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Iteration and the minimal resultant

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Abstract. Let K be an algebraically closed field that is complete with respect to a non-Archimedean absolute value, and let $\varphi \in K(z)$ have degree $d \geq 2$. We characterize maps for which the minimal resultant of an iterate φ^n is given by a simple formula in terms of d, n, and the minimal resultant of φ. Three characterizations of such maps are given, one measure-theoretic and two in terms of the indeterminacy locus $I(d)$ in the parameter space $\mathbb{P}^{2d+1}$ of (possibly degenerate) rational maps.

As an application, we are able to give a new explicit formula involving the Arakelov-Green's function attached to φ. We end by illustrating our results with some explicit examples.

Contents

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1. Introduction

Let K be a complete, algebraically closed non-Archimedean valued field with non-trivial absolute value $|\cdot|$. We will denote the ring of integers by $\mathcal{O}$, with maximal ideal $\mathfrak{m}$. The residue field will be written $k = \mathcal{O}/\mathfrak{m}$. If $\mathrm{char}(k) = 0$ let $q_{\mathfrak{m}} = e$ be the base of the natural logarithm; otherwise let $q_{\mathfrak{m}}$ be the residue characteristic. Let $\mathrm{ord}(x) = -\log_{q_{\mathfrak{m}}} |x|$.

Let $\varphi \in K(z)$ have degree $d \geq 2$. A homogeneous lift of φ is a pair of coprime homogeneous polynomials $\Phi = [F, G]$, say

$$F(X,Y) = a_d X^d + \ldots + a_0 Y^d$$
$$G(X,Y) = b_d X^d + \ldots + b_0 Y^d \ ,$$

with the property that $\varphi(z) = \frac{F(z,1)}{G(z,1)}$. A lift $[F, G]$ is said to be normalized if $\max(|a_i|, |b_i|) = 1$. We will often identify the map φ with a point in $\mathbb{P}^{2d+1}$ via the identification $\varphi \mapsto [a_d : \ldots : a_0 : b_d : \ldots : b_0] =: [a : b]$, which is clearly independent of the choice of lift.

The resultant $\mathrm{Res}(F, G)$ of a lift of φ is a homogeneous polynomial in the coefficients of F, G of degree $2d$, which we can also regard as a function of $\mathbb{P}^{2d+1}$ using the identification above. We will write R_φ for the ord value of the resultant of a normalized lift of φ. The *minimal resultant* is a conjugacy invariant of φ given

$$R_{[\varphi]} := \min_{\gamma \in \mathrm{PGL}_2(K)} R_{\varphi^\gamma} \ (\geq 0) \ ,$$

where $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$ is the usual conjugacy action. (A priori this should be an infimum, but see [9]). We say that φ has good reduction if $R_\varphi = 0$, and that φ has potential good reduction if $R_{[\varphi]} = 0$.

The minimal resultant has appeared in the work of several other authors. Silverman [11] gives an overview of the minimal resultant and asked questions about the existence of a global minimal model and about Northcott-type properties related to the minimal resultant. These questions were subsequently explored in work Rumely [8] and of Stout and Towsley [13]. Szpiro, Tepper, and the second author [14] have explored the connections between the minimality of the resultant and semistability in the sense of GIT, as has Rumely [9]. The first author has explored how the conjugates attaining the minimal resultant vary for higher iterates of the map [7].

In this paper, we are interested in understanding how the minimal resultant of an iterate φ^n relates to the minimal resultant of the original map. The resultant form itself behaves nicely under iteration: it is a power of the resultant of the original map, where the exponent is given by a simple formula in terms of n and d (see Lemma 3.1 below). Two things, however, get in the way of the *minimal* resultant from behaving so nicely. The first is the normalization that may have to take place in order to ensure that not all coefficients vanish under reduction: even if the coefficients for a lift of φ are normalized, the coefficients obtained by iteration need not be. The second is the potential change of coordinates that takes place to give the minimal valuation for the resultant, which need not be the same for every iterate.

We will draw on two tools for resolving these issues. The first is the connection between semi-stability and the minimality of the resultant, mentioned above. The second is a notion of indeterminacy introduced by DeMarco in [3, 4]; the indeterminacy locus $I(d) \subseteq \mathbb{P}^{2d+1}$ is the locus where the rational map $\Gamma_n : \mathbb{P}^{2d+1} \dashrightarrow \mathbb{P}^{2d^n+1}$ induced by iterating φ is undefined for some n.

These tools will be applied in particular to the reduction of φ: given a normalized lift $[F, G]$ of φ, corresponding to a point $[a : b] \in \mathbb{P}^{2d+1}$, let $[\tilde{a} : \tilde{b}] \in \mathbb{P}^{2d+1}(k)$ define the coordinates of a rational map $\varphi_{\mathfrak{m}}$ on $\mathbb{P}^1(k)$; we emphasize that $\varphi_{\mathfrak{m}}$ may not be a morphism, as $[\tilde{a} : \tilde{b}]$ may give rise to polynomials that share a common factor.

Our first main result is

Theorem 1.1. *Fix $n > 1$. The following are equivalent*:

(1) *The minimal resultant iteration formula*

$$\frac{1}{d(d-1)} \cdot R_{[\varphi]} = \frac{1}{d^n(d^n-1)} \cdot R_{[\varphi^n]} \tag{1.1}$$

holds.

(2) *In any coordinate system in which φ has semistable reduction, we have that $\varphi_{\mathfrak{m}} \notin I(d)$ and φ^n has semistable reduction as well.*

Condition 2 of the theorem can be stated in algebrogeometric terms: there is a natural diagram of graded rings:

$$\begin{array}{ccc} A_d^{\mathrm{SL}_2} & \longrightarrow & A_d \\ \uparrow & & \uparrow \\ A_{d^n}^{\mathrm{SL}_2} & \longrightarrow & A_{d^n} \end{array}$$

Here, $A_D = \mathbb{Z}[a_0, \ldots a_D, b_0, \ldots, b_D]$ is the free $\mathbb{Z}$-algebra generated by indeterminants corresponding to the coefficients of a pair of homogenous polynomials of degree D; $A_D^{\mathrm{SL}_2}$ is the SL_2 invariant subring. The vertical maps are given by the iteration morphism, which preserves SL_2 invariance because iteration commutes with the group action. If we apply Proj to the entire diagram then we get, passing from top right to bottom left, a morphism that is defined on an open set U_n of $\mathbb{P}^{2d+1}$:

$$U_n \to (\mathcal{M}_{d^n})^{ss}$$

Here the space $\mathcal{M}_{d^n}^{ss}$ is by definition $\mathrm{Proj}(A_{d^n}^{\mathrm{SL}_2})$, which has been shown (see [10]) to be a categorical quotient in the sense of geometric invariant theory. If we now base change to k, to get a diagram of varieties, then U_n consists of all maps that lie outside of $I(d)$, are semi-stable, and for which the n-th iterate is semi-stable. Condition 2 then says that there exists of choice of coordinates for which the reduction $\varphi_{\mathfrak{m}}$ is in U_n, the complement of the indeterminacy locus of the rational map $\mathbb{P}^{2d+1} \to (\mathcal{M}_{d^n})^{ss}$.

The second main result we will prove gives a geometric condition that can be useful in checking whether the minimal resultant iteration formula (1.1) holds asymptotically. The barycenter, $\mathrm{Bary}(\mu_\varphi)$ referred to in the statement of the theorem is a distinguished subset of the Berkovich projective line $\mathbf{P}^1_K$ that is 'balanced' with respect to the dynamics of f; its formal definition will be given in Section 4.

Theorem 1.2. *The following are equivalent*:

(1) *The minimal resultant iteration formula* (1.1) *holds for all n.*
(2) *The minimal resultant iteration formula* (1.1) *holds for infinitely many n.*
(3) *There exists a point $\zeta \in \mathrm{Bary}(\mu_\varphi)$ for which $\zeta = \gamma(\zeta_G)$ and $(\varphi^\gamma)_{\mathfrak{m}} \notin I(d)$.*

The proof of Theorem 1.2 involves a straightforward application of the work of DeMarco-Faber [5], along with previous work of first author.

One might like to get some sense of how many maps satisfy the equivalent conditions of Theorem 1.2. A natural way to measure this would be to take the closure in the moduli space $\mathcal{M}_d(K)$ of the set of such maps, and look at its dimension. This set trivially contains maps with potential good reduction. Silverman notes in [12] that the set of maps with potential good reduction includes monic integral polynomials, which gives it dimension at least $d-1$. He then improves ([12, Proposition 12]) this lower bound to d (he works over a number field, but the argument given works in our setting as well). So we have a lower bound of d.

As an application of Theorem 1.1, we are able to compute the minimal value of the diagonal Arakelov-Green's function $g_\varphi(x,x)$ (defined in Section 4) for maps φ satisfying the minimal resultant iteration formula for all n; in particular, we obtain

Corollary 1.3. *If φ satisfies the minimal resultant iteration formula for all n, then*

$$\min_{x \in \mathbf{P}^1_K} g_\varphi(x,x) = \frac{1}{d(d-1)} R_{[\varphi]} \ .$$

While it is tempting to believe that such a formula might hold in general, it turns out that this is not true: in a separate article the first author will show that, for a flexible Lattès map ψ_m associated to multiplication-by-m on a Tate curve with uniformizing parameter q, the min is given by $\min_{x\in\mathbf{P}^1_K} g_{\psi_m}(x,x) = -\frac{1}{24}\log|q|$, while when m is even $\frac{1}{d(d-1)} R_{[\psi_m]} = -\frac{1}{6}\log|q| + c(m)\log|q|$ for an explicit function $c(m)$ that depends on m (see Theorem 6.1 below).

Baker has shown [1] that $\min_{x\in\mathbf{P}^1_K} g_\varphi(x,x) > 0$ if and only if φ fails to have potential good reduction, and used this to show the finiteness of points of small height for non-isotrivial maps defined over function fields [1, Theorem 1.6]. It would be interesting to see whether the explicit computation given here can improve any of his bounds.

The outline for this paper is as follows: In Section 2 we introduce the necessary background regarding parameter space and reduction of rational maps. In Section 3 we establish preliminary lemmas concerning the resultant, semistability, and the indeterminacy locus $I(d)$, and at the end of this section we prove Theorem 1.1. Following this, in Section 4 we recall some background on the Berkovich projective line and prove Theorem 1.2. In Section 5 we prove Corollary 1.3, and we close in Section 6 with some examples.

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2. Notation and background

2.1. Iteration on parameter Space. Over any base, morphisms of degree d on $\mathbb{P}^1$ are parameterized by the coefficients of two homogeneous polynomials of degree d without common roots. This last condition is equivalent to the non-vanishing of the resultant of the two polynomials, and so the space of rational maps of degree d is the complement of the resultant hypersurface, an open subscheme of a projective space: $\mathrm{Rat_d} \subset \mathbb{P}^{2d+1}$. Points in $\mathbb{P}^{2d+1}$ that are not in $\mathrm{Rat_d}$ correspond to pairs of homogeneous polynomials $[\widetilde{F}, \widetilde{G}]$ with a common factor $\widetilde{A}$; canceling the common factor yields a "degenerate" map $\widetilde{\varphi}$ of lower degree.

Iteration of a rational map defines a morphism $\Gamma_n : \mathrm{Rat_d} \to \mathrm{Rat}_{d^n}$. This map extends to a rational map on the projective spaces:

$$\Gamma_n : \mathbb{P}^{2d+1} \dashrightarrow \mathbb{P}^{2d^n+1}.$$

In [3], DeMarco showed that, for every n, this map is defined outside of a set $I(d)$ of co-dimension $d+1$, and described precisely what this locus looks like. Though working over $\mathbb{C}$, DeMarco gives a completely algebraic characterization of the indeterminacy locus [3, Lemma 2.1] that works over base $\mathbb{Z}$. Her characterization of $I(d)$ as a set [3, Lemma 2.2] then works over an algebraically closed field.

Proposition 2.1. *The set on which $\Gamma_n : \mathbb{P}^{2d+1} \dashrightarrow \mathbb{P}^{2d^n+1}$ is undefined consists, for every n, of the maps such that $\widetilde{\varphi}$ is constant and this constant is a root of $\widetilde{A}$.*

Proof. See [3, Lemma 2.2]. □

Crucially, $I(d)$ as a set doesn't depend on n. Throughout this paper, we will primarily be concerned with whether or not a rational map defined over the residue field lies in $I(d)$; as such, we will most often view $I(d) \subseteq \mathbb{P}^{2d+1}(k)$.

2.2. Reduction and the resultant. Let $\varphi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ be a rational map of degree at most d. Then φ can be represented by a point $[a, b] = [a_d, ..., a_0, b_d, ..., b_0] \in \mathbb{P}^{2d+1}(K)$ in projective space; we let

$$F(X, Y) = a_d X^d + ... + a_0 Y^d \ , \ G(X, Y) = b_d X^d + ... + b_0 Y^d$$

be homogeneous polynomials of degree d that represent φ. If φ is a morphism, we say that the representation F, G is *normalized* if each coefficient has absolute value at most one, and at least one coefficient has absolute value 1. Any representative can be made into a normalized representative if we divide through by the coefficient with the largest absolute value; on the other hand, normalized representatives are not unique: scaling by any unit will preserve normalization.

Notation 1. Given a normalized representative F, G of a morphism φ, we define the *reduction of* φ to be the rational map of $\mathbb{P}^1(k)$ given

$$\varphi_{\mathfrak{m}} := [\tilde{F}, \tilde{G}] \ ,$$

where $\tilde{F}, \tilde{G}$ are the polynomials over k obtained by reducing the coefficients of F, G. On the parameter space $\mathbb{P}^{2d+1}(K)$, this corresponds to reducing coordinates modulo $\mathfrak{m}$; if φ corresponds to the point $[a, b] \in \mathbb{P}^{2d+1}(K)$, the point corresponding to the reduction map is denoted $[\tilde{a}, \tilde{b}] \in \mathbb{P}^{2d+1}(k)$.

Notation 2. The reduction is said to be *degenerate* if the polynomials $\tilde{F}, \tilde{G}$ have a common factor. In this case, we write $\tilde{A}$ =gcd$(\tilde{F}, \tilde{G})$. Let $\tilde{F} = \tilde{A} \cdot \tilde{F}_0$ and $\tilde{G} = \tilde{A} \cdot \tilde{G}_0$. The factors of $\tilde{A}$ are referred to as the *holes* of $\varphi_{\mathfrak{m}}$. The *residue map* $\widetilde{\varphi}$ of φ is the morphism of $\mathbb{P}^1(k)$ given by

$$\widetilde{\varphi} := [\tilde{F}_0, \tilde{G}_0] \ .$$

If the polynomials $\tilde{F}, \tilde{G}$ do not have a common factor, the residue map is defined to be the morphism $[\tilde{F}, \tilde{G}]$ of $\mathbb{P}^1(k)$; in this case, φ has good reduction.

Notation 3. Given a rational map $\varphi \in \mathrm{Rat}_d(K)$, let R_φ denote the ord-value of the resultant of a normalized lift of φ. Likewise, let $R_{[\varphi]}$ denote the minimal resultant, which gives the minimal value of R_{φ^γ} among all $\mathrm{PGL}_2(K)$-conjugates of φ.

Notation 4. We let ρ_d denote the resultant form, i.e. the homogenous polynomial of degree $2d$ in $2d + 2$ indeterminants that correspond to the generic coefficients of two homogenous polynomials of degree d; as mentioned above, the non-vanishing of the resultant determines $\mathrm{Rat_d}$ as an open subscheme of $\mathbb{P}^{2d+1}$.

3. Preliminary lemmas

3.1. The resultant under iteration. Our ultimate goal is to understand when the minimal resultant transforms "nicely" under iteration. Therefore the following lemma about how the resultant transforms under iteration is essential to what follows. Its proof is straightforward, so we have included it here.

Fix an integer d, and let ρ_d be the resultant form. Let

$$N_n = \frac{d^n(d^n-1)}{d(d-1)}.$$

Lemma 3.1. *If (a,b) are the $2d+2$ coefficients of two homogeneous polynomials of degree d, and (a_n,b_n) are the $2d^n+2$ coefficients of the two homogeneous polynomials of degree d^n obtained by iteration n times, then $\rho_{d^n}(a_n,b_n) = \rho_d(a,b)^{N_n}$.*

Proof. This follows from an exercise in [11] that gives the resultant for a composition of pairs of homogeneous polynomials in two variables: let f, g be of degree n_1 and F, G of degree n_2. Then if $R = F(f,g)$ and $S = G(f,g)$, then

$$\rho_{n_1 n_2}(R,S) = \rho_{n_1}(f,g)^{n_2}\rho_{n_2}(F,G)^{n_1^2}$$

We now apply this when F, G are the homogenous polynomials corresponding to (a,b) and F_n, G_n are the homogeneous polynomials of degree d^n obtained by the n-th iteration. Then

$$\begin{aligned}\rho_{d^{n+1}}(F_{n+1},G_{n+1}) &= \rho_d(F,G)^{d^n}\rho_{d^n}(F_n,G_n)^{d^2}\\ &= \rho_d(F,G)^{d^n+d^2(d^{n-1}(d^{n-1}+\cdots+1))}\\ &= \rho_d(F,G)^{d^n(d^{n+1}-1)/(d-1)}\end{aligned}$$

□

Now we move to understand when R_φ transforms nicely under iteration, without any assumptions yet about minimality:

Proposition 3.2. *The following are equivalent*:

(1) *For every n we have $R_{\varphi^n} = N_n \cdot R_\varphi$.*
(2) *For some $n > 1$, we have $R_{\varphi^n} = N_n \cdot R_\varphi$.*
(3) *The reduction $\varphi_{\mathfrak{m}}$ lies outside of $I(d)$.*

Proof. Clearly the first condition implies the second.

Now assume that for some $n \in \mathbb{N}$ we have $R_{\varphi^n} = N_n \cdot R_\varphi$. In [8, Equation (2.3)], it is shown that for a given n one has

$$R_{\varphi^n} = \operatorname{ordRes}(F^n, G^n) - 2d^n \min_{0\le i,j\le d}(\operatorname{ord}(a_i^n), \operatorname{ord}(b_j^n)) \ ,$$

where a_i^n, b_j^n are the coefficients of the coordinate polynomials of $\varphi^n = [F^n, G^n]$. Assuming that we start with normalized F, G, and using the iteration formula for the resultant given in Lemma 3.1 above, we find

$$R_{\varphi^n} = N_n R_\varphi - 2d^n \min_{0\le i,j\le d^n}(\operatorname{ord}(a_i^n), \operatorname{ord}(b_j^n)) \ .$$

Thus $R_{\varphi^n} = N_n R_\varphi$ holds if and only if $\min_{0\le i,j\le d^n}(\mathrm{ord}(a_i^n), \mathrm{ord}(b_j^n)) = 0$, and since $\mathrm{ord}(a_i^n)$, $\mathrm{ord}(b_i^n) \ge 0$, this is equivalent to the fact that some $\widetilde{a_i}^n, \widetilde{b_i}^n$ is non-zero. Since reduction commutes with iteration when iteration is defined, this is equivalent to saying that $\Gamma_n(\varphi_{\mathfrak{m}})$ is well-defined. By [3] this is equivalent to saying that that $\varphi_{\mathfrak{m}}$ is outside of $I(d)$. So the second condition implies the third.

In fact, assuming the third condition, the chain of equivalences in the preceeding paragraph implies that $R_{\varphi^n} = N_n \cdot R_\varphi$ for any choice of n; hence the third condition implies the first, and we are done. □

3.2. Semi-stability. To address the question of minimality, we will invoke a connection between semistability and minimality of the resultant.

In [10], Silverman studied the GIT quotient $\mathcal{M}_d$ of $\mathrm{Rat_d}$ by the conjugation action of SL_2. Crucial to this construction is the semi-stable locus $(\mathbb{P}^{2d+1})^{ss}$, which is an open subscheme of $\mathbb{P}^{2d+1}$ that contains $\mathrm{Rat_d}$. Intuitively, it is the largest subscheme of $\mathbb{P}^{2d+1}$ on which a quotient scheme makes sense. The following is a useful explicit way to think of the semi-stable locus. Let $A_d = \mathbb{Z}[a, b]$. $\mathcal{M}_d$ and $\mathrm{Rat_d}$ are affine schemes, defined over $\mathbb{Z}$, and the map between them is given by the map of rings $(A_d)^{\mathrm{SL}_2}_{(\rho_d)} \to (A_d)_{(\rho_d)}$ (the superscript indicates SL_2 invariant functions). The quotient space $\mathcal{M}_d^{ss}$ is $\mathrm{Proj}(A_d^{\mathrm{SL}_2})$ and $(\mathbb{P}^{2d+1})^{ss}$ is simply the largest open subset of $\mathbb{P}^{2d+1}$ on which the inclusion of graded rings $A_d^{\mathrm{SL}_2} \to A_d$ induces a morphism of schemes. In this way, the semi-stable points are the complement of the indeterminacy locus for the quotient map.

Proposition 3.3. *φ has semi-stable reduction if and only if $R_\varphi = R_{[\varphi]}$.*

Proof. See [9, Theorem 7.4], which is stated in the language of Berkovich spaces. The forward implication had been established earlier in [14, Theorem 3.3], for maps of $\mathbb{P}^n$ with $n \ge 1$. □

We are now ready to prove Theorem 1.1:

Proposition 3.4. *Let K be a complete, algebraically closed non-Archimedean valued field, and let $\varphi \in K(z)$ have degree $d \ge 2$. Fix $n > 1$.*

The minimal resultant iteration formula $R_{[\varphi^n]} = N_n \cdot R_{[\varphi]}$ holds if and only if for any coordinate system where φ has semistable reduction, we have that $\varphi_{\mathfrak{m}} \notin I(d)$ and φ^n has semistable reduction as well.

Proof. Let $n > 1$. Suppose first that

$$R_{[\varphi^n]} = N_n \cdot R_{[\varphi]}\ , \tag{3.1}$$

and fix coordinates so that φ has semistable reduction. Let $\varphi = [F, G]$ be a normalized lift of φ, with

$$F(X, Y) = a_d X^d + ... + a_0 Y^d\ ,\ G(X, Y) = b_d X^d + ... + b_0 Y^d\ .$$

By Proposition 3.3, we find

$$R_{[\varphi]} = \mathrm{ord}(\mathrm{Res}(F,G)) \ .$$

Again applying the formula in [8, Equation 2.3] and applying the iteration formula from Lemma 3.1 we have

$$R_{\varphi^n} = N_n \,\mathrm{ordRes}(F,G) - 2d^n \min_{0\le i,j\le d^n}(\mathrm{ord}(a_i^n), \mathrm{ord}(b_j^n)) \ .$$

Now, suppose φ^n does not have semistable reduction. Then by Proposition 3.3, $R_{[\varphi^n]} < R_{\varphi^n}$, and we find

$$\begin{aligned} N_n \cdot \mathrm{ordRes}(F,G) &= N_n \cdot R_{[\varphi]} \\ &= R_{[\varphi^n]} \\ &< R_{\varphi^n} \\ &= N_n \mathrm{ordRes}(F,G) - 2d^n \min_{0\le i,j\le d^n} \min(\mathrm{ord}(a_i^n), \mathrm{ord}(b_j^n)) \ . \end{aligned}$$

Cancelling the common factor of $N_n \cdot \mathrm{ordRes}(F,G)$ and reversing the inequality gives

$$0 > 2d^n \min_{0\le i,j\le d^n} \min(\mathrm{ord}(a_i^n), \mathrm{ord}(b_j^n)) \ ; \tag{3.2}$$

but recall that our lift $\varphi = [F,G]$ of φ was normalized, and the coefficients a_i^n, b_j^n are polynomial combinations of the coefficients of F, G. Taking polynomial combinations cannot decrease the ord value, hence (3.2) is a contradiction. We conclude that φ^n has semistable reduction as well.

In particular, (3.1) now reads

$$R_{\varphi^n} = R_{[\varphi^n]} = N_n \cdot R_{[\varphi]} = N_n \cdot R_\varphi \ ,$$

and so by Proposition 3.2 we conclude that $\varphi_{\mathfrak{m}} \notin I(d)$. This completes the proof of the forward implication of the proposition.

For the reverse implication, suppose that we have chosen a coordinate system in which φ and φ^n have semistable reduction, and also for which $\varphi_{\mathfrak{m}} \notin I(d)$. Combining Propositions 3.2 and 3.3 gives

$$R_{[\varphi^n]} = R_{\varphi^n} = N_n \cdot R_\varphi = N_n \cdot R_{[\varphi]} \ ,$$

which is the asserted equality. □

4. Barycenters and minimal resultant locus

In this section we establish Theorem 1.2 which gives a geometric condition for determining when the minimal resultant iteration formula (1.1) holds for all n. To do this, we first recall some facts about the Berkovich projective line $\mathbf{P}^1_K$ and probability measures on $\mathbf{P}^1_K$.

4.1. The Berkovich projective line. Let K be an algebraically closed field that is complete with respect to a non-trivial absolute value. The Berkovich projective line $\mathbf{P}^1_K$ over K is defined as the set of (equivalence classes[1] of) multiplicative seminorms on $K[X, Y]$ which extend the absolute value on K. There are four types of points in $\mathbf{P}^1_K$:

- A point $[a : b] \in \mathbb{P}^1(K)$ gives rise to a seminorm $G \mapsto |G(a, b)|$; these are called *type I* points. This identification gives an inclusion $\mathbb{P}^1(K) \hookrightarrow \mathbf{P}^1_K$.
- A closed disc $D(a, r) \subseteq K$ gives rise to a seminorm by
$$G \mapsto \sup_{z \in D(a,r)} |G(z, 1)| \ ;$$
these are called *type II* points if $r \in |K^\times|$, and are called *type III* points otherwise.
- Points of type IV correspond to sequences of type II or type III points, but their precise definition is not needed here. See [2, Chapters 1, 2].

When $K = \mathbb{C}$, it is a consequence of Gelfand's theorem that $\mathbf{P}^1_K = \mathbb{P}^1(\mathbb{C})$.

There are two topologies that one usually considers on $\mathbf{P}^1_K$. The first is the *weak topology*: it is the weakest topology so that the maps $\mathbf{P}^1_K \ni \zeta \mapsto [G]_\zeta$ are continuous for all $G \in K[X, Y]$. In this topology, $\mathbf{P}^1_K$ is a compact connected space, though it is not in general metrizable. The *strong topology* arises from a metric σ defined on $\mathbf{H}^1_K := \mathbf{P}^1_K \setminus \mathbb{P}^1(K)$ (and extended to all of $\mathbf{P}^1_K$ by setting $\sigma(x, y) = \infty$ whenever $x \in \mathbb{P}^1(K)$ and $y \in \mathbf{P}^1_K \backslash \{x\}$). In the strong topology, $\mathbf{P}^1_K$ is no longer compact, but it carries the structure of an $\mathbb{R}$-tree. Consequently, $\mathbf{P}^1_K$ is uniquely path connected.

The tangent space at a point $\zeta \in \mathbf{P}^1_K$ consists of equivalence classes of paths emanating from ζ; it is denoted by T_ζ, and since $\mathbf{P}^1_K$ is uniquely path connected its elements are in bijection with $\mathbf{P}^1_K \setminus \{\zeta\}$. We will often write $B_\zeta(\vec{v})^-$ for the connected component corresponding to a given $\vec{v} \in T_\zeta$. When ζ has type II, T_ζ is also in bijection with $\mathbb{P}^1(k)$: if $\zeta = \zeta_G$ this identification can be realized by identifying the tangent directions with open subdiscs $D(a, 1)^- \subseteq D(0, 1)$ or with the complement $\mathbb{P}^1(K) \setminus D(0, 1)$.The general case follows by change of coordinates. Having made such an identification, we write $\vec{v}_a \in T_\zeta$ for the vector corresponding to $a \in \mathbb{P}^1(k)$.

The automorphisms of $\mathbb{P}^1(K)$ extend to automorphisms of $\mathbf{P}^1_K$; more generally, the action of a rational map φ on $\mathbb{P}^1(K)$ extends to a proper, continuous map $\varphi : \mathbf{P}^1_K \to \mathbf{P}^1_K$. A description of this action can be found in [2, Chapter 2].

4.2. Canonical measures. A rational map $\varphi \in K(z)$ of degree $d \geq 2$, where K is either $\mathbb{C}$ or a complete algebraically closed non-Archimedean field, induces an invariant measure μ_φ on $\mathbf{P}^1_K$ characterized by the pullback

[1]See [2, Section 2.2] for the precise definition of the equivalence relation.

formula $\frac{1}{d}\varphi^*\mu_\varphi = \mu_\varphi$ and the fact that μ_φ does not charge the exceptional set of φ. It can be realized as the limit of the pullbacks $\frac{1}{d^n}(\varphi^n)^*\nu$ for any probability measure ν on $\mathbf{P}^1_K$ that does not charge the exceptional set ([6] Théorème A).

DeMarco defined analogous measures for degenerate rational maps: working with homogeneous lifts $\Phi = [F, G]$, let $A = \gcd(F, G)$ and write $\Phi = \widetilde{A}\cdot\widetilde{\varphi}$ as above. When $0 < \deg(\widetilde{\varphi}) < d$, the canonical invariant measure introduced by DeMarco is

$$\mu_{\varphi_\mathfrak{m}} = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} \sum_{\substack{\widetilde{\varphi}^n(z)=h \\ A(h)=0}} \delta_z \ ,$$

while if $\deg(\widetilde{\varphi}) = 0$ then

$$\mu_{\varphi_\mathfrak{m}} = \frac{1}{d} \sum_{A(h)=0} \delta_h \ .$$

Both are probability measures on $\mathbb{P}^1(K)$, and one can check that they satisfy $\mu_{\varphi_\mathfrak{m}{}^n} = \mu_{\varphi_\mathfrak{m}}$ provided $\varphi_\mathfrak{m} \notin I(d)$. DeMarco shows

Lemma 4.1. [4, Propositions 3.2, 3.3] *Let φ be a (possibly degenerate) rational map of degree at most d.*

Suppose that d is even and $\varphi \notin I(d)$. Then φ^n is stable for all $n \geq 1$ if and only if $\mu_{\varphi_\mathfrak{m}}(\{z\}) \leq \frac{1}{2}$ for all $z \in \mathbb{P}^1(K)$.

Suppose that d is odd and $\varphi \notin I(d)$. Then φ^n is semistable for all $n \geq 1$ if and only if $\mu_{\varphi_\mathfrak{m}}(\{z\}) \leq \frac{1}{2}$ for all $z \in \mathbb{P}^1(K)$. Furthermore, if $\mu_{\varphi_\mathfrak{m}}(\{z\}) < \frac{1}{2}$ for all $z \in \mathbb{P}^1(K)$, then φ^n is stable for all $n \geq 1$.

We reiterate that although [4, Proposition 3.2, 3.3] were originally stated for rational maps over $\mathbb{C}$, they hold over an arbitrary algebraically closed field.

4.3. Reduced measures. Let $\varphi \in K(z)$ have degree $d \geq 2$, where K is a complete, algebraically closed non-Archimedean field. Let μ_φ be the canonical invariant measure on $\mathbf{P}^1_K$ described above. Throughout this section, assume that φ fails to have good reduction. In this case, μ_φ induces a measure on $\mathbb{P}^1(k)$ as follows: identifying tangent directions $\vec{v}_a \in T_{\zeta_G}$ with points $a \in \mathbb{P}^1(k)$, let

$$\widetilde{\mu_\varphi}(\{a\}) := \mu_\varphi(B_{\zeta_G}(\vec{v}_a)^-) \ .$$

Note that this is a special case of a Γ-measure, which were introduced by DeMarco-Faber [5]; here, $\Gamma = \{\zeta_G\}$.

The map φ induces another measure on $\mathbb{P}^1(k)$: since φ does not have good reduction, $\varphi_\mathfrak{m}$ is a degenerate rational map defined over k, and the corresponding measure $\mu_{\varphi_\mathfrak{m}}$ for degenerate maps introduced in the previous section is a probability measure on $\mathbb{P}^1(k)$.

Proposition 4.2. [5, Theorem C and Proposition 5.1] *Suppose* φ *does not have good reduction. If* $\varphi_{\mathfrak{m}} \notin I(d)$*, then* $\widetilde{\mu_\varphi} = \mu_{\varphi_{\mathfrak{m}}}$.

Proof. By [5, Proposition 5.1], $\varphi_{\mathfrak{m}} \notin I(d)$ is equivalent to saying that the pair $(\varphi, \{\zeta_G\})$ is analytically stable (see [5] for a definition of analytically stable). This, together with the assumption that φ does not have good reduction, allows us to apply [5, Theorem C]. As DeMarco and Faber point out just after Proposition 5.1, the stationary measure arising from the Markov process in Theorem C recovers the formula for $\mu_{\varphi_{\mathfrak{m}}}$. □

4.4. Barycenters and semi-stability. Given a probability measure ν on $\mathbf{P}^1_K$, where K is a complete, algebraicaly closed non-Archimedean valued field, Rivera-Letelier defined the barycenter of ν to be

$$\mathrm{Bary}(\nu) = \{\zeta \in \mathbf{P}^1_K \ : \ \nu(B_\zeta(\vec{v})^-) \le \frac{1}{2} \text{ for all } \vec{v} \in T_\zeta\} \ .$$

This set is always non-empty, and will either be a point or a segment [7, Proposition 6].

Another distinguished subset of $\mathbf{P}^1_K$ is the minimal resultant locus of a rational map φ. If $\varphi \in K(z)$ has degree $d \ge 2$, the minimal resultant locus can be defined[2] as

$$\mathrm{MinResLoc}(\varphi) = \overline{\{\zeta \in \mathbf{P}^1_K \ : \ \zeta = \gamma(\zeta_G) \text{ for } \gamma \in \mathrm{PGL}_2(K) \text{ and } \varphi^\gamma \text{ has semistable reduction}\}}\ ,$$

where the closure is with respect to the strong topology. Rumely has shown that, as was the case with the barycenter, $\mathrm{MinResLoc}(\varphi)$ is always either a point or a segment. The first author has shown that the minimal resultant loci accumulate on the barycenter of μ_φ:

Proposition 4.3. [7, Proposition 5] *For any* $\epsilon > 0$*, there exists* N *so that*

$$\mathrm{MinResLoc}(\varphi^n) \subseteq \{\zeta \in \mathbf{P}^1_K \ : \ \sigma(\zeta, \mathrm{Bary}(\mu_\varphi)) < \epsilon\}$$

for all $n \ge N$.

We are now ready to prove Theorem 1.2, which we recall here:

Theorem. *The following are equivalent*:

(1) *The minimal resultant iteration formula*

$$\frac{1}{d(d-1)} \cdot R_{[\varphi]} = \frac{1}{d^n(d^n-1)} \cdot R_{[\varphi^n]}$$

holds for all n.

(2) *The minimal resultant iteration formula holds infinitely often.*

(3) *There exists a point* $\zeta \in \mathrm{Bary}(\mu_\varphi)$ *for which* $\zeta = \gamma(\zeta_G)$ *and* $(\varphi^\gamma)_{\mathfrak{m}} \notin I(d)$.

[2]This is different from, but equivalent to, Rumely's original definition of $\mathrm{MinResLoc}(\varphi)$. See [9, Theorem 7.4]

Proof. First, if φ has potential good reduction the result is immediate. So we assume that φ does not have potential good reduction.

The first condition clearly implies the second. Suppose now that the minimal resultant iteration formula holds infinitely often. Applying Theorem 1.1, if $\zeta = \gamma(\zeta_G)$ is any type II point with semistable reduction, we have that $(\varphi^\gamma)_{\mathfrak{m}} \notin I(d)$ and for infinitely many n, $(\varphi^\gamma)^n$ has semistable reduction as well. Geometrically, the latter condition says that $\zeta \in \mathrm{MinResLoc}(\varphi) \cap \mathrm{MinResLoc}(\varphi^n)$ for infinitely many n. By Proposition 4.3 this implies that $\zeta \in \mathrm{Bary}(\mu_\varphi)$, and hence (3) is established.

To show (3) implies (1), suppose that $\zeta \in \mathrm{Bary}(\mu_\varphi)$, that $\zeta = \gamma(\zeta_G)$ for some $\gamma \in \mathrm{PGL}_2(K)$, and that $(\varphi^\gamma)_{\mathfrak{m}} \notin I(d)$. Replacing φ by φ^γ, we may assume that $\zeta = \zeta_G$ and that $\varphi_{\mathfrak{m}} \notin I(d)$. The assumption that $\zeta_G \in \mathrm{Bary}(\mu_\varphi)$ implies that $\mu_\varphi(B_{\zeta_G}(\vec{v})^-) \leq \frac{1}{2}$ for all $\vec{v} \in T_{\zeta_G}$; by Proposition 4.2 this in turn implies that $\mu_{\varphi_{\mathfrak{m}}}(\{z\}) \leq \frac{1}{2}$ for all $z \in \mathbb{P}^1(k)$. The assumption that $\varphi_{\mathfrak{m}} \notin I(d)$ allows us to apply Lemma 4.1, which tells us that $\varphi_{\mathfrak{m}}$ has semistable reduction for all n. In particular Theorem 1.1 confirms that the minimal resultant iteration formula holds for all n. $\square$

5. An application to potential theory

If the equivalent conditions of Theorem 1.2 are satisfied, we are able to obtain a formula for the minimal value of the diagonal Arakelov-Green's function $g_\varphi(x,x)$.

Given a probability measure ν on $\mathbf{P}^1_K$, the (normalized) Arakelov-Green's function attached to ν is

$$g_\nu(x,y) = \int_{\mathbf{P}^1_K} -\log \delta(x,y)_\zeta d\nu(\zeta) + C \ ;$$

here, $\delta(x,y)_\zeta$ is the Hsia kernel which measures the distance between x and y relative to the basepoint ζ (see [2] Chapter 4 for the definition of the Hsia kernel, and [2] Chapter 8 for a discussion of the Arakelov-Green's function on $\mathbf{P}^1_K$).The constant C is chosen so that

$$\iint g_\nu(x,y) d\nu(x) d\nu(y) = 0 \ .$$

In the case that $\nu = \mu_\varphi$ is the equilibrium measure associated to φ, we simply write $g_\varphi(x,y) = g_{\mu_\varphi}(x,y)$.

The Arakelov-Green's function g_φ is the dynamical analogue of the Arakelov-Green's function associated to Haar measure on an elliptic curve. Baker [1] has used this function to show the finiteness of the set of points of small dynamical height for non-isotrivial maps defined over function fields. One of the key ingredients is a positivity result for the diagonal values in the case that φ does not have potential good reduction (see [1] Corollary 3.15). Our Corollary 1.3 gives an effective way to compute a lower bound for $g_\varphi(x,x)$ under the assumptions of Theorem 1.2.

Proof of Corollary 1.3. If φ satisfies the equivalent conditions of Theorem 1.2, then

$$\frac{1}{d^n(d^n-1)}R_{[\varphi^n]} = \frac{1}{d(d-1)}R_{[\varphi]} \ .$$

By [7, Corollary 3], we find that

$$\begin{aligned}\min_{x\in\mathbf{P}^1_K} g_\varphi(x,x) &= \lim_{n\to\infty}\frac{1}{d^n(d^n-1)}R_{[\varphi^n]}\\ &= \frac{1}{d(d-1)}R_{[\varphi]} \ .\end{aligned}$$

□

6. Examples

In this section we collect several examples illustrating the main Theorems.

Example 1. Let K be any complete, non-Archimedean valued field with residue characteristic not equal to 2, and let $\varphi_c(z) = z^2 + c$ for $|c| > 1$. For such maps, one can show that the support of the measure μ_{φ_c} lies in two directions away from $\zeta_{0,|c|^{1/2}}$, each direction having equal mass $\frac{1}{2}$; consequently $\zeta_{0,|c|^{1/2}} \in \mathrm{Bary}(\mu_{\varphi_c})$. Conjugating by $z \mapsto c^{1/2}z$ gives

$$\psi_c(z) = c^{1/2}z^2 + c^{1/2} \ .$$

A direct computation shows that $(\psi_c)_{\mathfrak{m}} \notin I(2)$, and so by Theorem 1.2 we know that the minimal resultant iteration formula holds for all n. One can show (e.g., by the algorithm in [8, Section 4]) that ψ_c has semistable reduction, so that

$$R_{[\varphi_c]} = R_{\psi_c} = \log|c| \ ,$$

and hence $R_{[\varphi_c^n]} = \frac{2^n(2^n-1)}{2}\log|c|$. One can also verify this formula directly using Rumely's crucial measures, but the calculations are more involved.

Example 2. (See [8, Example 2.3]) Let $p \geq 3$ be a prime number and let $K = \mathbb{C}_p$ be the p-adic complex numbers. Define

$$\varphi(z) = \frac{z^p - z}{p} \ .$$

It is known that μ_φ is Haar measure on $\mathbb{Z}_p$, and hence $\mathrm{Bary}(\mu_\varphi) = \{\zeta_G\}$. A direct computation shows that $\varphi_{\mathfrak{m}} \notin I(p)$, so that by Theorem 1.2 the minimal resultant iteration formula will hold for all n. One can also check that φ has semi-stable reduction; since the minimal resultant iteration formula holds for all n, it follows that φ^n has semi-stable reduction for all n.

Example 3. A forthcoming article by the first author shows exactly how the minimal resultant for Lattès maps transforms under iteration:

Theorem 6.1. *Suppose that K is a complete, algebraically closed, non-Archimedean valued field with residue characteristic not equal to 2 or 3. Let ψ_m be the multiplication-by-m Lattès map associated to a Tate curve E with uniformizing parameter q satisfying $0 < |q| < 1$. Then*

$$R_{[\psi_m]} = \begin{cases} -\frac{m^2(m^2-1)}{24} \log|q| \ , & m \text{ odd} \\ \left(\frac{m^5+m^4-2m^3}{8(m+1)} - \frac{m^2(m^2-1)}{6}\right) \log|q| \ , & m \text{ even} \end{cases} . \tag{6.1}$$

In particular, the iteration formula $R_{[(\psi_m)^n]} = \frac{(m^n)^2\left((m^n)^2-1\right)}{m^2(m^2-1)} R_{[\psi_m]}$ holds if and only if m is odd.

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