

## Searching for cycles in non-linear autonomous discrete dynamical systems

D. Dmitrishin, A. Stokolos and M. Tohaneanu

ABSTRACT. In the current paper we suggest a new robust algorithm to search for cycles of arbitrary length in non-linear autonomous discrete dynamical systems. With the help of the computer we were able to find (unstable) cycles for several basic maps of nonlinear science: Hénon, Holmes cubic, Ikeda, Lozi, Elhaj-Sprott. The theoretical part of the paper is based on properties of a new family of extremal polynomials that contains the Fejér and Suffridge polynomials. The associated combination of geometric complex analysis and discrete dynamics seems to be a new phenomenon, both theoretical and practical. A novelty of this paper is in the discovery of a close connection between two seemingly disconnected fields: extremal polynomials and cycles in dynamical systems.

### CONTENTS

1. Introduction	604
2. Average system	605
3. Stability analysis	606
4. Optimization problem	606
5. Case $T = 1, 2$ - Suffridge polynomials	607
6. Case $T \geq 3$ - generalized Suffridge polynomials	608
7. Experimental evidence	609
8. Some particular cases	612
9. The estimate for the product of cotangents	612
10. Applications to non-linear dynamics	619
Acknowledgments	624
References	624

---

Received February 9, 2019.

2010 *Mathematics Subject Classification*. 34H10,93D15,93C55,42A05.

*Key words and phrases*. Discrete dynamical systems, extremal polynomials, chaos control.

## 1. Introduction

**1.1. Settings.** Consider the discrete dynamical system

$$x_{n+1} = f(x_n), \quad f : A \rightarrow A, \quad A \subset \mathbb{R}^m \quad (1)$$

where  $A$  is a convex set that is invariant under  $f$ . Let us assume that the system has an unstable T-cycle  $(x_1^*, \dots, x_T^*)$ . We define the cycle multipliers  $\mu_1, \dots, \mu_m$  as the zeros of the characteristic polynomial

$$\det \left( \mu I - \prod_{j=1}^T Df(x_{T-j+1}^*) \right) = 0. \quad (2)$$

We will assume that the multipliers are located in a region  $M \subset \mathbb{C}$ . If all the multipliers are inside the unit disc  $\mathbb{D} = \{|z| < 1\}$  in the complex plane, then the cycle is locally asymptotically stable. If not, an infinitesimal perturbation of the cycle values can lead to behavior called “deterministic chaos” or just “chaos.” The term was coined by J.A. Yorke and T.Y. Li in the paper “Period Three Implies Chaos” (1975) [13] in which it was proved that any one-dimensional system which exhibits a regular cycle of period three will also display regular cycles of every other length, as well as completely chaotic cycles. The famous Sharkovsky’s theorem [19] includes this result as a special case (cf. [2], p. 79).

Various methods of chaos control have been developed since, starting with the groundbreaking work of Edward Ott, Celso Grebogi and James A. Yorke [16] (1990). In [16] the authors suggest a method to stabilize chaos by making only small time-dependent perturbations of an available system parameter. They also show that small time dependent changes in the control parameters of a nonlinear system can turn a previously chaotic trajectory into a stable, periodic motion. The method was developed further to find periodic orbits; a comparison of various versions can be found in [3].

The next step forward was done by Kestutis Pyragas [17] in 1992. He suggested a very simple linear scheme  $f(x_{n+1}) = f(x_n) + K(x_n - x_{n-1})$ . The Pyragas method has turned out to be very popular because it is easy to implement experimentally. It has been used in a large variety of systems in physics, chemistry, biology, medicine, and engineering [20, 18, 10].

In 1996 Toshimitsu Ushio [25] showed that the Pyragas method has several serious limitations. In particular it was shown that the admissible region of the multiplier is  $(-3, 1)$ , so it does not work for the whole range of negative values as one would like. Moreover, it was shown in [6] that going deeper in the prehistory by adding more delays does not improve the situation. If two multipliers of the system belong to the same connected component of the region of stability, then the distance between them will be at most 4, regardless of the number of delays.

In 1996 M. de Sousa Vieira and A.J. Lichtenberg [23] suggested a non-linear counterpart of the Pyragas method. In their “Conclusion and discussion” section they wrote “The generalization consists of feeding back the nonlinear mapping signal rather than a signal linearized around the fixed point. This increases the basin of attraction of the controlled signal and thus decreases the sensitivity to noise. *However, the range of parameters for which control can be achieved is limited.*”

Further work on the stabilization of cycles was done by J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier [21] and Ö.Morgül [14, 15]. In particular, the latter papers considered the problem of finding 2,4,5 and 6-cycles of the logistic map  $f(x) = \mu x(1 - x)$  for various values of  $\mu$ .

The goal of this paper is to provide a robust method to stabilize cycles of any length whose multipliers lie in the region  $(-\infty, 1)$ . We will be looking at the non-linear control (3) and find coefficients (that we conjecture to be optimal) that allow us to stabilize the cycle in polynomial time. Sections 2-4 provide a preliminary discussion and setup of the problem. In Section 5 we discuss the cases  $T = 1$  and  $T = 2$ , which have been rigorously studied by the authors and collaborators in [7], [8]. Sections 6-8 are devoted to discussing our choice of coefficients, and providing conjectures and experimental evidence of why we believe these conjectures to hold. In Section 9 we rigorously prove asymptotic bounds for the size of the multipliers that can be stabilized for a cycle of length  $T$  if we allow the length of prehistory  $N$  to go to infinity. More precisely, we obtain a polynomial-type bound of approximately  $N^2$  for the size of the multipliers, which is important in applications (as opposed to, say, an exponential bound). Finally, Section 10 is dedicated to numerical examples of how our method can be applied to well-known dynamical systems.

## 2. Average system

The standard approach developed in analysis to suppress oscillations is averaging. Let us apply this idea to stabilize unstable  $T$ -cycles, i.e. given a range for the multipliers  $\mu$  we want to stabilize the cycle by the following averaging procedure

$$x_{n+1} = \sum_{k=1}^N a_k f(x_{n-kT+T}), \quad \sum_{k=1}^N a_k = 1. \quad (3)$$

Note that the system (3) preserves the convex invariant set provided  $a_k > 0$ , and the  $T$ -cycles of the system (1), while offering great flexibility since we can choose the coefficients  $a_k$ .

Let us stress that condition  $|\mu_j| < 1$  implies stability, which allows us to find the cycle by a simple iterative procedure based on (1). However if  $|\mu_j| \geq 1$  then the iterative procedure does not work and should be modified, for example changed to Newton’s method. Our approach in this paper is to change the system rather than the method, which is different from the

standard approach.

Two natural questions arise: *Can stabilization be obtained with a bounded depth of prehistory  $N$ ? If so, what is the minimal depth necessary?*

### 3. Stability analysis

Note that the system (3) is not a dynamical system anymore; instead, it is a system with multiple delays. For a system with delays the notion of the characteristic equation is not so straightforward, as it requires an increase in dimensionality which involves a long and complicated expression for the characteristic equation (c.f. [8]). This complication prevented the development of the subject even in a scalar case for a long time (c.f. [15]). However, it turns out that this expression can be written in a magically compact form. Namely, the characteristic equation for the system (3) is

$$\prod_{j=1}^m \left[ \lambda^{T(N-1)+1} - \mu_j \left( \sum_{k=1}^N a_k \lambda^{N-k} \right)^T \right] = 0, \quad \mu_j \in M, \quad j = 1, \dots, m.$$

The proof for the scalar case  $m = 1$  is in [5], and for the vector case is in [12].

This particular form of the characteristic polynomial allows one to establish the following geometric criterion (c.f. [8, 5.1]) that was suggested by Alexei Solyanik [22].

**Lemma 3.1.** *The characteristic polynomial of the system (3) has all the roots inside the unit disc as long as the reciprocal values of the multipliers are outside the image of the closed unit disc  $\bar{\mathbb{D}}$  under the polynomial map  $F_T(z) = z(a_1 + \dots + a_N z^{N-1})^T$ , i.e.*

$$\frac{1}{\mu_j} \notin F_T(\bar{\mathbb{D}}), \quad j = 1, \dots, N,$$

or

$$\mu_j \in (\bar{\mathbb{C}} \setminus F_T(\bar{\mathbb{D}}))^*,$$

where  $z^* = 1/\bar{z}$ .

Note that in the case  $N = 1$  one gets  $F_T(z) = z$  and  $F_T(\bar{\mathbb{D}}) = \bar{\mathbb{D}}$  and thus  $(\bar{\mathbb{C}} \setminus F_T(\bar{\mathbb{D}}))^* = \mathbb{D}$  which is the standard criterion for stability in the open-loop system (1).

### 4. Optimization problem

In this paper we will assume that the multipliers lie on the half-axis  $(-\infty, 1)$ . In this case the problem of stabilization can be reduced to the following optimization problem: find

$$I_N^{(T)} = \sup_{\sum_{j=1}^N a_j = 1} \min_{t \in [0, \pi]} \{ \Re(F_T(e^{it})) : \Im(F_T(e^{it})) = 0 \}.$$

We refer the interested reader to [4] for more details on this topic.

Lemma 3.1 now implies that for the system (3) a robust stabilization (i.e. by the same control for all  $\mu_j \in (-\mu^*, 1)$ ) of any  $T$ -cycle is possible if

$$(\mu^*) \cdot |I_N^{(T)}| \leq 1. \quad (4)$$

We are left with the task of finding the polynomials that solve the optimization problem, and estimate  $|I_N^{(T)}|$ .

## 5. Case $T = 1, 2$ - Suffridge polynomials

For  $T = 1, 2$  the optimization problems were solved in [7] and [4] by means of Harmonic Analysis. Namely, for  $T = 1$

$$\begin{aligned} |I_N^{(1)}| &= \inf_{\sum_{j=1}^N a_j=1} \max_{t \in [0, \pi]} \left\{ -\sum_{j=1}^N a_j \cos jt : \sum_{j=1}^N a_j \sin jt = 0 \right\} \\ &= \tan^2 \frac{\pi}{2(N+1)} \sim \frac{\pi^2}{4N^2}. \end{aligned} \quad (5)$$

It is interesting to note that the optimal coefficients turn out to be the coefficients of Suffridge polynomials (see [24]).

$$a_j^{(1)} = A_N \left( 1 - \frac{j}{N+1} \right) \sin \frac{\pi j}{N+1}, \quad A_N = 2 \tan \frac{\pi}{2(N+1)}, \quad j = 1, \dots, N. \quad (6)$$

The Suffridge polynomials are extremely important in the theory of univalent polynomials [9]; however, their appearance in the context of cycles of dynamical systems was a surprise for the authors.

Similarly for  $T = 2$  one has

$$\begin{aligned} |I_N^{(2)}| &= \inf_{\sum_{j=1}^N a_j=1} \max_{t \in [0, \pi]} \left( -\sum_{j=1}^N a_j \sin(2j-1)t : \sum_{j=1}^N a_j \cos(2j-1)t = 0 \right)^2 \\ &= \frac{1}{N^2}. \end{aligned} \quad (7)$$

The optimal coefficients are odd coefficients of Fejér polynomials

$$a_j^{(2)} = A_N \left( 1 - \frac{2j-1}{2N} \right), \quad A_N = \frac{2}{N}, \quad j = 1, \dots, N. \quad (8)$$

Note that Fejér polynomials are central in classical Fourier series theory [26] and their appearance in the current context is also quite remarkable.

The images  $F_T(e^{it})$  of the unit circle under the optimal polynomial maps  $F_T(z)$  look very similar for  $T = 1$  and for  $T = 2$  (see Fig.1 and Fig.3). The only major difference between the two pictures is the behavior when  $t = \pi$ :

for even  $N$  the graph is tangent to the x-axis there, which does not happen for odd  $N$ .

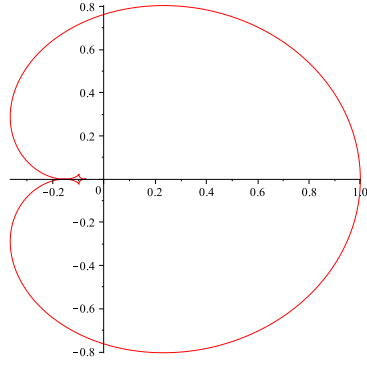


FIGURE 1.  $F_T(e^{it})$  for  $T=1$  and  $N=5$

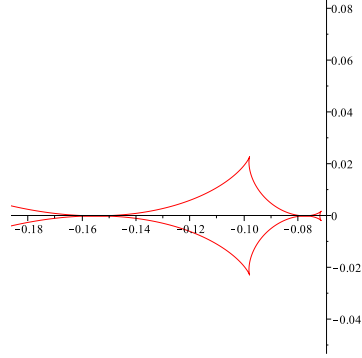


FIGURE 2. Fragment

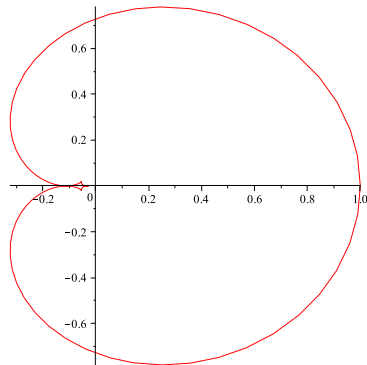


FIGURE 3.  $F_T(e^{it})$  for  $T=2$  and  $N=6$

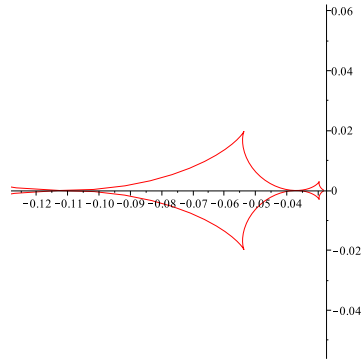


FIGURE 4. Fragment

### 6. Case $T \geq 3$ - generalized Suffridge polynomials

We will now introduce the remarkable family of extremal polynomials mentioned in the abstract. These polynomials generalize Suffridge polynomials for the case  $T \geq 3$  and are the same as the polynomials from the above section if  $T = 1, 2$ . In this section we provide explicit formulas for these polynomials.

Let  $q(z) := a_1 + \dots + a_N z^{N-1}$ . Define the set of points

$$\psi_j = \frac{\pi(2 + T(2j - 1))}{2 + (N - 1)T}, \quad j = 1, \dots, \frac{N}{2} \text{ (N-even)}, \left( \frac{N - 1}{2} \text{ (N-odd)} \right)$$

and the generating polynomials

$$\eta(z) = z(z+1) \prod_{j=1}^{\frac{N-2}{2}} (z - e^{i\psi_j})(z - e^{-i\psi_j}), \quad \text{N-even}$$

$$\eta(z) = z \prod_{j=1}^{\frac{N-1}{2}} (z - e^{i\psi_j})(z - e^{-i\psi_j}), \quad \text{N-odd.}$$

We now let

$$q(z) = \frac{KT}{2 + (N-1)T} \left( \left( \frac{1}{T} + N \right) \frac{\eta(z)}{z} - \eta'(z) \right), \quad (9)$$

where  $K$  is a normalizing factor so that  $q(1) = 1$ . A direct computation shows that

$$\frac{1}{K} = 2^{\frac{N-2}{2}} \prod_{j=1}^{\frac{N-2}{2}} (1 - \cos \psi_j), \quad N \text{ is even,}$$

$$\frac{1}{K} = 2^{\frac{N-3}{2}} \prod_{j=1}^{\frac{N-1}{2}} (1 - \cos \psi_j), \quad N \text{ is odd,}$$

as well as

$$q(-1) = P_N, \quad (10)$$

where

$$P_N = \frac{T}{2 + (N-1)T} \prod_{j=1}^{\frac{N-2}{2}} \cot^2 \frac{\psi_j}{2}, \quad N \text{ is even; } \quad P_N = \prod_{j=1}^{\frac{N-1}{2}} \cot^2 \frac{\psi_j}{2}, \quad N \text{ is odd.}$$

In order to compute the coefficients  $a_j$  we write  $\eta(z)$  in the standard form

$$\eta(z) = z \sum_{j=1}^N c_j z^{j-1}.$$

We then have

$$a_j = K \left( 1 - \frac{1 + (j-1)T}{2 + (N-1)T} \right) c_j. \quad (11)$$

## 7. Experimental evidence

Using MAPLE we plotted the image of the unit disc  $\mathbb{D}$  under the polynomial map  $F_T(z) = z(q(z))^T$  for various combinations of  $T$  and  $N$ , and all the plots look remarkably similar. Below we provide the images for a small value  $N = 5$  and two different values of  $T$ , a small one ( $T = 5$ ) and a large one ( $T = 1005$ ), see Fig 5 and Fig 7. We chose  $N = 5$  to be able to better observe the behavior of the curve  $F_T(e^{it})$  near the point  $t = \pi$ , as for large  $N$  this is harder.

A typical case of the plot is in Fig. 5 below. One can observe several cusps, which suggests that the roots of the derivatives of the polynomials  $F_T(z)$  are on the boundary of the unit disc  $\mathbb{D}$ , which was a crucial step in proving univalence of Suffridge polynomials. For completeness we also provide the inverse image  $F_T(\bar{\mathbb{D}})^*$ , which is the unbounded region in Fig. 6. We note in particular that the interval of multipliers  $(\mu^*, 1)$  does not intersect the interior of  $F_T(\bar{\mathbb{D}})^*$ , although it does intersect the boundary. This can be easily remedied by considering  $F_T^\epsilon = \frac{F_T(z) + \epsilon z}{1 + \epsilon}$  and sending  $\epsilon$  to 0, so considering the original  $F_T$  is justified for computer simulations.

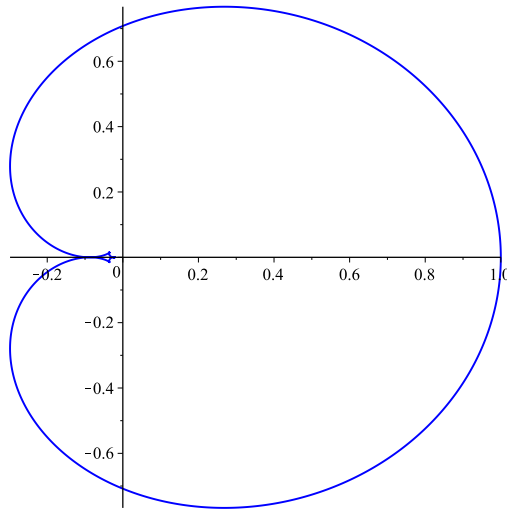


FIGURE 5.  $F_5(\bar{\mathbb{D}})$ ,  $N=5$

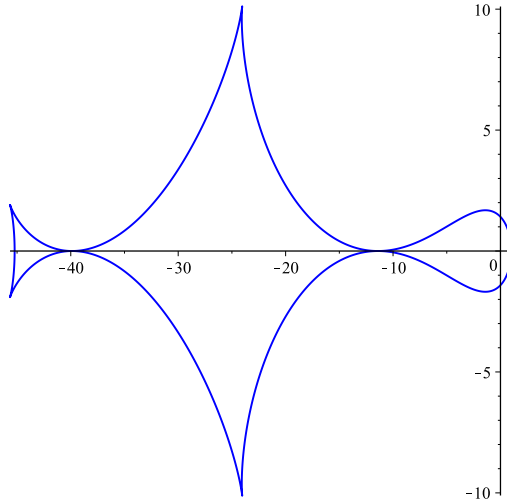
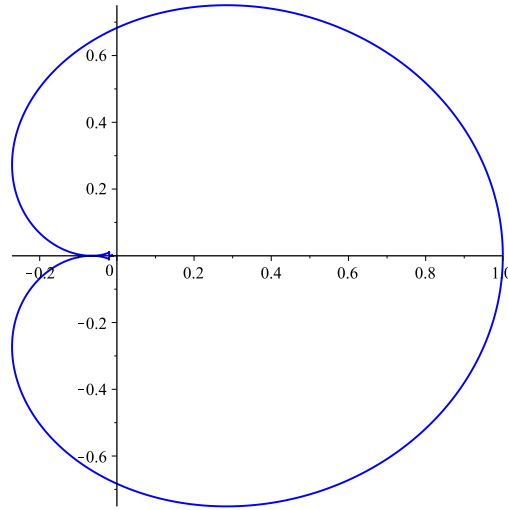
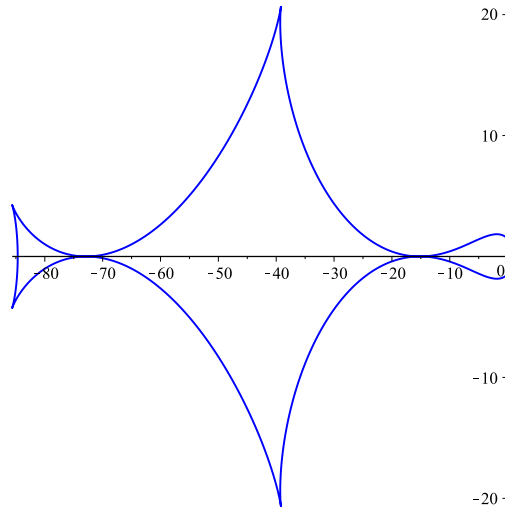


FIGURE 6.  $(\bar{\mathbb{C}} \setminus F_5(\bar{\mathbb{D}}))^*$ ,  $N=5$



FIGURE 7.  $F_{1005}(\bar{\mathbb{D}})$ ,  $N=5$ FIGURE 8.  $(\bar{\mathbb{C}} \setminus F_{1005}(\bar{\mathbb{D}}))^*$ ,  $N=5$ 

Based on the observations above we can conjecture that the polynomials  $F_T(z) = z(q(z))^T$  map the curve  $z = e^{it}$ ,  $0 < t < \pi$  into a curve in the closed upper half plane  $\{\Im(z) \geq 0\}$ . Moreover, the image of the curve should not have any self-intersections. Thus, we conjecture

**Conjecture A:** *The polynomials  $F_T(z) = z(q(z))^T$  are univalent.*

With(10) in mind we also conjecture

**Conjecture B:** *Let  $T \geq 3$  and  $N \geq 1$  be integers. Then  $|I_N^{(T)}| \leq (P_N)^T$ .*

The presence of cusps in Fig.5 and Fig.7 indicates that the zeros of the derivative of polynomials are on the unit circle. Thus we can strengthen Conjecture A to the claim that the polynomials  $F_T(z)$  might be quasi-extremal in the Genthner-Ruscheweyh-Salinas sense (see Definition 11 and Theorem 12 in [11]).

## 8. Some particular cases

**8.1. Case  $T = 1$ .** In this case

$$\psi_j = \frac{\pi(2j+1)}{N+1}, \quad \eta(z) = \sum_{j=1}^N \sin \frac{\pi j}{N+1} z^j,$$

and we get the Suffridge polynomials

$$F_T(z) = zq(z) = Kz \sum_{j=1}^N \left(1 - \frac{j}{N+1}\right) \sin \frac{\pi j}{N+1} z^{j-1},$$

where

$$K = 2 \tan \frac{\pi}{2(N+1)}.$$

Furthermore,  $P_N = \tan^2 \frac{\pi}{2(N+1)}$  which agrees with (6).

**8.2. Case  $T = 2$ .** In this case

$$\psi_j = \pi \frac{2j}{N}, \quad \eta(z) = \sum_{j=1}^N z^j,$$

and we get the Fejér polynomial

$$F_2(z) = zq(z)^2 = K^2 z \left( \sum_{j=1}^N \left(1 - \frac{2j-1}{2N}\right) z^{j-1} \right)^2, \quad K = \frac{1}{N}.$$

Furthermore,  $(P_N)^2 = \frac{1}{N^2}$ , which agrees with (8).

## 9. The estimate for the product of cotangents

The previous section provides the asymptotic behavior for  $|I_N^{(T)}|$  in the cases  $T = 1, 2$ . A natural question is whether we can find the asymptotic behavior for  $P_N$  when  $T \geq 3$ . The goal of this section is to give an affirmative answer. We rigorously prove the following result:

**Theorem 9.1.** *Let  $T \geq 3$ . Then  $P_N \approx N^{-\frac{2}{T}}$  as  $N \rightarrow \infty$ . More precisely, we get that*

$$\lim_{N \rightarrow \infty} P_N N^{\frac{2}{T}} = \pi^{\frac{2-T}{T}} \left( \Gamma \left( \frac{T+2}{2T} \right) \right)^2. \quad (12)$$

Let us begin with the following very simple lemma which will turn out to be extremely useful:

**Lemma 9.1.** *For any  $0 < \gamma < \frac{\pi}{2}$  the function*

$$f : [0, \frac{\pi}{2} - \gamma) \rightarrow \mathbb{R}, \quad f(x) = \frac{\tan x}{\tan(\gamma + x)}$$

*is increasing on  $[0, \frac{1}{2}(\frac{\pi}{2} - \gamma)]$  and decreasing on  $[\frac{1}{2}(\frac{\pi}{2} - \gamma), \frac{\pi}{2} - \gamma)$*

**Proof.** A simple computation gives that

$$\frac{d}{dx} f(x) = \frac{\sin \gamma \cos(2x + \gamma)}{\sin^2(x + \gamma) \cos^2(x + \gamma) \cos^2 x}$$

and the conclusion of the lemma follows immediately.  $\square$

**Proof.** We will start the proof of our main theorem with the case when  $N$  is odd. We first note that, since  $\cot(\frac{\pi}{2} - x) = \tan x$ , we can rewrite

$$P_N = \prod_{j=1}^{\frac{N-1}{2}} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}.$$

Let us now fix  $m \ll N$ , and define

$$P_N^{(m,1)} = \prod_{j=m+1}^{\frac{N-1}{2}-m} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}.$$

The strategy will be to first estimate  $P_N^{(m,1)}$  from above and below, let  $N \rightarrow \infty$ , and then let  $m \rightarrow \infty$ .

To accomplish the first task, let us first note that we can rewrite  $P_N^{(m,1)}$  in the form

$$\begin{aligned} P_N^{(m,1)} &= \prod_{j=m+1}^{\lfloor \frac{N+1}{4} \rfloor - 1} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \cdot \prod_{j=\lfloor \frac{N+1}{4} \rfloor}^{\lfloor \frac{N+1}{4} \rfloor + 1} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \\ &\quad \cdot \prod_{j=\lfloor \frac{N+1}{4} \rfloor + 2}^{\frac{N-1}{2}-m} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}. \end{aligned}$$

Now let

$$\begin{aligned} P_N^{(m,k)} &= \prod_{j=m+1}^{\lfloor \frac{N+1}{4} \rfloor - 1} \frac{\tan \pi \frac{2(k-1)+(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2k+(2j-1)T}{2(2+(N-1)T)}} \cdot \prod_{j=\lfloor \frac{N+1}{4} \rfloor}^{\lfloor \frac{N+1}{4} \rfloor + 1} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \\ &\quad \cdot \prod_{j=\lfloor \frac{N+1}{4} \rfloor + 2}^{\frac{N-1}{2}-m} \frac{\tan \pi \frac{2(1-k)+(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2(2-k)+(2j-1)T}{2(2+(N-1)T)}}. \end{aligned}$$

where  $1 \leq k \leq T$ . Note that when  $k = 1$  we indeed recover the correct formula for  $P_N^{(m,1)}$ .

The first key observation is that

$$\prod_{k=1}^T P_N^{(m,k)} = \frac{\tan \pi \frac{(2m+1)T}{2(2+(N-1)T)}}{\tan \pi \frac{(2\lfloor \frac{N+1}{4} \rfloor - 1)T}{2(2+(N-1)T)}} \left( \prod_{j=\lfloor \frac{N+1}{4} \rfloor}^{\lfloor \frac{N+1}{4} \rfloor + 1} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \right)^T \cdot \frac{\tan \pi \frac{2+(2\lfloor \frac{N+1}{4} \rfloor + 1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(N-2m-2)T}{2(2+(N-1)T)}}. \quad (13)$$

The second key observation is that

$$P_N^{(m,k)} \leq P_N^{(m,k+1)}, \quad 1 \leq k \leq T-1. \quad (14)$$

Indeed, this is a simple consequence of Lemma 9.1 applied for each term in the product with  $\gamma = \pi \frac{2}{2(2+(N-1)T)}$ .

As a consequence of (13), (14) and the fact that

$$\tan \pi \frac{(2m+1)T}{2(2+(N-1)T)} = \frac{1}{\tan \pi \frac{2+(N-2m-2)T}{2(2+(N-1)T)}},$$

we obtain

$$(P_N^{(m,1)})^T \leq \tan^2 \pi \frac{(2m+1)T}{2(2+(N-1)T)} \cdot \frac{\tan \pi \frac{2+(2\lfloor \frac{N+1}{4} \rfloor + 1)T}{2(2+(N-1)T)}}{\tan \pi \frac{(2\lfloor \frac{N+1}{4} \rfloor - 1)T}{2(2+(N-1)T)}} \cdot \left( \prod_{j=\lfloor \frac{N+1}{4} \rfloor}^{\lfloor \frac{N+1}{4} \rfloor + 1} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \right)^T.$$

Since

$$\lim_{N \rightarrow \infty} N^2 \tan^2 \pi \frac{(2m+1)T}{2(2+(N-1)T)} = \left(\frac{\pi}{2}\right)^2 \cdot (2m+1)^2$$

and

$$\lim_{N \rightarrow \infty} \frac{\tan \pi \frac{2+(2\lfloor \frac{N+1}{4} \rfloor + 1)T}{2(2+(N-1)T)}}{\tan \pi \frac{(2\lfloor \frac{N+1}{4} \rfloor - 1)T}{2(2+(N-1)T)}} \left( \prod_{j=\lfloor \frac{N+1}{4} \rfloor}^{\lfloor \frac{N+1}{4} \rfloor + 1} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \right)^T = 1,$$

we obtain that

$$\limsup_{N \rightarrow \infty} N^{\frac{2}{T}} P_N^{(m,1)} \leq \left( \left(\frac{\pi}{2}\right)^2 \cdot (2m+1)^2 \right)^{\frac{1}{T}}.$$

Moreover, since

$$P_N = \prod_{j=1}^m \left( \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \right)^2 P_N^{(m,1)}$$

and

$$\lim_{N \rightarrow \infty} \prod_{j=1}^m \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} = \prod_{j=1}^m \frac{(2j-1)T}{2+(2j-1)T},$$

we obtain that for any fixed  $m$

$$\limsup_{N \rightarrow \infty} P_N N^{\frac{2}{T}} \leq \left(\frac{\pi}{2}\right)^{\frac{2}{T}} (2m+1)^{\frac{2}{T}} \prod_{j=1}^m \left(\frac{(2j-1)T}{2+(2j-1)T}\right)^2 := \alpha_m. \quad (15)$$

We use a similar argument to bound  $\limsup_{N \rightarrow \infty} P_N N^{\frac{2}{T}}$ . Let

$$Q_N^{(m,1)} = \prod_{j=m+1}^{\frac{N-1}{2}-m} \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}.$$

Now, define  $Q_N^{(m,k)}$  by the equation

$$Q_N^{(m,k)} = \prod_{j=m+1}^{\lfloor \frac{N+1}{4} \rfloor} \frac{\tan \pi \frac{2(1-k)+(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2(2-k)+(2j-1)T}{2(2+(N-1)T)}} \prod_{j=\lfloor \frac{N+1}{4} \rfloor + 1}^{\frac{N-1}{2}-m} \frac{\tan \pi \frac{2(k-1)+(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2k+(2j-1)T}{2(2+(N-1)T)}},$$

where  $1 \leq k \leq T$ . Note that when  $k = 1$  we indeed recover the correct formula for  $Q_N^{(m,1)}$ .

As above, one can show that

$$Q_N^{(m,1)} \geq Q_N^{(k)}, \quad 1 \leq k \leq T$$

and

$$\prod_{k=1}^T Q_N^{(m,k)} = \frac{\tan \pi \frac{2+(2m-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2\lfloor \frac{N+1}{4} \rfloor - 1)T}{2(2+(N-1)T)}} \frac{\tan \pi \frac{(2\lfloor \frac{N+1}{4} \rfloor + 1)T}{2(2+(N-1)T)}}{\tan \pi \frac{(N-2m)T}{2(2+(N-1)T)}}.$$

Since

$$\lim_{N \rightarrow \infty} N^2 \tan^2 \pi \frac{2+(2m-1)T}{2(2+(N-1)T)} = \left(\frac{\pi}{2}\right)^2 \cdot (2m-1)^2$$

and

$$\lim_{N \rightarrow \infty} \frac{\tan \pi \frac{(2\lfloor \frac{N+1}{4} \rfloor + 1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2\lfloor \frac{N+1}{4} \rfloor - 1)T}{2(2+(N-1)T)}} = 1,$$

we obtain that

$$\liminf_{N \rightarrow \infty} N^{\frac{2}{T}} Q_N^{(m,1)} \geq \left(\left(\frac{\pi}{2}\right)^2 \cdot (2m-1)^2\right)^{\frac{1}{T}}.$$

Moreover, since

$$P_N = \prod_{j=1}^m \left(\frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}\right)^2 Q_N^{(m,1)}$$

and

$$\lim_{N \rightarrow \infty} \prod_{j=1}^m \frac{\tan \pi \frac{(2j-1)T}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} = \prod_{j=1}^m \frac{(2j-1)T}{2+(2j-1)T},$$

we obtain that for any fixed  $m$

$$\liminf_{N \rightarrow \infty} P_N N^{\frac{2}{T}} \geq \left(\frac{\pi}{2}\right)^{\frac{2}{T}} (2m-1)^{\frac{2}{T}} \prod_{j=1}^m \left(\frac{(2j-1)T}{2+(2j-1)T}\right)^2 := \beta_m. \quad (16)$$

From (15) and (16) we obtain that

$$\beta_m \leq \liminf_{N \rightarrow \infty} P_N N^{\frac{2}{T}} \leq \limsup_{N \rightarrow \infty} P_N N^{\frac{2}{T}} \leq \alpha_m.$$

It is now easy to check that  $\beta_m$  is an increasing sequence,  $\alpha_m$  is a decreasing sequence, and  $\lim_{m \rightarrow \infty} \frac{\alpha_m}{\beta_m} = 1$ . Thus we obtain that  $\lim_{N \rightarrow \infty} P_N N^{\frac{2}{T}}$  exists and equals  $\lim_{m \rightarrow \infty} \alpha_m$ . To finish the case of  $N$  odd, we are left to prove that

$$\lim_{m \rightarrow \infty} \alpha_m = \pi^{\frac{2-T}{T}} \left(\Gamma\left(\frac{T+2}{2T}\right)\right)^2. \quad (17)$$

We can write

$$\alpha_m = \pi^{\frac{2}{T}} \left(m + \frac{1}{2}\right)^{\frac{2}{T}} \left(\frac{\prod_{j=1}^m (2jT - T)}{\prod_{j=1}^m (2jT - (T-2))}\right)^2.$$

Since  $\Gamma(x) = x\Gamma(x-1)$ , one easily obtains:

$$\prod_{j=1}^m (jr - s) = r^m \Gamma\left(m + \frac{r-s}{r}\right) / \Gamma\left(\frac{r-s}{r}\right), \quad 0 \leq s \leq r-1.$$

By applying the above formula with  $r = 2T$ ,  $s = T$  for the numerator, and  $r = 2T$ ,  $s = T-2$  for the denominator we obtain

$$\alpha_m = \pi^{\frac{2}{T}} \left(m + \frac{1}{2}\right)^{\frac{2}{T}} \left(\frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma\left(m + \frac{T+2}{2T}\right)} \frac{\Gamma\left(\frac{T+2}{2T}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2.$$

We now use the asymptotic for the  $\Gamma$  function

$$\Gamma(z) = e^{-z} z^{z-1/2} \sqrt{2\pi} (1 + O(1/z))$$

to obtain

$$\begin{aligned} \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma\left(m + \frac{T+2}{2T}\right)} &= \frac{e^{-(m+\frac{1}{2})} (m + \frac{1}{2})^{(m+\frac{1}{2})-1/2} \sqrt{2\pi} (1 + O(1/m))}{e^{-(m+\frac{T+2}{2T})} (m + \frac{T+2}{2T})^{(m+\frac{T+2}{2T})-1/2} \sqrt{2\pi} (1 + O(1/m))} \\ &= e^{\frac{1}{T}} \left(m + \frac{1}{2}\right)^{-\frac{1}{T}} \left(\frac{m + \frac{1}{2}}{m + \frac{T+2}{2T}}\right)^{m+\frac{1}{T}} + O(1/m) \\ &= \left(m + \frac{1}{2}\right)^{-\frac{1}{T}} + O(1/m) \end{aligned}$$

since

$$\lim_{m \rightarrow \infty} \left( \frac{m + \frac{1}{2}}{m + \frac{T+2}{2T}} \right)^{m + \frac{1}{T}} = \lim_{m \rightarrow \infty} \left( 1 - \frac{\frac{1}{T}}{m + \frac{T+2}{2T}} \right)^{m + \frac{1}{T}} = e^{-\frac{1}{T}}.$$

We thus obtain

$$\lim_{m \rightarrow \infty} \alpha_m = \pi^{\frac{2}{T}} \left( \frac{\Gamma(\frac{T+2}{2T})}{\Gamma(\frac{1}{2})} \right)^2$$

and (17) follows since  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ .

The proof for  $N$  even follows in a similar manner. As above, we can rewrite

$$P_N = \frac{T}{2 + (N-1)T} \tilde{P}_N,$$

where

$$\tilde{P}_N = \prod_{j=1}^{\frac{N-2}{2}} \frac{\tan \pi \frac{2jT}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}.$$

For any  $m \ll N$ , let

$$\tilde{P}_N^{(m,1)} = \prod_{j=m+1}^{\frac{N-2}{2}-m} \frac{\tan \pi \frac{2jT}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}}.$$

Define

$$\tilde{P}_N^{(m,k)} = \prod_{j=m+1}^{\lfloor \frac{N}{4} \rfloor} \frac{\tan \pi \frac{-(k-1)+2jT}{2(2+(N-1)T)}}{\tan \pi \frac{-(k-1)+2+(2j-1)T}{2(2+(N-1)T)}} \prod_{j=\lfloor \frac{N}{4} \rfloor + 1}^{\frac{N-2}{2}-m} \frac{\tan \pi \frac{(k-1)+2jT}{2(2+(N-1)T)}}{\tan \pi \frac{(k-1)+2+(2j-1)T}{2(2+(N-1)T)}}$$

and

$$\begin{aligned} \tilde{Q}_N^{(m,k)} &= \prod_{j=m+1}^{\lfloor \frac{N}{4} \rfloor - 1} \frac{\tan \pi \frac{(k-1)+2jT}{2(2+(N-1)T)}}{\tan \pi \frac{(k-1)+2+(2j-1)T}{2(2+(N-1)T)}} \cdot \prod_{j=\lfloor \frac{N}{4} \rfloor + 1}^{\lfloor \frac{N}{4} \rfloor + 1} \frac{\tan \pi \frac{2jT}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \\ &\quad \cdot \prod_{j=\lfloor \frac{N}{4} \rfloor + 1}^{\frac{N-2}{2}-m} \frac{\tan \pi \frac{(k-1)+2jT}{2(2+(N-1)T)}}{\tan \pi \frac{(k-1)+2+(2j-1)T}{2(2+(N-1)T)}}, \end{aligned}$$

where  $1 \leq k \leq 2T$ . We observe that  $\tilde{P}_N^{(m,1)} = \tilde{Q}_N^{(m,1)}$  and by applying Lemma 9.1 as in the odd case,

$$\tilde{Q}_N^{(m,k)} \leq \tilde{P}_N^{(m,1)} \leq \tilde{P}_N^{(m,k)}, \quad 1 \leq k \leq 2T.$$

Moreover, we have

$$\prod_{k=1}^{2T} \tilde{P}_N^{(m,k)} = \prod_{i=0}^{T-3} \frac{1}{\tan^2 \pi \frac{-i-T+(2m+1)T}{2(2+(N-1)T)}} F(N),$$

and

$$\prod_{k=1}^{2T} \tilde{Q}_N^{(m,k)} = \prod_{i=0}^{T-3} \frac{1}{\tan^2 \pi \frac{i+2+(2m+1)T}{2(2+(N-1)T)}} G(N)$$

with  $\lim_{N \rightarrow \infty} F(N) = \lim_{N \rightarrow \infty} G(N) = 1$ .

Since

$$\lim_{N \rightarrow \infty} N^{2(2-T)} \prod_{i=0}^{T-3} \frac{1}{\tan^2 \pi \frac{-i-T+(2m+1)T}{2(2+(N-1)T)}} F(N) = \left(\frac{2}{\pi}\right)^{2(T-2)} \prod_{i=0}^{T-3} \frac{T^2}{(2mT-i)^2},$$

$$\lim_{N \rightarrow \infty} N^{2(2-T)} \prod_{i=0}^{T-3} \frac{1}{\tan^2 \pi \frac{i+2+(2m+1)T}{2(2+(N-1)T)}} G(N) = \left(\frac{2}{\pi}\right)^{2(T-2)} \prod_{i=0}^{T-3} \frac{T^2}{(2mT+T+i+2)^2}$$

we obtain that

$$\limsup_{N \rightarrow \infty} N^{\frac{2-T}{T}} \tilde{P}_N^{(m,1)} \leq \left[ \left(\frac{2}{\pi}\right)^{2(T-2)} \prod_{i=0}^{T-3} \frac{T^2}{(2mT-i)^2} \right]^{\frac{1}{2T}},$$

and

$$\liminf_{N \rightarrow \infty} N^{\frac{2-T}{T}} \tilde{P}_N^{(m,1)} \geq \left[ \left(\frac{2}{\pi}\right)^{2(T-2)} \prod_{i=0}^{T-3} \frac{T^2}{(2mT+T+i+2)^2} \right]^{\frac{1}{2T}}.$$

Moreover, since

$$P_N = \frac{T}{2+(N-1)T} \prod_{j=1}^m \left( \frac{\tan \pi \frac{2jT}{2(2+(N-1)T)}}{\tan \pi \frac{2+(2j-1)T}{2(2+(N-1)T)}} \right)^2 \tilde{P}_N^{(m,1)}$$

we obtain for every  $m$  that

$$\tilde{\beta}_m \leq \liminf_{N \rightarrow \infty} P_N N^{\frac{2}{T}} \leq \limsup_{N \rightarrow \infty} P_N N^{\frac{2}{T}} \leq \tilde{\alpha}_m$$

with

$$\tilde{\beta}_m = \prod_{j=1}^m \left( \frac{2jT}{2+(2j-1)T} \right)^2 \pi^{\frac{2-T}{T}} \prod_{i=0}^{T-3} \left[ \frac{T}{mT+(T+i+2)/2} \right]^{\frac{1}{T}},$$

$$\tilde{\alpha}_m = \prod_{j=1}^m \left( \frac{2jT}{2+(2j-1)T} \right)^2 \pi^{\frac{2-T}{T}} \prod_{i=0}^{T-3} \left[ \frac{T}{mT-i/2} \right]^{\frac{1}{T}}.$$

Since

$$\lim_{m \rightarrow \infty} m^{\frac{T-2}{T}} \prod_{i=0}^{T-3} \left[ \frac{T}{mT+(T+i+2)/2} \right]^{\frac{1}{T}} = \lim_{m \rightarrow \infty} m^{\frac{T-2}{T}} \prod_{i=0}^{T-3} \left[ \frac{T}{mT-i/2} \right]^{\frac{1}{T}} = 1$$

it follows that  $\lim_{m \rightarrow \infty} \frac{\tilde{\alpha}_m}{\tilde{\beta}_m} = 1$ , and it is enough to prove that

$$\lim_{m \rightarrow \infty} m^{\frac{2-T}{T}} \prod_{j=1}^m \left( \frac{2jT}{2+(2j-1)T} \right)^2 \pi^{\frac{2-T}{T}} = \pi^{\frac{2-T}{T}} \left( \Gamma \left( \frac{T+2}{T} \right) \right)^2.$$



But this follows just like in the case of  $N$  odd by using

$$\prod_{j=1}^m 2jT = (2T)^m \Gamma(m+1),$$

$$\prod_{j=1}^m (2 + (2j-1)T) = (2T)^m \frac{\Gamma(m + \frac{T+2}{2T})}{\Gamma(\frac{T+2}{2T})},$$

$$\lim_{m \rightarrow \infty} \frac{\Gamma(m+1)}{\Gamma(m + \frac{T+2}{2T})} m^{\frac{2-T}{2T}} = 1.$$

□

Note that if  $T = 1$  then  $\Gamma(3/2) = \sqrt{\pi}/2$  and if  $T = 2$  then  $\Gamma(2) = 1$ . Substituting in (12) yields  $\pi^2/(4N^2)$  for  $T = 1$ , which is the asymptotic behavior in (5), and exactly  $1/N^2$  which is the value in (7). This justifies the following

**Conjecture C:** *Let  $T \geq 3$  and  $N \geq 1$  be integers. Then  $|I_N^{(T)}| = (P_N)^T$ .*

The proven theorem gives a reasonable approximation to the interval  $(-(P_N)^T, 1)$  that is conjectured to be the optimal range of multipliers  $(-\mu^*, 1)$ . It is remarkable that asymptotically the range for the admissible multipliers grows as  $N^2$ , regardless of the value of  $T$ . The dependence on  $T$  is in the constant. Note that the proof of the relation (12) is quite nontrivial; we did not find any similar estimates in the literature, and we could not apply any standard proof due to very delicate cancellations of the factors in the product.

It is also interesting to compare the left and right sides of the relation (12). Numeric simulations indicates that they are pretty close even for small values of  $T$  and  $N$ . Say, for  $T = 5$  and  $N = 5$  we have  $F_T(-1) = 0.02211102001 \approx 0.01752033601$ . For  $T=105$  and  $N=55$ ,  $F_T(-1) = 0.00006979604353 \approx 0.00006734173127$ . For  $T=1005$  and  $N=25$ ,  $F_T(-1) = 0.0003388694786 \approx 0.0003126559570$ . For  $T=1005$  and  $N=35$ ,  $F_T(-1) = 0.0001689295955 \approx 0.0001595183454$ . For  $T=1005$  and  $N=55$ ,  $F_T(-1) = 0.00006699551479 \approx 0.00006459833822$ .

## 10. Applications to non-linear dynamics

Scalar discrete systems always have real multipliers, so our method can be applied. We provided an example of finding an 8-cycle for the logistic map in [8]. In this paper we choose to provide several examples of cycle detection in the vector case. Since we do not know whether the multipliers are real negative there is no guarantee that applying averaging we will find a cycle. Thus, we simply apply the averaging procedure and check if we found the cycle of the given length. We first run several hundreds iterations of the open loop system, and then we switch to averaging using the produced

chaotic orbits as initial values.

**10.1. Hénon Map.** The Hénon Map is defined by

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + by_n, \\ y_{n+1} = x_n. \end{cases} \quad |b| < 1, a > 0.$$

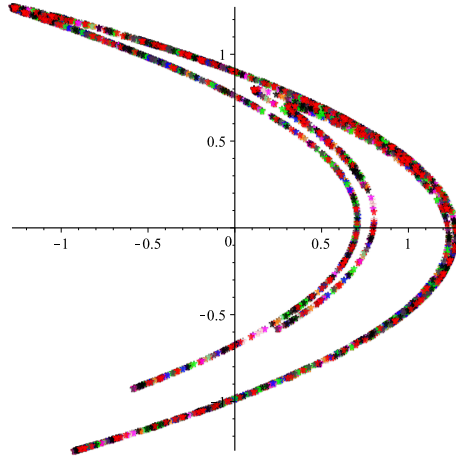


FIGURE 9. Hénon map

Fig.9 displays the strange attractor of the map, with different colors corresponding to orbits with the different initial values.

The Hénon map was well studied by mathematicians (c.f. for example [1]). It is standard to pick  $a = 1.4, b = 0.3$ , in which case there are two equilibria:  $(x_1, y_1) = (0.631354477, 0.631354477)$  and  $(x_2, y_2) = (-1.3199135566, -1.3199135566)$  with the corresponding multipliers

$$\mu_{1,2}(x_1, y_1) \in \{0.1559463223, -1.923738858\}$$

and

$$\mu_{1,2}(x_2, y_2) \in \{-0.09202956204, 3.259822098\}$$

We remark that the second equilibrium has a positive multiplier greater than 1, in which case our averaging method does not work.

We have applied 400 iterations to the initial system (1) and then the system (1) was replaced with the averaging system (3) for  $T = 1, N = 8$ . Fig. 10 displays the orbits of system (3) for  $n = 590, \dots, 690$  iterations.

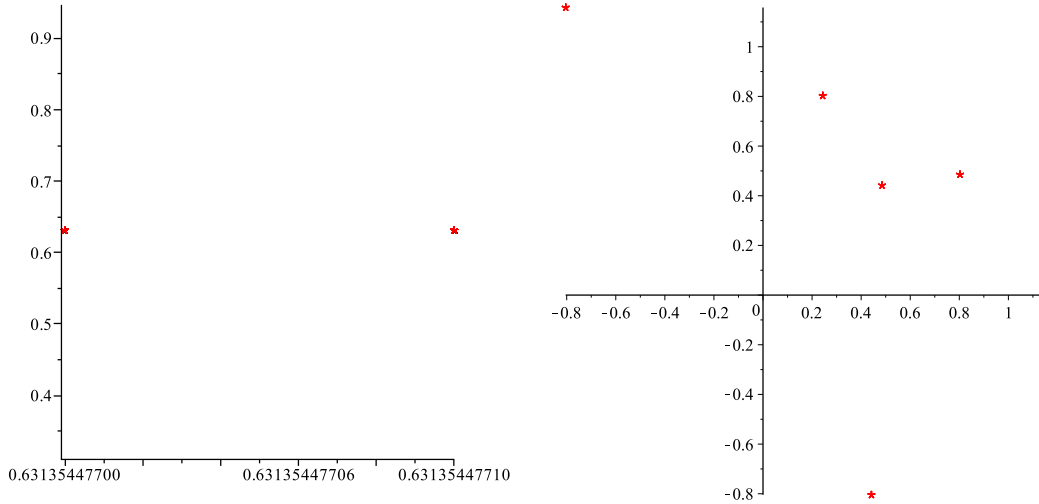


FIGURE 10. Hénon Map equilibrium and 6-cycle

One can observe only two spots, which is actually only one since the difference is very small. The spots are red because in MAPLE code the red color was assigned to the last initial value, but all the other colors converge to this equilibrium. Fig.10 also displays a 6-cycle of the map.

**10.2. Lozi Map.** The Lozi Map is a two-dimensional map similar to the Hénon map but with a different nonlinear term. It is given by the equations

$$\begin{cases} x_{n+1} = 1 - \alpha|x_n| + y_n \\ y_{n+1} = \beta x_n. \end{cases}$$

The standard choice of parameters is  $\alpha = 1.4, \beta = 0.3$ .

Fig.11 displays the strange attractor of the map. Fig.12 displays the 6 and 8 cycles in the Lozi map found by the averaging method with  $N = 6$ .

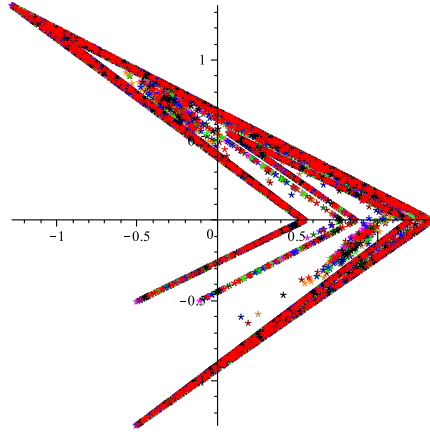


FIGURE 11. Lozi map

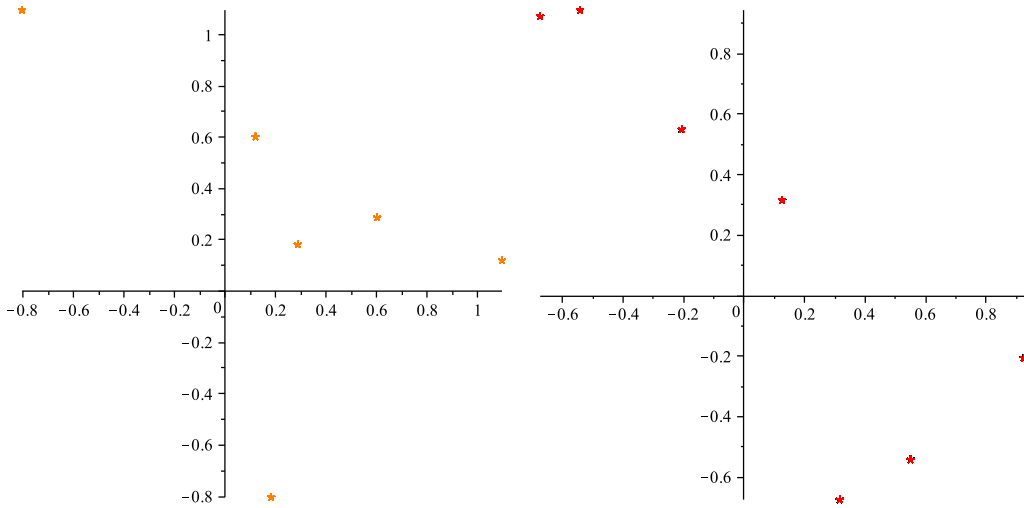


FIGURE 12. 6 and 8 cycles in the Lozi map

**10.3. Ikeda map.** The Ikeda map is given by

$$\begin{cases} x_{n+1} = 1 + 0.9 \left( x_n \cos \left( 0.4 - \frac{6}{1+x_n^2+y_n^2} \right) - y_n \sin \left( 0.4 - \frac{6}{1+x_n^2+y_n^2} \right) \right), \\ y_{n+1} = 0.9 \left( x_n \sin \left( 0.4 - \frac{6}{1+x_n^2+y_n^2} \right) + y_n \cos \left( 0.4 - \frac{6}{1+x_n^2+y_n^2} \right) \right). \end{cases}$$

Figure 13 displays the initial Ikeda map and the 5-cycle found by averaging the initial system starting from  $n=400$  to  $n=7800$  and  $N = 6$ .

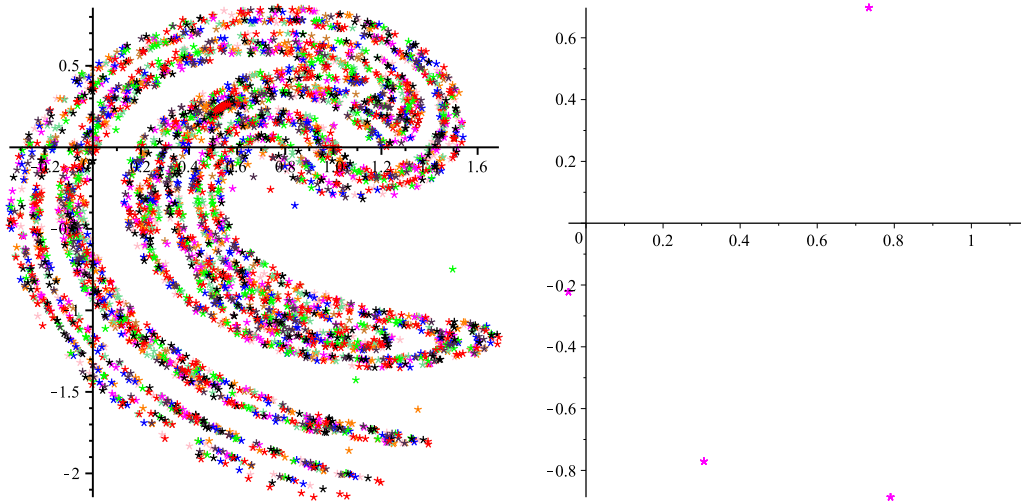


FIGURE 13. 5-cycle in the Ikeda Map

**10.4. Elhaj-Sprott map.** The Elhaj-Sprott map is given by the system

$$\begin{cases} x_{n+1} = 1 - 4 \sin(x_n) + 0.9y_n \\ y_{n+1} = \beta x_n. \end{cases}$$

After averaging with parameters  $N = 2$ ,  $T = 4$  we find a 4-cycle after 3900 iterations

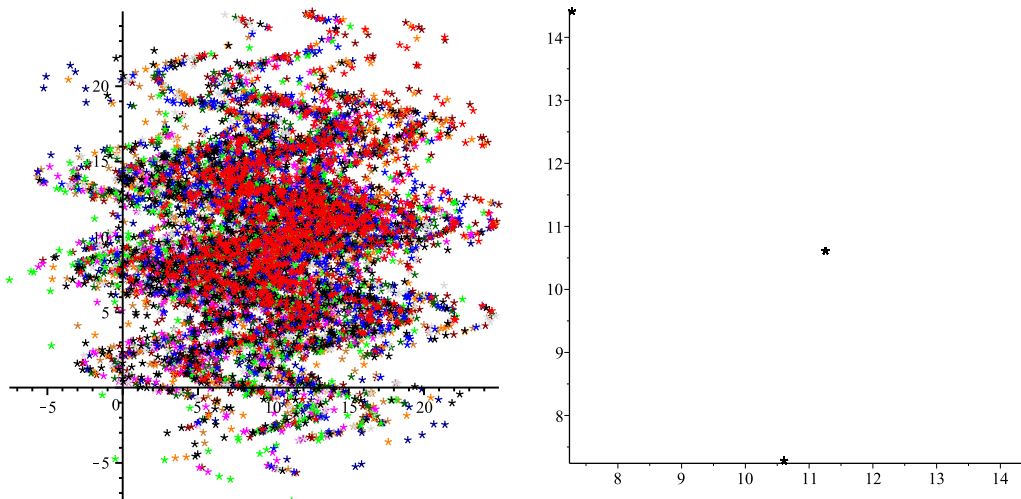


FIGURE 14. 4-cycle in the Elhaj-Sprott map

**10.5. Holmes cubic map.** The Holmes cubic map is given by the equations

$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = 0.2x_n + 2.77y_n - y_n^3. \end{cases}$$

Below we average with parameters  $N = 2, T = 2$ . The 2-cycle is detected after 5700 iterations.

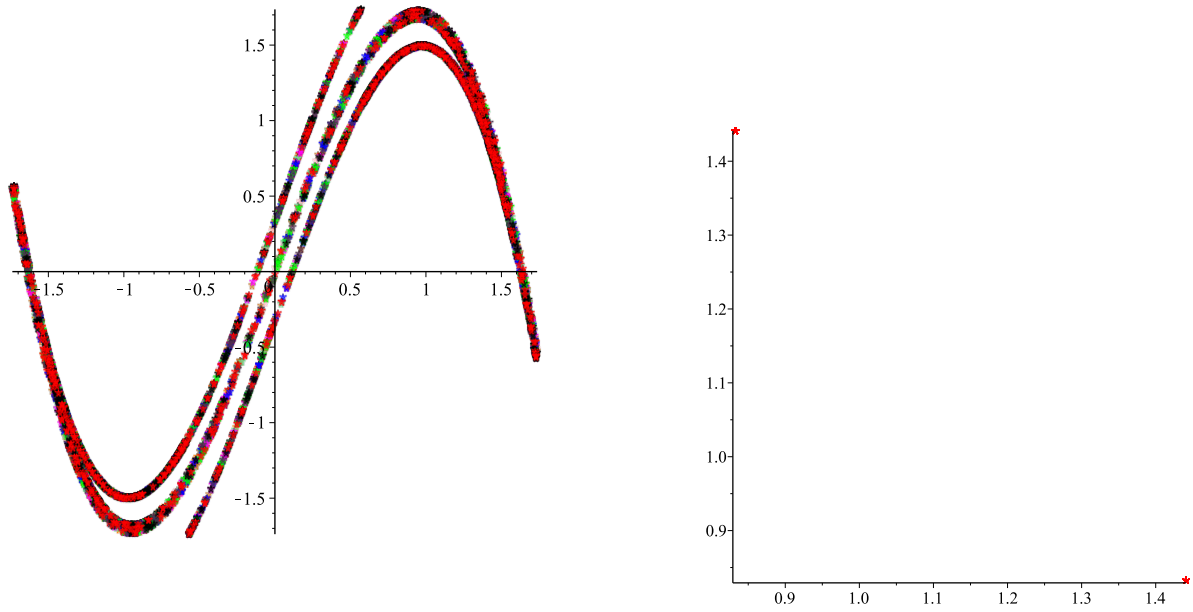


FIGURE 15. 2-cycle in Holmes cubic map

## Acknowledgments

We are thankful to an anonymous referee for carefully reading the paper and finding several typos. M.T. was supported in part by the NSF grant DMS-1636435.

## References

- [1] BENEDICKS, MICHAEL; CARLESON, LENNART. The dynamics of the Hénon maps. *Ann. of Math. (2)* **133** (1991), no. 1, 73–169. [MR1087346](#), [Zbl 0724.58042](#), doi:[10.2307/2944326](#). 620
- [2] BORWEIN, JONATHAN; BAILEY, DAVID. Mathematics by experiment: Plausible reasoning in the 21st Century. Second edition. *A K Peters, Ltd., Wellesley, MA*, 2008. xii+377 pp. ISBN: 978-1-56881-442-1. [MR2473161](#), [Zbl 1163.00002](#). 604

- [3] DAVIDCHACK, RUSLAN L.; LAI, YING-CHENG; KLEBANOFF, AARON; BOLLT, ERIK M. Towards complete detection of unstable periodic orbits in chaotic systems. *Phys. Lett. A.* **287** (2001), no. 1–2, 99–104. [MR1857249](#), [Zbl 0971.37041](#), doi:[10.1016/S0375-9601\(01\)00463-7](#). 604
- [4] DMITRISHIN, DMITRIY; HAGELSTEIN, PAUL; KHAMITOVA, ANNA; KORENOVSKIY, ANATOLIII; STOKOLOS, ALEXANDER. Fejér polynomials and control of nonlinear discrete systems. Preprint, 2018. To appear in *Constr. Approx.* [arXiv:1804.04537](#). 607
- [5] DMITRISHIN, DMITRIY; HAGELSTEIN, PAUL A.; KHAMITOVA, ANNA; STOKOLOS, ALEXANDER M. On the stability of cycles by delayed feedback control. *Linear Multilinear Algebra* **64** (2016), no. 8, 1538–1549. [MR3503366](#), [Zbl 1342.93062](#), [arXiv:1501.04573](#), doi:[10.1080/03081087.2015.1102833](#). 606
- [6] DMITRISHIN, DMITRIY; HAGELSTEIN, PAUL; KHAMITOVA, ANNA; STOKOLOS, ALEXANDER. Limitations of robust stability of a linear delayed feedback control. *SIAM J. Control Optim.* **56** (2018), no. 1, 148–157. [MR3742682](#), [Zbl 1386.93131](#), doi:[10.1137/16M1055980](#). 604
- [7] DMITRISHIN, DMITRIY; KHAMITOVA, ANNA. Methods of harmonic analysis in nonlinear dynamics. *C.R. Acad. Sci. Paris* **351** (2013), no. 9–10, 357–370. [MR3072162](#), [Zbl 1270.42001](#), doi:[10.1016/j.crma.2013.05.009](#). 605, 607
- [8] DMITRISHIN, DMITRIY; KHAMITOVA, ANNA; STOKOLOS, ALEXANDER M.; TOHANEANU, MIHAI. Finding cycles in nonlinear autonomous discrete dynamical systems. *Harmonic analysis, partial differential equations, Banach spaces, and operator theory, Vol. 2*, 199–237. Assoc. Women Math. Ser., 5. *Springer, Cham*, 2017. [MR3688144](#), [Zbl 1382.93027](#), doi:[10.1007/978-3-319-51593-9\\_8](#). 605, 606, 619
- [9] DUREN, PETER L. Univalent functions. Grundlehren der Mathematischen Wissenschaften, 259. *Springer-Verlag, New York*, 1983. xiv+382 pp. ISBN: 0-387-90795-5. [MR0708494](#), [Zbl 0514.30001](#). 607
- [10] GAUTHIER, DANIEL J. Resource letter: Controlling chaos. *Am. J. Phys.* **71** (2003), no. 8, 750–759. doi:[10.1119/1.1572488](#). 604
- [11] GENTHNER, CAREN N.; RUSCHEWEYH, STEPHAN; SALINAS, LUIS. A criterion for quasi-simple plane curves. *Comput. Methods Funct. Theory* **2** (2002), no. 1, 281–291. [MR2001604](#), [Zbl 1043.30003](#), doi:[10.1007/BF03321022](#). 612
- [12] KHAMITOVA, ANNA D. Characteristic polynomials for a cycle of non-linear discrete systems with time delays. *Vestn. St.-Peterbg. Univ. Ser. 10 Prikl. Mat. Inform. Protsesty Upr.* (2016), no. 4, 104–115. [MR3639416](#). 606
- [13] LI, TIEN-YIEN; YORKE, JAMES A. Period three implies chaos. *Amer. Math. Monthly* **82** (1975), no. 10, 985–992. [MR0385028](#), [Zbl 0351.92021](#), doi:[10.2307/2318254](#). 604
- [14] MORGÜL, ÖMER. A new generalization of delayed feedback control. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **19** (2009), no. 1, 365–377. [MR2512162](#), [Zbl 1170.93354](#), doi:[10.1142/S0218127409022920](#). 605
- [15] MORGÜL, ÖMER. Further stability results for a generalization of delayed feedback control. *Nonlinear Dynam.* **70** (2012), no. 2, 1255–1262. [MR2992132](#), [Zbl 1268.93063](#), doi:[10.1007/s11071-012-0530-z](#). 605, 606
- [16] OTT, EDWARD; GREBOGI, CELSO; YORKE, JAMES A. Controlling chaos. *Phys. Rev. Lett.* **64** (1990), no. 11, 1196–1199. [MR1041523](#), [Zbl 0964.37501](#), doi:[10.1103/PhysRevLett.64.1196](#). 604
- [17] PYRAGAS, KESTUTIS A. Continuous control of chaos by self-controlling feedback. *Phys. Lett.* 170A (1992), 421–428. doi:[10.1016/0375-9601\(92\)90745-8](#). 604
- [18] PYRAGAS, KESTUTIS A. Delayed feedback control of chaos. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **364** (2006), no. 1846, 2309–2334. [MR2251252](#), [Zbl 1152.93477](#), doi:[10.1098/rsta.2006.1827](#). 604
- [19] ŠARKOVSKIĬ, OLEKSANDR M. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Math. Ž.* **16** (1964), 61–71. [MR0159905](#), [Zbl 0122.17504](#). 604

- [20] SCHÖLL, ECKEHARD; SCHUSTER, HEINZ GEORG (EDS.). Handbook of chaos control. Second edition. *Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim*, 2008. xxx+819 pp. ISBN: 978-3-527-40605-0. MR2404092, Zbl 1130.93001, doi:10.1002/9783527622313. 604
- [21] SOCOLAR, JOSHUA E. S.; SUKOW, DAVID W.; GAUTHIER, DANIEL J. Stabilizing unstable periodic orbits in fast dynamical systems. *Phys. Rev. E* **50** (1994), 3245–3248. doi:10.1103/PhysRevE.50.3245. 605
- [22] SOLYANIK, ALEXEY. A-stabilization and the ranges of complex polynomials on the unit disk. Preprint, 2017. arXiv:1701.04784. 606
- [23] DE SOUSA VIEIRA, M.; LICHTENBERG, ALLAN J. Controlling chaos using nonlinear feedback with delay. *Phys. Rev. E* **54** (1996), 1200–1207. doi:10.1103/PhysRevE.54.1200. 605
- [24] SUFFRIDGE, TED J. On univalent polynomials. *J. London Math. Soc.* **44** (1969), 496–504. MR0235107, Zbl 0167.06302, doi:10.1112/jlms/s1-44.1.496. 607
- [25] USHIO, TOSHIMITSU. Limitation of delayed feedback control in nonlinear discrete-time systems. *IEEE Trans. Circ. Syst.* **43** (1996), no. 9, 815–816. doi:10.1109/81.536757. 604
- [26] ZYGMUND, ANTONI. Trigonometric series. Vol. I, II. Third edition. Cambridge Mathematical Library. *Cambridge University Press, Cambridge*, 2002. xii; Vol. I: xiv+383 pp.; Vol. II: viii+364 pp. ISBN: 0-521-89053-5. MR1963498, Zbl 1084.42003. 607

(D. Dmitrishin) DEPARTMENT OF MATHEMATICS, ODESSA NATIONAL POLYTECHNIC UNIVERSITY, ODESSA, UKRAINE  
[dmitrishin@opu.ua](mailto:dmitrishin@opu.ua)

(A. Stokolos) DEPARTMENT OF MATHEMATICS, GEORGIA SOUTHERN UNIVERSITY, STATESBORO GA, USA  
[astokolos@georgiasouthern.edu](mailto:astokolos@georgiasouthern.edu)

(M. Tohaneanu) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON KY, USA  
[mihai.tohaneanu@uky.edu](mailto:mihai.tohaneanu@uky.edu)

This paper is available via <http://nyjm.albany.edu/j/2019/25-28.html>.