

Combinatorial bases of principal subspaces of modules for twisted affine Lie algebras of type $A_{2l-1}^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$

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ABSTRACT. We construct combinatorial bases of principal subspaces of standard modules of level $k \geq 1$ with highest weight $k\Lambda_0$ for the twisted affine Lie algebras of type $A_{2l-1}^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$. Using these bases we directly calculate characters of principal subspaces.

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1. Introduction

The study of principal subspaces of standard modules for untwisted affine Lie algebras was initiated in [FS] by Feigin and Stoyanovsky and was later extended by Georgiev in [G]. Motivated by the work of J. Lepowsky and M. Primc in [LP] and extending the earlier work of Feigin and Stoyanovsky, Georgiev constructed bases of principal subspaces of certain standard modules of the affine Lie algebra of type $A_l^{(1)}$. These bases were described by using certain coefficients of vertex operators, which are called quasi-particles. From quasi-particle bases, they directly obtained the characters (i.e. multi-graded dimensions) of principal subspaces. The work of Feigin and Stoyanovsky and Georgiev has since been extended in many ways by other authors (cf. [Ar], [Ba], [BPT], [Bu1]–[Bu3], [FFJMM], [J1]–[J2], [JP], [Kan], [Kaw1]–[Kaw2], [Ko1]–[Ko3], [MPe], [P], [T1]–[T3], and many others).

The study of principal subspaces of basic modules for twisted affine Lie algebras was initiated in [CalLM4], where a general setting was given and the principal subspace of the basic $A_2^{(2)}$ -module was studied. This work was later extended in [CalMPe] and [PS1]–[PS2] to study the principal subspaces of the basic modules for all the twisted affine Lie algebras, and in [PSW] to a certain lattice setting. In each of these works the authors, using certain ideas from the untwisted affine Lie algebra setting found in [CapLM1]–[CapLM2] and [CalLM1]–[CalLM3] (see also [Cal1]–[Cal2], [S1]–[S2]), showed that the principal subspaces under consideration had certain presentations (i.e. could be defined in terms of certain generators and relations). Using these presentations, the authors constructed exact sequences among the principal subspaces and in this way obtained recursions satisfied by the characters of principal subspaces. Solving these recursions yields the characters of the principal subspaces of the basic modules for the twisted affine Lie algebras in each work.

Let ν be a Dynkin diagram automorphism of order v of a Lie algebra of type X where X is A_{2l-1} with $l \geq 2$, D_l with $l \geq 4$, or E_6 . Denote by $\hat{\nu}$ the lifting of ν to the basic $X^{(1)}$ -module $V_L \simeq L(\Lambda_0)$ constructed as a vertex operator algebra (cf [LL]) and the twisted V_L -module V_L^T , which is isomorphic to the basic vacuum module, which we denote $L^{\hat{\nu}}(\Lambda_0)$, of the twisted affine Lie algebra $X^{(v)}$ (cf. [L1], [CalLM4]). The aim of this work is to determine the characters of the principal subspaces of the level $k \geq 1$ vacuum $X^{(v)}$ -module $L^{\hat{\nu}}(k\Lambda_0)$, which we denote by $W_{L_k}^T$, for the twisted affine Lie algebras of type $A_{2l-1}^{(2)}$, for $l \geq 2$, $D_l^{(2)}$, for $l \geq 4$, $E_6^{(2)}$ and $D_4^{(3)}$, extending certain results found in [PS1]–[PS2]. The approach we use, however, is different than the approach found in [PS1]–[PS2]. By using vertex operator techniques we construct combinatorial bases of principal subspaces which are twisted analogues to those found in [G], and from which we obtain the characters of principal subspaces. In our proofs, we use level k analogues of certain maps originally developed and used in [CalLM4], [CalMPe], and

[PS1]–[PS2] analogously to how they were used in [G]. We note importantly that, in the untwisted setting, certain modes of intertwining operators play an important role in the proofs of linear independence of these bases. In our twisted affine Lie algebra setting, we instead use other maps developed for the twisted setting in [CaLM4].

More specifically, in this paper, following [G] and using certain results in [Li], we construct bases using the coefficients

$$x_{r\alpha_i}^{\hat{\nu}}(m) = \text{Res}_z \{ z^{m+r-1} x_{r\alpha_i}^{\hat{\nu}}(z) \},$$

of the twisted vertex operators

$$x_{r\alpha_i}^{\hat{\nu}}(z) = Y^{\hat{\nu}}(x_{\alpha_i}(-1)^r \mathbf{1}, z),$$

which we, in complete analogy with the untwisted case, call twisted quasi-particles of color i , charge r and energy $-m$. Similar to the untwisted case, (see [Bu1]–[Bu3], [JP]), first we prove certain relations for twisted quasi-particles of the form $x_{r\alpha_i}^{\hat{\nu}}(m)x_{r'\alpha_i}^{\hat{\nu}}(m')$ for $r \leq r'$ and $x_{r\alpha_i}^{\hat{\nu}}(m)x_{r'\alpha_j}^{\hat{\nu}}(m')$, where $1 \leq r, r' \leq k$, which we call relations among twisted quasi-particles. With these relations, along with the relations $x_{(k+1)\alpha_i}^{\hat{\nu}}(z) = 0$, we build twisted quasi-particle spanning sets of the principal subspaces $W_{L_k}^T$. The resulting bases are analogous to the quasi-particle bases of principal subspaces in the case of untwisted affine Lie algebras of type ADE in the sense that energies of twisted quasi-particles in the twisted quasi-particle spanning sets satisfy similar difference conditions, which are generalizations of difference two conditions found in [FS] and [G]. As in the untwisted case found in [G], in the proof of linear independence of our spanning sets we consider the principal subspace as a subspace of tensor product of k principal subspaces of basic modules. This enables us to use the above-mentioned maps obtained from the construction of level one twisted modules for lattice vertex operator algebras from [CaLM4] and [PS1]–[PS2]. Finally, we note that in this paper we do not consider the case of the principal subspaces $W_{L_k}^T$ for the twisted affine Lie algebra of type $A_{2l}^{(2)}$, which will be considered in future work.

Our main result in this work is as follows: denote by $\text{ch } W_{L_k}^T$ the character of the principal subspace $W_{L_k}^T$. The characters of the principal subspaces are:

Theorem 1.1. *We have for $A_{2l-1}^{(2)}$:*

$$\text{ch } W_{L_k}^T = \sum_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ \dots \\ r_{l-1}^{(1)} \geq \dots \geq r_{l-1}^{(k)} \geq 0}} \frac{q^{\frac{1}{2} \sum_{i=1}^{l-1} \sum_{s=1}^k r_i^{(s)2} - \frac{1}{2} \sum_{i=2}^{l-1} \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}}}{\prod_{i=1}^{l-1} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(1)} - r_i^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(k)}}}} \prod_{i=1}^{l-1} y_i^{r_i^{(1)} + \dots + r_i^{(k)}}$$

$$\cdot \sum_{r_l^{(1)} \geq \dots \geq r_l^{(k)} \geq 0} \frac{q^{\sum_{s=1}^k r_l^{(s)2} - \sum_{s=1}^k r_{l-1}^{(s)} r_l^{(s)}}}{(q)_{r_l^{(1)} - r_l^{(2)}} \cdots (q)_{r_l^{(k)}}} y_l^{r_l^{(1)} + \dots + r_l^{(k)}}$$

for $D_l^{(2)}$:

$$\begin{aligned} \text{ch } W_{L_k}^T &= \sum_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ \dots \\ r_{l-2}^{(1)} \geq \dots \geq r_{l-2}^{(k)} \geq 0}} \frac{q^{\sum_{i=1}^{l-2} \sum_{s=1}^k r_i^{(s)2} - \sum_{i=2}^{l-2} \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}}}{\prod_{i=1}^{l-2} (q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q)_{r_i^{(k)}}}} \prod_{i=1}^{l-2} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \\ &\cdot \sum_{r_{l-1}^{(1)} \geq \dots \geq r_{l-1}^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{s=1}^k r_{l-1}^{(s)2} - \sum_{s=1}^k r_{l-2}^{(s)} r_{l-1}^{(s)}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(1)} - r_{l-1}^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(k)}}}} y_{l-1}^{r_{l-1}^{(1)} + \dots + r_{l-1}^{(k)}} \end{aligned}$$

for $E_6^{(2)}$:

$$\begin{aligned} &\text{ch } W_{L_k}^T = \\ &= \sum_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ r_2^{(1)} \geq \dots \geq r_2^{(k)} \geq 0}} \frac{q^{\frac{1}{2} \sum_{i=1}^2 \sum_{s=1}^k r_i^{(s)2} - \sum_{s=1}^k r_1^{(s)} r_2^{(s)}}}{\prod_{i=1,2} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(1)} - r_i^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(k)}}}} \prod_{i=1,2} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \\ &\cdot \sum_{\substack{r_3^{(1)} \geq \dots \geq r_3^{(k)} \geq 0 \\ r_4^{(1)} \geq \dots \geq r_4^{(k)} \geq 0}} \frac{q^{\sum_{i=3}^4 \sum_{s=1}^k r_i^{(s)2} - \sum_{s=1}^k (r_2^{(s)} r_3^{(s)} + r_3^{(s)} r_4^{(s)})}}{\prod_{i=3,4} (q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q)_{r_i^{(k)}}}} \prod_{i=3,4} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \end{aligned}$$

and for $D_4^{(3)}$:

$$\begin{aligned} \text{ch } W_{L_k}^T &= \sum_{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0} \frac{q^{\frac{1}{3} \sum_{s=1}^k r_1^{(s)2}}}{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_{r_1^{(1)} - r_1^{(2)}} \cdots (q^{\frac{1}{3}}; q^{\frac{1}{3}})_{r_1^{(k)}}}} y_1^{r_1^{(1)} + \dots + r_1^{(k)}} \\ &\cdot \sum_{r_2^{(1)} \geq \dots \geq r_2^{(k)} \geq 0} \frac{q^{\sum_{s=1}^k r_2^{(s)2} - \sum_{s=1}^k r_1^{(s)} r_2^{(s)}}}{(q)_{r_2^{(1)} - r_2^{(2)}} \cdots (q)_{r_2^{(k)}}}} y_2^{r_2^{(1)} + \dots + r_2^{(k)}}. \end{aligned}$$

2. Preliminaries

In this section, we very closely follow the setting developed in [PS1]–[PS2] (cf. also [L1] and [CalLM4]), and recall many details from these works.

Let \mathfrak{g} be a finite dimensional simple Lie algebra of type A_{2l-1} , D_l , or E_6 , with root lattice

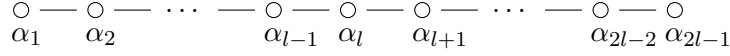
$$L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_D,$$

where D is the rank of \mathfrak{g} , with its standard nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Also, let

$$\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}.$$

We take the following labelings of the Dynkin diagrams of our Lie algebras:

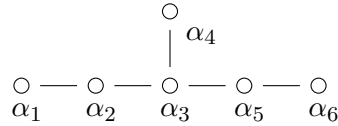
Type A_{2l-1} :



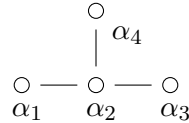
Type D_l :



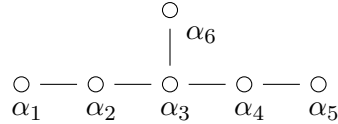
Type E_6 :



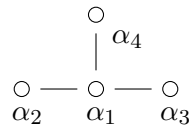
In the case of D_4 , we use the labeling:



Remark 2.1. We note here that in [PS2], the labeling used for E_6 was:



and in [PS1], the labeling used for D_4 was:



and we change the labeling in this work for notational simplicity. Later in the work, we will see that the roles of operators corresponding to α_4 and α_6 in the case of E_6 and the roles of operators corresponding to α_1 and α_2 in the case of D_4 will be swapped compared to their counterparts found in [PS2] and [PS1].

2.1. Dynkin diagram automorphisms. Let ν be a Dynkin diagram automorphism of \mathfrak{g} of order ν , extended to all of \mathfrak{h} . In the case that $\nu = 2$, we let $\eta = -1$ be a primitive second root of unity and set $\eta_0 = \eta$, and in the case that $\nu = 3$ we let η be a cube root of unity and set $\eta_0 = -\eta$. Following [CalLM4] and [PS1]–[PS2], we consider two central extensions of L by the group $\langle \eta_0 \rangle$, denoted by \hat{L} and \hat{L}_ν , with commutator maps C_0 and C and associated normalized 2-cocycles ϵ_{C_0} and ϵ_C , respectively:

$$1 \longrightarrow \langle \eta_0 \rangle \longrightarrow \hat{L} \xrightarrow{-} L \longrightarrow 1$$

and

$$1 \longrightarrow \langle \eta_0 \rangle \longrightarrow \hat{L}_\nu \xrightarrow{\bar{\nu}} L \longrightarrow 1$$

We define commutator maps C_0 and C by

$$\begin{aligned} C_0 : L \times L &\rightarrow \mathbb{C}^\times \\ (\alpha, \beta) &\mapsto (-1)^{\langle \alpha, \beta \rangle} \end{aligned}$$

and

$$C(\alpha, \beta) = \prod_{j=0}^{v-1} (-\eta^j)^{\langle \nu^j \alpha, \beta \rangle}.$$

Following [L1] and [CallM4], we let

$$\begin{aligned} e : L &\rightarrow \hat{L} \\ \alpha &\mapsto e_\alpha \end{aligned}$$

be a normalized section of \hat{L} so that

$$e_0 = 1$$

and

$$\overline{e_\alpha} = \alpha \quad \text{for all } \alpha \in L,$$

satisfying

$$e_\alpha e_\beta = \epsilon_{C_0}(\alpha, \beta) e_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in L.$$

We choose our 2-cocycle to be

$$\epsilon_{C_0}(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{if } i \leq j \\ (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{if } i > j \end{cases}$$

The 2-cocycles ϵ_C and ϵ_{C_0} are related by (see Equation 2.21 of [CallM4])

$$\epsilon_{C_0}(\alpha, \beta) = \prod_{-\frac{v}{2} < j < 0} (-\eta^{-j})^{\langle \nu^{-j} \alpha, \beta \rangle} \epsilon_C(\alpha, \beta).$$

We now lift the isometry ν of L to an automorphism $\hat{\nu}$ of \hat{L} such that

$$\overline{\hat{\nu}a} = \nu \overline{a} \quad \text{for } a \in \hat{L}.$$

and choose $\hat{\nu}$ so that

$$\hat{\nu}a = a \quad \text{if } \nu \overline{a} = \overline{a},$$

and thus $\hat{\nu}^2 = 1$ if ν has order 2 and $\hat{\nu}^3 = 1$ if ν has order 3. Indeed, set

$$\hat{\nu}e_\alpha = \psi(\alpha)e_{\nu\alpha}$$

where $\psi : L \rightarrow \langle \eta \rangle$ is defined by

$$\psi(\alpha) = \begin{cases} \epsilon_{C_0}(\alpha, \alpha) & \text{if } L \text{ is type } A_{2l-1} \\ 1 & \text{if } L \text{ is type } D_l \text{ and the order of } \nu \text{ is } 2 \\ (-1)^{r_3 r_4} \epsilon_{C_0}(\alpha, \alpha) & \text{if } L \text{ is type } E_6 \text{ and } \alpha = \sum_{i=1}^6 r_i \alpha_i \\ (-1)^{r_2 r_3} \epsilon_{C_0}(\alpha, \alpha) & \text{if } L \text{ is of type } D_4, \alpha = \sum_{i=1}^4 r_i \alpha_i \\ & \text{and the order of } \nu \text{ is } 3. \end{cases}$$

From [PS1]–[PS2], we have that:

$$\epsilon_{C_0}(\nu\alpha, \nu\beta) = \begin{cases} \epsilon_{C_0}(\beta, \alpha) & \text{if } L \text{ is type } A_{2l-1} \\ \epsilon_{C_0}(\alpha, \beta) & \text{if } L \text{ is type } D_l \\ (-1)^{r_4 s_3 + r_3 s_4} \epsilon_{C_0}(\beta, \alpha) & \text{if } L \text{ is type } E_6, \alpha = \sum_{i=1}^6 r_i \alpha_i \\ & \text{and } \beta = \sum_{i=1}^6 s_i \alpha_i \\ (-1)^{r_2 s_3 + r_3 s_2} \epsilon_{C_0}(\beta, \alpha) & \text{if } L \text{ is of type } D_4, \alpha = \sum_{i=1}^3 r_i \alpha_i \\ & \text{and } \beta = \sum_{i=1}^3 s_i \alpha_i. \end{cases}$$

As in [PS1]–[PS2], we have that

$$\hat{\nu}(e_{\alpha_i}) = e_{\nu\alpha_i}$$

for each simple root α_i .

2.2. The lattice vertex operator V_L and its twisted module V_L^T .

We assume that the reader is familiar with the construction of the lattice vertex operator algebra V_L (cf. [FLM1] and [LL]), and recall some important details of this construction. In particular, we follow Section 2 of [CalLM4].

We view \mathfrak{h} as an abelian Lie algebra, and let

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with the usual bracket, and let

$$\hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

We have that

$$V_L \cong S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L]$$

linearly. We extend $\hat{\nu}$ to an automorphism of V_L , which we also call $\hat{\nu}$, by $\hat{\nu} = \nu \otimes \hat{\nu}$.

Let

$$\mathfrak{h}_{(m)} = \{x \in \mathfrak{h} \mid \nu(x) = \eta^m x\}.$$

We have that

$$\mathfrak{h} = \coprod_{m \in \mathbb{Z}/v\mathbb{Z}} \mathfrak{h}_{(m)}.$$

We form the twisted affine Lie algebra

$$\hat{\mathfrak{h}}[\nu] = \coprod_{m \in \mathbb{Z}} \mathfrak{h}_{(m)} \otimes t^{m/v}$$

where

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} c$$

for $m, n \in \frac{1}{v}\mathbb{Z}$ and $\alpha \in \mathfrak{h}_{(vm)}$ and $\beta \in \mathfrak{h}_{(vn)}$ and c is central. The Lie algebra $\hat{\mathfrak{h}}[\nu]$ is $\frac{1}{v}\mathbb{Z}$ -graded by weights:

$$\text{wt}(\alpha \otimes t^m) = -m \quad \text{and} \quad \text{wt}(c) = 0.$$

Define the Heisenberg subalgebra

$$\hat{\mathfrak{h}}[\nu]_{\frac{1}{v}\mathbb{Z}} = \prod_{\substack{m \in \frac{1}{v}\mathbb{Z} \\ m \neq 0}} \mathfrak{h}_{(vm)} \otimes t^m \oplus \mathbb{C}c$$

of $\hat{\mathfrak{h}}[\nu]$, the subalgebras

$$\hat{\mathfrak{h}}[\nu]^{\pm} = \prod_{\substack{m \in \frac{1}{v}\mathbb{Z} \\ \pm m > 0}} \mathfrak{h}_{(vm)} \otimes t^m$$

of $\hat{\mathfrak{h}}[\nu]_{\frac{1}{v}\mathbb{Z}}$, and the induced module

$$S[\nu] = U\left(\hat{\mathfrak{h}}[\nu]\right) \otimes_{\prod_{m \geq 0} \mathfrak{h}_{(vm)} \otimes t^m \oplus \mathbb{C}c} \mathbb{C} \cong S\left(\hat{\mathfrak{h}}[\nu]^{-}\right),$$

which is \mathbb{Q} -graded such that

$$\text{wt}(1) = \frac{1}{4v^2} \sum_{j=1}^{v-1} j(v-j) \dim \mathfrak{h}_{(j)}.$$

Following [L1] and [CalLM4], we set

$$N = (1 - P_0)\mathfrak{h} \cap L,$$

where P_0 is the projection of \mathfrak{h} onto $\mathfrak{h}_{(0)}$. In particular, we define

$$\alpha_{(0)} = P_0 \alpha = \frac{1}{v} \sum_{i=0}^v \nu^i \alpha$$

In the case that $v = 2$, we have that

$$N = \prod_{i=1}^D \mathbb{Z}(\alpha_i - \nu \alpha_i)$$

and when $v = 3$ we have that:

$$N = \{r_1 \alpha_1 + r_3 \alpha_3 + r_4 \alpha_4 \in L \mid r_1 + r_3 + r_4 = 0\}.$$

Using Proposition 6.2 of [L1], let \mathbb{C}_τ denote the one dimensional \hat{N} -module \mathbb{C} with character τ and write

$$T = \mathbb{C}_\tau.$$

Consider the induced \hat{L}_ν -module

$$U_T = \mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\hat{N}]} T \cong \mathbb{C}[L/N],$$

which is graded by weights and on which \hat{L}_ν , $\mathfrak{h}_{(0)}$, and z^h for $h \in \mathfrak{h}_{(0)}$ all naturally act. Set

$$V_L^T = S[\nu] \otimes U_T \cong S(\hat{\mathfrak{h}}[\nu]^-) \otimes \mathbb{C}[L/N],$$

which is naturally acted upon by \hat{L}_ν , $\hat{\mathfrak{h}}_{\frac{1}{v}\mathbb{Z}}$, $\mathfrak{h}_{(0)}$, and z^h for $h \in \mathfrak{h}$.

For each $\alpha \in \mathfrak{h}$ and $m \in \frac{1}{v}\mathbb{Z}$ define the operators on V_L^T

$$\alpha_{(vm)} \otimes t^m \mapsto \alpha^{\hat{\nu}}(m),$$

where $\alpha_{(vm)}$ is the projection of α onto $\mathfrak{h}_{(vm)}$, and set

$$\alpha^{\hat{\nu}}(z) = \sum_{m \in \frac{1}{v}\mathbb{Z}} \alpha^{\hat{\nu}}(m) z^{-m-1}.$$

Of most importance will be the $\hat{\nu}$ -twisted vertex operators acting on V_L^T for each $e_\alpha \in \hat{L}$

$$Y^{\hat{\nu}}(\iota(e_\alpha), z) = v^{-\frac{\langle \alpha, \alpha \rangle}{2}} \sigma(\alpha) E^-(-\alpha, z) E^+(-\alpha, z) e_\alpha z^{\alpha_{(0)} + \frac{\langle \alpha_{(0)}, \alpha_{(0)} \rangle}{2} - \frac{\langle \alpha, \alpha \rangle}{2}},$$

as defined in [L1], where

$$E^\pm(-\alpha, z) = \exp \left(\sum_{m \in \pm \frac{1}{v}\mathbb{Z}_+} \frac{-\alpha_{(vm)}(m)}{m} z^{-m} \right),$$

and

$$\sigma(\alpha) = 1 \text{ when } v = 2$$

and

$$\sigma(\alpha) = (1 - \eta^2)^{\langle \nu \alpha, \alpha \rangle} \text{ when } v = 3$$

for $\alpha \in \mathfrak{h}$. For $m \in \frac{1}{v}$ and $\alpha \in L$ define the component operators $x_\alpha^{\hat{\nu}}(m)$ by

$$Y^{\hat{\nu}}(\iota(e_\alpha), z) = \sum_{m \in \frac{1}{v}\mathbb{Z}} x_\alpha^{\hat{\nu}}(m) z^{-m - \frac{\langle \alpha, \alpha \rangle}{2}} = x_\alpha^{\hat{\nu}}(z).$$

We note here that V_L^T is a $\hat{\nu}$ -twisted module for V_L , and in particular it satisfies the twisted Jacobi identity:

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y^{\hat{\nu}}(u, x_1) Y^{\hat{\nu}}(w, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y^{\hat{\nu}}(w, x_2) Y^{\hat{\nu}}(u, x_1) \\ = x_2^{-1} \frac{1}{v} \sum_{j \in \mathbb{Z}/v\mathbb{Z}} \delta \left(\eta^j \frac{(x_1 - x_0)^{1/v}}{x_2^{1/v}} \right) Y^{\hat{\nu}}(Y(\hat{\nu}^j u, x_0) w, x_2) \end{aligned}$$

for $u, w \in V_L$.

2.3. Twisted affine Lie algebras. We now construct the twisted affine Lie algebras of type $A_{2l-1}^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$, and give them an action on V_L^T . Define the vector space

$$\mathfrak{g} = \mathfrak{h} \oplus \prod_{\alpha \in \Delta} \mathbb{C}x_\alpha,$$

where $\{x_\alpha\}$ is a set of symbols, and Δ is the set of roots corresponding to L .

We give the vector space \mathfrak{g} the structure of a Lie algebra via the bracket defined

$$[h, x_\alpha] = \langle h, \alpha \rangle x_\alpha, \quad [\mathfrak{h}, \mathfrak{h}] = 0$$

where $h \in \mathfrak{h}$ and $\alpha \in \Delta$ and

$$[x_\alpha, x_\beta] = \begin{cases} \epsilon_{C_0}(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0 \\ \epsilon_{C_0}(\alpha, \beta)x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

We note that \mathfrak{g} is a Lie algebra isomorphic to one of type A_{2l-1} , D_l , or E_6 depending on the choice of L (cf. [FLM2]). We also extend the bilinear form $\langle \cdot, \cdot \rangle$ to \mathfrak{g} by

$$\langle h, x_\alpha \rangle = \langle x_\alpha, h \rangle = 0$$

and

$$\langle x_\alpha, x_\beta \rangle = \begin{cases} \epsilon_{C_0}(\alpha, -\alpha) & \text{if } \alpha + \beta = 0 \\ 0 & \text{if } \alpha + \beta \neq 0 \end{cases}$$

Following [L1], [CaLM4], and [PS1]–[PS2], we use our extension of $\nu : L \rightarrow L$ to $\hat{\nu} : \hat{L} \rightarrow \hat{L}$ to lift the automorphism $\nu : \mathfrak{h} \rightarrow \mathfrak{h}$ to an automorphism $\hat{\nu} : \mathfrak{g} \rightarrow \mathfrak{g}$ by setting

$$\hat{\nu}x_\alpha = \psi(\alpha)x_{\nu\alpha}$$

for all $\alpha \in \Delta$. Here, we are using our particular choices of $\hat{\nu}$ (extended to $\mathbb{C}\{L\}$) and section e .

For $m \in \mathbb{Z}$ set

$$\mathfrak{g}_{(m)} = \{x \in \mathfrak{g} \mid \hat{\nu}(x) = \eta^m x\}.$$

Form the $\hat{\nu}$ -twisted affine Lie algebra associated to \mathfrak{g} and $\hat{\nu}$:

$$\hat{\mathfrak{g}}[\hat{\nu}] = \prod_{m \in \frac{1}{v}\mathbb{Z}} \mathfrak{g}_{(vm)} \otimes t^m \oplus \mathbb{C}c$$

with

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m\delta_{m+n,0}c$$

and

$$[c, \hat{\mathfrak{g}}[\hat{\nu}]] = 0,$$

for $m, n \in \frac{1}{v}\mathbb{Z}$, $x \in \mathfrak{g}_{(vm)}$, and $y \in \mathfrak{g}_{(vn)}$. Adjoining the degree operator d to $\hat{\mathfrak{g}}[\hat{\nu}]$, we define

$$\tilde{\mathfrak{g}}[\hat{\nu}] = \hat{\mathfrak{g}}[\hat{\nu}] \oplus \mathbb{C}d,$$

where

$$[d, x \otimes t^n] = nx \otimes t^n,$$

for $x \in \mathfrak{g}_{(vn)}$, $n \in \frac{1}{v}\mathbb{Z}$ and $[d, c] = 0$. The Lie algebra $\tilde{\mathfrak{g}}[\hat{\nu}]$ is isomorphic to $A_{2l-1}^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$, or $D_4^{(3)}$ depending on the choice of L and ν , and is $\frac{1}{v}\mathbb{Z}$ -graded. We give V_L^T the structure of a $\hat{\mathfrak{g}}[\hat{\nu}]$ -module by:

Theorem 2.1. (*Theorem 3.1*[CalLM4], *Theorem 9.1* [L1], *Theorem 3* [FLM1]) *The representation of $\hat{\mathfrak{h}}[\nu]$ on V_L^T extends uniquely to a Lie algebra representation of $\hat{\mathfrak{g}}[\hat{\nu}]$ on V_L^T such that*

$$(x_\alpha)_{(vm)} \otimes t^m \mapsto x_\alpha^{\hat{\nu}}(m)$$

for all $m \in \frac{1}{v}\mathbb{Z}$ and $\alpha \in L$. Moreover V_L^T is irreducible as a $\hat{\mathfrak{g}}[\hat{\nu}]$ -module.

2.4. Gradings. As in Section 2 of [CalLM4] (also Section 6 of [L1]) we have a tensor product grading on V_L^T given by the action of $L^{\hat{\nu}}(0)$, where

$$Y^{\hat{\nu}}(\omega, z) = \sum_{m \in \mathbb{Z}} L^{\hat{\nu}}(m) z^{-m-2},$$

which we call the *weight grading*. In particular, we have

$$\begin{aligned} \text{wt}(1) &= \frac{l-1}{16}, \text{ for } A_{2l-1} \\ \text{wt}(1) &= \frac{1}{16}, \text{ for } D_l \text{ when } v=2 \\ \text{wt}(1) &= \frac{1}{8}, \text{ for } E_6, \\ \text{wt}(1) &= \frac{1}{9}, \text{ for } D_4 \text{ when } v=3. \end{aligned}$$

From [PS1]–[PS2], we recall that

$$\text{wt}(x_\alpha^{\hat{\nu}}(m)) = -m - 1 + \frac{1}{2} \langle \alpha, \alpha \rangle$$

for $m \in \frac{1}{v}\mathbb{Z}$ and $\alpha \in L$.

We endow V_L^T with *charge gradings* (see [PS1]–[PS2]). In particular, in the case of A_{2l-1} , we have

$$\text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle 2 \langle \alpha, (\lambda_1)_{(0)} \rangle, \dots, 2 \langle \alpha, (\lambda_{l-1})_{(0)} \rangle, \langle \alpha, (\lambda_l)_{(0)} \rangle \rangle. \quad (2.1)$$

In the case of D_l when $v=2$ we have

$$\text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle \langle \alpha, (\lambda_1)_{(0)} \rangle, \dots, \langle \alpha, (\lambda_{l-2})_{(0)} \rangle, 2 \langle \alpha, (\lambda_{l-1})_{(0)} \rangle \rangle. \quad (2.2)$$

In the E_6 case we have

$$\text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle 2 \langle \alpha, (\lambda_1)_{(0)} \rangle, 2 \langle \alpha, (\lambda_2)_{(0)} \rangle, \langle \alpha, (\lambda_3)_{(0)} \rangle, \langle \alpha, (\lambda_4)_{(0)} \rangle \rangle, \quad (2.3)$$

and finally in the case of D_4 when $v=3$ we have

$$\text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle 3 \langle \alpha, (\lambda_1)_{(0)} \rangle, \langle \alpha, (\lambda_2)_{(0)} \rangle \rangle. \quad (2.4)$$

We note that the λ_i here are the fundamental weights of the underlying finite dimensional Lie algebra \mathfrak{g} , dual to the roots:

$$\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$$

with $1 \leq i, j \leq \text{rank}(\mathfrak{g})$.

2.5. Higher levels. The primary focus of this section so far has been the construction of V_L^T , the basic module for the twisted affine Lie algebra $\hat{\mathfrak{g}}[\hat{\nu}]$. In the remainder of this work we will consider the standard $\tilde{\mathfrak{g}}[\hat{\nu}]$ -module $L^{\hat{\nu}}(k\Lambda_0)$ of level $k \geq 1$ with highest weight $k\Lambda_0$ such that $\langle k\Lambda_0, c \rangle = k$ and $\langle \Lambda_0, \mathfrak{h}_{(0)} \rangle = 0 = \langle \Lambda_0, d \rangle$. We note that when $k = 1$, we have $L^{\hat{\nu}}(\Lambda_0) \cong V_L^T$.

Since $L^{\hat{\nu}}(k\Lambda_0)$ is a faithful $\hat{\nu}$ -twisted $L(k\Lambda_0)$ -module (see [Li]), where $L(k\Lambda_0)$ denotes level k standard module of untwisted affine Lie algebra $\tilde{\mathfrak{g}}$ which has a structure of a vertex operator algebra, from Proposition 2.10 in [Li] follows that for $r \in \mathbb{N}$ and $x_\alpha \in \mathfrak{g}$

$$Y^{\hat{\nu}}(x_\alpha(-1)^r \mathbf{1}, z) = Y^{\hat{\nu}}(x_\alpha(-1)\mathbf{1}, z)^r$$

is a twisted vertex operator.

We will also use the commutator formula for twisted vertex operators

$$\left[x_\alpha^{\hat{\nu}}(z_1), x_{n'\beta}^{\hat{\nu}}(z_2) \right] = \tag{2.5}$$

$$= \sum_{j \geq 0} \frac{(-1)^j}{j!} \left(\frac{d}{dz_1} \right)^j z_2^{-1} \frac{1}{v} \sum_{q \in \mathbb{Z}/v\mathbb{Z}} \delta \left(\eta^q \frac{z_1^{\frac{1}{v}}}{z_2^{\frac{1}{v}}} \right) Y^{\hat{\nu}}(\hat{\nu}^q x_\alpha(j) x_\beta(-1)^{n'} \mathbf{1}, z_2).$$

In this work, we realize $L^{\hat{\nu}}(k\Lambda_0)$ as a submodule of the tensor product of k copies of the basic module $V_L^T \cong L^{\hat{\nu}}(\Lambda_0)$ as follows:

$$L^{\hat{\nu}}(k\Lambda_0) \cong U(\tilde{\mathfrak{g}}[\hat{\nu}]) \cdot v_L \subset V_L^T \otimes \cdots \otimes V_L^T = V_L^{T \otimes k},$$

where $v_L = \mathbf{1}_T \otimes \mathbf{1}_T \otimes \cdots \otimes \mathbf{1}_T$ is a highest weight vector of $L^{\hat{\nu}}(k\Lambda_0)$ and where $\mathbf{1}_T$ is a highest weight vector of V_L^T (cf. [Kac]).

It is known that $V_L^{\otimes k} = V_L \otimes \cdots \otimes V_L$ has a structure of vertex operator algebra. If we denote by $\hat{\nu}$ the automorphism $\hat{\nu} \otimes \cdots \otimes \hat{\nu}$ of the vertex operator algebra $V_L^{\otimes k}$, then we have $\hat{\nu}^v = 1$ and one can also define vertex operators corresponding to elements $v_1 \otimes \cdots \otimes v_k \in V_L^{T \otimes k}$ as tensor products of twisted vertex operators on the appropriate tensor factors $Y^{\hat{\nu}}(v_1, z) \otimes \cdots \otimes Y^{\hat{\nu}}(v_k, z)$. In this way $V_L^{T \otimes k}$ becomes an irreducible $\hat{\nu}$ -twisted module for the vertex operator algebra $V_L^{\otimes k}$ (cf. [Li]), with

$$Y^{\hat{\nu}}(x_\alpha(-1) \cdot (\mathbf{1} \otimes \cdots \otimes \mathbf{1}), z) = \sum_{m \in \frac{1}{v}\mathbb{Z}} x_\alpha^{\hat{\nu}}(m) z^{-m-1}.$$

3. Principal subspaces

In this section we define the notion of principal subspace of $L^{\hat{\nu}}(k\Lambda_0)$ and twisted quasi-particles which we will use in the description of our bases.

First, denote by

$$\mathfrak{n} = \prod_{\alpha \in \Delta_+} \mathbb{C}x_\alpha,$$

the ν -stable Lie subalgebra of \mathfrak{g} (the nilradical of a Borel subalgebra), its $\hat{\nu}$ -twisted affinization

$$\hat{\mathfrak{n}}[\hat{\nu}] = \prod_{m \in \frac{1}{\nu}\mathbb{Z}} \mathfrak{n}_{(\nu m)} \otimes t^m \oplus \mathbb{C}c$$

and the subalgebra of $\hat{\mathfrak{n}}[\hat{\nu}]$

$$\bar{\mathfrak{n}}[\hat{\nu}] = \prod_{m \in \frac{1}{\nu}\mathbb{Z}} \mathfrak{n}_{(\nu m)} \otimes t^m.$$

Following [FS] (see also [CalLM4], [CalMPE], and [PS1]–[PS2]), we define the principal subspace $W_{L_k}^T$ of $L^{\hat{\nu}}(k\Lambda_0)$ as:

$$W_{L_k}^T = U(\bar{\mathfrak{n}}[\hat{\nu}]) \cdot v_L.$$

3.1. Properties of $W_{L_k}^T$. We now recall some important properties of the operators $x_{\nu\alpha}^{\hat{\nu}}(m)$ on $W_{L_k}^T$. We recall from [PS1]–[PS2] that

$$\begin{aligned} x_{\nu\alpha}^{\hat{\nu}}(m) &= x_{\alpha}^{\hat{\nu}}(m) \text{ for } m \in \mathbb{Z} \\ x_{\nu\alpha}^{\hat{\nu}}(m) &= -x_{\alpha}^{\hat{\nu}}(m) \text{ for } m \in \frac{1}{2} + \mathbb{Z}. \end{aligned}$$

when $\nu = 2$ and that

$$\begin{aligned} x_{\alpha_3}^{\hat{\nu}}(m) &= x_{\alpha_1}^{\hat{\nu}}(m) \text{ for } m \in \mathbb{Z} \\ x_{\alpha_3}^{\hat{\nu}}(m) &= \eta x_{\alpha_1}^{\hat{\nu}}(m) \text{ for } m \in \frac{1}{3} + \mathbb{Z} \\ x_{\alpha_3}^{\hat{\nu}}(m) &= \eta^2 x_{\alpha_1}^{\hat{\nu}}(m) \text{ for } m \in \frac{2}{3} + \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} x_{\alpha_4}^{\hat{\nu}}(m) &= x_{\alpha_1}^{\hat{\nu}}(m) \text{ for } m \in \mathbb{Z} \\ x_{\alpha_4}^{\hat{\nu}}(m) &= \eta^2 x_{\alpha_1}^{\hat{\nu}}(m) \text{ for } m \in \frac{1}{3} + \mathbb{Z} \\ x_{\alpha_4}^{\hat{\nu}}(m) &= \eta x_{\alpha_1}^{\hat{\nu}}(m) \text{ for } m \in \frac{2}{3} + \mathbb{Z} \end{aligned}$$

when $\nu = 3$. In particular, we need only choose one representative from the orbit of each simple root α_i when working with the operators $x_{\alpha_i}^{\hat{\nu}}(m)$.

For every simple root α_i consider the one-dimensional subalgebra of \mathfrak{g}

$$\mathfrak{n}_{\alpha_i} = \mathbb{C}x_{\alpha_i},$$

and their $\hat{\nu}$ -twisted affinizations

$$\bar{\mathfrak{n}}_{\alpha_i}[\hat{\nu}] = \prod_{m \in \frac{1}{\nu}\mathbb{Z}} \mathfrak{n}_{\alpha_i(\nu m)} \otimes t^m.$$

In the case of $A_{2l-1}^{(2)}$ we define a special subspace of $\bar{\mathfrak{n}}[\hat{\nu}]$

$$U = U(\bar{\mathfrak{n}}_{\alpha_l}[\hat{\nu}])U(\bar{\mathfrak{n}}_{\alpha_{l-1}}[\hat{\nu}]) \cdots U(\bar{\mathfrak{n}}_{\alpha_1}[\hat{\nu}]).$$

Similary, for $D_l^{(2)}$ we define

$$U = U(\bar{\mathfrak{n}}_{\alpha_{l-1}}[\hat{\nu}])U(\bar{\mathfrak{n}}_{\alpha_{l-2}}[\hat{\nu}]) \cdots U(\bar{\mathfrak{n}}_{\alpha_1}[\hat{\nu}]),$$

in the case of $E_6^{(2)}$ we define

$$U = U(\bar{\mathfrak{n}}_{\alpha_4}[\hat{\nu}])U(\bar{\mathfrak{n}}_{\alpha_3}[\hat{\nu}])U(\bar{\mathfrak{n}}_{\alpha_2}[\hat{\nu}])U(\bar{\mathfrak{n}}_{\alpha_1}[\hat{\nu}]),$$

and for $D_4^{(3)}$ we define

$$U = U(\bar{\mathfrak{n}}_{\alpha_2}[\hat{\nu}])U(\bar{\mathfrak{n}}_{\alpha_1}[\hat{\nu}]).$$

The next lemma can be proved by using properties stated above and by aruing as in Lemma 3.1 of [G].

Lemma 3.1. *In all of the above cases, we have that*

$$W_{L_k}^T = U \cdot v_L.$$

3.2. Twisted quasi-particles. For each simple root α_i , $r \in \mathbb{N}$, and $m \in \frac{1}{\nu}\mathbb{Z}$ define the twisted quasi-particle of *color* i , *charge* r and *energy* $-m$ by

$$x_{r\alpha_i}^{\hat{\nu}}(m) = \text{Res}_z \{z^{m+r-1} x_{r\alpha_i}^{\hat{\nu}}(z)\},$$

where

$$x_{r\alpha_i}^{\hat{\nu}}(z) = \sum_{m \in \frac{1}{\nu}\mathbb{Z}} x_{r\alpha_i}^{\hat{\nu}}(m) z^{-m-r}$$

is the twisted vertex operator

$$x_{r\alpha_i}^{\hat{\nu}}(z) = Y^{\hat{\nu}}(x_{\alpha_i}(-1)^r \mathbf{1}, z).$$

As in the untwisted case (see [G], [Bu1]–[Bu3]), we build twisted quasi-particle monomials from twisted quasi-particles. We say that the monomial

$$\begin{aligned} b &= b^{\hat{\nu}}(\alpha_l) \cdots b^{\hat{\nu}}(\alpha_1) = \\ &= x_{n_{r_l^{(1)}, l} \alpha_l}^{\hat{\nu}}(m_{r_l^{(1)}, l}) \cdots x_{n_{1, l} \alpha_l}^{\hat{\nu}}(m_{1, l}) \cdots x_{n_{r_1^{(1)}, 1} \alpha_1}^{\hat{\nu}}(m_{r_1^{(1)}, 1}) \cdots x_{n_{1, 1} \alpha_1}^{\hat{\nu}}(m_{1, 1}), \end{aligned}$$

is of *charge-type*

$$\mathcal{R}' = \left(n_{r_l^{(1)}, l}, \dots, n_{1, l}; \dots; n_{r_1^{(1)}, 1}, \dots, n_{1, 1} \right),$$

where $0 \leq n_{r_i^{(1)}, i} \leq \dots \leq n_{1, i}$, *dual-charge-type*

$$\mathcal{R} = \left(r_l^{(1)}, \dots, r_l^{(s_l)}; \dots; r_1^{(1)}, \dots, r_1^{(s_1)} \right),$$

where $r_i^{(1)} \geq r_i^{(2)} \geq \dots \geq r_i^{(s_i)} \geq 0$, and *color-type*

$$(r_l, \dots, r_1),$$

where

$$r_i = \sum_{p=1}^{r_i^{(1)}} n_{p,i} = \sum_{t=1}^{s_i} r_i^{(t)} \quad \text{and} \quad s_i \in \mathbb{N},$$

if for every color i , $1 \leq i \leq l$, $(n_{r_i^{(1)},i}, \dots, n_{1,i})$ and $(r_i^{(1)}, r_i^{(2)}, \dots, r_i^{(s)})$ are mutually conjugate partitions of r_i (cf. [Bu1]–[Bu3], [G]).

For two monomials b and \bar{b} with charge-types \mathcal{R}' and $\overline{\mathcal{R}'} = (\bar{n}_{\bar{r}_l^{(1)},l}, \dots, \bar{n}_{1,1})$ and with energies $(m_{r_l^{(1)},l}, \dots, m_{1,1})$ and $(\bar{m}_{\bar{r}_l^{(1)},l}, \dots, \bar{m}_{1,1})$ (which we write so that energies of twisted quasi-particles of the same color and the same charge form an increasing sequence of integers from right to the left), respectively, we write $b < \bar{b}$ if one of the following conditions holds:

1. $\mathcal{R}' < \overline{\mathcal{R}'}$
2. $\mathcal{R}' = \overline{\mathcal{R}'}$ and $(m_{r_l^{(1)},l}, \dots, m_{1,1}) < (\bar{m}_{\bar{r}_l^{(1)},l}, \dots, \bar{m}_{1,1})$,

where we write $\mathcal{R}' < \overline{\mathcal{R}'}$ if there exists $u \in \mathbb{N}$ such that $n_{1,i} = \bar{n}_{1,i}$, $n_{2,i} = \bar{n}_{2,i}$, \dots , $n_{u-1,i} = \bar{n}_{u-1,i}$, and either $u = \bar{r}_i^{(1)} + 1$ or $n_{u,i} < \bar{n}_{u,i}$, starting from color $i = 1$. In the case that $\mathcal{R}' = \overline{\mathcal{R}'}$, we apply this definition to the energies to similarly define $(m_{r_l^{(1)},l}, \dots, m_{1,1}) < (\bar{m}_{\bar{r}_l^{(1)},l}, \dots, \bar{m}_{1,1})$.

4. Combinatorial bases

In this section we prove relations among twisted quasi-particles which we will use in the construction of our combinatorial bases for $W_{L_k}^T$.

4.1. Relations among twisted quasi-particles. For every color i we have the following relations:

$$x_{(k+1)\alpha_i}^{\hat{\nu}}(z) = 0 \tag{4.1}$$

and

$$x_{r\alpha_i}^{\hat{\nu}}(z)v_L \in W_{L_k}^T[[z]] \tag{4.2}$$

when $\hat{\nu}\alpha_i = \alpha_i$,

$$x_{r\alpha_i}^{\hat{\nu}}(z)v_L \in z^{-\frac{1}{2}}W_{L_k}^T\left[\left[z^{\frac{1}{2}}\right]\right] \tag{4.3}$$

when $\hat{\nu}\alpha_i \neq \alpha_i$ and $v = 2$, and

$$x_{r\alpha_i}^{\hat{\nu}}(z)v_L \in z^{-\frac{2}{3}}W_{L_k}^T\left[\left[z^{\frac{1}{3}}\right]\right] \tag{4.4}$$

when $\hat{\nu}\alpha_i \neq \alpha_i$ and $v = 3$, which all follow immediately from the fact that

$$x_{r\alpha_i}^{\hat{\nu}}(m)v_L = 0$$

whenever $m \geq 0$. We will use also the following relations among quasi-particles of the same color:

Lemma 4.1. *Let $1 \leq n \leq n'$ be fixed.*

a) *If $\hat{\nu}\alpha_i = \alpha_i$ and $M, j \in \mathbb{Z}$ are fixed, the $2n$ monomials from the set*

$$A = \{x_{n\alpha_i}^{\hat{\nu}}(j)x_{n'\alpha_i}^{\hat{\nu}}(M-j), x_{n\alpha_i}^{\hat{\nu}}(j-1)x_{n'\alpha_i}^{\hat{\nu}}(M-j+1), \dots \\ \dots, x_{n\alpha_i}^{\hat{\nu}}(j-2n+1)x_{n'\alpha_i}^{\hat{\nu}}(M-j+2n-1)\}$$

can be expressed as a linear combination of monomials from the set

$$\left\{x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m') : m + m' = M\right\} \setminus A$$

and monomials which have as a factor the quasi-particle $x_{(n'+1)\alpha_i}^{\hat{\nu}}(j')$, $j' \in \mathbb{Z}$.

b) *If $\hat{\nu}\alpha_i \neq \alpha_i$ and $M, j \in \frac{1}{v}\mathbb{Z}$ are fixed, the $2n$ monomials from the set*

$$B = \{x_{n\alpha_i}^{\hat{\nu}}(j)x_{n'\alpha_i}^{\hat{\nu}}(M-j), x_{n\alpha_i}^{\hat{\nu}}(j - \frac{1}{v})x_{n'\alpha_i}^{\hat{\nu}}(M - j + \frac{1}{v}), \dots \\ \dots, x_{n\alpha_i}^{\hat{\nu}}(j - \frac{2n-1}{v})x_{n'\alpha_i}^{\hat{\nu}}(M - j + \frac{2n-1}{v})\}$$

can be expressed as a linear combination of monomials from the set

$$\left\{x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m') : m + m' = M\right\} \setminus B$$

and monomials which have as a factor the quasi-particle $x_{(n'+1)\alpha_i}^{\hat{\nu}}(j')$, $j' \in \frac{1}{v}\mathbb{Z}$.

Proof. The proof of part a) is identical to the proof of Lemma 4.4 in [JP]. Part b) can be proven analogously. First, for fixed $M \in \frac{1}{v}\mathbb{Z}$ in the coefficient of $z^{-M-n-n'-N}$ of the formal series

$$\frac{1}{N!} \left(\frac{d^N}{dz^N} x_{n\alpha_i}^{\hat{\nu}}(z) \right) x_{n'\alpha_i}^{\hat{\nu}}(z) = \tag{4.5} \\ = \sum_{M \in \frac{1}{v}\mathbb{Z}} \left(\sum_{\substack{m, m' \in \frac{1}{v}\mathbb{Z} \\ m+m'=M}} \binom{-m-n}{N} x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m') \right) z^{-M-n-n'-N}$$

we separate the $2n$ monomials with $m = \frac{j}{v}, \frac{j-1}{v}, \dots, \frac{j-2n-1}{v}$ for some fixed $j \in \mathbb{Z}$

$$\begin{aligned} & \binom{-\frac{j}{v}-n}{N} x_{n\alpha_i}^{\hat{\nu}}\left(\frac{j}{v}\right) x_{n'\alpha_i}^{\hat{\nu}}\left(M - \frac{j}{v}\right) + \\ & + \binom{-\frac{j}{v}-n+\frac{1}{v}}{N} x_{n\alpha_i}^{\hat{\nu}}\left(\frac{j}{v} - \frac{1}{v}\right) x_{n'\alpha_i}^{\hat{\nu}}\left(M - \frac{j}{v} + \frac{1}{v}\right) + \\ & + \dots + \binom{-\frac{j}{v}-n+\frac{2n-2}{v}}{N} x_{n\alpha_i}^{\hat{\nu}}\left(\frac{j}{v} - \frac{2n-2}{v}\right) x_{n'\alpha_i}^{\hat{\nu}}\left(M - \frac{j}{v} + \frac{2n-2}{v}\right) + \\ & + \binom{-\frac{j}{v}-n+\frac{2n-1}{v}}{N} x_{n\alpha_i}^{\hat{\nu}}\left(\frac{j}{v} - \frac{2n-1}{v}\right) x_{n'\alpha_i}^{\hat{\nu}}\left(M - \frac{j}{v} + \frac{2n-1}{v}\right) + \end{aligned}$$

$$+ \sum_{\substack{m, m' \in \frac{1}{v}\mathbb{Z} \\ m+m'=M \\ m \neq \frac{j}{v}, \frac{j-1}{v}, \dots, \frac{j-2n-1}{v}}} \binom{-m-n}{N} x_{n\alpha_i}^{\hat{\nu}}(m) x_{n'\alpha_i}^{\hat{\nu}}(m').$$

Next, note that for $N = 0, 1, \dots, 2n-1$, we also have

$$\frac{1}{N!} \left(\frac{d^N}{dz^N} x_{n\alpha_i}^{\hat{\nu}}(z) \right) x_{n'\alpha_i}^{\hat{\nu}}(z) = A_1^{\hat{\nu}}(z) x_{(n'+1)\alpha_i}^{\hat{\nu}}(z) + A_2^{\hat{\nu}}(z) \frac{d}{dz} x_{(n'+1)\alpha_i}^{\hat{\nu}}(z), \quad (4.6)$$

where $A_1^{\hat{\nu}}(z)$ and $A_2^{\hat{\nu}}(z)$ are formal Laurent series whose coefficients are polynomials in $x_{\alpha_i}^{\hat{\nu}}(m)$, $m \in \frac{1}{v}\mathbb{Z}$ (the proof of which is the same as the proof of Lemma 4.2 in [JP]).

Now, by comparing powers of z , from (4.5) and (4.6) we obtain a system of $2n$ equations whose left-hand side consists of linear combinations of the $2n$ unknowns

$$x_{n\alpha_i}^{\hat{\nu}}(-p-n) x_{n'\alpha_i}^{\hat{\nu}}(M+p+n), x_{n\alpha_i}^{\hat{\nu}}(-p-n-\frac{1}{v}) x_{n'\alpha_i}^{\hat{\nu}}(M+p+n+\frac{1}{v}), \dots \\ \dots, x_{n\alpha_i}^{\hat{\nu}}(-p-n-\frac{2n-1}{v}) x_{n'\alpha_i}^{\hat{\nu}}(M+p+n+\frac{2n-1}{v}),$$

where $p = -\frac{j}{v} - n$, and whose right-hand side is equal to expressions in terms of higher quasi-particle monomials (with respect to ordering defined in previous section) and other quasi-particle monomials from the set $\{x_{n\alpha_i}^{\hat{\nu}}(m) x_{n'\alpha_i}^{\hat{\nu}}(m') : m+m'=M\} \setminus B$. Such an expression exists and is unique if the coefficient matrix

$$\begin{bmatrix} \binom{p}{0} & \binom{p+\frac{1}{v}}{0} & \dots & \binom{p+\frac{2n-2}{v}}{0} & \binom{p+\frac{2n-1}{v}}{0} \\ \binom{p}{1} & \binom{p+\frac{1}{v}}{1} & \dots & \binom{p+\frac{2n-2}{v}}{1} & \binom{p+\frac{2n-1}{v}}{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{p}{2n-1} & \binom{p+\frac{1}{v}}{2n-1} & \dots & \binom{p+\frac{2n-2}{v}}{2n-1} & \binom{p+\frac{2n-1}{v}}{2n-1} \end{bmatrix}$$

is invertible.

By using the Chu-Vandermonde identity

$$\sum_{q=0}^N \binom{a}{q} \binom{b}{N-q} = \binom{a+b}{N},$$

we can write the coefficient matrix as a product of two invertible matrices

$$\begin{bmatrix} \binom{p}{0} & 0 & \dots & 0 & 0 \\ \binom{p}{1} & \binom{p}{0} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{p}{2n-1} & \binom{p}{2n-2} & \dots & \binom{p}{1} & \binom{p}{0} \end{bmatrix} \cdot \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \dots & \binom{2n-2}{0} & \binom{2n-1}{0} \\ \binom{0}{1} & \binom{1}{1} & \dots & \binom{2n-2}{1} & \binom{2n-1}{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{0}{2n-1} & \binom{1}{2n-1} & \dots & \binom{2n-2}{2n-1} & \binom{2n-1}{2n-1} \end{bmatrix}.$$

To see that the second matrix in the product is invertible, we first, starting from row $2n - 1$, multiply row q of the determinant of the matrix by $\frac{(q-1)v-1}{qv}$, where $2 \leq q \leq 2n - 1$, and then add to row $q + 1$, which will give us

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & \frac{1}{v} & \frac{2}{v} & \dots & \frac{2n-2}{v^2} \binom{2n-2}{2} & \frac{2n-1}{v^2} \binom{2n-1}{2} \\ 0 & 0 & \frac{1}{v^2} & \dots & \frac{1}{v^2} \binom{2n-2}{2} & \frac{1}{v^2} \binom{2n-1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{(2n-1)v} \binom{2}{\frac{2}{v}} & \dots & \frac{2n-3}{(2n-1)v} \binom{2n-2}{\frac{2}{v}} & \frac{2n-2}{(2n-1)v} \binom{2n-1}{\frac{2}{v}} \end{vmatrix}.$$

We proceed with this process of reducing the determinant of our matrix to the determinant of an upper triangular matrix and assume that after $r - 1$ steps we have

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{v} & \dots & \frac{r-1}{v} & \frac{r}{v} & \dots & \frac{2n-1}{v} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{v^r} & \dots & \frac{1+r}{v^r} \binom{2n-1}{r} \\ 0 & 0 & \dots & 0 & \frac{1}{r(r-1)v^{r-1}} \binom{r}{\frac{r}{v}} & \dots & \frac{(2n-1)\dots(2n-r)}{v^{r-1}(r+1)\dots 3} \binom{r+1}{\frac{r}{v}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{(2n-1)\dots(2n-r-1)v^{r-1}} \binom{r}{2n-1-\frac{r}{v}-1} & \dots & \frac{(2n-1)\dots(2n-r)}{v^{r-1}(2n-1)\dots(2n-1-r)} \binom{2n-1}{2n-1-\frac{r}{v}-1} \end{vmatrix}.$$

Now, starting from row $2n - 1$, we multiply row q by $\frac{(q-r)v-r}{qv}$, where $r + 1 \leq q \leq 2n - 1$, and add to row $q + 1$ and obtain

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{v} & \dots & \frac{r-1}{v} & \frac{r}{v} & \frac{r+1}{v} & \dots & \frac{2n-1}{v} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{v^r} & \frac{r+1}{v^r} & \dots & \frac{r+1}{v^r} \binom{2n-1}{r} \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{v^{r+1}} & \dots & \frac{1}{v^{r+1}} \binom{2n-1}{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{(2n-1)\dots(2n-r-2)v^r} \binom{r+1}{2n-1-\frac{r}{v}-2} & \dots & \frac{(2n-1)\dots(2n-r-1)}{v^r(2n-1)\dots(2n-1-r)} \binom{2n-1}{2n-1-\frac{r}{v}-2} \end{vmatrix}.$$

After $2n - 1$ steps we get the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{v} & \frac{2}{v} & \dots & \frac{2n-1}{v} \\ 0 & 0 & \frac{1}{v^2} & \dots & \frac{1}{v^2} \binom{2n-1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{v^{2n-1}} \end{vmatrix} = v^{-n(2n-1)},$$

from which our claim follows. \square

Remark 4.1. We note here that a more general form of this matrix appeared in [PSW], where it was called a “generalized Pascal matrix”. In [PSW], this matrix was shown to be invertible but its determinant was not explicitly computed.

From Lemma 4.1 it follows that if $\hat{\nu}\alpha_i = \alpha_i$, then the $2n$ monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m')v_L, x_{n\alpha_i}^{\hat{\nu}}(m-1)x_{n'\alpha_i}^{\hat{\nu}}(m'+1)v_L, \dots$$

$\dots, x_{n\alpha_i}^{\hat{\nu}}(m-2n+2)x_{n'\alpha_i}^{\hat{\nu}}(m'+2n-2)v_L, x_{n\alpha_i}^{\hat{\nu}}(m-2n+1)x_{n'\alpha_i}^{\hat{\nu}}(m'+2n-1)v_L$
such that $n < n'$ can be expressed as a (finite) linear combination of the monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(j)x_{n'\alpha_i}^{\hat{\nu}}(j')v_L \text{ such that } j \leq m-2n, \quad j' \geq m'+2n$$

and monomial vectors with a factor quasi-particle $x_{(n+1)\alpha_i}^{\hat{\nu}}(j_1)$, $j_1 \in \mathbb{Z}$. If $n = n'$, then the monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m')v_L \text{ such that } m'-2n \leq m \leq m'$$

can be expressed as a linear combination of monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(j)x_{n'\alpha_i}^{\hat{\nu}}(j')v_L, \text{ such that } j \leq j' - 2n$$

and monomial vectors with a factor quasi-particle $x_{(n+1)\alpha_i}^{\hat{\nu}}(j_1)$, $j_1 \in \mathbb{Z}$.

If $\hat{\nu}\alpha_i \neq \alpha_i$, then the $2n$ monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m')v_L, x_{n\alpha_i}^{\hat{\nu}}(m-1)x_{n'\alpha_i}^{\hat{\nu}}(m'+1)v_L, \dots$$

$\dots, x_{n\alpha_i}^{\hat{\nu}}(m-2n+2)x_{n'\alpha_i}^{\hat{\nu}}(m'+2n-2)v_L, x_{n\alpha_i}^{\hat{\nu}}(m-2n+1)x_{n'\alpha_i}^{\hat{\nu}}(m'+2n-1)v_L$
with $n < n'$ can be expressed as a (finite) linear combination of the monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(j)x_{n'\alpha_i}^{\hat{\nu}}(j')v_L, \text{ such that } j \leq m - \frac{2n}{v}, \quad j' \geq m' + \frac{2n}{v}$$

and monomial vectors with a factor quasi-particle $x_{(n+1)\alpha_i}^{\hat{\nu}}(j_1)$, $j_1 \in \frac{1}{v}\mathbb{Z}$. If $n = n'$, then the monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(m)x_{n'\alpha_i}^{\hat{\nu}}(m')v_L, \text{ such that } m' - \frac{2n}{v} \leq m \leq m'$$

can be expressed as a linear combination of the monomial vectors

$$x_{n\alpha_i}^{\hat{\nu}}(j)x_{n'\alpha_i}^{\hat{\nu}}(j')v_L, \text{ such that } j \leq j' - \frac{2n}{v}$$

and monomial vectors with a factor quasi-particle $x_{(n+1)\alpha_i}^{\hat{\nu}}(j_1)$, $j_1 \in \frac{1}{v}\mathbb{Z}$.

The following lemma describes relations among quasi-particles with different colors:

Lemma 4.2. *Let $1 \leq n, n' \leq k$ be fixed. Then:*

a) *if $v = 2$ and $\langle \alpha, \beta \rangle = -1 = \langle \nu\alpha, \beta \rangle$, we have:*

$$(z_1 - z_2)^{\min\{n, n'\}} x_{n\alpha}^{\hat{\nu}}(z_1) x_{n'\beta}^{\hat{\nu}}(z_2) = (z_1 - z_2)^{\min\{n, n'\}} x_{n'\beta}^{\hat{\nu}}(z_2) x_{n\alpha}^{\hat{\nu}}(z_1), \quad (4.7)$$

b) if $v = 2$ and $\langle \alpha, \beta \rangle = -1$, $\langle \nu\alpha, \beta \rangle = 0$, we have:

$$(z_1^{\frac{1}{2}} - z_2^{\frac{1}{2}})^{\min\{n, n'\}} x_{n\alpha}^{\hat{\nu}}(z_1) x_{n'\beta}^{\hat{\nu}}(z_2) = (z_1^{\frac{1}{2}} - z_2^{\frac{1}{2}})^{\min\{n, n'\}} x_{n'\beta}^{\hat{\nu}}(z_2) x_{n\alpha}^{\hat{\nu}}(z_1), \quad (4.8)$$

c) if $v = 3$ for $\alpha = \alpha_1$, $\beta = \alpha_2$, we have:

$$(z_1 - z_2)^{\min\{n, n'\}} x_{n\alpha}^{\hat{\nu}}(z_1) x_{n'\beta}^{\hat{\nu}}(z_2) = (z_1 - z_2)^{\min\{n, n'\}} x_{n'\beta}^{\hat{\nu}}(z_2) x_{n\alpha}^{\hat{\nu}}(z_1). \quad (4.9)$$

Proof. The proof follows from the commutator formula for twisted vertex operators (2.5) and from properties of the δ -function. \square

By using relations (4.1)–(4.4) among twisted quasi-particles, Lemma 4.1 and relations (4.2)–(4.9), we define the following sets:

- for $A_{2l-1}^{(2)}$:

$$B = \bigcup_{\substack{n_{r_1^{(1)},1} \leq \dots \leq n_{1,1} \leq k \\ \dots \\ n_{r_l^{(1)},l} \leq \dots \leq n_{1,l} \leq k}} \left(\text{or, equivalently, } \bigcup_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ \dots \\ r_l^{(1)} \geq \dots \geq r_l^{(k)} \geq 0}} \right)$$

$$\{b = b(\alpha_l) \cdots b(\alpha_1) =$$

$$= x_{n_{r_n^{(1)},l} \alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{1,l} \alpha_l}^{\hat{\nu}}(m_{1,l}) \cdots x_{n_{r_1^{(1)},1} \alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1} \alpha_1}^{\hat{\nu}}(m_{1,1}) :$$

$$\left. \begin{array}{l} m_{p,i} \in \frac{1}{2}\mathbb{Z}, \quad 1 \leq p \leq r_i^{(1)}, \quad 1 \leq i \leq l-1; \\ m_{p,i} \leq -\frac{n_{p,i}}{2} + \frac{1}{2} \sum_{q=1}^{r_i^{(1)}} \min\{n_{q,i-1}, n_{p,i}\} - \sum_{p>p'>0} \min\{n_{p,i}, n_{p',i}\}, \\ \quad 1 \leq p \leq r_i^{(1)}, \quad 1 \leq i \leq l-1; \\ m_{p+1,i} \leq m_{p,i} - n_{p,i} \text{ if } n_{p+1,i} = n_{p,i}, \quad 1 \leq p \leq r_i^{(1)} - 1, \quad 1 \leq i \leq l-1; \\ m_{p,l} \in \mathbb{Z}, \quad 1 \leq p \leq r_l^{(1)}; \\ m_{p,l} \leq -n_{p,l} + \sum_{q=1}^{r_l^{(1)}} \min\{n_{q,l-1}, n_{p,l}\} - \sum_{p>p'>0} 2 \min\{n_{p,l}, n_{p',l}\}, \\ \quad 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l} \leq m_{p,l} - 2n_{p,l} \text{ if } n_{p,l} = n_{p+1,l}, \quad 1 \leq p \leq r_l^{(1)} - 1 \end{array} \right\},$$

- for $D_l^{(2)}$:

$$B = \bigcup_{\substack{n_{r_1^{(1)},1} \leq \dots \leq n_{1,1} \leq k \\ \dots \\ n_{r_{l-1}^{(1)},l-1} \leq \dots \leq n_{1,l-1} \leq k}} \left(\text{or, equivalently, } \bigcup_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ \dots \\ r_{l-1}^{(1)} \geq \dots \geq r_{l-1}^{(k)} \geq 0}} \right)$$

$$\{b = b(\alpha_{l-1}) \cdots b(\alpha_1) =$$

$$= x_{n_{r_{l-1}^{(1)},l-1} \alpha_{l-1}}^{\hat{\nu}}(m_{r_{l-1}^{(1)},l-1}) \cdots x_{n_{1,l-1} \alpha_{l-1}}^{\hat{\nu}}(m_{1,l-1}) \cdots$$

$$\cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(m_{1,1}) :$$

$$\left. \begin{array}{l} m_{p,i} \in \mathbb{Z}, \quad 1 \leq p \leq r_i^{(1)}, \quad 1 \leq i \leq l-2; \\ m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_i^{(1)}-1} \min\{n_{q,i-1}, n_{p,i}\} - \sum_{p>p'>0} 2 \min\{n_{p,i}, n_{p',i}\}, \\ \quad 1 \leq p \leq r_i^{(1)}, \quad 1 \leq i \leq l-2; \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i}, \quad 1 \leq p \leq r_i^{(1)} - 1, \quad 1 \leq i \leq 2; \\ m_{p,l-1} \in \frac{1}{2}\mathbb{Z}, \quad 1 \leq p \leq r_{l-1}^{(1)}; \\ m_{p,l-1} \leq -\frac{n_{p,l-1}}{2} + \sum_{q=1}^{r_{l-1}^{(1)}-2} \min\{n_{q,l-2}, n_{p,l-1}\} - \\ \quad - \sum_{p>p'>0} \min\{n_{p,l-1}, n_{p',l-1}\}, \quad 1 \leq p \leq r_{l-1}^{(1)}; \\ m_{p+1,l-1} \leq m_{p,l-1} - n_{p,l-1} \text{ if } n_{p,l-1} = n_{p+1,l-1}, \quad 1 \leq p \leq r_{l-1}^{(1)} - 1 \end{array} \right\},$$

• for $E_6^{(2)}$:

$$B = \bigcup_{\substack{n_{r_1^{(1)},1} \leq \dots \leq n_{1,1} \leq k \\ n_{r_4^{(1)},4} \leq \dots \leq n_{1,4} \leq k}} \left(\text{or, equivalently, } \bigcup_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ r_4^{(1)} \geq \dots \geq r_4^{(k)} \geq 0}} \right)$$

$$\{b = b(\alpha_4) \cdots b(\alpha_1) =$$

$$= x_{n_{r_4^{(1)},4}\alpha_4}^{\hat{\nu}}(m_{r_4^{(1)},4}) \cdots x_{n_{1,4}\alpha_4}^{\hat{\nu}}(m_{1,4}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(m_{1,1}) :$$

$$\left. \begin{array}{l} m_{p,i} \in \frac{1}{2}\mathbb{Z}, \quad 1 \leq p \leq r_i^{(1)}, \quad 1 \leq i \leq 2; \\ m_{p,i} \leq -\frac{n_{p,i}}{2} + \frac{1}{2} \sum_{q=1}^{r_i^{(1)}-1} \min\{n_{q,i-1}, n_{p,i}\} - \sum_{p>p'>0} \min\{n_{p,i}, n_{p',i}\}, \\ \quad 1 \leq p \leq r_i^{(1)}, \quad 1 \leq i \leq l-1; \\ m_{p+1,i} \leq m_{p,i} - n_{p,i} \text{ if } n_{p+1,i} = n_{p,i}, \quad 1 \leq p \leq r_i^{(1)} - 1, \quad 1 \leq i \leq 2; \\ m_{p,i} \in \mathbb{Z}, \quad 1 \leq p \leq r_i^{(1)}, \quad 3 \leq i \leq 4; \\ m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_i^{(1)}-1} \min\{n_{q,i-1}, n_{p,i}\} - \sum_{p>p'>0} 2 \min\{n_{p,i}, n_{p',i}\}, \\ \quad 1 \leq p \leq r_i^{(1)}, \quad 3 \leq i \leq 4; \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p,i} = n_{p+1,i}, \quad 1 \leq p \leq r_i^{(1)} - 1, \quad 3 \leq i \leq 4 \end{array} \right\},$$

• for $D_4^{(3)}$:

$$B = \bigcup_{\substack{n_{r_1^{(1)},1} \leq \dots \leq n_{1,1} \leq k \\ n_{r_2^{(1)},2} \leq \dots \leq n_{1,2} \leq k}} \left(\text{or, equivalently, } \bigcup_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ r_2^{(1)} \geq \dots \geq r_2^{(k)} \geq 0}} \right)$$

$$\{b = b(\alpha_2)b(\alpha_1) =$$

$$= x_{n_{r_2^{(1)},2} \alpha_2}^{\hat{\nu}}(m_{r_2^{(1)},2}) \cdots x_{n_{1,2} \alpha_2}^{\hat{\nu}}(m_{1,2}) x_{n_{r_1^{(1)},1} \alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1} \alpha_1}^{\hat{\nu}}(m_{1,1}) :$$

$$\left. \begin{array}{l} m_{p,1} \in \frac{1}{3}\mathbb{Z}, \quad 1 \leq p \leq r_1^{(1)}; \\ m_{p,1} \leq -\frac{n_{p,1}}{3} - \frac{2}{3} \sum_{p > p' > 0} \min\{n_{p,1}, n_{p',1}\}, \quad 1 \leq p \leq r_1^{(1)}; \\ m_{p+1,1} \leq m_{p,1} - \frac{2}{3} n_{p,1} \text{ if } n_{p+1,1} = n_{p,1}, \quad 1 \leq p \leq r_1^{(1)} - 1; \\ m_{p,2} \in \mathbb{Z}, \quad 1 \leq p \leq r_2^{(1)}; \\ m_{p,2} \leq -n_{p,2} + \sum_{q=1}^{r_1^{(1)}} \min\{n_{q,1}, n_{p,2}\} - \sum_{p > p' > 0} 2 \min\{n_{p,2}, n_{p',2}\}, \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 1 \leq p \leq r_2^{(1)}; \\ m_{p+1,2} \leq m_{p,2} - 2n_{p,2} \text{ if } n_{p,2} = n_{p+1,2}, \quad 1 \leq p \leq r_2^{(1)} - 1 \end{array} \right\} ,$$

where $r_0^{(1)} := 0$.

Now, using the same proof as in [G], we have:

Proposition 4.1. *The set*

$$\mathcal{B} = \{bv_L : b \in B\} \tag{4.10}$$

spans the principal subspace $W_{L_k}^T$.

5. Proof of linear independence

To prove that the set \mathcal{B} is a linearly independent set, we will use certain maps defined on our principal subspace. The first of these maps is a generalization of the projection map found in [G] to our twisted setting, and the remaining maps were used in [PS1] and [PS2], (see also [L1] and [CaLM4]). The proof of the linear independence of \mathcal{B} is carried out by induction on the linear order on the twisted quasi-particle monomials.

5.1. The projection $\pi_{\mathcal{R}}$. As we mentioned earlier, for $k \geq 1$, we will realize $W_{L_k}^T$ as a subspace of the tensor product of k principal subspaces W_L^T of the basic V_L -module V_L^T .

$$W_{L_k}^T \subset W_L^T \otimes \cdots \otimes W_L^T \subset V_L^{T \otimes k}.$$

For a chosen dual-charge-type

$$\mathcal{R} = \left(r_l^{(1)}, \dots, r_l^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(k)} \right),$$

we define a projection $\pi_{\mathcal{R}}$ of $W_{L_k}^T$ on

$$W_{L_{(r_l^{(k)}, \dots, r_1^{(k)})}}^T \otimes \cdots \otimes W_{L_{(r_l^{(1)}, \dots, r_1^{(1)})}}^T,$$

where $W_{L_{(r_l^{(t)}, \dots, r_1^{(t)})}}^T$ denotes the subspace of W_L^T consisting of the vectors of charges $r_l^{(t)}, \dots, r_1^{(t)}$, for $1 \leq t \leq k$ (see 2.1–2.4). We will also consider a generalization of $\pi_{\mathcal{R}}$, which we denote by the same symbol, to the space of formal series with coefficients in $W_L^{T \otimes k}$.

Using the level 1 relation $x_{2\alpha_i}^{\hat{\nu}}(z) = 0$, with the projection $\pi_{\mathcal{R}}$ for fixed $1 \leq p \leq r_i^{(1)}$, we “place” $n_{p,i}$, generating functions $x_{\alpha_i}^{\hat{\nu}}(z)$ on first $n_{p,i}$ tensor factors:

$$x_{n_{p,i}\alpha_i}^{\hat{\nu}^{(k)}}(z_{p,i})\mathbf{1}_T \otimes x_{n_{p,i}\alpha_i}^{\hat{\nu}^{(k-1)}}(z_{p,i})\mathbf{1}_T \otimes \cdots \otimes x_{n_{p,i}\alpha_i}^{\hat{\nu}^{(2)}}(z_{p,i})\mathbf{1}_T \otimes x_{n_{p,i}\alpha_i}^{\hat{\nu}^{(1)}}(z_{p,i})\mathbf{1}_T,$$

where

$$0 \leq n_{p,i}^{(t)} \leq 1, \quad n_{p,i} = \sum_{t=1}^k n_{p,i}^{(t)}, \quad 1 \leq t \leq k.$$

Now, it follows that the projection of generating function

$$x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(z_{r_l^{(1)},l}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(z_{1,1}) v_L \quad (5.1)$$

of dual-charge-type \mathcal{R} and corresponding charge-type $\mathcal{R}' = (n_{r_l^{(1)},l}, \dots, n_{1,1})$ is

$$\begin{aligned} & \pi_{\mathcal{R}} x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(z_{r_l^{(1)},l}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(z_{1,1}) v_L \\ & \mathbb{C} x_{n_{r_l^{(k)},l}\alpha_l}^{\hat{\nu}^{(k)}}(z_{r_l^{(k)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}^{(k)}}(z_{1,l}) \cdots x_{n_{r_1^{(k)},1}\alpha_1}^{\hat{\nu}^{(k)}}(z_{r_1^{(k)},1}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}^{(k)}}(z_{1,1}) \mathbf{1}_T \\ & \quad \otimes \cdots \otimes \\ & \otimes x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}^{(1)}}(z_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}^{(1)}}(z_{1,l}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}^{(1)}}(z_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}^{(1)}}(z_{1,1}) \mathbf{1}_T, \end{aligned}$$

where $\mathbb{C} \in \mathbb{C}^*$, and

$$0 \leq n_{p,i}^{(t)} \leq 1, 1 \leq t \leq k, n_{p,i}^{(1)} \geq n_{p,i}^{(2)} \geq \cdots \geq n_{p,i}^{(k-1)} \geq n_{p,i}^{(k)}, n_{p,i} = \sum_{t=1}^k n_{p,i}^{(t)},$$

for every every p , $1 \leq p \leq r_i^{(1)}$, $1 \leq i \leq l$.

Also, from above it follows that the projection of bv_L , where $b \in B$ is the monomial

$$x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(m_{1,1}) \quad (5.2)$$

of charge-type \mathcal{R}' and dual-charge-type \mathcal{R} , is a coefficient of the generating function (5.1), which we will denote with $\pi_{\mathcal{R}}bv_L$.

5.2. The maps $\Delta^T(\lambda, -z)$. Following [CalLM4], [CalMPe], and [PS1]–[PS2] for $\gamma \in \mathfrak{h}_{(0)}$ and a character of the root lattice $\theta : L \rightarrow \mathbb{C}$, define

$$\begin{aligned} \tau_{\gamma,\theta} : \bar{\mathfrak{n}}[\hat{\nu}] & \rightarrow \bar{\mathfrak{n}}[\hat{\nu}] \\ x_{\alpha}^{\hat{\nu}}(m) & \mapsto \theta(\alpha)x_{\alpha}^{\hat{\nu}}(m + \langle \alpha_{(0)}, \gamma \rangle). \end{aligned}$$

We note here that in general, $\tau_{\gamma,\theta}$ is a linear map, and for suitably chosen characters θ , it becomes an automorphism of $\bar{\mathfrak{n}}[\hat{\nu}]$, which we extend to an automorphism of $U(\bar{\mathfrak{n}}[\hat{\nu}])$. In particular, we consider maps $\tau_{\gamma,\theta}$ where $\gamma =$

$\gamma_i = (\lambda_i)_{(0)} = \frac{1}{2}(\lambda_i + \nu\lambda_i)$, and where the corresponding characters $\theta = \theta_i$ are chosen as in [PS1]–[PS2] defined by

$$\theta_i(\alpha_j) = (-1)^{\langle \lambda^{(i)}, \alpha_j \rangle},$$

where we choose our subscripts i as follows:

- for A_{2l-1} , let $i = 1, \dots, l$
- for D_l when $v = 2$, let $i = 1, \dots, l-1$
- for D_4 when $v = 3$, let $i = 1, 2$
- E_6 , let $i = 1, 2, 3, 4$.

and define $\lambda^{(i)} = v(\lambda_i)_{(0)}$ for $v = 2, 3$ (see [PS2] and [PSW] for a more general definition of this symbol).

We also consider the related map $\Delta_c(\lambda, x)$ from [CalLM4], [CalMPe], and [PS1]–[PS2].

For $\lambda \in \{\lambda_i | 1 \leq i \leq l\}$, we define the map

$$\Delta^T(\lambda, -z) = (-1)^{\nu\lambda} z^{\lambda(0)} E^+(-\lambda, z).$$

and its constant term $\Delta_c^T(\lambda, -z)$. From [PS1]–[PS2], we have

$$\Delta_c^T(\lambda_i, -z)(x_{\beta}^{\hat{\nu}}(m) \mathbf{1}_T) = \tau_{\gamma_i, \theta_i}(x_{\beta}^{\hat{\nu}}(m)) \mathbf{1}_T. \quad (5.3)$$

More generally, we have linear maps

$$\begin{aligned} \Delta_c^T(\lambda_i, -z) : W_L^T &\rightarrow W_L^T \\ a \cdot \mathbf{1}_T &\mapsto \tau_{\gamma_i, \theta_i}(a) \cdot \mathbf{1}_T, \end{aligned}$$

where $a \in U(\bar{\mathfrak{n}}[\hat{\nu}])$.

Fix $s \leq k$ and consider the map

$$1 \otimes \cdots \otimes \Delta_c^T(\lambda_i, -z) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{s-1 \text{ factors}}.$$

Let $b \in B$ as in (5.2). It follows that

$$(1 \otimes \cdots \otimes 1 \otimes \Delta_c^T(\lambda_i, -z) \otimes 1 \otimes \cdots \otimes 1) \pi_{\mathcal{R}} b v_L$$

is the coefficient of

$$(1 \otimes \cdots \otimes \Delta_c^T(\lambda_i, -z) \otimes 1 \otimes \cdots \otimes 1) \pi_{\mathcal{R}} x_{n_{r_l^{(s)}, l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(s)}, l}) \cdots x_{s\alpha_1}^{\hat{\nu}}(z_{1,1}) v_L,$$

where, from (5.3), it follows that operator $\Delta_c^T(\lambda_i, -z)$ acts only on the s -th tensor row as:

$$\begin{aligned} &\otimes x_{n_{r_l^{(s)}, l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(s)}, l}) \cdots x_{n_{1, l} \alpha_l}^{\hat{\nu}}(z_{1, l}) \cdots x_{n_{r_i^{(s)}, i} \alpha_i}^{\hat{\nu}}(z_{r_i^{(s)}, i}) z_{r_i^{(s)}, i} \cdots \\ &\cdots x_{n_{1, i} \alpha_i}^{\hat{\nu}}(z_{1, i}) z_{1, i} \cdots x_{n_{r_1^{(s)}, 1} \alpha_1}^{\hat{\nu}}(z_{r_1^{(s)}, 1}) \cdots x_{\alpha_1}^{\hat{\nu}}(z_{1,1}) \mathbf{1}_T \otimes, \end{aligned}$$

for i such that $\nu\alpha_i = \alpha_i$ and as

$$\otimes x_{n_{r_l^{(s)}, l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(s)}, l}) \cdots x_{n_{1, l} \alpha_l}^{\hat{\nu}}(z_{1, l}) \cdots x_{n_{r_i^{(s)}, i} \alpha_i}^{\hat{\nu}}(z_{r_i^{(s)}, i}) z_{r_i^{(s)}, i}^{\frac{1}{v}} \cdots$$

$$\cdots x_{n_{1,i}^{(s)}\alpha_i}^{\hat{\nu}}(z_{1,i})z_{1,i}^{\frac{1}{v}} \cdots x_{n_{r_1^{(s)},1}^{(s)}\alpha_1}^{\hat{\nu}}(z_{r_1^{(s)},1}) \cdots x_{\alpha_1}^{\hat{\nu}}(z_{1,1}) \mathbf{1}_T \otimes,$$

if i is such that $\nu\alpha_i \neq \alpha_i$, where $0 \leq n_{p,i}^{(s)} \leq 1$, for $1 \leq p \leq r_i^{(s)}$. By taking the corresponding coefficients, we have

$$(1 \otimes \cdots \otimes 1 \otimes \Delta_c^T(\lambda_i, -z) \otimes 1 \otimes \cdots \otimes 1) \pi_{\mathcal{R}} b v_L = \pi_{\mathcal{R}} b^+ v_L,$$

where

$$b^+ = b(\alpha_l) \cdots b(\alpha_{i+1}) b^+(\alpha_i) b(\alpha_{i-1}) \cdots b(\alpha_1),$$

with

$$b^+(\alpha_i) = x_{n_{r_i^{(1)},i}^{(1)}\alpha_i}^{\hat{\nu}}(m_{r_i^{(1)},i} + 1) \cdots x_{n_{1,i}^{\hat{\nu}}\alpha_i}^{\hat{\nu}}(m_{1,i} + 1)$$

if $\nu\alpha_i = \alpha_i$ and with

$$b^+(\alpha_i) = x_{n_{r_i^{(1)},i}^{(1)}\alpha_i}^{\hat{\nu}}(m_{r_i^{(1)},i} + \frac{1}{v}) \cdots x_{n_{1,i}^{\hat{\nu}}\alpha_i}^{\hat{\nu}}(m_{1,i} + \frac{1}{v})$$

if $\nu\alpha_i \neq \alpha_i$.

5.3. The maps e_{α_i} . Finally, we recall the maps e_{α_i} , which satisfy

$$e_{\alpha_i} : V_L^T \rightarrow V_L^T$$

and their restriction to the principal subspace $W_L^T \subset V_L^T$ where

$$\begin{aligned} e_{\alpha_i} \cdot \mathbf{1}_T &= \frac{2}{\sigma(\alpha_i)} x_{\alpha_i}^{\hat{\nu}}(-1) \cdot \mathbf{1}_T \text{ if } \nu\alpha_i = \alpha_i \\ e_{\alpha_i} \cdot \mathbf{1}_T &= \frac{2}{\sigma(\alpha_i)} x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) \cdot \mathbf{1}_T \text{ if } \nu\alpha_i \neq \alpha_i, \end{aligned}$$

when $v = 2$ and

$$\begin{aligned} e_{\alpha_1} \cdot \mathbf{1}_T &= \frac{3}{\sigma(\alpha_1)} x_{\alpha_1}^{\hat{\nu}}\left(-\frac{1}{3}\right) \cdot \mathbf{1}_T \\ e_{\alpha_2} \cdot \mathbf{1}_T &= \frac{3}{\sigma(\alpha_2)} x_{\alpha_2}^{\hat{\nu}}(-1) \cdot \mathbf{1}_T \end{aligned}$$

if $v = 3$, and commute with our operators $x_{\beta}^{\hat{\nu}}(n)$ via

$$e_{\alpha_i} x_{\beta}^{\hat{\nu}}(n) = C(\alpha_i, \beta) x_{\beta}^{\hat{\nu}}(n - \langle \beta_{(0)}, \alpha_i \rangle) e_{\alpha_i}.$$

Now, assume that we have a monomial

$$b = b(\alpha_l) \cdots b(\alpha_1) x_{s\alpha_1}^{\hat{\nu}}\left(-\frac{s}{2}\right) \in B$$

$$b = x_{n_{r_l^{(1)},l}^{(1)}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{r_1^{(1)},1}^{(1)}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}^{\hat{\nu}}(m_{2,1}) x_{s\alpha_1}^{\hat{\nu}}\left(-\frac{s}{2}\right),$$

of dual-charge-type

$$\mathcal{R} = (r_l^{(1)}, \dots, r_l^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(s)}, 0, \dots, 0)$$

and a projection $\pi_{\mathcal{R}} b v_L$, which is a coefficient of

$$\pi_{\mathcal{R}} x_{n_{r_l^{(1)},l}^{(1)}\alpha_l}^{\hat{\nu}}(z_{r_l^{(1)},l}) \cdots x_{n_{2,1}\alpha_1}^{\hat{\nu}}(z_{1,1}) \mathbf{1}_T \otimes \cdots$$

$$\begin{aligned}
& \cdots \otimes \mathbf{1}_T \otimes x_{\alpha_1}^{\hat{\nu}} \left(-\frac{1}{2}\right) \mathbf{1}_T \otimes \cdots \otimes x_{\alpha_1}^{\hat{\nu}} \left(-\frac{1}{2}\right) \mathbf{1}_T \\
= & \mathbb{C} x_{n_{r_l^{(k)},l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(k)},l}) \cdots x_{n_{1,l} \alpha_l}^{\hat{\nu}}(z_{1,l}) \cdots x_{n_{r_1^{(k)},1} \alpha_1}^{\hat{\nu}}(z_{r_1^{(k)},1}) \cdots x_{n_{2,1} \alpha_1}^{\hat{\nu}}(z_{1,1}) \mathbf{1}_T \\
& \otimes \cdots \otimes \\
& x_{n_{r_l^{(s+1)},l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(s+1)},l}) \cdots x_{n_{1,l} \alpha_l}^{\hat{\nu}}(z_{1,l}) \cdots \\
& \cdots x_{n_{r_1^{(s+1)},1} \alpha_1}^{\hat{\nu}}(z_{r_1^{(s+1)},1}) \cdots x_{n_{2,1} \alpha_1}^{\hat{\nu}}(z_{2,1}) \mathbf{1}_T \\
& x_{n_{r_l^{(s)},l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(s)},l}) \cdots x_{n_{1,l} \alpha_l}^{\hat{\nu}}(z_{1,l}) \cdots x_{n_{r_1^{(s)},1} \alpha_1}^{\hat{\nu}}(z_{r_1^{(s)},1}) \cdots x_{n_{2,1} \alpha_1}^{\hat{\nu}}(z_{2,1}) e_{\alpha_1} \mathbf{1}_T \\
& \otimes \cdots \otimes \\
& \otimes x_{n_{r_l^{(1)},l} \alpha_l}^{\hat{\nu}}(z_{r_l^{(1)},l}) \cdots x_{n_{1,l} \alpha_l}^{\hat{\nu}}(z_{1,l}) \cdots x_{n_{r_1^{(1)},1} \alpha_1}^{\hat{\nu}}(z_{r_1^{(1)},1}) \cdots x_{n_{2,1} \alpha_1}^{\hat{\nu}}(z_{2,1}) e_{\alpha_1} \mathbf{1}_T,
\end{aligned}$$

where $\mathbb{C} \in \mathbb{C}^*$.

If we move the operator $1 \otimes \cdots \otimes 1 \otimes \underbrace{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}}_{s \text{ factors}}$ all the way to the left we will get a projection $\pi_{\mathcal{R}-b'v_L}$, where

$$\mathcal{R}^- = (r_l^{(1)}, \dots, r_l^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(s)} - 1, 0, \dots, 0)$$

and

$$b' = b(\alpha_l) \cdots b'(\alpha_2) b'(\alpha_1),$$

with

$$\begin{aligned}
b'(\alpha_1) &= x_{n_{r_1^{(1)},1} \alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1} + n_{r_1^{(1)},1}^{(1)} + \cdots + n_{r_1^{(1)},1}^{(s)}) \cdots \\
& \cdots x_{n_{2,1} \alpha_1}^{\hat{\nu}}(m_{2,1} + n_{2,1}^{(1)} + \cdots + n_{2,1}^{(s)})
\end{aligned}$$

and

$$\begin{aligned}
b'(\alpha_2) &= x_{n_{r_2^{(1)},2} \alpha_2}^{\hat{\nu}}(m_{r_2^{(1)},2} - \frac{n_{r_2^{(1)},2}^{(1)} + \cdots + n_{r_2^{(1)},2}^{(s)}}{2}) \cdots \\
& \cdots x_{n_{1,2} \alpha_j}^{\hat{\nu}}(m_{1,2} - \frac{n_{1,2}^{(1)} + \cdots + n_{1,2}^{(s)}}{2}).
\end{aligned}$$

Above we only considered the case when $v = 2$ and $\nu \alpha_1 \neq \alpha_1$, but note here that the remaining cases are similar.

5.4. A proof of linear independence. Here we will prove our main theorem:

Theorem 5.1. *The set*

$$\mathcal{B} = \{bv_L : b \in B\}$$

is a basis of the principal subspace $W_{L_k}^T$.

Proof. By Proposition 4.1, the set of monomial vectors \mathcal{B} spans $W_{L_k}^T$, and so it remains to show that \mathcal{B} is linearly independent. To prove linear independence first assume that we have

$$bv_L = 0$$

where

$$\begin{aligned} b = & x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}}(m_{1,l}) \cdots \\ & \cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(m_{1,1}) \in B, \end{aligned}$$

of charge-type

$$\mathcal{R}' = \left(n_{r_l^{(1)},l}, \dots, n_{1,l}; \dots; n_{r_1^{(1)},1}, \dots, n_{1,1} \right)$$

and dual-charge-type

$$\mathcal{R} = \left(r_l^{(1)}, \dots, r_l^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(n_{1,1})} \right),$$

which determines the projection $\pi_{\mathcal{R}}$, so that we have

$$\pi_{\mathcal{R}}bv_L = 0. \tag{5.4}$$

We will assume that $\nu\alpha_1 \neq \alpha_1$ and we will let $s = n_{1,1}$. We apply $1 \otimes \cdots \otimes \Delta_c^T(\lambda_1, -z)^d \otimes \underbrace{1 \otimes \cdots \otimes 1}_{s-1 \text{ factors}}$ to (5.4), where

$$\Delta_c^T(\lambda_1, -z)^d = \underbrace{\Delta_c^T(\lambda_1, -z) \circ \cdots \circ \Delta_c^T(\lambda_1, -z)}_{d \text{ times}}$$

and where $d \in \mathbb{N}$ is selected so that after application of this map the twisted quasi-particle of color 1 and charge s has energy $-\frac{s}{v}$. From the considerations in Subsection 5.2, we have

$$\begin{aligned} & \pi_{\mathcal{R}} x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}}(m_{1,l}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}^+) \cdots \tag{5.5} \\ & \cdots x_{n_{2,1}\alpha_1}^{\hat{\nu}}(m_{2,1}^+) \left(\mathbf{1}_T \otimes \cdots \otimes \mathbf{1}_T \otimes x_{\alpha_1}^{\hat{\nu}}\left(-\frac{1}{v}\right) \mathbf{1}_T \otimes \cdots \otimes x_{\alpha_1}^{\hat{\nu}}\left(-\frac{1}{v}\right) \mathbf{1}_T \right) = 0, \end{aligned}$$

which is a projection of a monomial vector from \mathcal{B} . From (5.5) it follows that

$$\begin{aligned} & (1 \otimes \cdots \otimes 1 \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}) \pi_{\mathcal{R}} x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}}(m_{1,l}) \cdots \\ & \cdots x_{n_{r_2^{(1)},2}\alpha_2}^{\hat{\nu}}(m'_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}^{\hat{\nu}}(m'_{1,2}) \end{aligned}$$

$$x_{r_1^{(1)},1}^{\hat{\nu}} \alpha_1(m'_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}^{\hat{\nu}}(m'_{2,1})v_L = 0,$$

where

$$\mathcal{R}^- = (r_l^{(1)}, \dots, r_l^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(s)} - 1).$$

By injectivity of $1 \otimes \cdots \otimes 1 \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}$, we have

$$\pi_{\mathcal{R}^-} x_{r_l^{(1)},l}^{\hat{\nu}} \alpha_l(m_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}}(m_{1,l}) \cdots x_{n_{r_2^{(1)},2}\alpha_2}^{\hat{\nu}}(m'_{r_2^{(1)},2}) \cdots \quad (5.6)$$

$$\cdots x_{n_{1,2}\alpha_2}^{\hat{\nu}}(m'_{1,2}) x_{r_1^{(1)},1}^{\hat{\nu}} \alpha_1(m'_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}^{\hat{\nu}}(m'_{2,1})v_L = 0.$$

If $v = 2$, for every $2 \leq p \leq r_1^{(1)}$ such that $n_{p,1} = s' \leq s$ we have

$$m'_{p,1} = m_{p,1}^+ + s' \leq -\frac{n_{p,1}}{2} - \sum_{p>p'>0} \min\{n_{p,1}, n_{p',1}\} + s';$$

$$m'_{p+1,1} = m_{p,1}^+ + s' \leq m_{p,1}^+ - n_{p,1} + s' = m'_{p,1} - n_{p,1} \text{ if } n_{p+1,1} = n_{p,1};$$

$$m'_{p,2} = m_{p,2}^+ - \frac{s'}{2} \leq -\frac{n_{p,2}}{2} + \frac{1}{2} \sum_{q=1}^{r_1^{(1)}} \min\{n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} \min\{n_{p,1}, n_{p',1}\} - \frac{s'}{2};$$

$$m'_{p+1,2} = m_{p,2}^+ - \frac{s'}{2} \leq m_{p,2}^+ - n_{p,2} - \frac{s'}{2} = m'_{p,2} - n_{p,2} \text{ if } n_{p+1,2} = n_{p,2}.$$

If $v = 3$, for every $2 \leq p \leq r_1^{(1)}$ and for $n_{p,1} = s' \leq s$ we have

$$m'_{p,1} = m_{p,1}^+ + \frac{2s'}{3} \leq -\frac{n_{p,1}}{3} - \frac{2}{3} \sum_{p>p'>0} \min\{n_{p,1}, n_{p',1}\} + \frac{2s'}{3};$$

$$m'_{p+1,1} = m_{p,1}^+ + \frac{2s'}{3} \leq m_{p,1}^+ - \frac{2}{3}n_{p,1} + \frac{2s'}{3} = m'_{p,1} - \frac{2}{3}n_{p,1} \text{ if } n_{p+1,1} = n_{p,1};$$

$$m'_{p,2} = m_{p,2}^+ - s' \leq -n_{p,2} + \sum_{q=1}^{r_1^{(1)}} \min\{n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} 2 \min\{n_{p,1}, n_{p',1}\} - s';$$

$$m'_{p+1,2} = m_{p,2}^+ - s' \leq m_{p,2}^+ - n_{p,2} - s' = m'_{p,2} - n_{p,2} \text{ if } n_{p+1,2} = n_{p,2}.$$

This shows that in (5.6) we have the projection of a monomial vector from the set \mathcal{B} . In the case when $\nu\alpha_1 = \alpha_1$, with the above procedure will end with a monomial vector as in (5.6), which is also from the set \mathcal{B} , since for every $2 \leq p \leq r_1^{(1)}$ and for $n_{p,1} = s' \leq s$, we have

$$m'_{p,1} = m_{p,1}^+ + 2s' \leq -n_{p,1} - \sum_{p>p'>0} 2 \min\{n_{p,1}, n_{p',1}\} + 2s';$$

$$m'_{p+1,1} = m_{p,1}^+ + 2s' \leq m_{p,1}^+ - 2n_{p,1} + 2s' = m'_{p,1} - 2n_{p,1} \text{ if } n_{p+1,1} = n_{p,1};$$

$$m'_{p,2} = m_{p,2}^+ - s' \leq -n_{p,2} + \sum_{q=1}^{r_1^{(1)}} \min\{n_{q,1}, n_{p,2}\} - \sum_{p>p'>0} 2 \min\{n_{p,1}, n_{p',1}\} - s';$$

$$m'_{p+1,2} = m_{p,2}^+ - s' \leq m_{p,2}^+ - 2n_{p,2} - s' = m'_{p,2} - 2n_{p,2} \text{ if } n_{p+1,2} = n_{p,2}.$$

If we continue in this way, “removing” one by one twisted quasi-particles from the monomial b and by checking in each step that monomial vectors are in the set \mathcal{B} , after finitely many steps we arrive at $v_L = 0$, which is a contradiction.

Now, consider a linear combination of elements from \mathcal{B} satisfying

$$\sum_{a \in A} c_a b_a v_L = 0, \quad (5.7)$$

where $b_a \in B$ are monomials of the same color-type and $c_a \in \mathbb{C}$. We will assume that the monomial b of charge-type \mathcal{R}' and dual-charge-type \mathcal{R} is the smallest monomial in (5.7) with respect to our linear ordering. Recall that the dual-charge-type \mathcal{R} determines the projection $\pi_{\mathcal{R}}$ and note that from the definition of the projection it follows that all monomial vectors $b_a v_L$ in (5.7), with charge-type \mathcal{R}'_a such that $\mathcal{R}'_a > \mathcal{R}'$ (see (3.2)), will be annihilated. So, after applying $\pi_{\mathcal{R}}$ to (5.7), we will have

$$\sum_{a \in A} c_a \pi_{\mathcal{R}} b_a v_L = 0, \quad (5.8)$$

where all monomial vectors are of the same color-charge-type. Now we apply the above-described procedure of using the maps $1 \otimes \cdots \otimes \Delta_c^T(\lambda_1, -z)^d \otimes 1 \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes 1 \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}$ to the smallest element b . During this procedure, all monomial vectors b_a such that $b_a > b$ (see (3.2)) will be annihilated. From (5.8) now we have $\pi_{\mathcal{R}} c_a b v_L = 0$, which then implies $c_a = 0$. Repeating this procedure, after finitely many steps we will get that all coefficients c_a of (5.7) are zero, which proves the theorem. \square

6. Characters of principal subspaces

We define character of principal subspace W_L^T by

$$\text{ch} W_{L_k}^T = \sum_{m, r_1, \dots, r_l \geq 0} \dim W_{L_k(m, r_1, \dots, r_l)}^T q^m y_1^{r_1} \cdots y_l^{r_l},$$

where $W_{L_k(m, r_1, \dots, r_l)}^T$ is a weight subspace spanned by monomial vectors of weight $-m$ and color-type (r_1, \dots, r_l) .

We write the characters of principal subspaces in terms of dual-charge-type parts $r_i^{(s)}$. Therefore, to obtain characters first we rewrite the conditions on the energies of twisted quasi-particles of a basis \mathcal{B} in terms of the dual-charge-type. In the case of $A_{2l-1}^{(2)}$ for fixed color type (r_l, \dots, r_1) , charge-type $\mathcal{R}' = (n_{r_l^{(1)}}, \dots, n_{1,l}; \dots; n_{r_1^{(1)}}, \dots, n_{1,1})$ and dual-charge-type $\mathcal{R} = (r_l^{(1)}, \dots, r_l^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(k)})$ we have:

$$\sum_{p=1}^{r_i^{(1)}} \left(\sum_{p > p' > 0} \min\{n_{p,i}, n_{p',i}\} + \frac{1}{2} n_{p,i} \right) = \frac{1}{2} \sum_{s=1}^k r_i^{(s)2}, \quad 2 \leq i \leq l-1, \quad (6.1)$$

$$\sum_{p=1}^{r_l^{(1)}} \left(\sum_{p>p'>0} 2\min\{n_{p,l}, n_{p',l}\} + n_{p,l} \right) = \sum_{s=1}^k r_l^{(s)2}, \quad (6.2)$$

$$\sum_{p=1}^{r_i^{(1)}} \sum_{q=1}^{r_{i-1}^{(1)}} \frac{1}{2} \min\{n_{p,i}, n_{q,i-1}\} = \frac{1}{2} \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}, \quad 1 \leq i \leq l-1, \quad (6.3)$$

and

$$\sum_{p=1}^{r_l^{(1)}} \sum_{q=1}^{r_{l-1}^{(1)}} \min\{n_{p,l}, n_{q,l-1}\} = \sum_{s=1}^k r_{l-1}^{(s)} r_l^{(s)}. \quad (6.4)$$

Expressions (6.1)–(6.4) are proved by using induction on the level $k \in \mathbb{N}$ of the standard module. We would like to note that obtained expressions are similar to the expressions (5.9) and (5.12) in [G].

For $r \in \mathbb{N}$ set $(q^{\frac{1}{a}}; q^{\frac{1}{a}})_r = (1 - q^{\frac{1}{a}}) \cdots (1 - q^{\frac{r}{a}})$, for $a = 1, a = 2$, or $a = 3$. We have that

$$\frac{1}{(q^{\frac{1}{a}}; q^{\frac{1}{a}})_r} = \sum_{j \geq 0} p_r(j) q^{\frac{j}{a}}, \quad (6.5)$$

where $p_r(j)$ denotes the number of partitions of j with most r parts (cf. [An]).

Now, from the definition of the set \mathcal{B} by using (6.1)–(6.6), for $A_{2l-1}^{(2)}$ we have:

$$\begin{aligned} \text{ch } W_{L_k}^T = & \sum_{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0} \frac{q^{\frac{1}{2}r_1^{(1)2} + \dots + \frac{1}{2}r_1^{(k)2}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_1^{(1)} - r_1^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_1^{(k-1)} - r_1^{(k)}} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_1^{(k)}}} y_1^{r_1^{(1)} + \dots + r_1^{(k)}} \\ & \cdot \sum_{r_2^{(1)} \geq \dots \geq r_2^{(k)} \geq 0} \frac{q^{\frac{1}{2}r_2^{(1)2} + \dots + \frac{1}{2}r_2^{(k)2} - \frac{1}{2}(r_2^{(1)}r_1^{(1)} - \dots - r_2^{(k)}r_1^{(k)})}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_2^{(1)} - r_2^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_2^{(k-1)} - r_2^{(k)}} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_2^{(k)}}} y_2^{r_2^{(1)} + \dots + r_2^{(k)}} \\ & \cdot \dots \dots \dots \\ & \cdot \sum_{r_{l-1}^{(1)} \geq \dots \geq r_{l-1}^{(k)} \geq 0} \frac{q^{\frac{1}{2}r_{l-1}^{(1)2} + \dots + \frac{1}{2}r_{l-1}^{(k)2} - \frac{1}{2}(r_{l-1}^{(1)}r_{l-2}^{(1)} - \dots - r_{l-1}^{(k)}r_{l-2}^{(k)})}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(1)} - r_{l-1}^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(k-1)} - r_{l-1}^{(k)}} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(k)}}} y_{l-1}^{r_{l-1}^{(1)} + \dots + r_{l-1}^{(k)}} \\ & \cdot \sum_{r_l^{(1)} \geq \dots \geq r_l^{(k)} \geq 0} \frac{q^{r_l^{(1)2} + \dots + r_l^{(k)2} - r_l^{(1)}r_{l-1}^{(1)} - \dots - r_l^{(k)}r_{l-1}^{(k)}}{(q)_{r_l^{(1)} - r_l^{(2)}} \cdots (q)_{r_l^{(k-1)} - r_l^{(k)}} (q)_{r_l^{(k)}}} y_l^{r_l^{(1)} + \dots + r_l^{(k)}}. \end{aligned}$$

We calculate characters similarly for the other cases, therefore we will give only expressions for energies of basis twisted quasi-particles which we used.

In the case of $D_l^{(2)}$ for fixed color type (r_{l-1}, \dots, r_1) , charge-type $\mathcal{R}' = (n_{r_{l-1}^{(1)}}, \dots, n_{1, l-1}; \dots; n_{r_1^{(1)}}, \dots, n_{1, 1})$ and dual-charge-type

$$\mathcal{R} = (r_{l-1}^{(1)}, \dots, r_{l-1}^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(k)})$$

we have:

$$\sum_{p=1}^{r_i^{(1)}} \left(\sum_{p>p'>0} 2\min\{n_{p,i}, n_{p',i}\} + n_{p,i} \right) = \sum_{s=1}^k r_i^{(s)2}, \quad 1 \leq i \leq l-2, \quad (6.6)$$

$$\sum_{p=1}^{r_{l-1}^{(1)}} \left(\sum_{p>p'>0} 2\min\{n_{p, l-1}, n_{p', l-1}\} + \frac{1}{2}n_{p, l-1} \right) = \frac{1}{2} \sum_{s=1}^k r_{l-1}^{(s)2}, \quad (6.7)$$

$$\sum_{p=1}^{r_i^{(1)}} \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q, i-1}\} = \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}, \quad 2 \leq i \leq l-1. \quad (6.8)$$

In the case of $E_6^{(2)}$ for fixed color type (r_4, \dots, r_1) , charge-type $\mathcal{R}' = (n_{r_4^{(1)}}, \dots, n_{1, 4}; \dots; n_{r_1^{(1)}}, \dots, n_{1, 1})$ and dual-charge-type $\mathcal{R} = (r_4^{(1)}, \dots, r_4^{(k)}; \dots; r_1^{(1)}, \dots, r_1^{(k)})$ we have:

$$\sum_{p=1}^{r_i^{(1)}} \left(\sum_{p>p'>0} \min\{n_{p,i}, n_{p',i}\} + \frac{1}{2}n_{p,i} \right) = \frac{1}{2} \sum_{s=1}^k r_i^{(s)2}, \quad 1 \leq i \leq 2, \quad (6.9)$$

$$\sum_{p=1}^{r_i^{(1)}} \left(\sum_{p>p'>0} 2\min\{n_{p,i}, n_{p',i}\} + n_{p,i} \right) = \sum_{s=1}^k r_i^{(s)2}, \quad 3 \leq i \leq 4 \quad (6.10)$$

$$\sum_{p=1}^{r_i^{(1)}} \sum_{q=1}^{r_{i-1}^{(1)}} \frac{1}{2} \min\{n_{p,i}, n_{q, i-1}\} = \frac{1}{2} \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}, \quad 1 \leq i \leq 2, \quad (6.11)$$

and

$$\sum_{p=1}^{r_i^{(1)}} \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q, i-1}\} = \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}, \quad 3 \leq i \leq 4. \quad (6.12)$$

In the case of $D_4^{(3)}$ for fixed color type (r_2, r_1) , charge-type $\mathcal{R}' = (n_{r_2^{(1)}}, \dots, n_{1, 2}; n_{r_1^{(1)}}, \dots, n_{1, 1})$ and dual-charge-type $\mathcal{R} = (r_2^{(1)}, \dots, r_2^{(k)}; r_1^{(1)}, \dots, r_1^{(k)})$ we have:

$$\sum_{p=1}^{r_1^{(1)}} \left(\frac{2}{3} \sum_{p>p'>0} \min\{n_{p,1}, n_{p',1}\} + \frac{1}{3}n_{p,1} \right) = \frac{1}{3} \sum_{s=1}^k r_1^{(s)2}. \quad (6.13)$$

$$\sum_{p=1}^{r_2^{(1)}} \left(\sum_{p > p' > 0} 2\min\{n_{p,2}, n_{p',2}\} + n_{p,2} \right) = \sum_{s=1}^k r_2^{(s)2}, \quad (6.14)$$

and

$$\sum_{p=1}^{r_2^{(1)}} \sum_{q=1}^{r_1^{(1)}} \min\{n_{p,2}, n_{q,1}\} = \sum_{s=1}^k r_1^{(s)} r_2^{(s)}. \quad (6.15)$$

From the above equations, we now have:

Theorem 6.1. *For each of the affine Lie algebras $A_{2l-1}^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$, the principal subspace $W_{L_k}^T$ of $L^{\hat{\nu}}(k\Lambda_0)$ has multigraded dimension given by:*

- for $A_{2l-1}^{(2)}$:

$$\begin{aligned} \text{ch } W_{L_k}^T &= \sum_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ \dots \\ r_{l-1}^{(1)} \geq \dots \geq r_{l-1}^{(k)} \geq 0}} \frac{q^{\frac{1}{2} \sum_{i=1}^{l-1} \sum_{s=1}^k r_i^{(s)2} - \frac{1}{2} \sum_{i=2}^{l-1} \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}}}{\prod_{i=1}^{l-1} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(1)} - r_i^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(k)}}}} \prod_{i=1}^{l-1} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \\ &\cdot \sum_{r_l^{(1)} \geq \dots \geq r_l^{(k)} \geq 0} \frac{q^{\sum_{s=1}^k r_l^{(s)2} - \sum_{s=1}^k r_{l-1}^{(s)} r_l^{(s)}}}{(q)_{r_l^{(1)} - r_l^{(2)}} \cdots (q)_{r_l^{(k)}}}} y_l^{r_l^{(1)} + \dots + r_l^{(k)}} \end{aligned}$$

- for $D_l^{(2)}$:

$$\begin{aligned} \text{ch } W_{L_k}^T &= \sum_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ \dots \\ r_{l-2}^{(1)} \geq \dots \geq r_{l-2}^{(k)} \geq 0}} \frac{q^{\sum_{i=1}^{l-2} \sum_{s=1}^k r_i^{(s)2} - \sum_{i=2}^{l-2} \sum_{s=1}^k r_{i-1}^{(s)} r_i^{(s)}}}{\prod_{i=1}^{l-2} (q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q)_{r_i^{(k)}}}} \prod_{i=1}^{l-2} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \\ &\cdot \sum_{r_{l-1}^{(1)} \geq \dots \geq r_{l-1}^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{s=1}^k r_{l-1}^{(s)2} - \sum_{s=1}^k r_{l-2}^{(s)} r_{l-1}^{(s)}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(1)} - r_{l-1}^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_{l-1}^{(k)}}}} y_{l-1}^{r_{l-1}^{(1)} + \dots + r_{l-1}^{(k)}} \end{aligned}$$

- for $E_6^{(2)}$:

$$\begin{aligned} \text{ch } W_{L_k}^T &= \sum_{\substack{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0 \\ r_2^{(1)} \geq \dots \geq r_2^{(k)} \geq 0}} \frac{q^{\frac{1}{2} \sum_{i=1}^2 \sum_{s=1}^k r_i^{(s)2} - \sum_{s=1}^k r_1^{(s)} r_2^{(s)}}}{\prod_{i=1,2} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(1)} - r_i^{(2)}} \cdots (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{r_i^{(k)}}}} \prod_{i=1,2} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \\ &\cdot \sum_{\substack{r_3^{(1)} \geq \dots \geq r_3^{(k)} \geq 0 \\ r_4^{(1)} \geq \dots \geq r_4^{(k)} \geq 0}} \frac{q^{\sum_{i=3}^4 \sum_{s=1}^k r_i^{(s)2} - \sum_{s=1}^k (r_2^{(s)} r_3^{(s)} + r_3^{(s)} r_4^{(s)})}}{\prod_{i=3,4} (q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q)_{r_i^{(k)}}}} \prod_{i=3,4} y_i^{r_i^{(1)} + \dots + r_i^{(k)}} \end{aligned}$$

- for $D_4^{(3)}$:

$$\text{ch } W_{L_k}^T = \sum_{r_1^{(1)} \geq \dots \geq r_1^{(k)} \geq 0} \frac{q^{\frac{1}{3} \sum_{s=1}^k r_1^{(s)2}}}{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_{r_1^{(1)} - r_1^{(2)}} \cdots (q^{\frac{1}{3}}; q^{\frac{1}{3}})_{r_1^{(k)}}} y_1^{r_1^{(1)} + \dots + r_1^{(k)}} \\ \cdot \sum_{r_2^{(1)} \geq \dots \geq r_2^{(k)} \geq 0} \frac{q^{\sum_{s=1}^k r_2^{(s)2} - \sum_{s=1}^k r_1^{(s)} r_2^{(s)}}}{(q)_{r_2^{(1)} - r_2^{(2)}} \cdots (q)_{r_2^{(k)}}} y_2^{r_2^{(1)} + \dots + r_2^{(k)}} .$$

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