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Some new results about the smallest ideal of βS

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Abstract. We provide what we believe is the first nontrivial algebraic description of the smallest ideal of the Stone-Čech compactification of a discrete semigroup. Specifically, given sets A and B, let $S = \mathbf{R}(A,B) = A \times B$ with the discrete topology and the rectangular semigroup operation $(a,b)\cdot(c,d) = (a,d)$. Then the smallest ideal of βS is isomorphic (but not homeomorphic) to $\mathbf{R}(\beta A, \beta B)$. We also determine exactly the topological center of βS. The minimal left ideals of βS all have isolated points. We derive several results about βS that must hold for any semigroup S if the minimal left ideals have isolated points.

Contents

1. Introduction

If S is a discrete space, we take the Stone-Čech compactification βS of S to be the set of ultrafilters on S with the point $x \in S$ being identified with the principal ultrafilter $\{A \subseteq S : x \in A\}$. For every subset A of S, let $\overline{A} = \{p \in \beta S : A \in p\}$. The topology of βS is defined by choosing the sets of the form $\overline{A}$ as a base for the open sets. Then βS is a compact Hausdorff space, and, for every subset A of S, $\overline{A}$ is a clopen subset of βS equal to $c\ell_{\beta S}(A)$. If $A \subseteq S$, then $A^* = \overline{A} \setminus A$.

By the defining property of the Stone-Čech compactification, every function f mapping S to a compact Hausdorff space C, has a continuous extension mapping βS to C. So $\lim_{s\to p} f(s)$, where s denotes an element of S, is defined for every $p \in \beta S$. See [5, Section 3.5].

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If $(S,\cdot)$ is a semigroup, then the operation extends to βS so that $(\beta S,\cdot)$ is a right topological semigroup with S contained in its topological center. That is, for each $p\in\beta S$, the function $\rho_p:\beta S\to\beta S$ defined by $\rho_p(q)=q\cdot p$ is continuous and for each $x\in S$, the function $\lambda_x:\beta S\to\beta S$ defined by $\lambda_x(q)=x\cdot q$ is continuous. So, if p and q are elements of βS, $p\cdot q=\lim\limits_{s\to p}\lim\limits_{t\to q} s\cdot t$, where s and t denote elements of S. (This is because, given $s\in S$, $\lim_{t\to q} s\cdot t=s\cdot q$ since λ_s is continuous and $\lim_{s\to p} s\cdot q=p\cdot q$ since ρ_q is continuous.) A subset A of S is a member of $p\cdot q$ if and only if $\{s\in S: s^{-1}A\in q\}\in p$, where $s^{-1}A=\{t\in S: s\cdot t\in A\}$. Equivalently, $A\in p\cdot q$ if and only if $\{s\in S: s\cdot q\in \overline{A}\}\in p$. See [5, Part I] for an elementary introduction to the structure of $(\beta S,\cdot)$.

As does any compact Hausdorff right topological semigroup, $(\beta S,\cdot)$ has a smallest two sided ideal, $K(\beta S)$, which is the union of all of the minimal left ideals of βS and is also the union of all of the minimal right ideals of βS. If R is a minimal right ideal of βS and L is a minimal left ideal of βS, then $R\cap L$ is a group and any two such groups are isomorphic. This group is called the structure group of βS. Every left ideal of βS contains a minimal left ideal and every right ideal of βS contains a minimal right ideal. (Since minimal left ideals are compact, the first assertion is immediate. The second takes some work [5, Theorem 2.7].)

From a combinatorial point of view, the smallest ideal of βS is very important. For example, a subset C of S is *central* if and only if it is a member of an idempotent in $K(\beta S)$. As a consequence of that description, one knows that whenever S is partitioned into finitely many cells, one of them must be central. And central sets are guaranteed to have substantial combinatorial structure; for example, in $(\mathbb{N},+)$, a central set must contain solutions to every partition regular system of homogeneous linear equations with rational coefficients.

There are many examples where the algebraic structure of $K(\beta S)$ is known precisely. If S is a right zero semigroup (i.e., $x\cdot y=y$ for all x and y in S), then so is βS and if S is left zero, then so is βS. (See [5, Exercises 4.2.1 and 4.2.2].) In either case, $K(\beta S)=\beta S$.

For any linearly ordered set S, define $x\vee y=\max\{x,y\}$ and $x\wedge y=\min\{x,y\}$. Then $K(\beta\mathbb{N},\wedge)=\{1\}$ and $K(\beta\mathbb{N},\vee)=\mathbb{N}^*=\beta\mathbb{N}\setminus\mathbb{N}$ with the operation $p\vee q=q$ for $p,q\in\mathbb{N}^*$. That is, $K(\beta\mathbb{N},\vee)=\mathbb{N}^*$ with the right zero operation. More generally, if κ is an infinite cardinal, κ_d is κ with the discrete topology, and $C=\{p\in\beta\kappa_d:\text{every member of } p \text{ is cofinal in } \kappa\}$, then $K(\kappa_d,\vee)=C$ and C is right zero. (If κ is regular, then C is the set of uniform ultrafilters on κ.)

These examples (with the possible exception of $(\mathbb{N},\wedge)$) are all nice things to know. But they are all easy exercises to establish. In this paper we present an explicit algebraic description of $K(\beta S)$, where S is an arbitrary rectangular semigroup. While the proofs are not difficult, we don't think that they qualify as easy exercises.

Definition 1.1. Let A and B be nonempty sets. The *rectangular semigroup generated by* A *and* B is the set $\mathbf{R}(A,B) = A \times B$ with the discrete topology and the operation $\cdot$ defined by $(a,b)\cdot(c,d) = (a,d)$.

Rectangular semigroups arise in $K(\beta S)$ if one knows that the maximal groups in $K(\beta S)$ are trivial. By the Structure Theorem [5, Theorem 1.64], $K(\beta S)$ can be described, as a set, as $X \times G \times Y$, where X is the set of idempotents in a chosen minimal left ideal of βS, Y is the set of idempotents in a chosen right ideal of βS and G is the structure group. Since X is a left zero semigroup and Y is a right zero semigroup, $X \times Y$ is a rectangular semigroup. In the case in which all the elements in $K(\beta S)$ are idempotent, $|G| = 1$ and $K(\beta S)$ is algebraically isomorphic to $\mathbf{R}(X,Y)$.

Further, rectangular semigroups are known to play a significant role in the structure of $K(\beta\mathbb{N},+)$ (whose maximal groups are far from trivial, since each one contains a copy of the free group on $2^{\mathfrak{c}}$ generators [5, Corollary 7.37]). In [8], Yevhen Zelenyuk solved a long standing open problem by showing that there exist idempotents in $K(\beta\mathbb{N},+)$ whose sum is an idempotent not equal to either of them by showing that $K(\beta\mathbb{N},+)$ contains copies of $\mathbf{R}(A,A)$ for any finite nonempty set A. Later, it was shown in [7, Corollary 3.13] that if $|A| = 2^{\mathfrak{c}}$, then $K(\beta\mathbb{N},+)$ contains an algebraic copy of $\mathbf{R}(A,A)$; and if D is any subsemigroup of $K(\beta\mathbb{N},+)$ consisting of idempotents, then D is isomorphic to a subsemigroup of $\mathbf{R}(A,A)$. It follows that, if S is any infinite discrete cancellative semigroup, then $\mathbf{R}(A,A)$ occurs in βS, because βS contains semigroups topologically isomorphic to $\mathbb{H}$, a subsemigroup of $\beta\mathbb{N}$ which contains all the idempotents of $\beta\mathbb{N}$ [5, Theorem 6.32].

In Section 2, we determine a simple explicit description of an isomorphism from $K\big(\beta\big(\mathbf{R}(A,B)\big)\big)$ onto $\mathbf{R}(\beta A,\beta B)$. We also obtain substantial information about the minimal left ideals, minimal right ideals, and the topological center.

It turns out that all minimal left ideals of $\beta\big(\mathbf{R}(A,B)\big)$ have isolated points. Because of the relationship with minimal dynamical systems, there is substantial interest in minimal left ideals. For example, for any discrete semigroup S, the minimal closed invariant subsets of the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$ are precisely the minimal left ideals of βS and given a minimal left ideal L of βS, up to a homeomorphism respecting the action of S, $(L, \langle\lambda_{s|L}\rangle_{s\in S})$ is the unique universal minimal dynamical system for S [5, Lemma 19.6 and Theorem 19.10]. We shall see in Theorem 3.13, that the assumption that a minimal left ideal of βS has an isolated point, is equivalent to a strong recurrence property of the dynamical systems $(\langle T_s\rangle_{s\in S}, X)$. In Section 3 we obtain several results about semigroups S with the property that the minimal left ideals have isolated points.

2. The algebraic structure of $K(\beta(\mathrm{R}(A, B)))$

We do not require that the sets A and B be infinite, but if they are both finite, our results are trivial. In the following theorem, we list some of the simple basic properties of $K\big(\beta\big(\mathbf{R}(A,B)\big)\big)$.

Theorem 2.1. *Let A and B be nonempty discrete sets and let $S = \mathbf{R}(A,B)$. Let $\pi_1 : \beta S \to \beta A$ and $\pi_2 : \beta S \to \beta B$ be the continuous extensions of the projection functions. For $x \in \beta A$ and $y \in \beta B$, let $R_x = \{p \in \beta S : \pi_1(p) = x\}$ and let $L_y = \{p \in \beta S : \pi_2(p) = y\}$.*

(1) *For every $p, q, r \in \beta S$, $p \cdot q \cdot r = p \cdot r$.*
(2) *Let $p \in \beta S$. The following statements are equivalent.*
 (a) *$p \in K(\beta S)$.*
 (b) *p is an idempotent.*
 (c) *p is a product in βS.*
 In particular, $S \subseteq K(\beta S)$.
(3) *For $p, q \in \beta S$, $\pi_1(p \cdot q) = \pi_1(p)$ and $\pi_2(p \cdot q) = \pi_2(q)$.*
(4) *For $x \in \beta A$ and $y \in \beta B$, R_x is a right ideal of βS and L_y is a left ideal of βS.*
(5) *For every $x \in \beta A$ and every $y \in \beta B$, $|R_x \cap L_y \cap K(\beta S)| = 1$.*
(6) *If $x \in \beta A$, $R_x \cap K(\beta S)$ is a minimal right ideal of βS.*
(7) *If $y \in \beta B$, $L_y \cap K(\beta S)$ is a minimal left ideal of βS.*
(8) *For every $x \in \beta A$, R_x is a minimal right ideal of βS if and only if $|R_x \cap L_y| = 1$ for every $y \in \beta B$.*
(9) *For every $y \in \beta B$, L_y is a minimal left ideal of βS if and only if $|R_x \cap L_y| = 1$ for every $x \in \beta A$.*
(10) *The mapping $p \mapsto \big(\pi_1(p), \pi_2(p)\big)$ is an algebraic isomorphism from $K(\beta S)$ onto $\mathbf{R}(\beta A, \beta B)$.*

Proof. (1) If $p, q, r \in \beta S$, $p \cdot q \cdot r = \lim_{s \to p} \lim_{t \to q} \lim_{u \to r} s \cdot t \cdot u = \lim_{s \to p} \lim_{t \to q} \lim_{u \to r} s \cdot u = \lim_{s \to p} \lim_{t \to q} s \cdot r = \lim_{s \to p} s \cdot r = p \cdot r$, where s, t, u denote elements of S.

(2) (a) implies (c) because $\beta S \cdot \beta S$ is an ideal of βS. By (1), (c) implies (b). And (1) also establishes that (b) implies (a) because, if $p = p \cdot p \in \beta S$ and $q \in K(\beta S)$, then $p \cdot p = p \cdot q \cdot p \in K(\beta S)$.

In particular, $S \subseteq K(\beta S)$ because every element of S is idempotent.

(3) Note that for $s, t \in S$, $\pi_1(s \cdot t) = \pi_1(s)$ and $\pi_2(s \cdot t) = \pi_2(t)$. If $p, q \in \beta S$ and s and t denote members of S, $\pi_1(p \cdot q) = \lim_{s \to p} \lim_{t \to q} \pi_1(s \cdot t) = \lim_{s \to p} \lim_{t \to q} \pi_1(s) = \lim_{s \to p} \pi_1(s) = \pi_1(p)$ and $\pi_2(p \cdot q) = \lim_{s \to p} \lim_{t \to q} \pi_2(s \cdot t) = \lim_{s \to p} \lim_{t \to q} \pi_2(t) = \lim_{s \to p} \pi_2(q) = \pi_2(q)$.

(4) Let $x \in \beta A$ and $y \in \beta B$. Then $\pi_1 : S \to A$ and $\pi_2 : S \to B$ are surjective so $R_x \neq \emptyset$ and $L_y \neq \emptyset$. By (3), R_x is a right ideal of βS and L_y is a left ideal of βS.

(5) Let p_1 and p_2 be elements of $R_x \cap L_y \cap K(\beta S)$. We shall show that $p_1 = p_2$.

Let $P_1 \in p_1$ and $P_2 \in p_2$. It suffices to show that $P_1 \cap P_2 \neq \emptyset$. Let $P_1' = \{s \in S : s \cdot p_1 \in \overline{P}_1\}$ and $P_2' = \{s \in S : s \cdot p_2 \in \overline{P}_2\}$. Since p_1 and p_2 are idempotent, $P_1' \in p_1$ and $P_2' \in p_2$. This implies that $\pi_1[P_1'] \cap \pi_1[P_2'] \neq \emptyset$, because both these sets are members of x. So we can choose $a \in A$ and $b_1, b_2 \in B$ such that $(a, b_1) \in P_1'$ and $(a, b_2) \in P_2'$.

Let $P_1'' = \{t \in S : (a, b_1)t \in P_1\}$ and $P_2'' = \{t \in S : (a, b_2)t \in P_2\}$. Then $P_1'' \in p_1$ and $P_2'' \in p_2$. So $\pi_2[P_1''] \cap \pi_2[P_2''] \neq \emptyset$, because both these sets are members of y. So we can choose $c_1, c_2 \in A$ and $d \in B$ such that $(c_1, d) \in P_1''$ and $(c_2, d) \in P_2''$. Then $(a, b_1)(c_1, d) = (a, d) \in P_1$ and $(a, b_2)(c_2, d) = (a, d) \in P_2$.

(6) If $x \in \beta A$, R_x contains a minimal right ideal R. We claim that $R = R_x \cap K(\beta S)$. To see this, assume that $p \in (R_x \cap K(\beta S)) \setminus R$. If $y = \pi_2(p)$, $p \in R_x \cap L_y \cap K(\beta S)$. Since $R \cap L_y \neq \emptyset$, there is a point q in this set. So $R_x \cap L_y \cap K(\beta S)$ contains two distinct points p and q – contradicting (5).

(7) This is proved in the same way as (6).

(8) Let $x \in \beta A$. If R_x is a minimal right ideal in βS, then $R_x \subseteq K(\beta S)$, and so $|R_x \cap L_y| = 1$ for every $y \in \beta B$ by (5).

Conversely, suppose that $|R_x \cap L_y| = 1$ for every $y \in \beta B$. If R_x is not a minimal right ideal in βS, then $R_x \not\subseteq K(\beta S)$ by (6). So there exists $p \in R_x \setminus K(\beta S)$. Let $\pi_2(p) = y$. Since L_y meets the right ideal $R_x \cap K(\beta S)$, there exists a point $q \in R_x \cap L_y \cap K(\beta S)$. Then p and q are distinct points in $R_x \cap L_y$ – a contradiction.

(9) This is proved in the same way as (8).

(10) Let $\phi : K(\beta S) \to \mathbf{R}(\beta A, \beta B)$ be defined by $\phi(p) = (\pi_1(p), \pi_2(p))$. For every $x \in \beta A$ and every $y \in \beta B$, R_x and L_y are respectively a right ideal and a left ideal in βS (by (4)), and so $R_x \cap L_y \cap K(\beta S) \neq \emptyset$. So ϕ is surjective. It follows from (5) that ϕ is injective, and it follows from (3) that ϕ is a homomorphism.

□

As a set, $\mathbf{R}(\beta A, \beta B) = \beta A \times \beta B$, and can naturally be given the product topology. It is reasonable to ask whether $K(\beta S)$ is also homeomorphic to $\mathbf{R}(\beta A, \beta B)$ with this topology. We shall see in Theorem 2.3 below that $K(\beta S) \neq \beta S$. Since $S \subseteq K(\beta S)$, $K(\beta S)$ is dense in βS, and so $K(\beta S)$ is not compact.

In fact, more generally, for any semigroup S, $K(\beta S)$ cannot be homeomorphic to $\beta A \times \beta B$ for any infinite discrete spaces A and B. This is because by [4, Exercises 4K1 and 14Q1], the product of two infinite compact F-spaces cannot be an F-space and if $K(\beta S)$ were homeomorphic to $\beta A \times \beta B$, it would be a compact subset of βS, and therefore an F-space.

Theorem 2.2. *Let A and B be nonempty sets, let $S = \mathbf{R}(A, B)$, let $a \in A$ and let $b \in B$. Then R_a is a minimal right ideal of βS and L_b is a minimal left ideal of βS. Consequently, R_a is a right zero semigroup and L_b is a left* *zero semigroup. Furthermore, the minimal left ideals of βS have isolated points.*

Proof. We show that $R_a \subseteq K(\beta S)$ so that by Theorem 2.1(6), R_a is a minimal left ideal of βS. Let $p \in R_a$. By Theorem 2.1(2) it suffices to show that p is a product. Since $p \in R_a$, $\{a\} \times B = \pi_1^{-1}[\{a\}] \in p$. Let $b \in B$. We show that $(a, b) \cdot p = p$ by showing that $p \subseteq (a, b) \cdot p$ so let $P \in p$. Then $(\{a\} \times B) \cap P \subseteq (a, b)^{-1}P$ so $P \in (a, b) \cdot p$.

Similarly, L_b is a minimal left ideal of βS.

That R_a is a right zero semigroup and L_b is a left zero semigroup now follows from the isomorphism defined in Theorem 2.1(10).

To see that the minimal left ideals of βS have isolated points, let $a \in A$ and $y \in \beta S$. Let $L = L_y \cap K(\beta S)$, which is a minimal left ideal of βS by Theorem 2.1(7). Then R_a meets L in a unique point, by Theorem 2.1(5). This is an isolated point of L, because $R_a = \pi_1^{-1}[\{a\}]$ is an open subset of βS. $\square$

We now determine precisely the conditions under which R_x and L_y are minimal right or left ideals.

Theorem 2.3. *Assume that A and B are infinite, $A \subseteq B$ or $B \subseteq A$, $S = \mathbf{R}(A, B)$, and $x \in (A \cap B)^*$. Then there are points in $R_x \cap L_x$ that are not products. In particular, R_x is not a minimal right ideal of βS and L_x is not a minimal left ideal of βS.*

Proof. Pick a net $\langle a_\iota \rangle_{\iota \in D}$ in $A \cap B$ which converges to x. Pick a cluster point p of the net $\langle (a_\iota, a_\iota) \rangle_{\iota \in D}$ in βS. Then $\pi_1(p) = \pi_2(p) = x$ so $p \in R_x \cap L_x$. We shall show that p is not a product in βS so by Theorem 2.1(2), neither R_x nor L_x is contained in $K(\beta S)$.

Suppose $p = q \cdot r$ for some q and r in βS. Let $D = \{(a, a) : a \in A \cap B\}$. Then $D \in p$ so $\{s \in S : s^{-1}D \in r\} \in q$. Pick $s = (a, b) \in S$ such that $s^{-1}D \in r$. Since $\pi_2(r) = \pi_2(p) = x \in B^*$, $A \times \{a\} \notin r$. Pick $t = (c, d) \in S \setminus (A \times \{a\})$ such that $t \in s^{-1}D$. Then $s \cdot t = (a, d) \in D$ and $d \neq a$, a contradiction. $\square$

Recall that if κ is a cardinal, an ultrafilter p is *κ-complete* if and only if whenever $\mathcal{A} \subseteq p$ and $|\mathcal{A}| < \kappa$, one has that $\bigcap \mathcal{A} \in p$. Thus, if $0 < \kappa \leq \omega$, every ultrafilter is κ-complete. A cardinal κ is *Ulam-measurable* if there exists an ω^+-complete non-principal ultrafilter on a set of size κ. (For any cardinal κ, κ^+ is the smallest cardinal greater than κ.) And κ is *measurable* if there exists a κ-complete non-principal ultrafilter on a set of size κ. By [3, Theorem 8.31], the first Ulam-measurable cardinal is the first uncountable measurable cardinal. By the theorem on page 193 of [3], any measurable cardinal is strongly inaccessible (so the existence of uncountable meaurable cardinals cannot be established in ZFC).

It is a routine exercise to establish that an ultrafilter p on a set B is κ-complete if and only if whenever $|X| < \kappa$ and $f : B \to X$, there is some member of p on which f is constant.

Theorem 2.4. *Let A and B be nonempty sets, let $S = \mathbf{R}(A, B)$, let $\kappa = |B|$, and let $x \in \beta A$. The following statements are equivalent.*

(a) *x is κ^+-complete.*
(b) *R_x is a minimal right ideal of βS.*

Proof. (a) $\Rightarrow$ (b). Assume that x is κ^+-complete. We shall show that, if $y \in \beta B$, $|R_x \cap L_y| = 1$. It will then follow from Theorem 2.1(8) that R_x is a minimal right ideal of βS.

So assume that there are two disinct points p_1 and p_2 in $R_x \cap L_y$. Then we can partition S into disjoint sets D_1 and D_2 for which $D_1 \in p_1$ and $D_2 \in p_2$. Observe that, for each $b \in B$, the sets $\{a \in A : (a, b) \in D_1\}$ and $\{a \in A : (a, b) \in D_2\}$ form a disjoint partition of A. Let $B_1 = \{b \in B : \{a \in A : (a, b) \in D_1\} \in x\}$ and $B_2 = \{b \in B : \{a \in (a, b) \in D_2\} \in x\}$. Then B_1 and B_2 form a disjoint partition of B. We may assume without loss of generality that $B_1 \in y$.

For every $b \in B_1$, $\{a \in A : (a, b) \in D_2\} \notin x$. Since $|B_1| \leq \kappa$, it follows that $\bigcup_{b \in B_1} \{a \in A : (a, b) \in D_2\} = \pi_1[\pi_2^{-1}[B_1] \cap D_2] \notin x$. Since $B_1 \in y$, $\pi_2^{-1}[B_1] \in p_2$ and so $\pi_2^{-1}[B_1] \cap D_2 \in p_2$. This contradicts our asssumption that $\pi_1(p_2) = x$.

(b) $\Rightarrow$ (a) Assume that R_x is a minimal right ideal of βS and suppose that x is not κ^+-complete. Pick $f : A \to B$ such that f is not constant on any member of x. We extend f to a continuous function from βA to βB, which we shall also denote by f. Let $y = f(x)$. Since f is not constant on any member of x we have $y \in B^*$. Let $\langle a_\iota \rangle_{\iota \in D}$ be a net in A which converges to x in βA. Let q be a cluster point of the net $\big\langle \big(a_\iota, f(a_\iota)\big) \big\rangle_{\iota \in D}$ in βS. Since $\pi_1(q) = x$ we have that $q \in R_x$ so $q \in K(\beta S)$ and thus, by Theorem 2.1(2), q is an idempotent. Note that $\pi_2(q) = y$.

Let $E = \big\{\big(a, f(a)\big) : a \in A\big\}$. Then $E \in q = q \cdot q$ so $\{s \in S : s^{-1}E \in q\} \in q$. Pick $s = (a, b) \in S$ such that $s^{-1}E \in q$. Since $\pi_2(q) = y \neq f(a)$, $\{t \in S : \pi_2(t) \neq f(a)\} \in q$. Pick $t = (c, d) \in s^{-1}E$ such that $d \neq f(a)$. Then $(a, d) = (a, b) \cdot (c, d) \in E$, a contradiction. $\square$

Theorem 2.5. *Let A and B be nonempty sets, let $S = \mathbf{R}(A, B)$, let $\mu = |B|$, and let $y \in \beta B$. The following statements are equivalent.*

(a) *y is μ^+-complete.*
(b) *L_y is a minimal left ideal of βS.*

Proof. The proof that (a) $\Rightarrow$ (b) is a right-left switch of the corresponding part of the proof of Theorem 2.4.

(b) $\Rightarrow$ (a) Assume that L_y is a minimal left ideal and suppose that y is not μ^+-complete. Pick $f : B \to A$ such that f is not constant on any member of y. Denote also by f the continuous extension of f taking βB

to βA. Observe that $f(y) \in A^*$. Let $\langle b_\iota\rangle_{\iota\in D}$ be a net in B converging to y. Let q be a cluster point in βS of $\langle (f(b_\iota), b_\iota)\rangle_{\iota\in D}$. Then $\pi_2(q) = y$ so $q \in L_y \subseteq K(\beta S)$ and so by Theorem 2.1(2), $q = q\cdot q$. Note that $\pi_1(q) = f(y)$. Let $E = \{(f(b), b) : b \in B\}$. Then $E \in q$ and so $\{s \in S : s^{-1}E \in q\} \in q$. Choose $s = (a, b)$ such that $s^{-1}E \in q$.

Let $C = \{(c, d) \in S : f(d) \neq a\}$. Then $C \in q$ because otherwise we would have $f\big(\pi_2(q)\big) = f(y) = a$ – contradicting the fact that $f(y) \in A^*$. So we can choose $(c, d) \in C \cap s^{-1}E$. But then $s \cdot (c, d) = (a, d) \notin E$ – a contradiction.

□

Recall that the topological center of a right topological semigroup T is $\{p \in T : \lambda_p$ is continuous$\}$. We now characterize the topological center of $\mathbf{R}(A, B)$.

Theorem 2.6. *Let A and B be nonempty sets, let $S = \mathbf{R}(A, B)$, let $p \in \beta S$, let $\kappa = |B|$, and let $x = \pi_1(p)$. The following statements are equivalent.*

(a) *λ_p is continous.*
(b) *For every $q \in \beta S$, the operation on βS is jointly continuous at (p, q).*
(c) *x is κ^+-complete.*
(d) *For all $y \in \beta B$, $|R_x \cap L_y| = 1$.*
(e) *R_x is a minimal right ideal of βS.*
(f) *$p \cdot \beta S$ is closed in βS.*
(g) *$R_x \cap K(\beta S)$ is closed in βS.*

Proof. The equivalence of (c) and (e) is Theorem 2.4, the equivalence of (d) and (e) is Theorem 2.1(8), (b) trivially implies (a), and (a) trivially implies (f).

(d) ⇒ (b). Assume that for all $y \in \beta B$, $|R_x \cap L_y| = 1$ and let $q \in \beta S$. It suffices to let $\langle (p_\iota, q_\iota)\rangle_{\iota\in D}$ be a net in $\beta S \times \beta S$ converging to (p, q) and show that $\langle p_\iota \cdot q_\iota\rangle_{\iota\in D}$ converges to $p\cdot q$. Note that $\langle p_\iota\rangle_{\iota\in D}$ converges to p and $\langle q_\iota\rangle_{\iota\in D}$ converges to q. We will show that the only cluster point of $\langle p_\iota \cdot q_\iota\rangle_{\iota\in D}$ is $p \cdot q$.

So let r be a cluster point of $\langle p_\iota \cdot q_\iota\rangle_{\iota\in D}$. Passing to a subnet, we may presume that $\langle p_\iota \cdot q_\iota\rangle_{\iota\in D}$ converges to r. Now $\langle \pi_1(p_\iota \cdot q_\iota)\rangle_{\iota\in D}$ converges to $\pi_1(r)$. But for $\iota \in D$, $\pi_1(p_\iota \cdot q_\iota) = \pi_1(p_\iota)$ and $\langle \pi_1(p_\iota)\rangle_{\iota\in D}$ converges to $\pi_1(p) = x$. So $\pi_1(r) = \pi_1(p \cdot q) = \pi_1(p) = x$.

Let $y = \pi_2(q)$. As above, one has $\pi_2(r) = \pi_2(p \cdot q) = \pi_2(q) = y$. So $\{r, p \cdot q\} \subseteq R_x \cap L_y$ and thus $r = p \cdot q$.

(a) ⇒ (c). Assume that λ_p is continuous and suppose that x is not κ^+-complete. Let $f : A \to B$ be a function which is not constant on any member of x and denote also by f its continuous extension from βA to βB. Define $g : A \to S$ by $g(a) = \big(a, f(a)\big)$ and denote also by g its continuous extension from βA to βS. Put $y = f(x)$ and $q = g(x)$. We claim that

$y = \pi_2\big(g\big(\pi_1(p)\big)\big)$. To see this, $\pi_2\big(g\big(\pi_1(p)\big)\big) = \pi_2\big(g(x)\big) = \pi_2\big(g(\lim_{a\to x} a)\big) = \lim_{a\to x} \pi_2\big(a, f(a)\big) = y$, where a denotes an element of A.

Since λ_p is continuous, $p\cdot q = \lim_{a\to x} p\cdot\big(a, f(a)\big)$, where a denotes an element of A.

Linearly order B by $<$. Since f is not constant on any member of x, y is not principal, so for any $c \in B$, either $\{b \in B : b < c\} \in y$ or $\{b \in B : b > c\} \in y$. So

(i) $\{c \in B : \{b \in B : b < c\} \in y\} \in y$ or
(ii) $\{c \in B : \{b \in B : b > c\} \in y\} \in y$.

We may assume that (i) holds, because, if it does not, it would hold with the ordering $<$ on B reversed. So let $C = \{c \in B : \{b \in B : b < c\} \in y\}$ and assume that $C \in y$.

Let $H = \{(a, d) \in S : d < f(a)\}$. We show now that $H \in p\cdot q$. Since $\pi_2\big(g\big(\pi_1(p)\big)\big) = y$, $C \in y$, and for $(a, b) \in S$, $\pi_2\big(g\big(\pi_1\big((a, b)\big)\big)\big) = f(a)$ we have that $P = \{(a, b) \in S : f(a) \in C\} \in p$. To see that $H \in p\cdot q$ it suffices to show that $P \subseteq \{s \in S : s^{-1}H \in q\}$, so let $s = (a, b) \in P$. Then $f(a) \in C$ so $\{d \in B : d < f(a)\} \in y$. Since $\pi_2(q) = y$, pick $Q \in q$ such that $\pi_2[Q] \subseteq \{d \in B : d < f(a)\}$. Given $(c, d) \in Q$, $(a, b)\cdot(c, d) = (a, d) \in H$ as required.

However, we also have $\lim_{a\to x} p\cdot\big(a, f(a)\big) = p\cdot q \in \overline{H}$. so $\{c \in A : p\cdot\big(c, f(c)\big) \in \overline{H}\} \in x$. Also $\pi_2\big(g(x)\big) = y$ and $C \in y$ so pick $X \in x$ such that $\pi_2\big[g[X]\big] \subseteq C$. Pick $c \in X$ such that $p\cdot\big(c, f(c)\big) \in \overline{H}$. Pick $P_1 \in p$ such that $P_1\cdot\big(c, f(c)\big) \subseteq H$. Since $c \in X$, $f(c) \in C$ so $\{d \in B : d < f(c)\} \in y$. Pick $P_2 \in p$ such that $\pi_2\circ g\circ\pi_1[P_2] \subseteq \{d \in B : d < f(c)\}$. Pick $(a, b) \in P_1 \cap P_2$. Since $(a, b) \in P_1$, $\big(a, f(c)\big) = (a, b)\cdot\big(c, f(c)\big) \in H$ so $f(c) < f(a)$. Since $(a, b) \in P_2$, $f(a) < f(c)$. This contradiction completes the proof.

(f)⇒(a) Assume that $p\cdot\beta S$ is closed in βS. By Theorem 2.1(2), $p\cdot\beta S \subseteq K(\beta S)$. Let $q \in \beta S$ and let $\langle q_\iota\rangle_{\iota\in D}$ be a net in βS coinverging to q. We shall show that $p{\cdot}q$ is the only cluster point of the net $\langle p{\cdot}q_\iota\rangle_{\iota\in D}$, so let r be a cluster point of $\langle p\cdot q_\iota\rangle_{\iota\in D}$. By passing to a subnet, we may presume that $\langle p\cdot q_\iota\rangle_{\iota\in D}$ converges to r. Since each $p\cdot q_\iota \in p\cdot\beta S$ which is closed, $r \in p\cdot\beta S \subseteq K(\beta S)$. For each $\iota \in D$, $\pi_1(p{\cdot}q_\iota) = \pi_1(p) = x$ so $\pi_1(r) = \pi_1(p{\cdot}q) = x$. Let $y = \pi_2(q)$. Then for $\iota \in D$, $\pi_2(p\cdot q_\iota) = \pi_2(q_\iota)$ so $\langle\pi_2(p\cdot q_\iota)\rangle_{\iota\in D}$ cohnverges to $\pi_2(q)$ so $\pi_2(r) = \pi_2(q) = y$ and $\pi_2(p\cdot q) = \pi_2(q) = y$. Then $p\cdot q$ and r are both in $R_x \cap L_y \cap K(\beta S)$. Since this set is a singleton, by Theorem 2.1(5), $p\cdot q = r$.

(g)⇒(a) Assume that $R_x \cap K(\beta S)$ is closed in βS. As in the preceding paragraph, let $\langle q_\iota\rangle_{\iota\in D}$ be a net in βS which converges to an element q in βS and assume that $\langle p\cdot q_\iota\rangle_{\iota\in D}$ converges to r. By Theorem 2.1(2), $p\cdot q_\iota \in R_x{\cap}K(\beta S)$ for every $\iota \in D$, and so $r \in K(\beta S)$. We have $\pi_1(p{\cdot}q) = \pi_1(r) = x$ and $\pi_2(p\cdot q) = \pi_2(r) = \pi_2(q)$. Let $y = \pi_2(q)$. Since $R_x \cap L_y \cap K(\beta S)$ is a singleton, by Theorem 2.1(5), $p\cdot q = r$.

(e)⇒ (g) If R_x is a minimal right ideal of βS, $R_x \cap K(\beta S) = R_x$, which is closed in βS since $R_x = \pi_1^{-1}[\{x\}]$. □

Note that the equivalent conditions of Theorem 2.5 neither imply that λ_p is continuous, nor are they implied by the assumption that λ_p is continuous. To see this assume that $|A| = \mu \geq \omega$ and $|B| = \kappa \geq \omega$. Then it is easy to choose p so that $\pi_1(p)$ is in A so is κ^+-complete and $\pi_2(p)$ is not μ^+-complete and vice versa.

Corollary 2.7. *Let A and B be nonempty sets, let $S = \mathbf{R}(A, B)$, and assume that $\kappa = |B|$ is infinite and $|A|$ is not Ulam-measurable. Then the topological center of βS is $\{p \in \beta S : \pi_1(p) \in A\}$.*

Proof. There are no κ^+-complete nonprincipal ultrafilters on A. □

As a consequence of Theorem 2.6, if x is not κ^+-complete, then none of $R_x \cap K(\beta S)$, $K(\beta S)$, or $p \cdot \beta S$ can be closed, where $\pi_1(p) = x$. We will see in Corollary 2.9 that if A and B are infinite, then $K(\beta S)$ is not a Borel set.

In the proof of the following theorem, we use the fact from [6, Lemma 3.1] that if S is any countably infinite discrete set and B is a Borel subset of βS, then there is a set $\mathcal{B}$ of compact subsets of βS with $|\mathcal{B}| \leq \mathfrak{c}$ such that $B = \bigcup \mathcal{B}$.

Theorem 2.8. *Let $S = \mathbf{R}(\mathbb{N}, \mathbb{N})$. Then $K(\beta S)$ is not Borel. If $x \in \mathbb{N}^*$, then $R_x \cap K(\beta S)$ is not Borel. If $p \in \beta S$ and $\pi_1(p) \in \mathbb{N}^*$, then $p \cdot \beta S$ is not Borel.*

Proof. Let $x \in \mathbb{N}^*$. We claim that if C is a compact subset of $R_x \cap K(\beta S)$, then π_2 assumes only finitely many values on C. Suppose instead we have a sequence $\langle p_n \rangle_{n=1}^{\infty}$ in C on which π_2 is injective. Let p be a cluster point of the sequence $\langle p_n \rangle_{n=1}^{\infty}$. Let $M = \{n \in \mathbb{N} : \pi_2(p_n) \neq \pi_2(p)\}$. Then $p \in c\ell\{p_n : n \in M\}$. Also $p \in K(\beta S)$ so by Theorem 2.1(2), $p \in \beta S \cdot p = c\ell(S \cdot p)$. Therefore by [5, Theorem 3.40] either $\{p_n : n \in M\} \cap (\beta S \cdot p) \neq \emptyset$ or $c\ell\{p_n : n \in M\} \cap (S \cdot p) \neq \emptyset$. In the first case we have some $m \in M$ and $q \in \beta S$ such that $p_m = q \cdot p$. But then $\pi_2(p_m) = \pi_2(q \cdot p) = \pi_2(p)$, so $m \notin M$. In the second case, we have some $r \in c\ell\{p_n : n \in M\}$ and some $s \in S$ such that $r = s \cdot p$. Now $R_x = \pi_1^{-1}[\{x\}]$ so is compact and thus $\pi_1(r) = x$. But $\pi_1(r) = \pi_1(s \cdot p) = \pi_1(s) \in \mathbb{N}$. This contradiction establishes the claim. Thus if C is a Borel subset of $R_x \cap K(\beta S)$, then π_2 takes on at most $\mathfrak{c}$ values on C. But for each $y \in \beta\mathbb{N}$, $R_x \cap L_y \cap K(\beta S) \neq \emptyset$ and so π_2 takes on $2^{\mathfrak{c}}$ values on $R_x \cap K(\beta S)$. Thus we have established that $R_x \cap K(\beta S)$ is not Borel. Since R_x is compact this also establishes that $K(\beta S)$ is not Borel.

Now let $p \in \beta S$ such that $x = \pi_1(p) \in \mathbb{N}^*$. By Theorem 2.1(2), $p \cdot \beta S \subseteq K(\beta S)$ so $p \cdot \beta S$ is a right ideal of βS contained in $R_x \cap K(\beta S)$, which is a minimal right ideal by Theorem 2.1(6), so $p \cdot \beta S = R_x \cap K(\beta S)$ which is not Borel. □

Corollary 2.9. *Let A and B be infinite sets and let $S = \mathbf{R}(A, B)$. Then $K(\beta S)$ is not Borel.*

Proof. Pick countably infinite $A' \subseteq A$ and $B' \subseteq B$ and let $S' = \mathbf{R}(A', B')$. By Theorem 2.1(2) $S \subseteq K(\beta S)$ so $\beta S' \cap K(\beta S) \neq \emptyset$. Thus by [5, Theorem 1.65] $K(\beta S') = \beta S' \cap K(\beta S)$. By Theorem 2.8, $K(\beta S')$ is not Borel, so since $\beta S'$ is compact, $K(\beta S)$ cannot be Borel. □

3. Semigroups with isolated points in minimal left ideals

We saw in the last section that the minimal left ideals in $\beta\mathbf{R}(A, B)$ have isolated points. In this section we establish several results that apply to any such semigroup. As we have already noted, since all minimal left ideals of βS are homeomorphic, if one of them has isolated points, they all do.

Lemma 3.1. *Let S be a semigroup, let L be a minimal left ideal of βS, let q be an isolated point of L, and let p be the identity of $q \cdot \beta S \cap L$. Then $\{s \in S : s \cdot q = q\} \in p$.*

Proof. Pick $A \subseteq S$ such that $\overline{A} \cap L = \{q\}$. Let $B = \{s \in S : s \cdot q = q\}$. We claim that $B \in p$. Suppose instead that $S \setminus B \in p$. Then $q = p \cdot q \in \overline{S \setminus B} \cdot q = c\ell\big((S \setminus B) \cdot q\big)$. Pick $s \in S \setminus B$ such that $s \cdot q \in \overline{A}$. Then $s \cdot q \in \overline{A} \cap L = \{q\}$, a contradiction. □

Theorem 3.2. *Let S be a semigroup and assume that βS has a minimal left ideal with an isolated point.*

(1) *If L is any minimal left ideal of βS and q is an isolated point of L, then the minimal right ideal $q \cdot \beta S$ is closed in βS.*
(2) *All maximal subgroups of βS are finite.*
(3) *If L is any minimal left ideal of βS and q is an isolated point of L, then every point of the group $q \cdot \beta S \cap L$ is isolated in L.*
(4) *Let R be a minimal right ideal of βS. If some $r \in R$ is isolated in the minimal left ideal $\beta S \cdot r$ then every $q \in R$ is isolated in $\beta S \cdot q$.*

Proof. (1) Pick $A \in q$ such that $\overline{A} \cap L = \{q\}$. Let p be the identity of the group $G = q \cdot \beta S \cap \beta S \cdot q$. Let $B = \{s \in S : s \cdot q = q\}$. By Lemma 3.1, $B \in p$. For $s \in B$, let $T_s = \{x \in \beta S : s \cdot x = x\}$ and let $T = \bigcap_{s \in B} T_s$. Then $q \in T$. Each T_s is a right ideal of βS and if $x \in \beta S \setminus T_s$, $D \in s \cdot x$, and $C \in x$ such that $\overline{D} \cap \overline{C} = \emptyset$, then $\overline{s^{-1}D} \cap \overline{C}$ is a neighborhood of x missing T_s. Thus T is a closed right ideal of βS. Further, for $x \in T$, ρ_x is constantly equal to x on B so $p \cdot x = x$. Therefore $T \subseteq p \cdot \beta S = q \cdot \beta S$, so $T = q \cdot \beta S$.

(2) Since the maximal subgroups of $K(\beta S)$ are all isomorphic, it suffices to show that some maximal subgroup is finite. Let L be a minimal left ideal of βS and let q be an isolated point in L. By (1) the minimal right ideal $q \cdot \beta S$ is closed in βS so the maximal group $G = q \cdot \beta S \cap L$ is closed in βS. Further, for each $x \in G$, the restriction of ρ_x to G is a homeomorphism so G is homogeneous. By [5, Theorem 6.38], βS does not contain an infinite compact homogeneous subspace so G is finite.

(3) Let L be a minimal left ideal of βS, let q be an isolated point of L, and let $G = q \cdot \beta S \cap L$. Let $x \in G$ and suppose that x is not isolated in L.

Pick a net $\langle x_\iota \rangle_{\iota \in D}$ in $L \setminus \{x\}$ which converges to x. Pick $y, z \in G$ such that $x \cdot y = q$ and $q \cdot z = x$. Then $\langle x_\iota \cdot y \rangle_{\iota \in D}$ is a net in L converging to q so eventually $x_\iota \cdot y = q$ and thus eventually $x_\iota \cdot y \cdot z = q \cdot z = x$. Now $x \cdot y \cdot z = x$ so $y \cdot z = p$, the identity of G. So eventually $x_\iota \cdot p = x$. But p is a right identity for L so eventually $x_\iota = x$, a contradiction.

(4) Let R be a minimal right ideal of βS. Assume we have $r \in R$ which is isolated in the minimal left ideal $J = \beta S \cdot r$. By (3), every point of $R \cap J$ is isolated in J. Now let $q \in R$, let $L = \beta S \cdot q$, and let p be the identity of $R \cap L$. By [5, Theorem 2.11(c)], the restriction of ρ_p to J is a homeomorphism from J onto L and by [5, Theorem 2.11(b)], $\rho_p[R \cap J] = R \cap L$. □

Lemma 3.3. *Let S be a semigroup, let L be a minimal left ideal of βS, let q be an isolated point of L, and let p be the identity of $q \cdot \beta S \cap L$. Then*

(1) $\{s \in S : s \cdot p = p\} \in p$.
(2) *If $s \cdot p = p$, then* $\{t \in S : s \cdot t = t\} \in p$.
(3) $\{s \in S : s \cdot p = p\} \in r$ *for every idempotent* $r \in p \cdot \beta S$.

Proof. (1) By Theorem 3.2(3), p is isolated in L. Also $p \cdot \beta S = r \cdot \beta S$ so Lemma 3.1 applies.

(2) [5, Theorem 3.35]

(3) For every $s \in S$ and every idempotent $r \in p \cdot \beta S$, $s \cdot p = p$ if and only if $s \cdot r = r$, because $p \cdot r = r$ and $r \cdot p = p$. So (3) follows from Lemma 3.1 and Theorem 3.2(4). □

Theorem 3.4. *Let S be a countably infinite semigroup. Assume that βS has a minimal left ideal L with an isolated point q, and the identity p of the group $q \cdot \beta S \cap L$ is in S^*. Then $K(\beta S)$ contains a compact right zero G_δ semigroup C. Furthermore, every idempotent in the minimal right ideal $p \cdot \beta S$ is in C.*

Proof. Let $B = \{s \in S : s \cdot p = p\}$ and for $s \in B$, let $B_s = \{t \in S : s \cdot t = t\}$. By Lemma 3.3, $B \in p$ and for each $s \in B$, $B_s \in p$. Let $C = \overline{B} \cap \bigcap_{s \in B} \overline{B_s}$. Then $p \in C$ so $C \cap K(\beta S) \neq \emptyset$. We claim that C is right zero. (Having shown that, given $q \in C$ we have that $p \cdot q = q$ so $q \in K(\beta S)$, so C is a G_δ subset of $K(\beta S)$.) So let $q, r \in C$. To see that $q \cdot r = r$, let $A \in r$. We claim that $B \subseteq \{s \in S : s^{-1}A \in r\}$. Let $s \in B$. We claim that $B_s \cap A \subseteq s^{-1}A$. Let $t \in B_s \cap A$. Then $s \cdot t = t$ so $t \in s^{-1}A$.

Let r be any idempotent in $p \cdot \beta S$. Since $r \cdot p = p$, $\{s \in S : s^{-1}B \in p\} \in r$. Further, if $s^{-1}B \in p$ and $t \in s^{-1}B \cap B$, then $p = s \cdot t \cdot p = s \cdot p$ so $s \in B$ and therefore $B \in r$. Furthermore, since $p \cdot r = r$, $s \cdot r = s \cdot p \cdot r = p \cdot r = r$ for every $s \in B$. It follows from [5, Theorem 3.35] that $B_s \in r$ for every $s \in B$. □

Theorem 3.5. *Let S be a countably infinite semigroup and assume that $K(\beta S) \subseteq S^*$. Then $K(\beta S)$ contains a compact G_δ right zero semigroup if and only if every minimal left ideal of βS has an isolated point.*

Proof. The sufficiency follows from Theorem 3.4.

For the necessity let C be a compact G_δ subset of $K(\beta S)$ such that $q\cdot r = r$ for all q and r in C. By [5, Theorem 3.36], the interior U of C in S^* is nonempty. Pick $p \in U$ and let $L = \beta S \cdot p$. We claim that $U \cap L = \{p\}$. Let $x \in U \cap L$. Since x is an idempotent in L, $p \cdot x = p$. Since $x \in C$, $p \cdot x = x$. □

The next result involves a notion of largeness stronger than the notion of being *strongly central*, which was introduced in [1]. A subset A of a semigroup S is strongly central if and only if for every minimal left ideal L of βS, there is an idempotent in $\overline{A} \cap L$. As the name would suggest, any strongly central set is *central* (which means it is a member of some idempotent in $K(\beta S)$). It must also be *syndetic* which is characterized by the fact that $\overline{A} \cap L \neq \emptyset$ for every minimal left ideal L of βS. We shall use the notion of a subset of a semigroup S being piecewise syndetic, which is equivalent to being a member of an ultrafilter in $K(\beta S)$. A combinatorial definition of this property can be found in [5, Definition 4.38].

We shall say that a subset A of S is *hypersyndetic* if and only if there is a minimal right ideal R of βS such that A is a member of every idempotent in R. This property is stronger than being strongly central. (It is at least superficially weaker than the property of being very strongly syndetic, introduced in [1, Definition 2.10].)

We shall see that, if the minimal left ideals of S have isolated points, S is rich in equations of the form $s \cdot t = t$.

Theorem 3.6. *Let S be a semigroup, let J be a minimal left ideal of βS, and let q be an isolated point of J. Let $R = q \cdot \beta S$ and let $E(R) = \{r \in R : r \cdot r = r\}$. Let $A = \big\{a \in S : \big(\forall r \in E(R)\big)(a \cdot r = r)\big\}$ and for $a \in A$, let $B_a = \{t \in S : a \cdot t = t\}$. Then $E(R) \subseteq \overline{A} \cap \bigcap_{a\in A} \overline{B_a}$. Therefore A is hypersyndetic and for each $a \in A$, B_a is hypersyndetic.*

Proof. Let $p \in E(R)$. We show that $A \in p$ and for each $a \in A$, $B_a \in p$. Given $r \in E(R)$, we have that $r \in p \cdot \beta S$ so $p \cdot r = r$. Thus, if $a \cdot p = p$, then $a \cdot r = r$ so $A = \{a \in S : a \cdot p = p\}$. Now q is isolated in $J = \beta S \cdot q$ so by Theorem 3.2(4), p is isolated in $\beta S \cdot p$ and consequently by Lemma 3.3(1), $A \in p$. And by Lemma 3.3(2), $B_a \in p$. □

Theorem 3.7. *Let S be a discrete semigroup. For each $s \in S$, let $B_s = \{t \in S : s \cdot t = t\}$.*

(1) *If each minimal left ideal of βS has an isolated point, there is a hypersyndetic subset B of S such that B_s is hypersyndetic for every $s \in B$.*
(2) *Each minimal left ideal of βS has an isolated point if and only if there is a piecewise syndetic subset B of S for which $\{B_s : s \in B\}$ has the finite intersection property.*

Proof. (1) follows immediately from Theorem 3.6. The necessity of (2) follows from (1). For the sufficiency, assume that B is a piecewise syndetic subset of S for which $\{B_s : s \in B\}$ has the finite intersection property. We can choose $p \in \overline{B} \cap K(\beta S)$. Note that $p \cdot \beta S$ is the minimal right ideal of βS to which p belongs. Choose $x \in \bigcap_{s \in B} \overline{B_s}$. Then $s \cdot x = x$ for every $s \in B$, and so $q \cdot x = x$ for every $q \in \overline{B}$. Let $R = \{y \in \beta S : (\forall q \in \overline{B})(q \cdot y = y)\}$. Since R is non-empty, it is a right ideal of βS. Also R is contained in the minimal right ideal $p \cdot \beta S$, because $p \in \overline{B}$, and therefore $R = p \cdot \beta S$. Since $p \in R \cap \overline{B}$, p is an idempotent. Let L be the minimal left ideal of βS which contains p. If $y \in \overline{B} \cap L$, then $y \cdot p = p$. Since $L \cdot p = L$, $y \in L \cdot p$ and so $y \cdot p = y$ and therefore $y = p$. Thus $L \cap \overline{B} = \{p\}$ and p is an isolated point of L. □

Definition 3.8. Let $(S, \cdot)$ be a semigroup with $|S| = \kappa$.

(a) A set $A \subseteq S$ is a *left solution set* if and only if there exist a and b in S such that $A = \{x \in S : a \cdot x = b\}$.
(b) A set $A \subseteq S$ is a *right solution set* if and only if there exist a and b in S such that $A = \{x \in S : x \cdot a = b\}$.
(c) For $a \in S$, $\text{Fix}(a) = \{x \in S : x \cdot a = a\}$.
(d) S is *weakly left cancellative* if and only if every left solution set is finite.
(e) S is *weakly right cancellative* if and only if every right solution set is finite.
(f) S is *very weakly left cancellative* if and only if whenever $\mathcal{A}$ is a set of left solution sets with $|\mathcal{A}| < \kappa$, one has $|\bigcup \mathcal{A}| < \kappa$.
(g) S is *very weakly right cancellative* if and only if whenever $\mathcal{A}$ is a set of right solution sets with $|\mathcal{A}| < \kappa$, one has $|\bigcup \mathcal{A}| < \kappa$.

In [2, Lemma 3.8(1)] it was shown that if S is both very weakly left cancellative and very weakly right cancellative and there is a finite bound on $\{\text{Fix}(a) : a \in S\}$, then the minimal left ideals of βS do not have isolated points. In Theorem 3.9 we strenghten the left hypothesis, but significantly weaken the right hypothesis.

Theorem 3.9. *Assume that S is weakly left cancellative and there is a finite bound on $|\text{Fix}(a)|$ for $a \in S$. Let L be a minimal left ideal of βS. Then $L \subseteq S^*$ and L has no isolated points in the relative topology.*

Proof. It suffices to show that there is some minimal left ideal with no isolated points, and any set with no isolated points is automatically contained in S^*. By [5, Theorem 4.31] S^* is left ideal of βS, and so there is a minimal left L of βS contained in S^*. If L has an isolated point, it follows from Theorem 3.6 that there is a very strongly central subset B of S for which $\{B_s : s \in B\}$ has the finite intersection property, where $B_s = \{t \in S : s \cdot t = t\}$. Since B is a member of an ultrafilter in L, B is infinite. Since $\{B_s : s \in B\}$ has the finite intersection property, $\{|\text{Fix}(a) : a \in S\}$ cannot be bounded. □

Corollary 3.10. *If S is weakly left cancellative and there is a finite bound on $|\mathrm{Fix}(a)|$ for $a \in S$, then $K(\beta S) \subseteq S^*$.*

Proof. $K(\beta S) = \bigcup\{L : L$ is a minimal left ideal of $\beta S\}$. □

We note that Theorem 3.9 is not a consequence of [2, Lemma 3.8(1)]. That is, the hypotheses of Theorem 3.9 can be satisfied by a semigroup S which is not very weakly right cancellative.

Theorem 3.11. *Let κ be an infinite cardinal. There is a semigroup S with $|S| = \kappa$ such that S is left cancellative, S is not very weakly right cancellative, and for each $a \in S$, $\mathrm{Fix}(a) = \emptyset$.*

Proof. Let $D = \{x_\sigma : \sigma \leq \kappa\} \cup \{y_{\sigma,\tau} : \tau < \sigma \leq \kappa\}$, where these symbols are all distinct. Let S be the set of all words over D with no occurrences of $y_{\sigma,\tau}x_\tau$. If $u, v \in S$ and it is not the case that there exist $\tau < \sigma \leq \kappa$ such that u ends in $y_{\sigma,\tau}$ and v begins with x_τ, then $u \cdot v$ is ordinary concatenation of words. If we have $\tau < \sigma \leq \kappa$ such that u ends in $y_{\sigma,\tau}$ and v begins with x_τ, then let $\sigma_1 = \sigma$, $\sigma_0 = \tau$, and assume we have $m \in \mathbb{N}$, and $\sigma_m > \sigma_{m-1} > \ldots > \sigma_1$ and $z \in S \cup \{\emptyset\}$ such that z does not end in y_{η,σ_m} for some $\eta > \sigma_m$ and $u = zy_{\sigma_m,\sigma_{m-1}} \cdots y_{\sigma_1,\tau}$. Pick $w \in S \cup \{\emptyset\}$ such that $v = x_\tau w$ and define $u \cdot v = zx_{\sigma_m}w$.

It is an exercise (unfortunately quite tedious) to verify that the operation defined is associative. It is an easy exercise to verify that S is left cancellative.

Let $A = \{s \in S : sx_0 = x_\kappa\}$. Then A is a right solution set and $\{y_{\kappa,\sigma}y_{\sigma,0} : 0 < \sigma < \kappa\} \subseteq A$ so $|A| = \kappa$ and thus S is not very weakly right cancellative.

Let $a \in S$ and suppose we have $u \in S$ such that $u \cdot a = a$. The concatenation of u with a is a word longer than a so we must have $u = zy_{\sigma_m,\sigma_{m-1}} \cdots y_{\sigma_1,\tau}$ and $a = x_\tau w$ as in the definition of the operation so $zx_{\sigma_m}w = x_\tau w$, a contradiction since $\sigma_m > \tau$. □

We see now that one can have $K(\beta S) \subseteq S^*$ and yet have infinite minimal left ideals with isolated points.

Theorem 3.12. *Let $S = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a < b\}$ and for (a, b) and (c, d) in S, define $(a, b) \cdot (c, d) = (a, b \vee d)$. Then $K(\beta S) \subseteq S^*$ and the minimal left ideals of βS are infinite and have isolated points.*

Proof. It is routine to verify that S is both weakly left cancellative and weakly right cancellative. Therefore by [5, Theorem 4.36], S^* is an ideal of βS so $K(\beta S) \subseteq S^*$.

Let $\pi_1 : \beta S \to \beta\mathbb{N}$ and $\pi_2 : \beta S \to \beta\mathbb{N}$ be the continuous extensions of π_1 and π_2 respectively.

Given $p, q \in \beta S$ we have that $\pi_1 \circ \rho_q$ and π_1 agree on S so $\pi_1(p \cdot q) = \pi_1(p)$. If $s, t \in S$ and $\pi_2(s) \leq \pi_2(t)$, then $\pi_2(s \cdot t) = \pi_2(t)$. Consequently if $\pi_2(q) \in \mathbb{N}^*$, then $\pi_2 \circ \rho_q$ is constantly equal to $\pi_2(q)$ on S so that for all p in βS, $\pi_2(p \cdot q) = \pi_2(q)$.

Let $x \in \beta\mathbb{N}$ and let $y \in \mathbb{N}^*$. Let $R_x = \{p \in \beta S : \pi_1(p) = x\}$ and let $L_y = \{p \in \beta S : \pi_2(p) = y\}$. Then $R_x \neq \emptyset$ since π_1 is surjective and $L_y \neq \emptyset$ since $\pi_2[S] = \mathbb{N} \setminus \{1\}$. Therefore R_x is a right ideal of βS and L_y is a left ideal of βS.

As in the proof of Theorem 2.1(3) we have that If $a \in \mathbb{N}$, $y \in \mathbb{N}^*$, and $p \in R_a \cap L_y$, then $R_a \cap L_y = \{p\}$ and $\overline{\{a\} \times B} \cap L_x = \{p\}$. In particular, minimal left ideals of βS have isolated points.

Since $R_a \cap R_b = \emptyset$ if $a \neq b$ we have that each minimal left ideal of βS is infinite. □

If S is as in Theorem 3.12, theorems analogous to those proved in the preceding section for $\beta(\mathbf{R}(A, B))$ can be proved for βS. For example, $p \mapsto (\pi_1(p), \pi_2(p))$ defines an algebraic isomorphism from $K(\beta S)$ onto $\mathbf{R}(\beta\mathbb{N}, \mathbb{N}^*)$.

$(X, \langle T_s \rangle_{s \in S})$ is called a dynamical system if S is a discrete semigroup, X is a compact Hausdorff space, for each $s \in S$, T_s is a continuous mapping from X to itself and, for each $s, t \in S$, $T_s \circ T_t = T_{s \cdot t}$. In this case, the map $s \mapsto T_s$ from S into ${}^X X$ extends to a continuous map from βS into ${}^X X$ where ${}^X X$ has the product topology. If $p \in \beta S$, we shall denote the image of p under this map by T_p. So T_p is a mapping from X to itself. For each $p, q \in \beta S$, $T_p \circ T_q = T_{p \cdot q}$ ([5, Remark 19.13]).

Theorem 3.13. *Let S be a discrete semigroup. Then the following statements are equivalent:*

(1) *Each minimal left ideal of βS has an isolated point;*
(2) *If $(X, \langle T_s \rangle_{s \in S})$ is a dynamical system and X is non-empty, there exists $x \in X$ such that $\{s \in S : T_s(x) = x\}$ is hypersyndetic.*

Proof. (1) ⇒ (2). Assume that (1) holds. Pick a minimal left ideal L of βS By Theorem 3.2(4) there is an idempotent p which is isolated in L. Let $A = \{s \in S : s \cdot p = p\}$ and let $R = p \cdot \beta S$. Then by Lemma 3.3(3), for each idempotent $r \in R$, $A \in r$.

Now let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system with $X \neq \emptyset$ and pick $y \in X$. Let $x = T_p(y)$ and note that $T_p(x) = T_p\big(T_p(y)\big) = T_p(y) = x$. To see that $\{s \in S : T_s(x) = x\}$ is hypersyndetic, it suffices to show that $A \subseteq \{s \in S : T_s(x) = x\}$, so let $s \in A$. Then $T_s(x) = T_s\big(T_p(y)\big) = T_{s \cdot p}(y) = T_p(y) = x$.

(2) ⇒ (1). Assume that (2) holds. Let L be a minimal left ideal of βS. By applying (2) to the dynamical system $(L, \langle \lambda_{s|L} \rangle_{s \in S})$, we see that there exists $q \in L$ such that $\{s \in S : s \cdot q = q\}$ is hypersyndetic. Let $B = \{s \in S : s \cdot q = q\}$ and for $s \in B$, let $B_s = \{t \in S : s \cdot t = t\}$. By [5, Theorem 3.35], $B_s \in q$ for every $s \in B$. It follows from Theorem 3.7(2) that L has an isolated point. □

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