

On the gonality of graphs and connections to orientable genus

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ABSTRACT. We find that hyperelliptic graphs in the sense of Baker and Norine are planar and examine connections between the gonality and orientable genus of a graph. We give a notion of a bielliptic graph and show that each of these must embed into a closed orientable surface of genus one. We also find, for all $g \geq 0$, trigonal graphs of orientable genus g , and give analogues for graphs of higher gonality.

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1. Introduction

The *gonality* of a graph can refer to many related notions inspired by the Brill-Noether theory of an algebraic curve. Baker and Norine [3] were the first to define it as the least degree of a non-constant harmonic morphism of graphs $G \rightarrow T$ where G is the graph of interest and T is a tree. Compare this to the definition of the gonality of an algebraic curve C : the least degree of a nonconstant morphism from C to \mathbb{P}^1 . We will discuss why this is a reasonable analogy in §5. Several other notions of gonality have been defined by other authors, including Caporaso [5] and Corneliison-Kato-Kool [6]. The last notion, *stable gonality*, allows refinements of G which do not change the *orientable genus* of G . This is the least genus of a closed orientable surface into which G embeds. This stable gonality is notable as it admits a spectral

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lower bound, i.e., in terms of the spectrum of the Laplacian of G [1, 6]. This is particularly appealing due to the connection with *Shimura curves*, which define graphs that reflect spectral data of modular forms and are typically non-planar (see §5). Could it be that there is a connection between stable gonality and orientable genus? In the following we say that a graph is *d-gonal* if its stable gonality is d . If $d = 2$, this ends up being equivalent to the notion of a *hyperelliptic graph* when the Euler characteristic of G is negative [3, §5]. The following shows that there is some connection between stable gonality and orientable genus.

Theorem 1.1. *All hyperelliptic graphs are planar.*

Recall that for a graph to be planar it is equivalent to having orientable genus zero. Similarly, we say a graph is *toroidal* if its orientable genus is at most 1. Consider that an algebraic curve is called *bielliptic* if it admits a degree 2 morphism to an algebraic curve of genus one. Similarly, we call a graph G bielliptic if it admits a degree 2 harmonic morphism to a graph G' of Euler characteristic zero (i.e., G' has a unique shortest cycle). We have the following.

Theorem 1.2. *All bielliptic graphs are toroidal.*

Since the bipartite graph $K_{3,3}$ is bielliptic, this is the best that could be hoped for. We are led to the following question, which is more intuitively stated with the correct analogue of genus: we will say that the (Euler) genus of a connected graph G' is $1 - \chi(G')$ where χ denotes the Euler characteristic. Therefore a graph of negative Euler characteristic has genus $g \geq 2$, etc.

Question 1.3. *If G is a graph which admits a degree 2 harmonic morphism to a graph G' of (Euler) genus g , is the orientable genus of G at most g ?*

An affirmative answer to this question would not be totally optimal - e.g., K_5 admits a degree 2 morphism to a genus 2 graph, but is toroidal. We know of no counterexamples to this statement and the proof of Theorem 4.3 suggests extensions but does not itself extend beyond the genus one case. There are also interesting differences between the cases of degree 2 harmonic morphisms and other cases.

Theorem 1.4. *If $d \geq 3$ with $d \not\equiv 2 \pmod{4}$ then there exist 3-connected d -gonal graphs of all orientable genera at least $(d/2 - 1)^2$.*

We mention 3-connectedness only to note that this is not the result of anomalous examples, but of genuine phenomena. The connection between gonality and orientable genus is therefore somewhat complicated, and ties in with some more properly graph-theoretic notions like treewidth. It would not be surprising if there were more interesting things that the Laplacian spectrum could tell us about the orientable genus. Already, it is understood that good planar immersions of graphs may be found using the eigenvectors of the Laplacian [10].

2. Preliminaries on the involutions of graphs and hyperelliptic graphs

We wish to study hyperelliptic graphs, which could be defined either in terms of harmonic morphisms of graphs or in terms of mixing involutions. We prefer the latter. To define this, we note that a connected graph G is given by its vertices $V(G)$ and its edges $E(G)$ (so that, e.g., its genus is $1 - \#V(G) + \#E(G)$). Any given edge e is drawn between two vertices x and y , and in this case we will say, $x \in e$ or $y \in e$. We will allow multiple edges to be drawn between the same pair of vertices. A *morphism of graphs* $f : G \rightarrow H$ is given by a pair of maps $f_V : V(G) \rightarrow V(H)$ and $f_E : E(G) \rightarrow E(H) \cup V(H)$, such that if $e \in E(G)$ and $x \in e$ then either $f_V(x) \in f_E(e)$ or $f_E(e) = f_V(x)$ [3, §2]. An isomorphism $f : G \rightarrow H$ is one that admits an inverse $f^{-1} : H \rightarrow G$, and an automorphism is an isomorphism $G \rightarrow G$.

Definition. A **mixing involution** on a graph G is a non-identity order-two automorphism $\iota : G \rightarrow G$ such that if e is an edge between x and y fixed by ι then $\iota(x) = y$.

A graph with a mixing involution ι and without loops¹ cannot have any edges e fixed by ι between ι -fixed vertices x and y . If G is a graph with loops, then the graph G' obtained by deleting those loops has the same orientable genus. Throughout this note, we will assume all graphs G are loopless.

We are now in the proper setting to consider **harmonic morphisms of graphs** [3, §2.1], an example of which is given by the quotient of a graph G by a mixing involution ι [3, §5.2]. The quotient G/ι has vertices of the form $\{v, \iota(v)\}$ such that v is a vertex of G , and edges of the form $\{e, \iota(e)\}$ such that the bounding vertices of e are inequivalent under ι . The canonical quotient morphism $q_\iota : G \rightarrow G/\iota$ sends vertices v to vertices $\{v, \iota(v)\}$ and edges e between v and w to $\{e, \iota(e)\}$ if $\iota(v) \neq w$ and to the quotient vertex $\{v, w = \iota(v)\}$ otherwise.

In the terminology of Baker-Norine, if G has at least 3 vertices, this map is a harmonic morphism of degree 2. All such morphisms on graphs with at least 3 vertices arise this way [3, Lemma 5.6]. If G has two vertices, then there is an obvious mixing involution and the quotient is a point, and thus a tree, and it is only because that map is constant that we do not say it has degree 2.

Definition. We say that a connected graph G admitting a mixing involution $\iota : G \rightarrow G$ such that G/ι is a tree is **hyperelliptic** and that ι is the corresponding **hyperelliptic involution**.

This is a slightly nonstandard definition in that we don't require the genus to be at least 2. Typically one stipulates that because when G is

¹meaning edges containing only one vertex

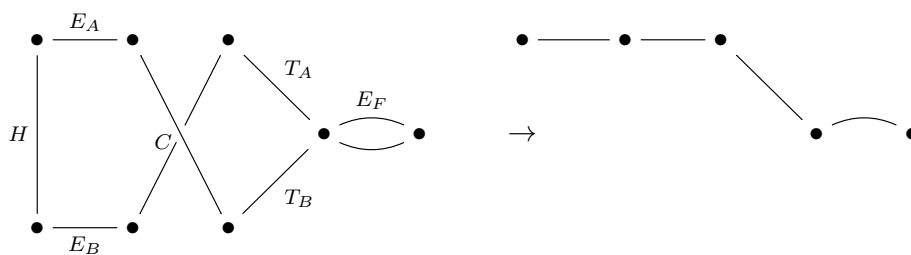


FIGURE 1. A hyperelliptic graph showing the different types of edges

2-edge-connected and has genus ≥ 2 , such an involution must be unique [3, Corollary 5.15]. Thankfully we can reduce to the 2-edge-connected case without pain by contracting all its bridges [3, Corollary 5.11]. There are no 2-edge connected trees, and the only 2-edge connected genus one graphs are cycles, which are planar. In this note, all graphs G with a mixing involution ι will be 2-edge connected unless noted.

A graph with all its bridges contracted has the same orientable genus as the original graph. Of course we will allow other graphs to not be 2-edge connected. Indeed G/ι will often be a tree in what follows.

Suppose now G is a graph which is 2-edge-connected, loopless, and has a mixing involution ι . A given vertex can be either fixed or moved by ι . We let F denote the set of vertices which are fixed by ι . By definition, all other vertices are permuted, and there must be an even number of these. Let A and B be any disjoint sets of permuted vertices: we let a_1, \dots, a_n be the elements of A , so $B = \{b_1 = \iota(a_1), \dots, b_n = \iota(a_n)\}$.

The edges of G must therefore fall into one of the following categories with respect to this partition.

- The set E_A of edges from A to itself.
- The set $E_B = \iota(E_A)$ of edges from B to itself.
- The set E_F of edges from F to itself.
- The “horizontal edges” H from some a_i to b_i .
- The “cross edges” C from some a_i to some b_j such that $i \neq j$.
- The “transfer edges” T_A from F to A .
- The “transfer edges” $T_B = \iota(T_A)$ from F to B .

Figure 1 shows the minimal example of a hyperelliptic graph with all seven of these types, along with the quotient by its hyperelliptic involution. The rightward arrow indicates this quotient morphism.

A first approximation to thinking about how these graphs embed into an orientable surface is to embed the graphs with the involution into \mathbb{R}^3 with the antipodal involution $x \mapsto -x$ and to build the orientable surface around it. This is not exactly right, for instance because there is only one

fixed point in \mathbb{R}^3 of the antipodal map, while a hyperelliptic graph can have arbitrarily many fixed vertices.

The basic idea remains though. To make everything precise, recall that when we speak of an embedding of a graph $\rho : G \rightarrow M$ with M a real manifold, we mean an embedding of its geometric realization, which associates to each edge a homeomorphic copy of the unit interval $[0, 1]$, and to each endpoint a vertex. We note that for the purposes of graph embeddings, we can assume without loss of generality that there are no horizontal edges.

Lemma 2.1. *For any graph with a mixing involution (G, ι) , there is a graph (G', ι') without horizontal edges and with the same embedding genus.*

Proof. We know $V(G) = A \cup B \cup F$ and $E(G) = E_A \cup E_B \cup T_A \cup T_B \cup H \cup C \cup E_F$. We can create a refined graph G' without horizontal edges as follows. If $h \in H$, then say $a_h \in h \cap A$, $b_h \in h \cap B$. We then make a new vertex f_h to be in the fixed vertices of ι' in G' . This vertex will have precisely two edges in G' going through it: $t(a_h)$ connecting a_h to f_h and $t(b_h)$ connecting b_h to f_h , which of course have to be exchanged by ι' . Then G' must have $V(G') = A \cup B \cup F \cup \{f_h : h \in H\}$ and $E(G') = E_A \cup E_B \cup T_A \cup T_B \cup C \cup E_F \cup \{t(a_h) : h \in H\} \cup \{t(b_h) : h \in H\}$. Note that this operation does not introduce bridges, for if removing $t(a_h)$ or $t(b_h)$ disconnected G' then so too would removing h disconnect G . The action of ι on G induces another mixing involution on G' , which we call ι' , and we see that G, G' have homeomorphic geometric realizations. \square

We focus on a particular type of embedding which can handle an arbitrary number of fixed vertices.

Definition. *An involutive embedding of a graph G with a mixing involution ι is an embedding $\rho : G \rightarrow \mathbb{R}^2$ whose image is a piecewise smooth subset of \mathbb{R}^2 with the following condition on ι . If $(x, y) = \rho(z) \in \rho(G)$ then $\rho(\iota(z)) = (-x, y)$.*

There are many involutions of \mathbb{R}^2 that would work just as well, but this one allows us to keep our intuition about “horizontal edges,” or rather the pairs of edges we get from horizontals. Note that each involutive embedding ρ gives an explicit identification of the geometric realization of the tree G/ι with the set $\rho(G) \cap \{(x, y) : x \geq 0\}$. Since G is compact as a topological space, its image will be compact and we will often think of G as embedding into the compact subset $[-1, 1]^2 \in \mathbb{R}^2$.

Once you find any involutive embedding ρ of G , you can find many, for instance by scaling, translating vertically, or by spherical inversion at a point $(a, 0) \notin \rho(G)$. To this end we will define the functions

$$\begin{aligned} \tau_a : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (x, y - a), \end{aligned}$$

and

$$\begin{aligned} \sigma_a : \mathbb{R}^2 - \{(0, a)\} &\rightarrow \mathbb{R}^2 - \{(0, a)\} \\ (x, y) &\mapsto \left(\frac{x}{x^2 + (y - a)^2}, \frac{y}{x^2 + (y - a)^2} \right). \end{aligned}$$

We could define various scaling functions as well, but instead we will use the single map

$$\begin{aligned} \alpha : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto \left(\frac{2}{\pi} \arctan(x), \frac{2}{\pi} \arctan(y) \right), \end{aligned}$$

which is a diffeomorphism $\mathbb{R}^2 \rightarrow (-1, 1)^2$ respecting the involution ι .

Any involutive embedding (or in fact any embedding $\rho : G \rightarrow \mathbb{R}^2$ defines an embedding $G \rightarrow S^2$. In fact, it will define a CW decomposition of S^2 as follows.

Definition. *An interior face of an involutive embedding ρ is a connected component of $S^2 - \rho(G)$.*

Each interior face F° is homeomorphic to the open unit disc D° , as G is connected and $\rho(G)$ is piecewise smooth. The closure in S^2 of any interior face F° will be referred to as a **face** F , and will be homeomorphic to the closed unit disc D . The boundary of any face $F - F^\circ$ will therefore be homeomorphic to $D - D^\circ$, or S^1 . We single out the face containing the point $\infty \in S^2$ as “the outside face” F_∞ . We close this set of preliminaries by noting that every interior face of an involutive embedding of a hyperelliptic graph must have a point of the form $(0, y)$.

Lemma 2.2. *If $\rho : G \rightarrow \mathbb{R}^2$ is an involutive embedding of (G, ι) and F° is an interior face of ρ without a point of the form $(0, y)$, then G/ι is not a tree.*

Proof. If $S \subset \mathbb{R}^2$ is a set, then let us use $\iota(S)$ to mean $\{(-x, y) : (x, y) \in S\}$. If there is no such point, then $F^\circ \cap \iota(F^\circ)$ is empty. Up to the action of ι , assume $F^\circ \subset \{(x, y) : x > 0\}$. Therefore $\partial F = F - F^\circ \subset \{(x, y) : x \geq 0\} \cap \rho(G)$, which is homeomorphic to the geometric realization of G/ι . This one-dimensional topological space contains ∂F , which is homeomorphic to S^1 , so G/ι contains a cycle. □

In light of Lemma 2.2, we can move any face F to the outside face by selecting any y_F such that $(0, y_F) \in F^\circ$ and applying σ_{y_F} .

3. Subgraphs and involutive embeddings

Let us begin with an informal description of how we make an involutive embedding of a hyperelliptic graph G . We retain all notation of the previous section, and will induct on the size of $\#A = \#B$. Suppose we want to create an involutive embedding for G with $\#A = n$, and we assume we can create

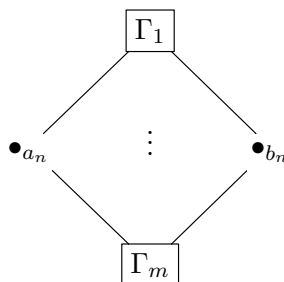


FIGURE 2. The inductive step for creating an involutive embedding.

an involutive embedding for graphs with fewer exchanged vertices. We can then perform the following rough operations, as depicted in Figure 2.

- (1) Remove one element a_n from $A \subset V(G)$, $b_n = \iota(a_n)$, and all edges containing a_n and b_n , setting them aside.
- (2) Let $\{\Gamma_i\}_{1 \leq i \leq m}$ be the ι -orbits of connected components of the remaining vertices and edges. Find involutive embeddings ρ_i of each into $[-1, 1]^2$ by our inductive hypothesis.
- (3) By connectedness, there must be an edge e_i connecting a_n to each Γ_i . Apply σ_{y_i} for some y_i so that $\sigma_{y_i}\rho_i$ puts the other end of e_i on the outside face. By the action of ι , the edge connecting b_n to Γ_i will be $\iota(e_i)$.
- (4) Apply α to embed each Γ_i into $(-1, 1)^2$. Translate each embedding $\Gamma_i \rightarrow (-1, 1)^2$ into different boxes with τ_{1-2i} and draw piecewise smooth edges from a_n, b_n to the Γ_i .

We now seek to formalize this inductive process. The hyperelliptic condition places severe restrictions on the edges that can occur between fixed vertices. Specifically, the subgraph (F, E_F) of G with all vertices fixed by ι will be a finite collection of connected components and each will be a “string of sausage links” of the following form:



Lemma 3.1. *The connected components of the subgraph (F, E_F) are either single vertices or chains of vertices f_1, \dots, f_r such that between f_i and f_{i+1} there are exactly two edges and between f_i and f_j there are no edges if $|i - j| > 1$.*

Proof. Let $e \in E_F$ and let f, f' be the bounding vertices of e . Since ι fixes f, f' and ι is mixing, we must have $\iota(e) \neq e$. Therefore there are at least 2 edges between f and f' . If we suppose to the contrary that there was a third edge e' then $\iota(e')$ would be distinct from e' again by the mixing property. But also since $e' \neq e$ and $e' \neq \iota(e)$ we must also have $\iota(e') \neq e$

and $e' \neq \iota(e)$. The quotient graph G/ι would then have a cycle ee' and since the hyperelliptic involution is unique we have a contradiction.

Therefore between any two vertices f, f' in our subgraph (F, E_F) there are either zero or two edges. If f, f', f'' each have two edges between them, then in the quotient, we would have a cycle $e(f, f')e(f', f'')e(f'', f)$. The result follows. \square

We see therefore that (F, E_F) is planar, and if we wish to provide an explicit involutive embedding of (F, E_F) into \mathbb{R}^2 it suffices to give an involutive embedding of the connected components, each of which must be a string as above.

Lemma 3.2. *If (G, ι) is a connected hyperelliptic graph with each vertex fixed by ι , then it admits an involutive embedding.*

Proof. The graph $(\{f_1, \dots, f_{m+1}\}, E_{\{f_1, \dots, f_{m+1}\}})$ admits an explicit involutive embedding. For the vertices we take $f_i \mapsto (0, \pi(i - 1))$. For the edges, we take the smooth functions $t \mapsto (\pm \sin(t), t)$ for $t \in [0, m\pi]$. \square

Note that the boundary of each face is piecewise smooth. For all compact subsets of the plane D with nonempty connected interior D° and boundary δ given as a connected, piecewise smooth parametrized curve, and for all finite subsets $S \subset \delta$, $D^\circ \cup S$ is path-connected by simple piecewise linear paths.

In a similar style to Lemma 2.2, we establish another useful fact about our informal setup. As before, we let (G, ι) be a connected hyperelliptic graph containing two vertices $a \neq b$ such that $\iota(a) = b$. By Lemma 2.1, assume G has no horizontal edges, i.e., no edges e such that $\iota(e) = e$. We let $G_{a,b} = (V(G) - \{a, b\}, \{e \in E(G) : a \notin e \text{ and } b \notin e\})$, and we let $\{\Gamma_i\}_{i=1}^m$ be the ι -orbits of connected components of $G_{a,b}$. Note that each Γ_i is the disjoint union of one or two connected components, and if there are two, they must be exchanged by ι , so in any case Γ_i/ι is connected.

Lemma 3.3. *With $(G, \iota), a, b, G_{a,b}, \{\Gamma_i\}_{i=1}^m$ as above, if $\Gamma = \Gamma_i$ is not connected then it contains no fixed vertices and ι gives an isomorphism between each connected component and the tree Γ/ι .*

Proof. First we show that if Γ contains a fixed vertex f , then it is connected. If Γ contains only f then it must be connected. If not, let $v \neq f$ be a vertex, and let $\bar{\gamma}$ be the shortest path in the tree Γ/ι from the vertex $\{v, \iota(v)\}$ to the vertex $\{f, f\}$. Pick an edge e of Γ in the preimage of $\bar{\gamma}$ containing v . Since e is not a loop, there is a vertex $v_1 \neq v$ in e . If $v_1 = f$ then you're done. If not, pick an edge $e_1 \notin \{e, \iota(e)\}$ in the preimage of $\bar{\gamma}$ containing v_1 . Repeat this process until we have a path γ_v in Γ from v to f . Therefore if $w \notin \{f, v\}$ is a vertex, then $\gamma_v \gamma_w$ makes a path in Γ from v to w , showing Γ is connected.

Therefore each vertex v of Γ/ι has 2 vertices v_1, v'_1 in its preimage. Let K, K' be the connected components of Γ . We claim that neither can contain

both v_1 and v'_1 , for if not, let K contain both without loss of generality. Since K is connected, we must have a simple path of length n from v_1 to $v'_1 = v_{n+1}$ in K . Let v_2, \dots, v_n be the vertices in this path and e_1, \dots, e_n be the edges in this path with $e_i \ni \{v_i, v_{i+1}\}$. Since Γ/ι is a tree, the image of this path is a straight line of edges. Therefore $\iota(e_1) = e_n$, and so $\iota(v_2) = v_n$, and continuing along we see that $\iota(e_{1+i}) = e_{n-i}$, $\iota(v_{2+j}) = v_{n-j}$.

If n is even, say $n = 2k$ then $\iota(v_{k+1}) = v_{k+1}$ is a fixed vertex, which by the above argument do not exist in Γ . Therefore our path cannot be of even length. If n is odd, say $n = 2k + 1$ then $\iota(e_{k+1}) = e_{k+1}$, a horizontal edge, which are not in G by assumption. Therefore the vertices of Γ are equally partitioned into K and K' . The same argument shows that the edges are equally partitioned into K and K' , for if not we could pick endpoints of edges exchanged by ι as our v_1, v'_1 . We conclude that ι induces an isomorphism of graphs $K \rightarrow K'$ as they have matching vertices and edges. \square

Lemma 3.4. *With $(G, \iota), a, b, G_{a,b}, \{\Gamma_i\}_{i=1}^m$ as above, there exists for each $1 \leq i \leq m$ a unique pair of ι -permuted edges $e_i, e'_i \in E(G)$ such that $a \in e_i$, $b \in e'_i$ and the other endpoints of e_i, e'_i lie in Γ_i .*

Proof. Existence holds because G is connected. Uniqueness holds because if not, then let d_i, d'_i be another set. Let $r_i \in e_i \cap \Gamma_i$ and $s_i \in d_i \cap \Gamma_i$, and r'_i, s'_i similarly defined for e'_i, d'_i . Since Γ_i/ι is connected, there is a path p_i between the quotient vertices $\{r_i, r'_i\}$ and $\{s_i, s'_i\}$. It follows that $\{e_i, e'_i\} p_i \{d_i, d'_i\}$ is a cycle in G/ι . \square

The following Lemma will be the main tool in the inductive step of our main Theorem on hyperelliptic graphs.

Lemma 3.5. *With $(G, \iota), a, b, G_{a,b}, \Gamma_i, e_i, e'_i$ as above, for all $1 \leq i \leq m$ let v_i be the unique vertex in $e_i \cap \Gamma_i$ and $v'_i \in e'_i \cap \Gamma_i$. If for all i there is an involutive embedding $\rho_i : \Gamma_i \rightarrow \mathbb{R}^2$ such that*

- (1) $\rho_i(v_i)$ (and thus also $\rho_i(v'_i)$) is on the outside face of ρ_i , and
- (2) the x value of $\rho_i(v_i)$ is ≤ 0 ,

then there is an involutive embedding $\rho : G \rightarrow \mathbb{R}^2$.

Proof. If so, then $\zeta_i := \tau_{1-2i}\alpha\rho_i$ embeds each Γ_i into the product of intervals $(-1, 1) \times (2i - 2, 2i)$ with v_i sent to the outside face in the left half plane. Let M_i be the maximum over $(-1, 0]$ of the x -values of $\zeta_i(\Gamma_i)$. Since G has no horizontal edges, M_i is zero if and only if Γ_i has a fixed vertex by Lemma 3.3. Consider the set

$$\begin{aligned} \Sigma &= \bigcup_{i=1}^m (\zeta_i(\Gamma_i) \cup \{(M_i, t) : t \in [2i - 2, 2i]\}) \\ &\cup \bigcup_{j=0}^m (\{(t, 2j(t + 2)) : t \in [-2, -1]\} \cup \{(t, 2j) : t \in [-1, 0]\}). \end{aligned}$$

Let D_i° be the connected component of the point $(-3/2, i - 1/2)$ in $\mathbb{R}^2 - \Sigma$ and note that the closure D_i is compact with connected, piecewise smooth boundary. Let $D_i^* = D_i^\circ \cup \{\zeta_i(v_i), (-2, 0)\}$ and note that the only point of intersection between any D_i^* is the point $(-2, 0)$. Let ϵ_i be a simple piecewise smooth path from $(-2, 0)$ to $\zeta_i(v_i)$ contained within D_i^* . Let ϵ'_i be the image of ϵ_i under the map $(x, y) \mapsto (-x, y)$.

We therefore have an embedding ρ defined as follows:

- $\rho(a) = (-2, 0)$ and $\rho(b) = (2, 0)$.
- For all $1 \leq i \leq m$, $\rho(e_i) = \epsilon_i$, $\rho(e'_i) = \epsilon'_i$.
- If r is either a vertex or an edge of $G_{a,b}$, then $r \in \Gamma_i$ for exactly one i . For this i , $\rho(r) = \zeta_i(r)$.

We see that each edge is piecewise smooth, and that ρ is involutive because each ζ_i is involutive. □

4. Planarity and toroidality of graphs with involutions

Theorem 4.1. *All hyperelliptic graphs are planar.*

Proof. Let (G, ι) be a hyperelliptic connected graph, which without loss of generality has no loops and is 2-edge connected. Let F be the set of vertices of G fixed by ι and let $A = A(G)$ be a maximal collection of vertices of G which are not equivalent under ι , $B = \iota(A)$. By Lemma 2.1, we may assume that G has no horizontal edges. The method of proof will be to show that one can choose the partition $V(G) = A \cup B \cup F$ in such a way that there are no cross edges.

Let $Ind(n)$ be the statement that for all connected hyperelliptic graphs H with $\#A(H) \leq n$, there is an involutive embedding of H . By Lemma 3.2, $Ind(0)$ is true. If we can show $Ind(n)$ holds for all n then our proof will be complete.

Suppose that $Ind(n)$ holds, $\#A = n + 1$, and let $a \in A$, $b = \iota(a)$. Let $G_{a,b} = (V(G) - \{a, b\}, \{e \in E(G) : a \notin e \text{ and } b \notin e\})$, and we let $\{\Gamma_i\}_{i=1}^m$ be the ι -orbits of connected components of $G_{a,b}$.

If Γ_i is connected, then the action of ι on Γ_i makes Γ_i into a connected hyperelliptic graph with at most n vertices exchanged by ι , so by $Ind(n)$, there is an involutive embedding $\bar{\rho}_i$ of Γ_i .

Since Γ_i/ι is connected, there are at most two connected components of Γ_i . If Γ_i is not connected, call them C_i, C'_i . Lemma 3.3 shows $C_i \cong C'_i \cong \Gamma_i/\iota$. We may therefore take $\bar{\rho}_i|_{C_i}$ to be any embedding of C_i into $\{(x, y) : x < 0\}$, and $\bar{\rho}_i|_{C'_i}$ such that $\bar{\rho}_i C'_i = \bar{\rho}_i \iota C_i$ is the image of $\bar{\rho}_i C_i$ under the map $(x, y) \mapsto (-x, y)$.

Therefore we have involutive embeddings of each Γ_i . By Lemma 3.4, there is a unique edge e_i from a to Γ_i and e'_i from b to Γ_i with respective other endpoints v_i, v'_i . Since we have assumed that G has no horizontal edges, we recall that C (the set of cross edges) is now nothing more than the set of edges from A to B .

We may then define a function $\psi_{a,b} : \{1, \dots, m\} \rightarrow \{0, 1\}$ by

$$\psi_{a,b}(i) = \begin{cases} 1 & e_i \in C \\ 0 & \text{else} \end{cases},$$

with respect to the partition of $V(G)$ as $A \cup B \cup F$. As $a \in e_i$, this means that $\psi_{a,b}(i) = 1$ if and only if there is a vertex of B lying in e_i . We will say that ι^0 is the identity map on G , and so the function $\iota^{\psi_{a,b}(i)}$ either fixes or flips Γ_i based on the partition of $V(G)$.

Finally, for each i , we pick a face F_i of $\bar{\rho}_i$ such that v_i lies in the boundary of F_i . We will pick for all i a real number y_i such that the point $(0, y_i)$ lies in the interior face F_i° , by Lemma 2.2. It follows that $\rho_i := \sigma_{y_i} \bar{\rho}_i \iota^{\psi_{a,b}(i)}$ is an involutive embedding of Γ_i such that $\rho_i(v_i)$ has x -value ≤ 0 and lies on the boundary of the outside face. We may therefore apply Lemma 3.5 to produce an involutive embedding of (G, ι) .

We have therefore shown that $Ind(n)$ implies $Ind(n+1)$, completing our induction. \square

We give a second proof of the planarity of hyperelliptic graphs.

Proof. By work of de Bruyn and Gijswijt [7], we know that for all graphs G , the stable gonality of G is bounded below by the treewidth of G . We know that G is hyperelliptic if and only if the stable gonality is 2. Since G is hyperelliptic, we find that it has treewidth 2, and therefore is a subgraph of a series-parallel graph [4], and is therefore planar. \square

There is also a third proof of this result due to Spencer Backman which characterizes the ear decomposition of a hyperelliptic graph and which pre-dates work of de Bruyn and Gijswijt but was not written up. While it may not seem so, these proofs work out to being very similar. Since G is hyperelliptic, G/ι is a tree. We may think of the inductive proof as rooting that tree and thus realizing it as a series-parallel graph. Note that our embedding ρ_G gives G/ι as $\rho_G(G) \cap \{(x, y) : x \leq 0\}$, so the source and sink vertices are respectively the a and b that we remove from G in the inductive step. The advantage of working so explicitly is that some natural improvements present themselves.

Lemma 4.2. *Suppose that (G, ι) is a hyperelliptic graph, $v_1 \neq v_2$ are vertices of G/ι . Then there is an involutive embedding $\rho : G \rightarrow (-1, 1)^2$ such that all vertices of G in the preimage of $\{v_1, v_2\}$ lie on a common face of ρ .*

Proof. We proceed by induction. To fix notation analogous to that of earlier sections, we will let $\bar{\rho}$ be an involutive embedding of (G, ι) constructed as in the proof of Theorem 4.1. Let A be the set of vertices of G with negative x -value, B positive, and F zero. Let $Ind'(n)$ be considered verified when the conditions of the lemma are verified for all hyperelliptic graphs G with $\#A = \#B \leq n$.

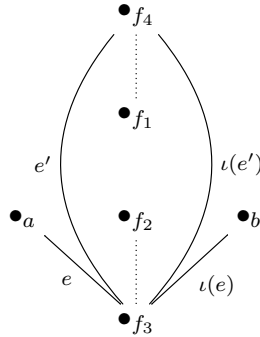


FIGURE 3. A depiction of a portion of an involutive embedding with fixed vertices sharing a face.

The number of vertices in the preimage of $\{v_1, v_2\}$ can be either 2, 3, or 4, and this will effect how we show $Ind'(n)$ implies $Ind'(n+1)$. In the latter two cases, without loss of generality there will be a pair (a, b) such that $a \in A, b \in B$ in the preimage of v_1 . As before, we let $\{\Gamma_i\}$ be the ι -orbits of connected components of $G_{a,b}$, with Γ_1 containing the preimages of v_2 . We let v be the unique vertex in Γ_1 connected to a , and note that $v \notin B$.

We find involutive embeddings ρ_2, \dots, ρ_m of $\Gamma_2, \dots, \Gamma_m$ keeping the x values of A negative and the vertices connecting to a, b on the outside face as in the proof of Theorem 4.1. We find ρ_1 of Γ_1 with both v and the preimages of v_2 on the outside face of Γ_1 by $Ind'(n)$. We then draw the edges between a and the Γ_i, b and the Γ_i to get an involutive embedding ρ of G . Then any face of ρ containing a preimage of v_2 and contained in the outside face of ρ_1 also contains a, b . This shows that ρ satisfies the conditions of $Ind'(n+1)$.

If on the other hand the preimage of $\{v_1, v_2\}$ contains only two vertices, both must be fixed under ι . Let f_1, f_2 be these vertices.

Suppose first that f_1, f_2 lie on two different connected components of $(F, E_F) \subset G$. If this is the case, note that G/ι is a tree, and so there is a unique simple path of edges and vertices from v_1 to v_2 in G/ι . If we remove any $a \in A$ which lies in the preimage of that path, then f_1, f_2 lie in distinct ι -orbits of connected components in $G_{a,b}$ where $b = \iota(a)$. Order the ι -orbits $\Gamma_1, \dots, \Gamma_m$ so that $f_1 \in \Gamma_1, f_2 \in \Gamma_m$. By Theorem 4.1 or Lemma 3.3 there is an involutive embedding of Γ_i into the product of intervals $[-1, 1] \times [2i - 2, 2i]$ with the vertex connecting a to Γ_i and thus Γ_i to b on the outside face. Therefore (modulo flipping Γ_1 or Γ_m upside down) drawing symmetric edges to a and b produces an involutive embedding of G with f_1, f_2 on the outside face.

Finally, suppose that f_1, f_2 lie on the same connected component K of (F, E_F) . Let e be an edge with one vertex in K and another, $a \in A$. Let

$b = \iota(a)$ and let f_3 be the other endpoint of e . Since $f_1 \neq f_2$, the number of vertices of K is at least 2, and so there is an edge $e' \in K$ connected to f_3 and its conjugate $\iota(e') \neq e'$. Let f_4 be the other endpoint of $e', \iota(e')$. Let $\Gamma_1, \dots, \Gamma_m$ be the ι -orbits of connected components of $G_{a,b}$, and reorder them so that $\Gamma_1 \supset K$. There are involutive embeddings of $\Gamma_2, \dots, \Gamma_m$. We pick them so that the connecting vertices to a, b are on the outside and the signs of the x -coordinates are the same as for $\bar{\rho}$. By $Ind'(n)$ there is an involutive embedding of Γ_1 keeping f_1, f_2 on the outside face F_∞ . Let the embedding of Γ_i be denoted ρ_i , and let $\gamma = \rho_1(e') \cup \rho_1(\iota(e'))$. Therefore f_1, f_2 do not lie inside of γ (in the sense of differential topology [9, §3.3]). Note that ρ_1 takes F and no other part of G to the line $\{0\} \times \mathbb{R}$. We can, without loss of generality, say that $\rho_1(f_4) = (0, y_4)$ has a higher y -value than $\rho_1(f_3) = (0, y_3)$. There is thus a bounded open interval $I = \{(0, y) : y_3 < y < y'_3 \leq y_4\}$ where $I \cap \rho_1(\Gamma_1)$ is empty. Since $\rho_1(\Gamma_1)$ is compact, there is an open subset U of \mathbb{R}^2 containing I and not intersecting $\rho_1(\Gamma_1)$. Therefore, if $y_3 < y < y'_3$ then $\alpha\sigma_y\rho_1$ is an involutive embedding with a face $\alpha\sigma_y(F_\infty)$ which is not outside of $\alpha\sigma_y(\gamma)$, and such that the outside face of $\alpha\sigma_y\rho_1$ contains $\alpha\sigma_y(U)$, thus its closure, and thus $\alpha\sigma_y(\rho_1(f_3))$. Therefore we may construct our involutive embedding ρ of G without altering the face $\alpha\sigma_y F_\infty$ containing f_1, f_2 . See Figure 3 for a depiction.

Having established that $Ind'(n)$ implies $Ind'(n+1)$, we note that if $\#A = 0$ then we have an embedding where all vertices lie on the outside face by Lemma 3.2. All other cases follow from our inductive assumption above. \square

Theorem 4.3. *Bielliptic graphs are toroidal.*

Proof. Let (G, ι) be a bielliptic graph. Without loss of generality, we assume G is 2-edge connected, and that the genus of G is at least 3, else G is already planar.

Since G/ι has genus one, there is an edge \bar{e} of G/ι such that $G/\iota - \bar{e}$ is a tree. Let $e, e' = \iota(e)$ be the preimages of \bar{e} in G and let $G_0 = G - \{e, e'\}$ with ι_0 the induced involution, whose quotient is $G/\iota - \bar{e}$. Let v_1, v_2 be the vertices in \bar{e} .

By Lemma 4.2 there is an involutive embedding $\rho_0 : G_0 \rightarrow (-1, 1)^2$ such that all pre-images of v_1, v_2 lie on the outside face of ρ_0 . We let A be the set of vertices of G_0 (and thus G) which get mapped under ρ_0 to the left half plane $\{(x, y) : x < 0\}$. For this choice of $A, B = \iota(A)$, there are no cross edges of G_0 . Let m be the minimum x value such that $(x, 0) \in \rho_0(G_0)$.

If e (and thus e') is not a cross edge, then G is still planar by drawing e in the left half plane and e' in the right. Let Σ be the set

$$\rho_0(G_0) \cup \{(t, \pm 1) : -1 \leq t \leq 0\} \cup \{(-1, t) : -1 \leq t \leq 1\} \cup \{(0, t) : -1 \leq t \leq 1\}.$$

Let u_1, u_2 respectively be preimages of v_1, v_2 such that $\rho_0(u_i)$ each have x -coordinates in $(-1, 0]$. For any x_0 in the interval $(-1, m)$, let D° be the connected component of the point $(m, 0)$ in $\mathbb{R}^2 - \Sigma_0$ and let D be $D^\circ \cup \{\rho_0(u_1), \rho_0(u_2)\}$. Let ϵ be any non-self-intersecting piecewise smooth path

from $\rho_0(u_1)$ to $\rho_0(u_2)$ in D and e' be the image of e under the map $(x, y) \mapsto (-x, y)$. We therefore find an involutive embedding $\rho : G \rightarrow \mathbb{R}^2$ such that $\rho = \rho_0$ away from e, e' , $\rho(e) = e$, $\rho(e') = e'$.

Now suppose that e is a cross edge, and thus the same is true for e' . Let u_1 be a preimage of v_1 with strictly negative x -value, $u'_1 = \iota u_1$ and likewise for u_2, u'_2 . We say $e \ni u_1, u'_2$ and $e' \ni u'_1, u_2$ without loss of generality.

Note that if we build ρ_0 inductively as in Lemma 4.2, then always the y -value of $\rho_0(u_1)$ is strictly less than the y -value of $\rho_0(u_2)$. With the application of some τ_y , we will assume that $\rho_0(u_1)$ has negative y -value and $\rho_0(u_2)$ has positive y -value. That is to say, each vertex in a preimage of \bar{e} is mapped to a different quadrant under ρ_0 . We will let

$$Q_N = \rho_0(u_2), Q_S = \rho_0(u'_1), Q_E = \rho_0(u'_2), Q_W = \rho_0(u_1),$$

and consider the points

$$R_N = (0, 1), R_S = (0, -1), R_E = (1, 0), R_W = (-1, 0).$$

Suppose can create paths γ_* between Q_* and R_* which do not intersect each other and do not meet $\rho_0(G_0)$ aside from Q_* . Then we may identify points on the boundary of $[-1, 1]^2$ to create the torus

$$\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T} = \frac{[-1, 1]^2}{\{(-1, y) \sim (1, y), (x, -1) \sim (x, 1) : x, y \in [-1, 1]\}}.$$

Then we have an embedding $\rho : G \rightarrow \mathbb{T}$ by $\rho(G_0) = \rho_0(G_0) \subset \mathbb{T}$ and $\rho(e) = \gamma_N \cup \gamma_S, \rho(e') = \gamma_E \cup \gamma_W$.

We therefore create the γ_* to complete the proof. Let Σ be the set

$$\rho_0(G_0) \cup \{(s, t), (t, s) : s \in \{0, \pm 1\}, -1 \leq t \leq 1\}.$$

We will also let P_N, P_S, P_E, P_W be points on the outside face of ρ_0 such that

$$P_N \in (-1, 0) \times (0, 1), \quad P_S \in (0, 1) \times (-1, 0),$$

and

$$P_E \in (0, 1) \times (0, 1), \quad P_W \in (-1, 0) \times (-1, 0).$$

Let U_* be the connected component of P_* in $\mathbb{R}^2 - \Sigma$ and then let $D_* = U_* \cup \{Q_*, R_*\}$. We may then take γ_* to be a piecewise smooth path between Q_* and R_* in D_* . □

Example. *The complete bipartite graph $G = K_{3,3}$ is well-known to be bi-elliptic. If $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ are the vertices of G , and the edges of G are only between the a 's and b 's, then there is an involution ι taking a_i to b_i and vice versa. Let us see how to use this involution to realize G as toroidal. The quotient G/ι is the cycle on 3 vertices v_1, v_2, v_3 . We refine the horizontal edges away by introducing 3 fixed vertices f_1, f_2, f_3 , producing a G' . Then from G' we obtain a hyperelliptic graph G_0 by removing the preimages of the edge between v_1 and v_3 . See Figure 4 for a depiction of these graphs.*

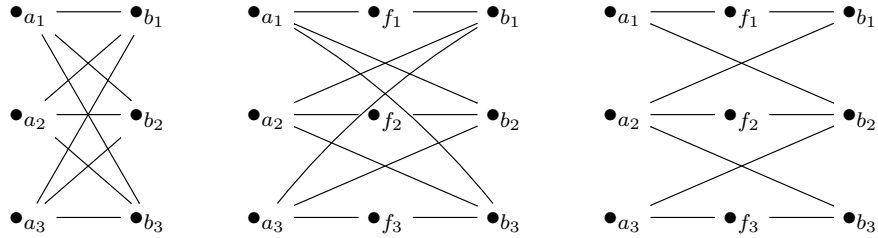


FIGURE 4. The complete graph $K_{3,3}$ with a refinement G' and a hyperelliptic $G_0 \subset G'$

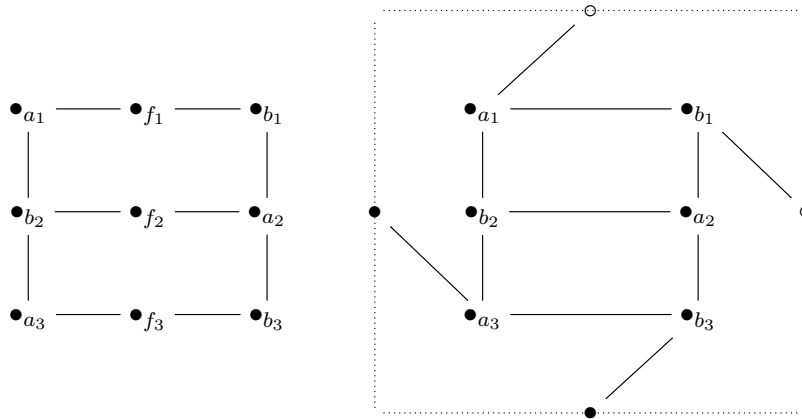


FIGURE 5. A planar embedding of G_0 gives a toroidal embedding of $G \cong K_{3,3}$

We may then use Theorem 4.1 to embed G_0 into the product of intervals $(-1,1)^2$. We may then re-insert the edges of G to obtain a toroidal embedding, as depicted in Figure 5.

One could imagine extending this to the case where G/ι has genus g , e.g. by removing g edges from G/ι and embedding the analogous G_0 into a $4g$ -gon in the plane, or something similar. That would depend on finding a sequence of points interchanged by ι which sequentially lie on common faces. This fails however, as we see in Figure 6 where G/ι has genus 2.

Nonetheless we note that this graph does indeed admit an embedding into a genus two surface! In particular, we get slightly lucky in that the above method shows how to embed this graph into the connected sum of two tori, albeit in a way that does not obviously generalize. Indeed, we do not know of an example of a graph with a mixing involution ι to a genus g graph which does not already embed into a genus g orientable surface.

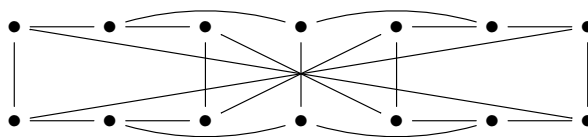


FIGURE 6. A graph with mixing involution quotient of Euler genus 2.

We conclude by noting that although our criterion for being toroidal has *something* to do with gonality, there is more that goes into the orientable genus than the gonality.

Theorem 4.4. *There are trigonal graphs of all possible orientable genera. Moreover, there are d -gonal graphs which are either planar or of all possible orientable genera $\geq (\frac{d}{2} - 1)^2$ whenever $d \not\equiv 2 \pmod 4$.*

Proof. First we note that there are d -gonal planar graphs for all d - simply take $n \geq d$ and note that the $d \times n$ grid graph has gonality d [7, Example 3.3].

Then note that for $3 \leq d \leq n$, the complete bipartite graph $K_{d,n}$ has orientable genus $\lceil \frac{(d-2)(n-2)}{4} \rceil$ [14]. If d is not 2 mod 4 then this can be any integral value at least $(\frac{d}{2} - 1)^2$. If $d \equiv 2 \pmod 4$ then the orientable genus can be any integer multiple of $\frac{d-2}{4}$ at least $(\frac{d}{2} - 1)^2$. We see that there is a clear degree d harmonic map from $K_{d,n}$ to a tree given by simply identifying the vertices in the size d subset. Therefore the gonality of $K_{d,n}$ is at most d . For the lower bound, note that the treewidth of $K_{d,n}$ is d , so this is a lower bound for gonality [7], and we find that $K_{d,n}$ is d -gonal. \square

We conclude by noting that in the complete bipartite graphs above, gonality, stable gonality, and treewidth all coincide. It is conjectured for the hypercube graph Q_n that there is a gap between the two which increases along with n [7, §3]. In that case, the orientable genus is large and the conjectural least degree map to a tree is given by successive quotients by involutions $Q_n \rightarrow Q_{n-1}$. Finally we give some open questions.

- Are there any other infinite families of graphs with gaps between gonality and treewidth which also have large orientable genus?
- What is the connection between the orientable genus and the spectrum of the Laplacian?

We think the latter ought to be better understood. After all, the spectrum of the $d \times n$ grid graph is very limited [8]: the eigenvalues can only be

$$\lambda_{j,k} = 4 \sin^2 \left(\frac{j\pi}{2n} \right) + 4 \sin^2 \left(\frac{k\pi}{2d} \right).$$

In particular, the spectral lower bound on gonality [6, Theorem C] for this example tends to 0 as $n \rightarrow \infty$.

5. Hyperelliptic graphs associated to Shimura curves

A graph theorist who has made their way through this paper might be filled with questions. Why should a tree correspond to \mathbb{P}^1 ? Why should any curve tell us about a graph or vice versa? It is difficult to explain without some algebraic geometry, but briefly, if X is a curve over a number field K , which is the fraction field of a valuation ring R with residue field k , then consider \mathcal{X} over R to be a regular semistable model for X .

Note that the reduction of \mathcal{X} need not be smooth, nor even irreducible. Let (C_1, \dots, C_v) be the irreducible components of $\mathcal{X}_{\bar{k}}$ and (P_1, \dots, P_e) be the set of singular points. Since \mathcal{X} is a semistable model, the only possible singularities are ordinary double points and each P_j is the intersection of at most two components. We form a graph from this data by associating to each C_i a vertex V_i and to each P_j at the intersection of components C_m, C_n , we associate an edge E_j between V_m and V_n . We need not work in the semistable case [11, Introduction] but this greatly simplifies the description. A great deal of arithmetic and geometry of X can be contained in the **dual graph** $G(\mathcal{X})$ produced above.

When X has a semistable model \mathcal{X} such that for $1 \leq i \leq v$, $C_i \cong \mathbb{P}_{\bar{k}}^1$, we have the most information transferred to the graph $G(\mathcal{X})$. This is the case known as “totally degenerate reduction.” For instance, if X is a curve with totally degenerate reduction then the genus of X is the (Euler) genus of $G(\mathcal{X})$, so X with totally degenerate reduction has genus zero if and only if $G(\mathcal{X})$ is a tree. For any graph G it is possible to find a curve $X(G)$ with totally degenerate reduction and G as the dual graph [2, Appendix B].

There is a very special set of curves for certain squarefree numbers D with totally degenerate reduction modulo $p \mid D$ called Shimura curves X^D . The differentials of these curves can be given by certain cuspidal modular forms for the congruence subgroup $\Gamma_0(D)$ of $\mathrm{SL}_2(\mathbb{Z})$, and the adjacency matrix for the dual graph at a prime $p \mid D$ is the matrix for the Hecke operator T_p acting on a certain subspace of modular forms for $\Gamma_0(D/p)$. The resulting graph is $(p+1)$ -regular and Ramanujan since cusp forms obey the Ramanujan bound. Families of Shimura curves therefore give prototypical families of expander graphs.

Ogg has determined all Shimura curves X^D which are hyperelliptic over $\overline{\mathbb{Q}}$ [13]. In particular, note that in each case D is the product of two primes and so there are only two primes of bad reduction to explore. In each case, the dual graph is also hyperelliptic. The following magma code verifies that all of these dual graphs are planar.

```
Dlist := [26,35,38,39,51,55,57,58,62,69,74,
82,86,87,93,94,95,111,119,134,146,159,194,206];
// Ogg's list of Shimura curves hyperelliptic over QQbar
del := function(x)
if x eq 0 then return 0;
```

```

else return 1;
end if;
end function;
ReducedDualGraph := function(p,q)
// Returns in magma format the dual graph of  $X^{\{pq\}}$  over  $\overline{\mathbb{F}p}$ 
// Rather, the "reduced dual graph" with parallel edges collapsed
M := BrandtModule(q,1);
d := Dimension(M);
Mx := MatrixRing(Integers(),d);
Bx := Mx!HeckeOperator(M,p);
for i in [1..d] do for j in [1..d] do
Bx[i,j] := del(Bx[i,j]);
end for; end for;
return Graph<2*Dimension(M)|BlockMatrix(2,2,[[Mx!0,Bx],[Bx,Mx!0]]>;
end function;
for D in Dlist do
G1 := ReducedDualGraph(PrimeDivisors(D)[1],PrimeDivisors(D)[2]);
G2 := ReducedDualGraph(PrimeDivisors(D)[2],PrimeDivisors(D)[1]);
D,IsPlanar(G1),IsPlanar(G2);
end for;

```

Similar lists exist for, e.g., bielliptic Shimura curves, each of which has $D \leq 546$. Similar code to the above suggests that if X^D has a dual graph (of its reduction modulo p for $p \mid D$) which is planar and has at least six vertices, then for $D \geq 500$ the complete list of (D, p) is

p	2	3	5	7	11	13	29
D	510, 546	510, 570, 690	690, 910, 1110	798, 910	1122	1365	667, 2958

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