebraic numbers [19, p. 167, Satz 9].

This paper is a part of a project, the aim of which is to build on locally convex spaces of functions, especially on the space of real analytic functions, a theory of concrete operators following the ideas of the theory of operators on Hilbert spaces. The main object of interest is the (appropriately defined) matrix associated to an operator. The idea to consider operators determined by matrices associated with them comes from the work of Domański and Langenbruch. In a series of papers [10], [11], [12] they created the theory of the so-called Hadamard multipliers on the space of real analytic functions. These are the operators the associated matrix of which is just diagonal. This research was continued by Domański, Langenbruch and Vogt [16] and also by Vogt in [39], [40] for spaces of distributions, in [38] for spaces of smooth functions and by Trybuła in [36] for spaces of holomorphic functions. In [14] and [15] Domański and Langenbruch showed that this theory provides the correct framework to study Euler's equation. This equation on temperate distributions was studied in [41] by Vogt. Golińska in [20] and [21] studies operators defined by Hankel matrices. The results of this paper are the analogs for the space of entire functions of the results obtained previously for the space of real analytic functions on the real line ([9], [23], [24], [25]).

The paper is divided into five sections. In the next one we obtain the aforementioned characterization of Toeplitz operators on $H(\mathbb{C})$. The third

[1]The formulation of this theorem was not correct in our previous paper [25].

section is devoted to the proof of Theorem 1. In the fourth we provide a proof of Theorem 2. Lastly, we study one-sided invertibility of Toeplitz operators, that is, we prove Theorem 3. We conclude the paper by applying our result to the Toeplitz operators, the symbols of which are (classes of) rational functions.

2. Basic example. Characterization of Toeplitz operators and the symbol space.

Let U be an open simply connected neighborhood of ∞ in the Riemann sphere $\mathbb{C}_\infty$. Assume that F is holomorphic in the punctured neighborhood $U \setminus \{\infty\}$ of the point ∞. We assign to the function F an operator $T_F \colon H(\mathbb{C}) \to H(\mathbb{C})$ and argue that it is reasonable to call the operator T_F a Toeplitz operator. Then we show, essentially following the arguments of [9, Theorem 1], that all Toeplitz operators on $H(\mathbb{C})$ are of the form T_F for some F of the form described above. We provide these (elementary) arguments to motivate our further study.

Let $f \in H(\mathbb{C})$ and let $z \in \mathbb{C}$. Choose a C^∞ smooth Jordan curve $\gamma \subset (U \setminus \{\infty\})$ such that the point z belongs to the interior $I(\gamma)$ (recall Jordan's theorem) of the curve γ and also the connected set $\mathbb{C} \setminus U$ is contained in $I(\gamma)$. That is, $\operatorname{Ind}_\gamma(z) = \operatorname{Ind}_\gamma(w) = 1$ for any $w \notin U$.

Put

$$(T_F f)(z) := \frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)\cdot f(\zeta)}{\zeta - z} d\zeta. \tag{2.1}$$

Naturally, for any $z \in \mathbb{C}$ we can choose such a curve γ. By Cauchy's theorem the value of integral (2.1) does not depend on γ. This implies that $(T_F f)(z)$ is well-defined for any $z \in \mathbb{C}$ and it is a holomorphic function.

Recall that $H(\mathbb{C})$ is a Fréchet space when equipped with the topology of uniform convergence on compact sets, i.e. the topology induced by all seminorms

$$\|f\|_K := \sup_{z\in K} |f(z)|,$$

with $K \subset \mathbb{C}$ compact. Our reference text as far as the theory of locally convex spaces is concerned is [28]. We refer the reader also to [33] and [35], where some information on Fréchet spaces can also be found.

The following fact is immediate

Proposition 2.1. *The operator $T_F \colon H(\mathbb{C}) \to H(\mathbb{C})$ is continuous.*

Let as above F be holomorphic in $U \setminus \{\infty\}$, where U is an open simply connected in $\mathbb{C}_\infty$ neighborhood of the infinity. Assume further that $z \in U$. It follows from Cauchy's theorem that

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_{outer}} \frac{F(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int_{\gamma_{inner}} \frac{F(\zeta)}{\zeta - z} d\zeta,$$

where $\gamma_{outer}, \gamma_{inner}$ are C^∞ smooth Jordan curves such that γ_{inner} is contained in the interior $I(\gamma_{outer})$ of γ_{outer} and $\mathbb{C} \setminus U \subset I(\gamma_{inner})$. Also, the point z belongs to the intersection of the interior $I(\gamma_{outer})$ of γ_{outer} and the exterior $E(\gamma_{inner})$ of the curve γ_{inner}. The orientation of the curves $\gamma_{inner}, \gamma_{outer}$ is induced by the natural orientation of $\mathbb{C}$. This implies that

$$F = F_+ + F_-, \tag{2.2}$$

where F_+ is entire and F_- is holomorphic in some neighborhood of ∞ and vanishes at ∞. Develop the functions F_+ and F_- into Laurent series

$$F_+(z) = \sum_{m=0}^{\infty} a_m z^m,$$
$$F_-(z) = \sum_{m=1}^{\infty} \frac{a_{-m}}{z^m}.$$

The second series converges for $|z| > R$ with an R large enough, while the first is convergent in $\mathbb{C}$.

Then

$$(T_F f)(z) = (F_+ \cdot f)(z) + \frac{1}{2\pi i}\int_\gamma \frac{F_-(z)\cdot f(z)}{\zeta - z} d\zeta \tag{2.3}$$

for an appropriate curve γ. We compute now $T_F f$ for $f(z) = z^n$. It is elementary that for $|z| < R$

$$\begin{aligned} \frac{1}{2\pi i}\int_{|\zeta|=2R} \frac{F_-(\zeta)\cdot \zeta^n}{\zeta - z} d\zeta &= \sum_{m=1}^{\infty} a_{-m} \cdot \frac{1}{2\pi i}\int_{|\zeta|=2R} \frac{\zeta^n}{\zeta - z}\frac{d\zeta}{\zeta^m} \\ &= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} a_{-m} \cdot \Big(\frac{1}{2\pi i}\int_{|\zeta|=2R} \frac{\zeta^n}{\zeta^{k+m+1}} d\zeta\Big) \cdot z^k \\ &= a_{-n} + a_{-n+1} z + \cdots + a_{-1} z^{n-1}. \end{aligned}$$

In view of (2.3) we have

$$\begin{aligned} (T_F \zeta^n)(z) &= a_{-n} + a_{-n+1} z + \cdots + a_{-1} z^{n-1} + z^n \Big(\sum_{m=0}^{\infty} a_m z^m\Big) \\ &= a_{-n} + a_{-n+1} z + \ldots \end{aligned}$$

Now create an infinite matrix M by putting the Taylor series coefficients of the entire functions $T_F \zeta^n, n \in \mathbb{N}$ in the consecutive columns:

$$M := \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots \\ a_1 & a_0 & a_{-1} & \ldots \\ a_2 & a_1 & a_0 & \ldots \\ \ldots & \ldots & \ldots & \ddots \end{pmatrix}. \tag{2.4}$$

This is an infinite Toeplitz matrix. Naturally, we may create such a matrix for any operator $T\colon H(\mathbb{C}) \to H(\mathbb{C})$. Indeed, let $T(\zeta^n)(z) = \sum_{m=0}^{\infty} a_{mn} z^m$ be the Taylor series development of the entire function $T(\zeta^n)$. Then the corresponding matrix is just $M_T := (a_{mn})_{m,n \in \mathbb{N}_0}$. We claim that if for a continuous linear operator $T\colon H(\mathbb{C}) \to H(\mathbb{C})$ the matrix M_T is a Toeplitz matrix, then T is the operator T_F for a function F, which is holomorphic in a punctured neighborhood of ∞.

Indeed, assume that $T\colon H(\mathbb{C}) \to H(\mathbb{C})$ is a continuous linear operator for which the matrix M_T is a Toeplitz matrix. Assume that M_T is given by (2.4) for some complex number $a_n, n \in \mathbb{Z}$. Put $F_+ := T1$. Then $F_+(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function. Consider now the continuous functional $\phi\colon H(\mathbb{C}) \to \mathbb{C}$ defined by the condition $\phi(f) := (Tf)(0)$. We have $\phi(z^n) = a_{-n}$. Recall that

$$H(\mathbb{C})_b' \cong H_0(\infty). \tag{2.5}$$

This characterization of the space dual to $H(\mathbb{C})$ is known as the Grothendieck-Köthe-da Silva duality [26, pp 372–378], also [2, Theorem 1.3.5]. The duality between $H(\mathbb{C})$ and $H_0(\infty)$ is given by the integral

$$\frac{1}{2\pi i}\int_\gamma f(z)G(z)dz, \tag{2.6}$$

where f is entire, G is holomorphic in some neighborhood U of ∞ (which as usual may be assumed simply connected in $\mathbb{C}_\infty$) and vanishes at ∞. The C^∞-smooth Jordan curve is chosen in such a way that $\mathbb{C} \setminus U \subset I(\gamma)$. Naturally, the value of (2.6) depends only on the equivalence class of G in the space of germs $H_0(\infty)$.

The subscript b in (2.5) indicates that the dual space of $H(\mathbb{C})$ is equipped with the strong topology, that is with the topology of uniform convergence on bounded sets of $H(\mathbb{C})$. We remark that in general, unlike in the Banach space case, there is no distinguished topology on the dual space of a locally convex space. We refer the reader to [28, Chapter 23] for a presentation of the duality theory of locally convex spaces.

Recall that for an entire function f, $\phi(f) = (Tf)(0)$. There exists therefore a function $G \in H_0(V)$, where V is a simply connected in $\mathbb{C}_\infty$ neighborhood of the infinity such that

$$\phi(f) = \frac{1}{2\pi i}\int_\gamma f(z)G(z)dz, \tag{2.7}$$

where $\gamma \subset \mathbb{C}$ is a C^∞ smooth Jordan curve such that $\mathbb{C} \setminus V \subset I(\gamma)$. For an $R > 0$ large enough we have $G(z) = \sum_{n=1}^{\infty} \frac{G_{-n}}{z^n}$ when $|z| > R$. It follows from (2.7) that $\phi(z^n) = G_{-(n+1)}$. That is, $a_{-n} = G_{-(n+1)}$. Set $F := F_+ + z \cdot G - a_0$. Then F is a function holomorphic in some punctured neighborhood of ∞ and the matrix of T_F is given by (2.4). This means that T and T_F are equal on polynomials. Hence, they are equal. We proved the following theorem:

Theorem 2.2. *The following conditions are equivalent:*

(i) $T\colon H(\mathbb{C}) \to H(\mathbb{C})$ *is a continuous linear operator, the matrix of which is given by* (2.4) *for some complex numbers* $a_n, n \in \mathbb{Z}$*;*

(ii) *There exists a function F, which is holomorphic in a punctured neighborhood $U \setminus \{\infty\}$ of ∞ in $\mathbb{C}_\infty$ such that $T = T_F$. Then*

$$a_n = \frac{1}{2\pi i}\int_\gamma F(z)z^{-n-1}dz,\ n \in \mathbb{Z},$$

where γ is a C^∞ smooth Jordan curve in $U \setminus \{\infty\}$ (the set U may be assumed to be simply connected in $\mathbb{C}_\infty$) such that $\mathbb{C} \setminus U \subset I(\gamma)$ and $0 \in I(\gamma)$.

The proof of Theorem 2.2 is essentially the same as [9, Theorem 1]. We feel it was better to repeat it, since the other results are based on this Theorem.

Observe that if $F_i, i = 1, 2$ are functions holomorphic in $U_i \setminus \{\infty\}$, U_i are neighborhoods of ∞ in $\mathbb{C}_\infty$, such that

$$F_1|_{U\setminus\{\infty\}} = F_2|_{U\setminus\{\infty\}} \tag{2.8}$$

for some open neighborhood U of the infinity, then $T_{F_1} = T_{F_2}$. In is natural therefore to define the symbol space $\mathfrak{S}(\mathbb{C})$ of Toeplitz operators on the space $H(\mathbb{C})$ as the inductive limit of the spaces $H(U \setminus \{\infty\})$, where U run through open neighborhoods of ∞,

$$\mathfrak{S}(\mathbb{C}) := \operatorname{lim\,ind} H(U \setminus \{\infty\}).$$

That is, $\mathfrak{S}(\mathbb{C})$ is the space of equivalence classes of functions holomorphic in some punctured neighborhood of ∞ with respect to the equivalence relation (2.8). The space $\mathfrak{S}(\mathbb{C})$ carries a natural locally convex topology as the inductive limit of Fréchet spaces.

Recall that the power series space of infinite type $\Lambda_\infty(n)$ is defined in the following way:

$$\Lambda_\infty(n) := \Big\{x \in \mathbb{C}^{\mathbb{N}_0} \colon \sum_{n=0}^{\infty} |x_n|^2 r^{2n} < \infty, \text{for all}\, r > 0\Big\}.$$

We refer the reader to [28, Chapter 29] for a presentation of the theory of power series spaces. We only need here the fact that $\Lambda_\infty(n) \cong H(\mathbb{C})$ as Fréchet spaces. In fact, an isomorphism is given by

$$\mathcal{T} \colon \Lambda_\infty(n) \to H(\mathbb{C}),\ (\mathcal{T}x)(z) := \sum_{n=0}^{\infty} x_n z^n.$$

This is explained in [28, p. 360, Example 29.4. (2)] – we slightly changed the notation and index sequences starting with zero rather than one. Observe that

$$\mathcal{T}(0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots)(z) = z^n.$$

Consider the matrix of a continuous linear operator $S \colon \Lambda_\infty(n) \to \Lambda_\infty(n)$ with respect to the standard Schauder basis. We immediately have the following theorem:

Theorem 2.3. *An infinite Toeplitz matrix* $(c_{mn})_{m,n\in\mathbb{N}_0} = (a_{m-n})$, *where* $(a_n)_{n\in\mathbb{Z}}$ *is a sequence of complex numbers, is a matrix of a continuous linear operator on the infinite power series space* $\Lambda_\infty(n)$ *with respect to the standard Schauder basis if and only if there exists a function* F *holomorphic in some punctured neighborhood* $U\setminus\{\infty\}$ *of* ∞, U *a simply connected open set in the Riemann sphere* $\mathbb{C}_\infty$, *such that*

$$a_n = \frac{1}{2\pi i}\int_\gamma F(z)z^{-n-1}dz,\ n\in\mathbb{Z}$$

where γ *is a* C^∞*-smooth Jordan curve in* U *such that* $\mathbb{C}\setminus U\subset I(\gamma)$ *and* $0\in I(\gamma)$.

3. Fredholm and Semi-Fredholm Toeplitz operators.

We shall say that a symbol $F\in\mathfrak{S}(\mathbb{C})$ *does not vanish* if F is the equivalence class in $\mathfrak{S}(\mathbb{C})$ of a function $\tilde F$ holomorphic in $U\setminus\{\infty\}$, U an open neighborhood of ∞ in $\mathbb{C}_\infty$, which does not vanish in $U\setminus\{\infty\}$. Otherwise we say that F *vanishes*.

We intend now to characterize the Toeplitz operators $T_F, F\in\mathfrak{S}(\mathbb{C})$, which are Fredholm operators, as the operators, the symbol of which does not vanish. That is, we prove Theorem 3.1.

Theorem 3.1. *A Toeplitz operator* $T_F\colon H(\mathbb{C})\to H(\mathbb{C}), F\in\mathfrak{S}(\mathbb{C})$ *is a Fredholm operator if and only if the symbol* F *does not vanish. In this case,*

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

For $F\in\mathfrak{S}(\mathbb{C})$ which does not vanish we need to define the number $\operatorname{winding} F$. Assume that F is the equivalence class of $\tilde F\in H(U\setminus\{\infty\})$, where U is an open simply connected in $\mathbb{C}_\infty$ neighborhood of ∞, such that $\tilde F$ does not vanish in $U\setminus\{\infty\}$. We put

$$\operatorname{winding} F := \operatorname{Ind}_{\tilde F\circ\gamma}(0) = \frac{1}{2\pi i}\int_\gamma \frac{\tilde F'(\zeta)}{\tilde F(\zeta)}d\zeta,$$

where γ is (any) C^∞ smooth Jordan curve in $U\setminus\{\infty\}$ such that $\mathbb{C}\setminus U\subset I(\gamma)$. We may always assume that γ is $|z|=R$ for an R large enough. It follows from Cauchy's Theorem that the definition is correct. It does not depend on γ and it does not depend on the representative $\tilde F$.

Our goal is to deduce the proof of Theorem 3.1 from the corresponding result for the Toeplitz operators on the spaces of germs on finite closed intervals of $\mathbb{C}$, [24, Theorem 5.3]. We need therefore to determine the adjoint operator T_F'. Recall that the dual space of $H(\mathbb{C})$ with the strong topology is isomorphic as a locally convex space with the space $H_0(\infty)$ of germs of holomorphic functions at ∞, which vanish at ∞, [26, pp 372–378], see also [2, Theorem 1.3.5]. We have

$$H_0(\infty) = \lim\operatorname{ind} H_0(V), \tag{3.1}$$

where V run through open neighborhoods of ∞ and $H_0(V)$ is the Fréchet space of all functions holomorphic in V, which vanish at ∞. Let $g \in H_0(\infty)$. Then g is the equivalence class in $\operatorname{lim\,ind} H_0(V)$ of a holomorphic function $\tilde{g}\colon V \to \mathbb{C}$, which vanishes at ∞. The set V may be assumed to be simply connected in $\mathbb{C}_\infty$. Let us recall that the duality between $H(\mathbb{C})$ and $H_0(\infty)$ is given by

$$(f, g) \mapsto \langle f, g\rangle = \frac{1}{2\pi i}\int_\gamma f(z)\tilde{g}(z)dz, \tag{3.2}$$

where γ is a C^∞ smooth Jordan curve in $V \setminus \{\infty\}$ such that $\mathbb{C} \setminus V \subset I(\gamma)$.

Incidentally, this and decomposition (2.2) imply that

$$\mathfrak{G}(\mathbb{C}) \cong H(\mathbb{C}) \oplus H_0(\infty) \cong H(\mathbb{C}) \oplus H(\mathbb{C})'.$$

We need to determine the adjoint operator $T_F'\colon H_0(\infty) \to H_0(\infty)$. Assume that F is holomorphic in $U \setminus \{\infty\}$, U an open simply connected in $\mathbb{C}_\infty$ neighborhood of ∞. Let g be the equivalence class of a function $\tilde{g}$ which is holomorphic in some simply connected neighborhood V of ∞ and vanishes at ∞. Let δ be a C^∞ smooth Jordan curve in $V\setminus\{\infty\}$ such that $\mathbb{C}\setminus V \subset I(\delta)$. Then for any $f \in H(\mathbb{C})$ the function $T_F f$ is entire and

$$\langle T_F f, g\rangle = \frac{1}{2\pi i}\int_\delta (T_F f)(z)\tilde{g}(z)dz = \frac{1}{2\pi i}\int_\delta \Big(\frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta\Big)\tilde{g}(z)dz.$$

Here, γ is a C^∞ smooth Jordan curve in $U \setminus \{\infty\}$ such that both $\delta \subset I(\gamma)$ and $\mathbb{C} \setminus U \subset I(\gamma)$. By Fubini's Theorem and Cauchy's integral formula

$$\langle T_F f, g\rangle = \frac{1}{2\pi i}\int_\gamma \Big(\frac{1}{2\pi i}\int_\delta \frac{\tilde{g}(z)}{\zeta - z}dz\Big) f(\zeta)\cdot F(\zeta)d\zeta = \frac{1}{2\pi i}\int_\gamma \tilde{g}(\zeta)\cdot f(\zeta)\cdot F(\zeta)d\zeta, \tag{3.3}$$

since $\tilde{g}$ vanishes at ∞.

We may assume that the intersection $U \cap V$ is simply connected in $\mathbb{C}_\infty$, in particular, it is connected. Let Δ be a C^∞ smooth Jordan curve in $(U \cap V) \setminus \{\infty\}$ such that $\mathbb{C} \setminus (U \cap V) \subset I(\Delta)$. For $\zeta \in E(\Delta)$ define

$$(S_F g)(\zeta) := \frac{1}{2\pi i}\int_\Delta \frac{F(z)\cdot \tilde{g}(z)}{\zeta - z}dz,$$

The function $S_F g$ is holomorphic in some neighborhood of ∞ and vanishes at ∞. Observe that we may assume by an appropriate choice of the curve Δ that $S_F g$ is holomorphic in $U \cap V$.

The equivalence class of this function in $\operatorname{lim\,ind} H_0(V)$ defines an element in $H_0(\infty)$. We have therefore defined the operator $S_F\colon H_0(\infty) \to H_0(\infty)$ – we abuse the notation and denote by S_F both the operator between the spaces $H_0(U \cap V)$ and on the inductive limit $H_0(\infty)$. One easily checks that S_F is indeed well-defined and continuous, when $H_0(\infty)$ is equipped with the topology of the inductive limit (3.1).

Let Γ be a C^∞ smooth Jordan curve in $(U\cap V)\setminus\{\infty\}$ such that $\mathbb{C}\setminus(U\cap V)\subset I(\Gamma)$. Then

$$\langle f, S_F g\rangle = \frac{1}{2\pi i}\int_\Gamma f(\zeta)(S_F g)(\zeta)d\zeta = \frac{1}{2\pi i}\int_\Gamma f(\zeta)\frac{1}{2\pi i}\int_\Delta \frac{F(z)\cdot\tilde{g}(z)}{\zeta - z}dz d\zeta,$$

where $\Delta\subset(U\cap V)\setminus\{\infty\}$ satisfies $\Gamma\subset E(\Delta)$ and $\mathbb{C}\setminus(U\cap V)\subset I(\Delta)$. Again Fubini's theorem and Cauchy's integral formula gives

$$\langle f, S_F g\rangle = \frac{1}{2\pi i}\int_\Delta F(z)\cdot\tilde{g}(z)\cdot f(z)dz.$$

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_\Delta F(z)\cdot\tilde{g}(z)\cdot f(z)dz = \frac{1}{2\pi i}\int_\gamma \tilde{g}(\zeta)\cdot f(\zeta)\cdot F(\zeta)d\zeta,$$

since the curves γ, Δ are homologous in $U\cap V$.

Proposition 3.2. *The operator $S_F\colon H_0(\infty)\to H_0(\infty)$ is the adjoint of the operator T_F in the sense of duality* (3.2)*.*

Consider now the space $H(0)$. That is, the space of all germs of holomorphic functions on the origin,

$$H(0) = \operatorname{lim\,ind} H(W).$$

Here, the open sets W run through open neighborhoods of 0. Consider a map

$$C\colon H_0(\infty)\to H(0)$$

defined in the following way: assume that $f\in H_0(\infty)$ is the equivalence class in $H_0(\infty)$ of a function $\tilde{f}$, which is holomorphic in a neighborhood V of ∞ and vanishes at infinity. Define Cf as the equivalence class in $H(0)$ of the function $\frac{1}{z}\tilde{f}(\frac{1}{z})$. One easily shows that the definition is correct. Furthermore, the map C factorizes through continuous maps between the Fréchet spaces $H_0(V)$ and $H(W)$, where V is an open neighborhood of ∞ and W is an open neighborhood of 0 (formally we use [28, Proposition 24.7]). This implies that C is a continuous map between $\operatorname{lim\,ind} H_0(V)$ and $\operatorname{lim\,ind} H(W)$, where again V run through open neighborhoods of ∞ and W run through open neighborhoods of 0.

Let $g\in H(0)$. Then g is the equivalence class in $\operatorname{lim\,ind} H(W)$ of a function $\tilde{g}\in H(W)$ for some open neighborhood of 0. Let $D\colon H(0)\to H_0(\infty)$ be defined for g as the equivalence class in $H_0(\infty)$ of the function $\frac{1}{z}\tilde{g}(\frac{1}{z})$. Then, by the same arguments, D is well-defined and continuous. One also easily checks that $D = C^{-1}$.

The space $H(0)$ is a special case of a space $H(K)$, $K\subset\mathbb{R}$ compact, which is the space of all germs of holomorphic functions over the set K. For this space we defined in [24] also Toeplitz operators. We now sketch this construction, since we want to deduce the proof of Theorem 3.1 from the corresponding result for the spaces $H(K)$. We specialize to the case

$K = \{0\}$. The symbol space, denoted $\mathcal{X}(0)$, is the defined as the inductive limit of the spaces $H(W \setminus \{0\})$, where W are open neighborhoods of 0,

$$\mathcal{X}(0) := \liminf H(W \setminus \{0\}).$$

Let $G \in \mathcal{X}(0)$. Then G is represented by some function $\tilde{G}$, which is holomorphic in $W \setminus \{0\}$, W an open neighborhood of 0. Let $g \in H(0)$ be the equivalence class in $H(0)$ of $\tilde{g}$ holomorphic in $H(V)$, V an open neighborhood of 0. Then $T_{G,0}g$ is defined as the equivalence class in $H(0)$ of the holomorphic function

$$\frac{1}{2\pi i}\int_\gamma \frac{\tilde{G}(\zeta)\cdot\tilde{g}(\zeta)}{\zeta - z}d\zeta,$$

where γ is a C^∞ smooth Jordan curve such that $0 \in I(\gamma)$ and $z \in I(\gamma)$.

This defines a continuous linear operator $T_{G,0}\colon H(0) \to H(0)$. Essentially the same arguments as in Section 2 show that it is reasonable to call the operator $T_{G,0}$ a Toeplitz operator. We provided the details in [24, Section 5]. Importantly, we proved the following theorem

Theorem 3.3 ([24], Theorem 5.3)**.** *Let $G \in \mathcal{X}(0)$. The operator*

$$T_{G,0}\colon H(0) \to H(0)$$

is a Fredholm operator if and only if there exists an open set $U \ni 0$ and a function $\tilde{G} \in H(U \setminus \{0\})$ such that $\tilde{G}$ does not vanish in $U \setminus \{0\}$ and G is the equivalence class in $\mathcal{X}(0)$ of $\tilde{G}$.

This theorem is formulated in [24] for the more general case of the spaces $H(K)$, K a compact interval contained in $\mathbb{R}$. Furthermore, an inspection of the proof in [24] shows that

$$\operatorname{index} T_{G,0} = -\operatorname{winding} G := -\operatorname{Ind}_{\tilde{G}\circ\gamma}(0) = -\frac{1}{2\pi i}\int_\gamma \frac{\tilde{G}'(\zeta)}{\tilde{G}(\zeta)}d\zeta,$$

where γ is a C^∞ smooth Jordan curve in $U \setminus \{0\}$ with $0 \in I(\gamma)$. We use this to prove Theorem 3.1.

Proof of Theorem 3.1. Assume that $F \in \mathfrak{S}(\mathbb{C})$ does not vanish. That is, F is the equivalence class of a function $\tilde{F}$ holomorphic in $U \setminus \{0\}$, where U is an open neighborhood of ∞, which does not vanish in U. Let $\tilde{G}(z) := \tilde{F}(\frac{1}{z})$ and define G as the equivalence class in $\mathcal{X}(0)$ of the function $\tilde{G}$. Consider the following composition of operators

$$H_0(\infty) \overset{C}{\to} H(0) \overset{T_{G,0}}{\to} H(0) \overset{C^{-1}}{\to} H_0(\infty),$$

i.e. the operator $S := C^{-1} \circ T_{G,0} \circ C$. We claim that $S = T'_F$. Indeed, one easily shows that close to ∞,

$$S(\frac{1}{\zeta^n})(z) = b_{-(n-1)}\frac{1}{z} + b_{-(n-2)}\frac{1}{z^2} + \cdots + b_0\frac{1}{z^n} + \ldots,$$

where $b_n, n \in \mathbb{Z}$ are the moments of G, i.e.

$$b_n = \frac{1}{2\pi i} \int_\gamma \tilde{G}(z) z^{-n-1} dz,$$

where γ is a C^∞ smooth Jordan curve in V, $\tilde{G} \in H(V)$, with $0 \in I(\gamma)$.

Similarly,

$$T_F^{'}(\frac{1}{\zeta^n})(z) = a_{n-1}\frac{1}{z} + a_{n-2}\frac{1}{z^2} + \cdots + a_0\frac{1}{z^n} + \ldots$$

It follows however from the definition of G that $b_n = a_{-n}, n \in \mathbb{Z}$. That is, the operators $T_F^{'}$ and S are equal on a set, which is linearly dense in $H_0(\infty)$. Indeed, this follows from Runge's theorem applied to the spaces $H_0(V)$, with V an open neighborhood of ∞ and the definition of $H_0(\infty)$ as $\lim\operatorname{ind} H_0(V)$.

The operators $T_F^{'}$ and S are therefore equal. Hence, since $T_{G,0}$ is a Fredholm operator by Theorem 3.3 and C, C^{-1} are isomorphisms, the operator $T_F^{'}$ is a Fredholm operator. Also,

$$\operatorname{index} T_F^{'} = \operatorname{index} S = \operatorname{index} T_{G,0}.$$

We have

$$\operatorname{index} T_{G,0} = -\frac{1}{2\pi i} \int_\gamma \frac{\tilde{G}^{'}(\zeta)}{\tilde{G}(\zeta)} d\zeta,$$

where $\gamma(t) = re^{it}, t \in [0, 2\pi]$ and $r > 0$ is sufficiently small. On the other hand,

$$\frac{1}{2\pi i} \int_\gamma \frac{\tilde{G}^{'}(\zeta)}{\tilde{G}(\zeta)} d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{[\tilde{F}(\frac{1}{\zeta})]^{'}}{\tilde{F}(\frac{1}{\zeta})} d\zeta = \frac{1}{2\pi i} \int_{\frac{1}{\gamma}} \frac{\tilde{F}^{'}(\zeta)}{\tilde{F}(\zeta)} d\zeta$$
$$= -\operatorname{Ind}_{\tilde{F}\circ\delta}(0) = -\operatorname{winding} F,$$

where $\delta(t) = \frac{1}{r}e^{it}$. It follows from [31, Satz 7.1] that, just as in the Banach space case, $\operatorname{index} T_F = -\operatorname{index} T_F^{'}$. We infer that

$$\operatorname{index} T_F = -\operatorname{winding} F,$$

which completes the proof.

The last argument requires however some comment. We used the fact that the operator $T_F^{'}$ is a ϱ-transformation [31, Definition 1.1]. An operator is a ϱ-transformation [31, Satz 1.1] if it is continuous and open onto its image (i.e. the image of an open set is open in the relative topology of the range) and both the kernel and the range of the operator are (continuously) complemented.

We justify the claim now that $T_F^{'}$ is a ϱ-transformation. Roughly speaking, we need an appropriate open mapping theorem.

Theorem 3.4 (Open mapping theorem, [28], Theorem 24.30)**.** *Let E and F be locally convex spaces. If E has a web and F is ultrabornological, then every continuous, linear, surjective map $S\colon E \to F$ is open.*

We refer to [28] for the proof of this profound result. Also, on p. 287 therein one can find the (rather technical) definition of a web, on p. 283 one finds discussion of ultrabornological spaces – these are just the spaces, the topology of which is induced by some inductive system of Banach spaces.

The space $H_0(\infty)$ carries the topology of an inductive system of Fréchet spaces and, as a result, it is bornological [28, Definition p. 281], since every Fréchet space is ultrabornological [28, Proposition 24.13, Remark 24.15 (b)]. Furthermore, the space $H_0(\infty)$ is complete [28, Proposition 25.7]. Hence, it is ultrabornological [28, Remark 24.15 (b)], another argument follows from the fact that the dual space of the complete Schwartz space $H(\mathbb{C})$ is ultrabornological [28, Proposition 24.23]. Also, the space $H_0(\infty)$ has a web [28, Lemma 24.28]. Thus, we can apply the Open mapping theorem for operators on $H_0(\infty)$. The arguments of [9, Proposition 5.1] show that a continuous linear operator, the range of which is a finite codimension has closed range.

The codimension of the range of $T_F^{'}$ in $H_0(\infty)$ is finite – we justified this above when we proved that $T_F^{'} = C^{-1} \circ T_{G,0} \circ C$. It follows from [22, Theorem 13.5.2] that the range of $T_F^{'}$ is bornological. As we argued above, the range of $T_F^{'}$ is closed in a complete space $H_0(\infty)$. Hence, the range is ultrabornological [28, Remark 24.15 (b)]. We can again apply the Open mapping theorem and conclude that $T_F^{'}$ is an open map onto its range.

The kernel of $T_F^{'}$ is of finite dimension, the range of $T_F^{'}$ is of finite codimension in $H_0(\infty)$. The fact that all linear topologies on finite dimensional linear spaces coincide [33, Theorem 1.21] and a standard application of the Hahn-Banach theorem show that the kernel and the range of $T_F^{'}$ are complemented in $H_0(\infty)$. This shows that $T_F^{'}$ is a ϱ-transformation.

Also, the space $H(\mathbb{C})$ is a Montel space and, as a result, it is reflexive [28, Remark 24.24 (a)], see also [35, Corollary p. 376]. We infer that $T_F^{''} = T_F$ and apply [31, Satz 7.1] in order to eventually conclude that $\operatorname{index} T_F^{'} = -\operatorname{index} T_F$. □

Our next goal is a characterization of semi-Fredholm Toeplitz operators. We start with the following theorem.

Theorem 3.5. *Assume that $F \in \mathfrak{S}(\mathbb{C})$ vanishes. Then the range of*

$$T_F \colon H(\mathbb{C}) \to H(\mathbb{C})$$

is closed.

Proof. Assume that F is the equivalence class in $\mathfrak{S}(\mathbb{C})$ of a function $\tilde{F} \in H(U \setminus \{\infty\})$, U simply connected in $\mathbb{C}_\infty$. Since F vanishes there is a sequence

$$|z_1| \le |z_2| \le \dots,$$

$z_n \to \infty$ such that $\tilde{F}(z_n) = 0$. We emphasize that these may be not all the zeros of $\tilde{F}$, some zeros may for example accumulate at bU. This is however of no importance. All we need is a sequence (z_n) with the above described properties.

Choose numbers $R_n, n \in \mathbb{N}$ such that at least n of the zeros z_n is contained in the disk $|z| < R_n$, no zero lies on the circle $|z| = R_m$ and $R_n < R_{n+1}$. For n large enough we have that the circle $|z| = R_n$ is contained in the set U. For simplicity we assume that this holds for every $n \in \mathbb{N}$. Let γ be any C^∞ smooth Jordan curve in U such that $\mathbb{C} \setminus U \subset I(\gamma)$ and $z_n \in E(\gamma), n \in \mathbb{N}$. By the Argument principle,

$$n \leq \frac{1}{2\pi i} \int_{|\zeta|=R_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)} d\zeta - \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)} d\zeta.$$

It follows that

$$\frac{1}{2\pi i} \int_{|\zeta|=R_n} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)} d\zeta \to \infty, \tag{3.4}$$

as $n \to \infty$.

Consider now the Hardy space $H^2(R_n\mathbb{T})$ on the circles $|z| = R_n$ and the Toeplitz operators on these spaces defined by the symbol $\tilde{F}$ restricted to $R_n\mathbb{T}$. Denote these operators by $T_{\tilde{F},R_n\mathbb{T}}$. Since $\tilde{F}$ does not vanish on $R_n\mathbb{T}$, the operator $T_{\tilde{F},R_n\mathbb{T}}$ is a Fredholm operator. It is a classical fact (see[7, Theorem 2.42 (b)]) that

$$\operatorname{index}\big(H^2(R_n\mathbb{T}) \to H^2(R_n\mathbb{T})\big) = -\frac{1}{2\pi i} \int_{R_n\mathbb{T}} \frac{\tilde{F}'(\zeta)}{\tilde{F}(\zeta)} d\zeta.$$

By (3.4) we infer that there is $N \in \mathbb{N}$ such that $\operatorname{index} T_{\tilde{F},R_n\mathbb{T}} < 0$ for $n \geq N$. By the Coburn-Simonenko theorem [7, Theorem 2.38], the operators $T_{\tilde{F},R_n\mathbb{T}} \colon H^2(R_n\mathbb{T}) \to H^2(R_n\mathbb{T})$ are injective for $n \geq N$.

Assume now that $T_F f_m$ tends to g in $H(\mathbb{C})$ as $m \to \infty$ (the space $H(\mathbb{C})$ is a Fréchet space, hence it suffices to consider sequences). We shall show that $g = T_F f$ for some entire function f. That is, the range of T_F is closed in $H(\mathbb{C})$. Since $T_F f_m$ is entire it belongs to $H^2(R_n\mathbb{T})$ for any $n \in \mathbb{N}$. Also, since $T_F f_m$ tends to g in $H(\mathbb{C})$, it tends uniformly to g on compact subsets of $\mathbb{C}$. This implies that $T_F f_m \to g$ in $H^2(R_n\mathbb{T})$. Naturally for $|z| < R_n$ we have

$$(T_F f_m)(z) = \frac{1}{2\pi i} \int_{R_n\mathbb{T}} \frac{\tilde{F}(\zeta) f_m(\zeta)}{\zeta - z} d\zeta = (T_{\tilde{F},R_n\mathbb{T}} f_m)(z). \tag{3.5}$$

We remark that that it is a standard fact that the functions in the Hardy space $H^2(R_n\mathbb{T})$ can be thought as either a function on $R_n\mathbb{T}$ or as a function in $|z| < R_n$ satisfying a certain growth condition. In integral (3.5) the function f_m is defined on $R_n\mathbb{T}$, while $|z| < R_n$.

Since $T_{\tilde{F},R_n\mathbb{T}}$ is a Fredholm operator, it has closed range in $H^2(R_n\mathbb{T})$. There exists therefore a function $h_n \in H^2(R_n\mathbb{T})$ such that $T_{\tilde{F},R_n\mathbb{T}} f_m \to T_{\tilde{F},R_n\mathbb{T}} h_n$ as $m \to \infty$. That is for any $n \in \mathbb{N}$ there is a function $h_n \in H^2(R_n\mathbb{T})$ such that $g(z) = (T_{\tilde{F},R_n\mathbb{T}} h_n)(z)$ for $|z| < R_n$. Consider now the functions h_m. We will glue them together. Observe that by a standard limit argument and Cauchy's theorem

$$\frac{1}{2\pi i}\int_{R_{n-1}\mathbb{T}}\frac{\tilde{F}(\zeta)\cdot h_{n+1}(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{R_{n-1}\mathbb{T}}\frac{\tilde{F}(\zeta)\cdot h_n(\zeta)}{\zeta - z}d\zeta$$

for $|z| < R_{n-1}$. However, as we shown for $n-1 \geq N$ the operators $T_{\tilde{F},R_{n-1}\mathbb{T}}$ are injective. This follows that h_n and h_{n+1} are equal in $|z| < R_{n-1}$. They define therefore an entire function f. From Cauchy's theorem it follows that $T_F f = g$. □

Corollary 3.6. *Let $F \in \mathfrak{S}(\mathbb{C})$. Then the operator $T_F\colon H(\mathbb{C}) \to H(\mathbb{C})$ is always a Φ_+-operator. In particular, the range of T_F is closed.*

Proof. Either F vanishes or not. If F vanishes then the range of T_F is closed by Theorem 3.5. If F does not vanish then T_F is a Fredholm operator. One easily shows mimicking [9, Proposition 5.1] that the range of T_F is closed.

Assume that F vanishes. Consider the operator $S := C^{-1} \circ T_{G,0} \circ C$ defined as in the proof of Theorem 3.1 with $T_{G,0}$ acting on $H(0)$. It follows from [24, Theorem 5.6] that the range of $T_{G,0}$ is dense in $H(0)$. This implies that the range of S is dense in $H_0(\infty)$. That is, the range of the adjoint operator T_F' is dense in $H_0(\infty)$. Since $H(\mathbb{C})$ is reflexive (see the proof of Theorem 3.1) we infer that T_F is injective.

To sum this up, either T_F is a Fredholm operator, in which case $\ker T_F$ is finite dimensional or it is injective. This implies that T_F is always a Φ_+ operator. □

We now show that if F vanishes then the operator T_F is not a Φ_--operator.

Theorem 3.7. *Assume that $F \in \mathfrak{S}(\mathbb{C})$ vanishes. Then the range of T_F is of infinite codimension in $H(\mathbb{C})$.*

Proof. Consider the symbol $F \in \mathfrak{S}(\mathbb{C})$ and assume that it is represented by $\tilde{F} \in H(U \setminus \{\infty\})$, where U is an open simply connected neighborhood of ∞ in $\mathbb{C}_\infty$. Also, let $z_n \in U$, $|z_1| \leq |z_2| \leq \dots$, $z_n \to \infty$ be some zeros of $\tilde{F}$ in U. Without loss of generality we may assume that there are no other zeros of $\tilde{F}$ in U.

Let $0 < r < R$ be chosen in such a way that the circles $|z| = r, R$ are contained in $U \setminus \{\infty\}$and $\mathbb{C} \setminus U \subset \{|z| < r\}$. By Cauchy's integral formula for any $f \in H(\mathbb{C})$,

$$\tilde{F}(z)\cdot f(z) = \frac{1}{2\pi i}\int_{|z|=R}\frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{|z|=r}\frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta$$

for $r < |z| < R$. For R large enough and $r < |z| < R$ we therefore have that

$$(T_F f)(z) = \tilde{F}(z)\cdot f(z) + \frac{1}{2\pi i}\int_{|z|=r}\frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta.$$

This implies that

$$\lim_{n\to\infty}(T_F f)(z_n) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{|z|=r}\frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z_n}d\zeta = 0,$$

since $\tilde{F}(z_n) = 0$. Thus, under the assumption that F vanishes, the range of T_F does not contain any polynomial. □

Corollary 3.8. *Let $F \in \mathfrak{S}(\mathbb{C})$. Then the operator $T_F\colon H(\mathbb{C}) \to H(\mathbb{C})$ is a Φ_--operator if and only if F does not vanish, in which case T_F is a Fredholm operator.*

Proof of Main Theorem 1. Corollary 3.6 implies (i). Corollary 3.8 implies (ii), (iii) is just Theorem 3.1. □

4. Invertible Toeplitz operators

Theorem 4.1. *Assume that $F \in \mathfrak{S}(\mathbb{C})$ and consider the Toeplitz operator*

$$T_F\colon H(\mathbb{C}) \to H(\mathbb{C}).$$

Then either $\ker T_F = \{0\}$ or $\ker T_F^{'} = \{0\}$.

Proof. Assume that $T_F f = 0$, where f is entire, and $T_F^{'} g = 0$ for some $g \in H_0(\infty)$. The germ $g \in H_0(\infty)$ is represented by a function $\tilde{g}$, which is holomorphic in an open simply connected neighborhood of ∞ denoted by W. The function $\tilde{g}$ vanishes at ∞.

Let F be the equivalence class in $\mathfrak{S}(\mathbb{C})$ of a function $\tilde{F} \in H(U \setminus \{\infty\})$, where U is an open simply connected neighborhood of ∞ in $\mathbb{C}_\infty$. Let γ and Γ by C^∞ smooth Jordan curves in $U \setminus \{\infty\}$ such that $\mathbb{C} \setminus U \subset I(\gamma)$ and $\gamma \subset I(\Gamma)$. By Cauchy's integral formula

$$(\tilde{F}\cdot f)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta$$

for $z \in E(\gamma) \cap I(\Gamma)$. Since $T_F f = 0$ we have for $z \in E(\gamma)$

$$(\tilde{F}\cdot f)(z) = -\frac{1}{2\pi i}\int_\gamma \frac{\tilde{F}(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta.$$

The right-hand side of the above equality defines a function, which is holomorphic not only in $\mathbb{C} \setminus U$ but in $\mathbb{C}_\infty \setminus U$ and vanishes at ∞. That is, the product $\tilde{F}\cdot f$ extends by zero to $\mathbb{C}_\infty \setminus U$.

On the other hand, again by Cauchy's integral formula,

$$(\tilde{F}\cdot \tilde{g})(z) = \frac{1}{2\pi i}\int_\Delta \frac{\tilde{F}(\zeta)\cdot \tilde{g}(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_\delta \frac{\tilde{F}(\zeta)\cdot \tilde{g}(\zeta)}{\zeta - z}d\zeta,$$

where δ, Δ are C^∞ smooth Jordan curves in $(U \cap W) \setminus \{\infty\}$ such that $\mathbb{C} \setminus (U \cap W) \subset I(\delta)$, $\delta \subset I(\Delta)$ and $z \in E(\delta) \cap I(\Delta)$ (we can assume that

$U \cap W$ is simply connected in $\mathbb{C}_\infty$). Since $T_F' g = 0$, we have

$$(\tilde{F} \cdot \tilde{g})(z) = \frac{1}{2\pi i} \int_\Delta \frac{\tilde{F}(\zeta) \cdot \tilde{g}(\zeta)}{\zeta - z} d\zeta \tag{4.1}$$

for $z \in I(\Delta)$. The right-hand side of (4.1) defines an entire function. This is a consequence of Cauchy's theorem, since for any $z \in \mathbb{C}$ we can find an appropriate curve such that $z \in I(\Delta)$ and the function $\tilde{F} \cdot \tilde{g}$ is holomorphic in a simply connected in $\mathbb{C}_\infty$ neighborhood of ∞. Thus the product $\tilde{F} \cdot \tilde{g}$ extends to an entire function.

Consider now the product $\tilde{F} \cdot f \cdot \tilde{g}$. As we shown, $\tilde{F} \cdot f$ belongs to $H_0(\mathbb{C}_\infty \setminus U)$. Since $\tilde{g} \in H_0(\mathbb{C}_\infty \setminus W)$, we infer that $\tilde{F} \cdot f \cdot \tilde{g}$ is holomorphic in some neighborhood of ∞ and vanishes at ∞. On the other hand, since $\tilde{F} \cdot \tilde{g}$ extends to an entire function and f is an entire function we obtain that $\tilde{F} \cdot f \cdot \tilde{g}$ extends to an entire function. To sum this up, the product $\tilde{F} \cdot f \cdot \tilde{g}$ extends to a function which is holomorphic in the sphere $\mathbb{C}_\infty$ and vanishes at ∞. By Liouville's theorem this function vanishes identically, which implies that either $f = 0$ or $g = 0$. □

Theorem 4.2. *Assume that $F \in \mathfrak{S}(\mathbb{C})$ and consider the Toeplitz operator*

$$T_F \colon H(\mathbb{C}) \to H(\mathbb{C}).$$

The operator T_F is invertible if and only if it is a Fredholm operator of index zero.

Proof. Naturally an invertible operator is a Fredholm operator of index zero. Assume therefore that $T_F \colon H(\mathbb{C}) \to H(\mathbb{C})$ is a Fredholm operator and $\operatorname{index} T_F = 0$. By the Open mapping theorem [28, Theorem 24.30] in order to show that T_F is invertible, it suffices to show that T_F is a bijection.

Assume that T_F is not an injection, then it follows from Theorem 4.1 that $\ker T_F' = \{0\}$. This means that the range of T_F is dense. Since the range of T_F is closed (see [9, Proposition 5.1]) the operator T_F is a surjection. We infer that $\operatorname{index} T_F > 0$, which is a contradiction.

Assume that T_F is not onto. By the Hahn-Banach theorem there exists a non-zero functional $\xi \in H(\mathbb{C})'$ which vanishes on the range of T_F. This implies that $T_F' \xi = 0$, which in view of Theorem 4.1 implies that $\ker T_F = \{0\}$. We again reach a conclusion that $\operatorname{index} T_F \neq 0$, which is a contradiction. Thus, if $\operatorname{index} T_F = 0$, then T_F is a bijection. □

5. One-sided invertibility

If $F \in \mathfrak{S}(\mathbb{C})$ vanishes then the operator $T_F \colon H(\mathbb{C}) \to H(\mathbb{C})$ is injective and of closed range. We shall prove the following theorem, which completes the description of the operator T_F:

Theorem 5.1. *If $F \in \mathfrak{S}(\mathbb{C})$ vanishes then the range of the operator T_F is not complemented in $H(\mathbb{C})$.*

In order to prove Theorem 5.1 we obtain a rather precise description of the range of the operator T_F.

Theorem 5.2. *Assume that $F \in \mathfrak{S}(\mathbb{C})$ vanishes. There exists a sequence of functionals $\xi_n \in H(\mathbb{C})'$ such that*

$$R(T_F) = \bigcap_{n=1}^{\infty} \ker \xi_n.$$

Proof. Assume that $F \in \mathfrak{S}(\mathbb{C})$ is represented by a function $\tilde{F} \in H(U \setminus \{\infty\})$, where as usual U is an open simply connected in $\mathbb{C}_\infty$ neighborhood of ∞. Since F vanishes there is a sequence of points $|z_1| \leq |z_2| \leq \dots$ such that $\tilde{F}(z_n) = 0, n \in \mathbb{N}$. Using Weierstrass theory the function $\tilde{F}$ can be factored in the following way

$$\tilde{F}(z) = \prod_{n=1}^{\infty} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot \tilde{F}_0(z),$$

for some natural numbers p_n, where m_n is the multiplicity of z_n and $\tilde{F}_0$ does not vanish in $U \setminus \{\infty\}$. Here, the symbol $E_p, p \in \mathbb{N}$ stands for the Weierstrass elementary factor. Furthermore, we may assume that winding $\tilde{F}_0 = 1$. Indeed, assume that winding $\tilde{F}_0 = k > 1$. Choose $l \in \mathbb{N}$ and $0 \leq \mu_l \leq m_l$

$$m_1 + \cdots + m_{l-1} + \mu_l = k - 1.$$

Set

$$\tilde{G}_0(z) := \frac{\tilde{F}_0(z)}{\prod_{n=1}^{l-1} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot E_{p_l}^{\mu_l}\left(\frac{z}{z_l}\right)}.$$

The function $\tilde{G}_0$ is holomorphic in a smaller simply connected neighborhood of ∞ and

$$\tilde{F}(z) = \prod_{n=1}^{\infty} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot \prod_{n=1}^{l-1} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot E_{p_l}^{\mu_l}\left(\frac{z}{z_l}\right) \cdot \tilde{G}_0(z).$$

Furthermore, $\tilde{G}_0$ does not vanish in some neighborhood of ∞ and by the Argument principle

$$\operatorname{winding} \tilde{G}_0 = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{G}_0'(\zeta)}{\tilde{G}_0(\zeta)} d\zeta = \operatorname{winding} \tilde{F}_0 - (m_1 + \cdots + m_{l-1} + \mu_l) = 1.$$

Here, γ is a C^∞ smooth Jordan curve in $U \setminus \{\infty\}$ such that $\mathbb{C} \setminus U \subset I(\gamma)$ and $z_1, \dots, z_l \in I(\gamma)$.

If winding $\tilde{F}_0 = k < 1$ then we redefine $\tilde{F}_0$ as

$$\tilde{G}_0(z) := \prod_{n=1}^{l-1} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot E_{p_l}^{\mu_l}\left(\frac{z}{z_n}\right) \cdot \tilde{F}_0(z),$$

where $m_1 + \cdots + m_{l-1} + \mu_l = k + 1$ with $0 \leq \mu_l \leq m_l$. Observe that $\tilde{G}_0$ does not vanish in some neighborhood of ∞. That is, the function $\tilde{G}_0$ defines a symbol in $\mathfrak{S}(\mathbb{C})$, which does not vanish. By the Argument principle we have winding $\tilde{G}_0 = 1$.

In view of Theorem 3.1 we have that $T_{\tilde{F}_0}$ is a Fredholm operator and $\operatorname{Ind} T_{\tilde{F}_0} = -1$. We abused the notation here and used the symbol $\tilde{F}_0$ not only to denote the function $\tilde{F}_0$ but also its equivalence class in $\mathfrak{S}(\mathbb{C})$. Since $\operatorname{Ind} T_{\tilde{F}_0} < 0$ it follows that $\ker T'_{\tilde{F}_0}$ is non-trivial. This implies in view of Theorem 4.1 that $T_{\tilde{F}_0}$ is injective. Hence, the range of $T_{\tilde{F}_0}$ is a closed subspace of codimension 1 in $H(\mathbb{C})$. There exists therefore a functional $\xi \in H(\mathbb{C})'$ such that $R(T_{\tilde{F}_0}) = \ker \xi$. The functional ξ is represented by a function $\varphi \in H_0(V)$, where V is an open simply connected neighborhood of ∞ in $\mathbb{C}_\infty$. That is, for any $f \in H(\mathbb{C})$,

$$\xi(f) = \frac{1}{2\pi i}\int_\gamma f(z)\varphi(z)dz,$$

where γ is a C^∞ smooth Jordan curve in $V \setminus \{\infty\}$ such that $\mathbb{C} \setminus V \subset I(\gamma)$. We have $\xi(T_{\tilde{F}_0}(f)) = 0$ for any $f \in H(\mathbb{C})$. Thus, it follows from formula (3.3), it follows that for any $f \in H(\mathbb{C})$ it holds that

$$\int_\gamma \tilde{F}_0(z) \cdot f(z) \cdot \varphi(z)dz = 0, \tag{5.1}$$

where γ is a C^∞ smooth Jordan curve in $(U\cap V)\setminus\{\infty\}$ with $\mathbb{C}\setminus(U\cap V) \subset I(\gamma)$ – as usual we may assume that $U \cap V$ is simply connected in $\mathbb{C}_\infty$. We claim that this implies that $\tilde{F}_0 \cdot \varphi$ is an entire function. Indeed, by Cauchy's integral formula for $z \in (U \cap V) \setminus \{\infty\}$

$$\begin{aligned}\tilde{F}_0(z)\cdot\varphi(z) &:= \frac{1}{2\pi i}\int_\Delta \frac{\tilde{F}_0(\zeta)\cdot\varphi(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_\delta \frac{\tilde{F}_0(\zeta)\cdot\varphi(\zeta)}{\zeta - z} \\ &= G_+(z) + G_-(z),\end{aligned}$$

where δ, Δ are C^∞ smooth Jordan curves in $(U\cap V)\setminus\{\infty\}$ with $\mathbb{C}\setminus(U\cap V) \subset I(\delta)$, $\delta \subset I(\Delta)$ and $z \in E(\delta) \cap I(\Delta)$. By Cauchy's theorem the function G_+ is entire and G_- is holomorphic in $|z| > R$ for R large enough and vanishes at ∞. Thus it develops into a Laurent series

$$G_-(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}.$$

It follows from (5.1) and Cauchy's theorem that $a_{-n} = 0$ for $n \in \mathbb{N}$. Indeed, (5.1) implies that

$$\int_\gamma G_-(z)z^n dz = 0, n = 0, 1, \ldots.$$

Thus, the product $\tilde{F}_0 \cdot \varphi$ extends to an entire function, which we shall denote by G. That is,

$$\varphi(z) = \frac{G(z)}{\tilde{F}_0(z)} \tag{5.2}$$

for $z \in (U \cap V) \setminus \{\infty\}$ with G entire.

Consider now the functionals corresponding to the functions $\frac{\varphi}{(z-z_n)^k}$ with $1 \le k \le m_n$. For any $f \in H(\mathbb{C})$,

$$\left\langle T_F f, \frac{\varphi}{(z-z_m)^k} \right\rangle = \frac{1}{2\pi i} \int_\gamma \frac{\prod_{n=1}^{\infty} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right)}{(z-z_m)^k} \cdot \tilde{F}_0(z) \cdot f(z) \cdot \varphi(z) dz$$
$$= \langle T_{\tilde{F}_0} g, \varphi \rangle,$$

for an entire function g – we again applied formula (3.3). This implies that for any $f \in H(\mathbb{C})$,

$$\left\langle T_F f, \frac{\varphi}{(z-z_m)^k} \right\rangle = 0, \tag{5.3}$$

since the functional represented by φ vanishes on $R(T_{\tilde{F}_0})$. We have shown that

$$R(T_F) \subset \bigcap_{n=1}^{\infty} \ker \xi_n, \tag{5.4}$$

where the functionals ξ_n correspond to the functions $\varphi, \frac{\varphi}{(z-z_n)^k}, 1 \le k \le m_n$. We now prove that the equality holds.

Assume that $\eta \in H(\mathbb{C})'$ vanishes on the range $R(T_F)$. The functional η is represented by a function $g \in H_0(V)$, where V is an open simply connected in $\mathbb{C}_\infty$ neighborhood of ∞. We have

$$\int_\gamma \tilde{F}(z) \cdot f(z) \cdot g(z) dz = 0$$

for any $f \in H(\mathbb{C})$. By the same arguments which lead to representation (5.2) we obtain that $\tilde{F} \cdot g$ extends to an entire function, say H. That is,

$$\prod_{n=1}^{\infty} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot \tilde{F}_0(z) \cdot g(z) = H(z)$$

for $z \in (U \cap V) \setminus \{\infty\}$ with H entire. Only a finite number of zeros z_n. say $z_1, \dots, z_m$, does not belong to $U \cap V$. We conclude that $H(z_n) = 0$ for $n > m$ and, as a result, these zeros can be divided out. Thus for a different entire function H we have

$$g(z) = \frac{H(z)}{\prod_{n=1}^{m} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot \tilde{F}_0(z)}$$

for $z \in U \cap V$. Consider now the Toeplitz operator T_S, where S is the equivalence class in $\mathfrak{S}(\mathbb{C})$ of the function

$$\prod_{n=1}^{m} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot \tilde{F}_0(z). \tag{5.5}$$

Naturally, for any $f \in H(\mathbb{C})$,

$$\langle T_S f, g\rangle = \frac{1}{2\pi i}\int_\gamma \frac{\prod_{n=1}^m E_{p_n}^{m_n}\left(\frac{z}{z_n}\right)\cdot \tilde{F}_0(z)}{\prod_{n=1}^m E_{p_n}^{m_n}\left(\frac{z}{z_n}\right)\cdot \tilde{F}_0(z)} \cdot f(z)\cdot H(z)dz = 0,$$

since f and H are entire. That is, $R(T_S) \subset \ker \eta$, where the functional $\eta \in H(\mathbb{C})'$ is represented by the function g.

The symbol S does not vanish, since the function (5.5) has only a finite number of zeros. In view of Theorem 3.1 the operator T_S is a Fredholm operator and $\operatorname{Ind} T_S = -1 - \sum_{n=1}^m m_n$. The same arguments which lead to (5.3) show that

$$\left\langle T_S, \frac{\varphi}{(z-z_j)^k}\right\rangle = 0, \tag{5.6}$$

for $j = 1, \ldots, m$ and $1 \le k \le m_j$. We claim that

$$R(T_S) = \bigcap_{n=1}^N \ker \xi_n,$$

where the functionals ξ_n correspond to φ and $\frac{\varphi}{(z-z_1)}, \ldots, \frac{\varphi}{(z-z_m)^{m_\nu}}$, i.e. $N = 1 + \sum_{n=1}^\nu m_n$. Indeed, these functions are linearly independent. Hence, the codimension in $H(\mathbb{C})$ of

$$\bigcap_{n=1}^N \ker \xi_n$$

is $1+\sum_{n=1}^\nu m_n$. It follows from Theorem 4.1 that the codimension of $R(T_S)$ is also equal to $1 + \sum_{n=1}^\nu m_n$. Since by (5.6)

$$R(T_S) \subset \bigcap_{n=1}^N \ker \xi_n,$$

the equality must hold.

We showed that if the functional η represented by the function g vanishes on $R(T_F)$, then

$$g \in \operatorname{span}\left\{\varphi, \frac{\varphi}{(z-z_1)}, \ldots, \frac{\varphi}{(z-z_\nu)^{m_\nu}}\right\} \tag{5.7}$$

for some $\nu \in \mathbb{N}$. The proof is completed by an application of the Hahn-Banach theorem. Assume that

$$f \in \left(\bigcap_{n=1}^\infty \ker \xi_n\right) \setminus R(T_F).$$

Since $R(T_F)$ is closed, there exists a functional $\eta \in H(\mathbb{C})'$ such that $\eta|_{R(T_F)} = 0$ and $\eta(f) \neq 0$. This however, in view of (5.7), is impossible. □

Assume that $F \in \mathfrak{S}(\mathbb{C})$ is represented by a function $F \in H(U \setminus \{\infty\})$ – we abuse the notation and use the same symbol to denote both the function and its equivalence class. The function F factorizes as

$$F(z) = \prod_{n=1}^{\infty} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot F_0(z)$$

and F_0 does not vanish in $U \setminus \{\infty\}$. The points z_n satisfy

$$0 < |z_1| \leq |z_2| \leq \dots .$$

According to Theorem 5.2 there exist functionals $\xi_n \in H(\mathbb{C})'$ such that

$$R(T_F) = \bigcap_{n=1}^{\infty} \ker \xi_n .$$

Each functional ξ_n is defined in the sense of duality (3.2) by a function $\varphi \in H_0(\infty)$ or some function $\frac{\varphi}{(z-z_k)^j}$, where $k \in \mathbb{N}$ and $1 \leq j \leq m_k$. Furthermore, as we have shown in (5.2) the function φ can be written as $\frac{G}{F_0}$, where G is entire.

Define a map $\Xi \colon H(\mathbb{C}) \to \omega$, where ω is the Fréchet space of all sequences by the formula

$$\Xi(f) := (\xi_n(f))_{n \in \mathbb{N}} .$$

The topology of ω is the pointwise convergence topology.

We shall prove the following theorem.

Theorem 5.3. *The map Ξ is surjective.*

The proof of this result is based on Eidelheit's theorem, which we recall for convenience of the reader.

Theorem 5.4 ([28], Theorem 26.27)**.** *Let E be a Fréchet space, $(U_k)_{k \in \mathbb{N}}$ be a fundamental system of zero neighborhoods in E and let $(A_j)_{j \in \mathbb{N}}$ be linearly independent, continuous linear forms on E. Then the infinite system of equations*

$$A_j x = y_j \textit{ for all } j \in \mathbb{N}$$

is solvable for each sequence $y \in \omega$ if, and only if, the following holds:

$$\dim((E')_{U_k^\circ} \cap \operatorname{span}\{A_j \colon j \in \mathbb{N}\}) < \infty \textit{ for all } k \in \mathbb{N}.$$

We explain now the notation used in Eidelheit's theorem. If $M \subset E$ is a non-empty subset of a locally convex space E, then the polar of $M \subset E$ is

$$M^\circ := \{\xi \in E' \colon |\xi(x)| \leq 1 \text{for all}\, x \in M\} \subset E' .$$

If B is an absolutely convex subset of E, then

$$E_B = \operatorname{span} B = \bigcup_{t>0} tB$$

and $\|\cdot\|_B$ is the Minkowski functional of B

$$\|x\|_B := \inf\{t > 0 \colon x \in tB\}.$$

The Minkowski functional is a norm in E_B (in particular, it is finite). Thus in Theorem 5.4 the symbol $(E^{'})_{U_k^\circ}$ stands for the span in $E^{'}$ of the set $U_k^\circ \subset E^{'}$, which is the polar of the zero neighborhood $U_k \subset E$. We remark here that Eidelhet's Theorem is used to prove E. Borel's Theorem, which says that for every sequence $(y_j) \subset \omega$ there is an $f \in C^\infty[-1,1]$ such that $f^{(j)}(0) = y_j, j \in \mathbb{N}$ (see [28, p. 324]).

Proof of Theorem 5.3. By assumption, $F, F_0 \in H(U \setminus \{\infty\})$, where U is an open simply connected neighborhood of ∞.

Choose a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of $\mathbb{C}$ such that $z_1, \ldots, z_n \in K_n$, $K_n \subset \{|z| \le |z_n|\}$ and $z_{n+1}, \ldots \notin K_n$. Without loss of generality we may assume that $|z_{n+1}| > |z_n|$. A fundamental system of seminorms in $H(\mathbb{C})$ is given by

$$\|f\|_k := \sup_{K_n} |f|$$

and a fundamental system of zero neighborhoods in $H(U)$ is

$$U_k := \{f \in H(\mathbb{C})\colon \|f\|_k \le 1\}.$$

For any $f \in H(\mathbb{C})$ it holds that $\frac{f}{\|f\|_k} \in U_k$. Hence if $\xi \in U_k^\circ$, then for any $f \in H(U)$,

$$|\xi(f)| \le \|f\|_k.$$

For a fixed $k \in \mathbb{N}$ we have

$$(H(\mathbb{C})^{'})_{U_k^\circ} \cap \operatorname{span}\{\xi_M\colon M \in \mathbb{N}\}$$
$$= \{\xi = \sum \alpha_M \xi_M \colon \exists_C \forall_{f\in H(\mathbb{C})} |\xi(f)| \le C\|f\|_k\}.$$

We need to show that this space is finite dimensional. The functionals ξ_M are linearly independent. Fix $k \in \mathbb{N}$ and consider functions of the form

$$g(z) := \prod_{n\neq k+1} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot f(z),$$

where $f \in H(\mathbb{C})$. Then $g \in H(\mathbb{C})$ and for any such a function

$$\left\langle T_{F_0} g, \frac{\varphi}{(z-z_i)^j} \right\rangle = \int_\gamma F_0(z) \cdot \prod_{n\neq k+1} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot f(z) \cdot \frac{G(z)}{F_0(z)} \cdot \frac{dz}{(z-z_i)^j} = 0,$$

when $i \neq k+1$ and $1 \le j \le m_i$. Consider the function

$$f_{m,l}(z) := (z - z_{k+1})^l \frac{E_m\left(\frac{z}{z_{k+1}}\right)}{(z - z_{k+1})},$$

where $m \in \mathbb{N}$ and $0 \le l \le m_{k+1} - 1$. Naturally,

$$\langle T_{F_0}\big(\prod_{n\neq k+1} E_{p_n}^{m_n}\left(\frac{z}{z_n}\right) \cdot f_{m,l}(z)\big), \frac{\varphi}{(z-z_i)^j} \rangle = 0,$$

if $i \neq k+1$ and $1 \leq j \leq m_i$. Also,

$$\begin{aligned}
&\langle T_{F_0}\big(\prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\cdot f_{m,m_{k+1}-1}(z)\big), \frac{\varphi}{(z-z_{k+1})^{m_{k+1}}}\rangle \\
&\qquad = \int_\gamma \prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\cdot G(z)\cdot \frac{E_m\Big(\frac{z}{z_{k+1}}\Big)}{z-z_{k+1}}\frac{dz}{(z-z_{k+1})} \\
&\qquad = 2\pi i \prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z_{k+1}}{z_n}\Big)\cdot G(z_{k+1})\cdot \lim_{z\to z_{k+1}} \frac{E_m\Big(\frac{z}{z_{k+1}}\Big)}{z-z_{k+1}}.
\end{aligned}$$

Obviously,

$$\lim_{z\to z_{k+1}} \frac{E_m\Big(\frac{z}{z_{k+1}}\Big)}{z-z_{k+1}} = \frac{1}{z_{k+1}}E_m'(1) = -\frac{1}{z_{k+1}}\exp(1+\cdots+\frac{1}{m}).$$

Observe that for $1 \leq j < m_{k+1}$

$$\begin{aligned}
&\Big\langle T_{F_0}\big(\prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\cdot f_{m,m_{k+1}-1}(z)\big), \frac{\varphi}{(z-z_{k+1})^{j}}\Big\rangle \\
&\qquad = 2\pi i\int_\gamma \prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\cdot G(z)\cdot \frac{E_m\Big(\frac{z}{z_{k+1}}\Big)}{z-z_{k+1}}(z-z_{k+1})^{m_{k+1}-1-j}dz = 0
\end{aligned}$$

by Cauchy's theorem, since the integrand is entire. Thus

$$\begin{aligned}
&\sum_M \alpha_M\xi_M\Big(T_{F_0}\big(\prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\cdot f_{m,m_{k+1}-1}(z)\big)\Big) \\
&\qquad = -2\pi i\alpha_N \prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z_{k+1}}{z_n}\Big)\cdot G(z_{k+1})\cdot \frac{\exp(1+\cdots+\frac{1}{m})}{z_{k+1}},
\end{aligned}$$

if ξ_N is defined by $\frac{\varphi}{(z-z_{k+1})^{m_{k+1}}}$. Unless $G(z_{k+1}) = 0$ or $\alpha_N = 0$ this expression tends to ∞ as $m \to \infty$. Choose now a C^∞ smooth Jordan curve such that $\gamma \subset U$, $\operatorname{dist}(\gamma, K_k) \geq \delta$, $\gamma \subset \{|z| < |z_{k+1}|\}$ (thus $z_{k+1}, \ldots \notin \overline{I(\gamma)}$). Then,

$$\begin{aligned}
&\Big\|T_{F_0}\big(\prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\cdot f_{m,m_{k+1}-1}(z)\big)\Big\|_k \\
&\qquad \leq \frac{1}{\delta}\sup_{z\in\gamma}\Big|F_0(z)\cdot(z-z_{k+1})^{m_{k+1}-1}\cdot \prod_{n\neq k+1} E_{p_n}^{m_n}\Big(\frac{z}{z_n}\Big)\Big|\cdot \sup_{z\in\gamma}\Big|\frac{E_m\Big(\frac{z}{z_{k+1}}\Big)}{z-z_{k+1}}\Big|.
\end{aligned}$$

Observe that we have

$$\sup_{z\in\gamma}\Big|E_m\Big(\frac{z}{z_{k+1}}\Big)\Big| \leq 1+\sup_{z\in\gamma}\Big|E_m\Big(\frac{z}{z_{k+1}}\Big)-1\Big| \leq 1+\sup_{z\in\gamma}\Big|\frac{z}{z_{k+1}}\Big|^{m+1} \leq C$$

for a uniform constant C. Assume now that

$$\sum \alpha_M \xi_M \in (H(U)')_{U_k^\circ},$$

then we infer that, if $G(z_{k+1}) \neq 0$, then for any $m \in \mathbb{N}$

$$|\alpha_N \exp(1 + \cdots + \frac{1}{m})| \leq C,$$

where N corresponds to $\frac{\varphi}{(z-z_{k+1})^{m_{k+1}}}$. This is possible only if $\alpha_N = 0$. In the same way we prove that $\alpha_{N-1} = 0$ simply by taking functions $f_{m,m_{k+1}-2}$ and eventually that all α_N which correspond to z_{k+1} vanish. We repeat the argument also for $z_{k+2}, \dots$. Observe there is at most a finite number of z_n such that $G(z_n) = 0$. Indeed, if $G(z_n) = 0$, then $\varphi(z_n) = 0$ by (5.2) and if the function G vanish on infinite number of points z_n, then ∞ is an accumulation point of zeros of φ. Since φ is holomorphic at ∞, this would imply that $\varphi \equiv 0$. The theorem is proved. □

Theorem 5.5. *If $F \in \mathfrak{S}(\mathbb{C})$ vanishes, then the operator $T_F\colon H(\mathbb{C}) \to H(\mathbb{C})$ is not left invertible.*

Proof. If $\ker \Xi$ is complemented in $H(\mathbb{C})$, then $\Xi\colon H(\mathbb{C}) \to \omega$ is right invertible. Indeed, let $P\colon H(\mathbb{C}) \to H(\mathbb{C})$ be a continuous projection onto $\ker \Xi$. Consider the closed space $Y := (I - P)H(\mathbb{C})$. It follows from Theorem 5.3 that the operator Ξ when restricted to Y is a bijection onto ω. There exists therefore a linear inverse map $\Omega\colon \omega \to Y$. We claim that Ω is continuous. This follows from an application of the Closed graph theorem [28, Theorem 24.31]. Note that ω is ultrabornological as a Fréchet space [28, Proposition 24.13 and Remark 24.15 (b)], it also has a web as a closed subspace of a webbed space [28, Lemma 24.28]. Now Ω is a right inverse of Ξ.

In order to prove the theorem it suffices to show that Ξ is not right invertible. Since Ξ is surjective by Theorem 5.3 this implies that $\ker \Xi$ is not complemented but, as we have shown in Theorem 5.2, $R(T_F) = \ker \Xi$. If T_F is left invertible and S is a left inverse, then $T_F \circ S$ is a continuous projection onto $R(T_F)$.

Assume that there exists a continuous linear map $E\colon \omega \to H(\mathbb{C})$ such that $\Xi \circ E = \mathrm{id}$. Consider the sequences $\delta_M \in \omega, M \in \mathbb{N}$, $\delta_M(n) = \delta_{nM}$. Define $g_M := E\delta_M$. Since $\Xi(g_M) = \Xi \circ E(\delta_M) = \delta_M \neq 0$, none of the functions g_M vanishes identically. There exists therefore $\tilde{z} \in \mathbb{C}$ such that $g_M(\tilde{z}) \neq 0$ for all $M \in \mathbb{N}$. Consider now the sequences $\Delta_M := \delta_M / g_M(\tilde{z})$, $M \in \mathbb{N}$. The space ω is equipped with the topology of pointwise convergence. Hence $\Delta_M \to 0$ in ω as $M \to \infty$. If E is continuous then $E\Delta_M$ tends to 0 in $H(\mathbb{C})$, in particular pointwise for any $z \in \mathbb{C}$. Since E is linear we have $(E\Delta_M)(\tilde{z}) = 1$. This is a contradiction. This in particular implies that Ξ is not right invertible and, as a result, $R(T_F)$ is not complemented in $H(\mathbb{C})$. Hence T_F is not left invertible. □

There is another argument which shows that $\ker \Xi$ is not complemented. In fact, the kernel of every surjection $A\colon E \to \omega$ from a Fréchet space with a continuous norm onto ω is not complemented. Otherwise E would contain a closed subspace without a continuous norm, which is impossible.

The proof of Theorem 3 is now standard.

Proof of Theorem 3. Assume that T_F with $F \in \mathfrak{S}(\mathbb{C})$ is left invertible, then the range of T_F is complemented. It follows from Theorem 5.5 that F does not vanish. It follows from Theorem 1 that T_F is a Fredholm operator. Also, if T_F is left invertible then T_F is injective. Hence, $\operatorname{index} T_F \leq 0$.

Assume now that T_F is a Fredholm operator such that

$$\operatorname{index} T_F = -\operatorname{winding} F \leq 0.$$

Then it follows from Theorem 2 that T_F is injective. Hence, T_F is an injective Fredholm operator. We shall prove that T_F is left invertible. If T_F is additionally surjective, then T_F is invertible by the Open mapping theorem [28, Theorem 24.30]. Assume therefore that the classes of some functions $g_1, \ldots, g_n \in H(\mathbb{C})$ span $H(\mathbb{C})/R(T_F)$. For every $f \in H(\mathbb{C})$ there are unique numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and a unique function $g \in H(\mathbb{C})$ such that

$$f = \alpha_1 g_1 + \cdots + \alpha_n g_n + T_F g. \tag{5.8}$$

We now define an operator $S\colon H(\mathbb{C}) \to H(\mathbb{C})$, which is a left inverse of T_F. Simply put $Sf = g$. The definition is correct, since representation (5.8) is unique. This also implies that S is linear. The fact that S is continuous follows easily from the Closed graph theorem [28, Theorem 24.31].

Observe that if F vanishes, then T_F is not surjective. Hence, it cannot be right invertible. The arguments for the right invertibility of surjective Fredholm operators are essentially the same as for the left invertibility. Therefore we omit it. The reader may consult [25, Theorem 2] for the details. □

We conclude the paper by providing an elementary and important example of Toeplitz operators on $H(\mathbb{C})$. Namely, let R be a rational function

$$R(z) = \frac{\prod_{i=1}^{m} (z - a_i)^{\alpha_i}}{\prod_{i=1}^{n} (z - b_i)^{\beta_i}},$$

where a_i and b_i are pairwise different and $a_i \neq b_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The function R defines a symbol in $\mathfrak{S}(\mathbb{C})$. Indeed, R is holomorphic in some punctured neighborhood of infinity. Furthermore, it does not vanish, since $a_1, \ldots, a_m$ are the only zeros of R. We apply our results to T_R and obtain the following theorem.

Theorem 5.6. *The following holds true for the Toeplitz operator*

$$T_R : H(\mathbb{C}) \to H(\mathbb{C}).$$

(i) *The operator T_R is a Fredholm operator and*

$$\operatorname{index} T_R = \sum_{i=1}^{n} \beta_i - \sum_{i=1}^{m} \alpha_i.$$

(ii) *The operator T_R is invertible if and only if*

$$\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{m} \alpha_i = 0.$$

(iii) *The operator T_R is left invertible if and only if*

$$\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{m} \alpha_i \leq 0.$$

(iv) *The operator T_R is right invertible if and only if*

$$\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{m} \alpha_i \geq 0.$$

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References

[1] Albanese, Angela A.; Bonet, José; Ricker, Werner J. The Cesàro operator on power series spaces. *Studia Math.* **240** (2018), no. 1, 47–68. MR3719466, Zbl 06828573, doi: 10.4064/sm8590-2-2017. 760

[2] Berenstein, Carlos A.; Gay, Roger. Complex analysis and special topics in harmonic analysis. *Springer-Verlag, New York*, 1995. x+482 pp. ISBN: 0-387-94411-7. MR1344448, Zbl 0837.30001, doi: 10.1007/978-1-4613-8445-8. 765, 767

[3] Bonet, José; Domański Paweł. A note on the spectrum of composition operators on spaces of real analytic functions. *Complex Anal. Oper. Theory* **11** (2017), no. 1, 161–174. MR3595980, Zbl 06692074, doi: 10.1007/s11785-016-0589-5. 760

[4] Bonet José; Lusky, Wolfgang; Taskinen, Jari. On boundedness and compactness of Toeplitz operators in weighted H^∞-spaces. *J. Funct. Anal.* **278** (2020), no. 10, 108456, 26 pp. MR4067994, Zbl 07173893, doi: 10.1016/j.jfa.2019.108456. 760

[5] Bonet, José; Taskinen, Jari. A note about Volterra operators on weighted Banach spaces of entire functions. *Math. Nachr.* **288** (2015), no. 11–12, 1216–1225. MR3377113, Zbl 1342.47062, doi: 10.1002/mana.201400099. 760

[6] Brown, Arlen; Halmos, Paul R. Algebraic properties of Toeplitz operators. *J. Reine Angew. Math.* **213** (1963/64), 89–102. MR0160136, Zbl 0116.32501, doi: 10.1515/crll.1964.213.89. 757

[7] Böttcher, Albrecht; Silbermann, Bernd. Analysis of Toeplitz operators. *Springer-Verlag, Berlin*, 1990. 512 pp. ISBN: 3-540-52147-X. MR1071374, Zbl 0732.47029, doi: 10.1007/978-3-662-02652-6. 757, 759, 773

[8] Coburn, Lewis A. Weyl's theorem for nonnormal operators. *Michigan Math. J.* **13** (1966), 285–288. MR0201969, Zbl 0173.42904, doi: 10.1307/mmj/1031732778. 759

[9] Domański, Paweł; Jasiczak, Michal. Toeplitz operators on the space of real analytic functions: the Fredholm property. *Banach J. Math. Anal.* **12** (2018), no. 1, 31–67. MR3745573, Zbl 06841264, doi: 10.1215/17358787-2017-0022. 756, 758, 760, 761, 762, 763, 766, 772, 774, 776

[10] Domański, Paweł; Langenbruch, Michael. Representation of multipliers on spaces of real analytic functions. *Analysis* (Munich) **32** (2012), no. 2, 137–162. MR3043719, Zbl 1288.46020, doi: 10.1524/anly.2012.1150. 762

[11] Domański, Paweł; Langenbruch, Michael. Algebra of multipliers on the space of real analytic functions of one variable. *Studia Math.* **212** (2012), no. 2, 155–171. MR3008439, Zbl 1268.46021, doi: 10.4064/sm212-2-4. 762

[12] Domański, Paweł; Langenbruch, Michael. Hadamard multipliers on spaces of real analytic functions. *Adv. Math.* **240** (2013), 575–612. MR3046319, Zbl 1288.46019, doi: 10.1016/j.aim.2013.01.015. 762

[13] Domański, Paweł; Langenbruch, Michael. Interpolation of holomorphic functions and surjectivity of Taylor coefficient multipliers. *Adv. Math.* **293** (2016), 782–855. MR3474335, Zbl 1336.35115, doi: 10.1016/j.aim.2016.02.028.

[14] Domański, Paweł; Langenbruch, Michael. Euler type partial differential operators on real analytic functions. *J. Math. Anal. Appl.* **443** (2016), no. 2, 652–674. MR3514312, Zbl 1347.35083, doi: 10.1016/j.jmaa.2016.05.018. 762

[15] Domański, Paweł; Langenbruch, Michael. Surjectivity of Euler type differential operators on spaces of smooth functions. *Trans. Amer. Math. Soc.* **372** (2019), no. 9, 6017–6086. MR4024514, Zbl 07124931, doi: 10.1090/tran/7367. 762

[16] Domański, Paweł; Langenbruch, Michael; Vogt, Dietmar. Hadamard type operators on spaces of real analytic functions in several variables. *J. Funct. Anal.* **269** (2015), no. 12, 3868–3913. MR3418073, Zbl 1341.46016, doi: 10.1016/j.jfa.2015.09.011. 762

[17] Douglas, Ronald G.; Sarason, Donald. Fredholm Toeplitz operators. *Proc. Amer. Math. Soc.* **26** (1970), 117–120. MR0259639, Zbl 0201.16801, doi: 10.1090/s0002-9939-1970-0259639-x. 759

[18] Flajolet, Philippe; Sedgewick, Robert. Mellin transforms and asymptotics: finite differences and Rice's integrals. Special volume on mathematical analysis of algorithms. *Theoret. Comput. Sci.* **144** (1995), no. 1–2, 101–124. MR1337755, Zbl 0869.68056, doi: 10.1016/0304-3975(94)00281-m. 761

[19] Gel'fond, Aleksandr O. Differenzenrechnung. Hochschulbücher für Mathematik, Bd. 41. *VEB Deutscher Verlag der Wissenscheften, Berlin*, 1958. viii+336 pp. MR0094608, Zbl 0080.07601. 761, 762

[20] Golińska, A. Hankel operators on the space of real analytic functions $\mathcal{A}(\mathbb{R})$. Preprint, 2017. 762

[21] Golińska, A. Classical operators on the space of real analytic functions. PhD thesis, Adam Mickiewicz University, 2019. 762

[22] Jarchow, Hans. Locally convex spaces. Mathematische Leitfäden. *B. G. Teubner, Stuttgart*, 1981. 548 pp. ISBN: 3-519-02224-9. MR0632257, Zbl 0466.46001, doi: 10.1007/978-3-322-90559-8. 772

[23] Jasiczak, Michał. Coburn–Simonenko theorem and invertibility of Toeplitz operators on the space of real analytic functions. *J. Operator Theory* **79** (2018), no. 2, 327–344. MR3803560, Zbl 1399.47092. 756, 759, 760, 761, 762

[24] Jasiczak, Michał. Semi-Fredholm Toeplitz operators on the space of real analytic functions. *Studia Math.* **252** (2020), no. 3, 213–250. MR4069992, Zbl 07178844, doi: 10.4064/sm170810-23-5. 756, 758, 759, 760, 761, 762, 767, 769, 770, 774

[25] Jasiczak, Michał; Golińska, Anna. One-sided invertibility of Toeplitz operators on the space of real analytic functions on the real line. *Integral Equations Operator Theory* **92** (2020), no. 1, Paper No. 6, 51 pp. MR4056774, Zbl 07191848, doi: 10.1007/s00020-020-2562-y. 756, 760, 761, 762, 785

[26] Köthe, Gottfried. Topological vector spaces. I. Die Grundlehren der mathematischen Wissenschaften, Band 159. *Springer-Verlag, New York Inc., New York*, 1969. xv+456 pp. MR0248498, Zbl 0179.17001, doi: 10.1007/978-3-642-64988-2. 765, 767

[27] Markushevich, A. I. Theory of functions of a complex variable. II. *Chelsea Publishing Co., New York*, 1977. xxii 325 pp. MR0444912, Zbl 0357.30002. 761, 762

[28] Meise, Reinhold; Vogt, Dietmar. Introduction to functional analysis. Oxford Graduate Texts in Mathematics, 2. *The Clarendon Press, Oxford University Press, New York*, 1997. x+437 pp. ISBN: 0-19-851485-9. MR1483073, Zbl 0924.46002. 758, 763, 765, 766, 769, 771, 772, 776, 781, 782, 784, 785

[29] Nikolski, Nikolai K. Operators, functions, and systems: an easy reading. 1. Hardy, Hankel, and Toeplitz. Mathematical Surveys and Monographs, 92. *American Mathematical Society, Providence, RI*, 2002. xiv+461 pp. ISBN: 0-8218-1083-9. MR1864396, Zbl 1007.47001, doi: 10.1090/surv/092. 757

[30] Nörlund, Niels E. Vorlesungen über Differenzenrechnung. Die Grundlehren der mathematischen Wissenschaften, Einzeldarstellungen, Bd. 13. *Springer, Berlin, Heidelberg*, 1924. JFM 50.0315.02, doi: 10.1007/978-3-642-50824-0. 761

[31] Pietsch, Albrecht. Zur Theorie der σ-Transformationen in lokalkonvexen Vektorräumen. *Math. Nachr.* **21** (1960), 347–369. MR0123185, Zbl 0095.30903, doi: 10.1002/mana.19600210604. 771, 772

[32] Rudin, Walter. Function theory in polydiscs. *W. A. Benjamin, Inc., New York-Amsterdam*, 1969. vii+188 pp. MR0255841, Zbl 0177.34101. 760

[33] Rudin, Walter. Functional analysis. Second edition. International Series in Pure and Applied Mathematics. *McGraw-Hill, Inc., New York*, 1991. xviii+424 pp. ISBN: 0-07-054236-8. MR1157815, Zbl 0867.46001. 763, 772

[34] Simonenko, Igor B. Certain general questions of the theory of the Riemann boundary value problem. *Izv. Akad. Nauk SSSR, Ser. Mat.* **32** (1968), 1138–1146; *Math. USSR Izv.* **2** (1968), 1091–1099. MR0235135, Zbl 0165.16703, doi: 10.1070/im1968v002n05abeh000706. 759

[35] Trèves. François. Topological vector spaces, distributions and kernels. *Academic Press, New York-London*, 1967. xvi+624 pp. MR0225131, Zbl 0171.10402. 763, 772

[36] Trybuła, Maria. Hadamard multipliers on spaces of holomorphic functions. *Integral Equations Operator Theory* **88** (2017), no. 2, 249–268. MR3669129, Zbl 1384.30013, doi: 10.1007/s00020-017-2369-7. 762

[37] Vasilevski, Nikolai L. Commutative algebras of Toeplitz operators on the Bergman space. Operator Theory: Advances and Applications, 185. *Birkhäuser Verlag, Basel, Boston*, 2008. xxx+417 pp. ISBN: 978-3-7643-8725-9. MR2441227, Zbl 1168.47057, doi: 10.1007/978-3-7643-8726-6. 757

[38] Vogt, Dietmar. Operators of Hadamard type on spaces of smooth functions. *Math. Nachr.* **288** (2015), no. 2–3, 353–361. MR3310518, Zbl 1314.47119, doi: 10.1002/mana.201300269. 762

[39] Vogt, Dietmar. Hadamard operators on $\mathcal{D}'(\mathbb{R}^d)$. *Studia Math.* **237** (2017), no. 2, 137–152. MR3620748, Zbl 1387.47019, arXiv:1511.08593, doi: 10.4064/sm8481-1-2017. 762

[40] Vogt, Dietmar. Hadamard operators on $\mathcal{D}'(\Omega)$. *Math. Nachr.* **290** (2017), no. 8–9, 1374–1380. MR3667002, Zbl 1379.46031, arXiv:1602.03078, doi: 10.1002/mana.201600305. 762

[41] Vogt, Dietmar. Surjectivity of Euler operators on temperate distributions. *J. Math. Anal. Appl.* **466** (2018), no. 2, 1393–1399. MR3825446, Zbl 06910523, doi: 10.1016/j.jmaa.2018.06.063. 762

[42] Zhu, Kehe. Analysis on Fock spaces. Graduate Texts in Mathematics, 263. *Springer, New York*, 2012. x+344 pp. ISBN: 978-1-4419-8800-3. MR2934601, Zbl 1262.30003, doi: 10.1007/978-1-4419-8801-0. 757

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