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Effective separability of finitely generated nilpotent groups

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Abstract. We give effective proofs of residual finiteness and conjugacy separability for finitely generated nilpotent groups. In particular, we give precise effective bounds for a function introduced by Bou-Rabee that measures how large the finite quotients that are needed to separate nonidentity elements of bounded length from the identity which improves the work of Bou-Rabee. Similarly, we give polynomial upper and lower bounds for an analogous function introduced by Lawton, Louder, and McReynolds that measures how large the finite quotients that are needed to separate pairs of distinct conjugacy classes of bounded word length using work of Blackburn and Mal'tsev.

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Part I. Introduction

We say that Γ is *residually finite* if for each nontrivial element $\gamma \in \Gamma$ there exists a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $\varphi(\gamma) \neq 1$. Mal'tsev [28] proved that if Γ is a residually finite finitely presentable group, then there exists a solution to the word problem of Γ. We say that Γ is *conjugacy separable* if for each nonconjugate pair of elements $\gamma, \eta \in \Gamma$ there exists a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $\varphi(\gamma)$ and $\varphi(\eta)$ are not conjugate. Mal'tsev [28] also proved that if Γ is a conjugacy separable finitely presentable group, then there exists a solution to the conjugacy problem of Γ.

Residual finiteness, conjugacy separability, subgroup separability, and other residual properties have been extensively studied and used to great effect in resolving important conjectures in geometry, such as the work of Agol on the Virtual Haken conjecture [1]. Much of the work in the literature has been to understand which groups satisfy various residual properties. For example, free groups, polycyclic groups, finitely generated nilpotent groups, surface groups, and fundamental groups of compact, orientable 3-manifolds have all been shown to be residually finite and conjugacy separable [3, 14, 18, 19, 34, 37]. Recently, there have been several papers that have made effective these separability properties for certain classes of groups. The main purpose of this article is to improve on the effective residual finiteness results of [4] and establish effective conjugacy separability results for the class of finitely generated nilpotent groups.

1. Main results

To state our results, we require some notation. For two nondecreasing functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there exists a $C \in \mathbb{N}$ such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. We write $f \approx g$ when $f \preceq g$ and $g \preceq f$. For a finitely generated nilpotent group Γ, we denote $T(\Gamma)$ to be the normal subgroup of finite order elements. For a finitely generated group Γ, we denote Γ_i as the i-th step of the lower central series.

1.1. Effective residual finiteness. For a finitely generated group Γ with a finite generating subset S, [4] (see also [35]) introduced a function $\mathrm{F}_{\Gamma,S}(n)$ on the natural numbers that quantifies residual finiteness. Specifically, the value of $\mathrm{F}_{\Gamma,S}(n)$ is the maximum order of a finite group needed to distinguish a nonidentity element from the identity as one varies over nonidentity elements in the n-ball. Numerous authors have studied the effective behavior of $\mathrm{F}_{\Gamma,S}(n)$ for a wide collection of groups (see [4, 5, 6, 7, 10, 21, 32, 35]).

As we will see in Subsection 2.2.1, the dependence of $\mathrm{F}_{\Gamma,S}(n)$ on S is mild; subsequently, we will suppress the dependence of F_Γ on the generating subset in this subsection.

For finitely generated nilpotent groups, Bou-Rabee [4, Thm 0.2] proved that $\mathrm{F}_\Gamma(n) \preceq (\log(n))^{h(\Gamma)}$ where $h(\Gamma)$ is the Hirsch length of Γ. Our first result establishes the precise effective behavior of $\mathrm{F}_\Gamma(n)$.

Theorem 1.1. *Let Γ be an infinite, finitely generated nilpotent group. There exists a $\psi_{\mathrm{RF}}(\Gamma) \in \mathbb{N}$ such that*

$$\mathrm{F}_\Gamma(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Additionally, one can compute $\psi_{\mathrm{RF}}(\Gamma)$ given a basis for $(\Gamma/T(\Gamma))_c$ where c is the step length of $\Gamma/T(\Gamma)$.

Given the nature of the study of the effective behavior of residual finiteness for some finitely generated group Γ, we must study the upper bounds and lower bounds of $\mathrm{F}_\Gamma(n)$ separately. However, the necessary tools used in the calculation of both bounds are developed in Part II. In the next few paragraphs, we describe the tools developed in these sections and how they are applied to the study of $\mathrm{F}_\Gamma(n)$ when Γ is an infinite, finitely generated nilpotent group.

§3 introduces admissible quotients of a torsion-free, finitely generated nilpotent group which are associated to central, nontrivial elements. These admissible quotients are the main tool of use in the evaluation of the upper and lower bounds for Theorem 1.1. This section develops properties of admissible quotients associated to nontrivial, central elements and introduces the idea of a maximal admissible quotient. These maximal admissible quotients capture the complexity of residual finiteness of torsion-free, finitely generated nilpotent groups. In particular, the Hirsch length of a maximal admissible quotient is a global invariant of a torsion-free, finitely generated nilpotent group Γ and is equal to the value $\psi_{\mathrm{RF}}(\Gamma)$.

§4 is devoted to developing tools that allow us to reduce the study of residual finiteness of an infinite, finitely generated nilpotent group Γ to the study of residual finiteness of a maximal admissible quotient $(\Gamma/T(\Gamma))/\Lambda$ of $\Gamma/T(\Gamma)$.

§4 and §5 allow us to give an overall strategy for the upper and lower bounds for $\mathrm{F}_\Gamma(n)$ when Γ is an infinite, finitely generated nilpotent group. We first demonstrate that $\mathrm{F}_\Gamma(n)$ is equivalent to $\mathrm{F}_{\Gamma/T(\Gamma)}(n)$. That allows us to assume that the nilpotent group in consideration is torsion-free. For the upper bound, we then proceed by induction on the step length of Γ which reduces us to the consideration of elements who have powers that are in the last nontrivial step of the lower central series. If $\gamma \in \Gamma$ is a nontrivial, central element with admissible quotient Γ/Λ_γ associated to γ, we have, by construction, that the image of γ is nontrivial in Γ/Λ. Then, via the Prime Number Theorem, we demonstrate that there exists a surjective homomorphism to a finite group $\varphi : \Gamma/\Lambda_\gamma \to Q$ such that $\varphi(\gamma) \neq 1$ and where

$$|Q| \leq (\log(\|\gamma\|_S))^{h(\Gamma/\Lambda_\gamma)}.$$

We finish by observing that if Γ/Λ is a maximal admissible quotient, then $h(\Gamma/\Lambda_\gamma) \leq h(\Gamma/\Lambda)$ for all central, nontrivial elements $\gamma \in \Gamma$. As an immediate consequence, the effective behavior of $\mathrm{F}_\Gamma(n)$ is bounded above by $(\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}$.

For the lower bound, we show that the elements that realize the lower bound for $\mathrm{F}_\Gamma(n)$ are central elements γ satisfying $\gamma \not\equiv 1 \mod \Lambda$ where Γ/Λ is a maximal admissible quotient of Γ. Thus, we need to study surjective homomorphisms to finite groups $\varphi : \Gamma \to Q$ with $\varphi(\gamma) \neq 1$. We first demonstrate that the study of the given homomorphism may be reduced to the study of the homomorphism $\widetilde{\pi \circ \varphi} : \Gamma/\Lambda \to Q/\varphi(\Lambda)$ where $\pi : Q \to Q/\varphi(\Lambda)$ is the natural projection and where $\widetilde{\pi \circ \varphi} : \Gamma/\Lambda \to Q/\varphi(\Lambda)$ is the homomorphism induced by $\pi \circ \varphi : \Gamma \to Q/\varphi(\Lambda)$. We then introduce necessary conditions on the homomorphism $\widetilde{\pi \circ \varphi}$ so that $|Q/\varphi(\Lambda)| \geq p^{\psi_{\mathrm{RF}}(\Gamma)}$. In particular, we use the Prime Number Theorem to pick a sequence of elements $\{\gamma_i\} \subseteq \Gamma$ such that the order of the minimal finite group that separates γ_i from the identity is bounded below by $C(\log(C\|\gamma_i\|))^{\psi_{\mathrm{RF}}(\Gamma)}$ for some $C \in \mathbb{N}$.

§5 gives a preview of the techniques used for the proof of Theorem 1.1 by explicitly calculating $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$ where $\mathrm{H}_{2m+1}(\mathbb{Z})$ is the $(2m+1)$-dimensional integral Heisenberg group. In particular, we use the techniques and tools developed in the previous sections.

The following is a consequence of the proof of Theorem 1.1.

Corollary 1.2. *Let Γ be a finitely generated nilpotent group. Then*

$$\mathrm{F}_\Gamma(n) \approx (\log(n))^{h(\Gamma)}$$

if and only if $h(Z(\Gamma/T(\Gamma))) = 1$.

We now introduce some terminology. Suppose that G is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We say that G is $\mathbb{Q}$-*defined* if $\mathfrak{g}$ admits a basis with rational structure constants. The *Mal'tsev completion* of a torsion-free, finitely generated nilpotent group Γ is a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group G such that Γ embeds into as a cocompact lattice.

The next theorem demonstrates that the effective behavior of $\mathrm{F}_\Gamma(n)$ is an invariant of the Mal'tsev completion of $\Gamma/T(\Gamma)$.

Theorem 1.3. *Suppose that Γ_1 and Γ_2 are two infinite, finitely generated nilpotent groups such that $\Gamma_1/T(\Gamma_1)$ and $\Gamma_2/T(\Gamma_2)$ have isomorphic Mal'tsev completions. Then $\mathrm{F}_{\Gamma_1}(n) \approx \mathrm{F}_{\Gamma_2}(n)$.*

The proof of Theorem 1.3 follows from an examination of a cyclic series that comes from a refinement of the upper central series and its interaction with the topology of the Mal'tsev completion.

Since the 3-dimensional integral Heisenberg group embeds into every infinite, nonabelian nilpotent group, Theorem 1.1, Theorem 1.3, [4, Thm 2.2], and [4, Cor 2.3] allow us to characterize $\mathbb{R}^d$ within the collection of connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie groups by the effective behavior of residual finiteness of a cocompact lattice.

Corollary 1.4. *Let G be a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group. Then G is Lie isomorphic to $\mathbb{R}^{\dim(G)}$ if and only if*

$$\mathrm{F}_\Gamma(n) \precnapprox (\log(n))^3$$

where $\Gamma \subseteq G$ is any cocompact lattice.

For the last result of this section, we need the following. We say that a group Γ is *irreducible* if there is no nontrivial splitting of Γ as a direct product.

Theorem 1.5.

(i) *For each $c \in \mathbb{N}$, there exists a $m(c) \in \mathbb{N}$ satisfying the following. For each $\ell \in \mathbb{N}$, there exists an irreducible, torsion-free, finitely generated nilpotent group Γ of step length c and $h(\Gamma) \geq \ell$ such that*

$$\mathrm{F}_\Gamma(n) \preceq (\log(n))^{m(c)}.$$

(ii) *Suppose that $\ell \neq 2$. There exists an irreducible, torsion-free, finitely generated nilpotent group Γ_ℓ such that*

$$\mathrm{F}_{\Gamma_\ell}(n) \approx (\log(n))^{\ell}.$$

(iii) *Suppose* $2 \leq c_1 < c_2$ *are natural numbers. For each* $\ell \in \mathbb{N}$*, there exist irreducible, torsion-free, finitely generated nilpotent groups* Γ_ℓ *and* Δ_ℓ *of step lengths* c_1 *and* c_2*, respectively, such that*

$$\mathrm{F}_{\Gamma_\ell}(n), \mathrm{F}_{\Delta_\ell}(n) \approx (\log(n))^{\ell \ \mathrm{lcm}(c_1+1, c_2+1)}.$$

(iv) *For natural numbers* $c > 1$ *and* $m \geq 1$*, there exists an irreducible, torsion-free, finitely generated nilpotent group* Γ *of step length* c *such that*

$$(\log(n))^m \preceq \mathrm{F}_\Gamma(n).$$

For Theorem 1.5(i), we consider free nilpotent groups of fixed step length and increasing rank. We make use of central products of filiform nilpotent groups for Theorem 1.5(ii)–(iv).

Using Theorem 1.5, we are able to relate the constant $\psi_{\mathrm{RF}}(\Gamma)$ with well known invariants of Γ when Γ is a finitely generated nilpotent group. Theorem 1.5(i) implies that $\psi_{\mathrm{RF}}(\Gamma)$ does not depend on the Hirsch length of Γ. Similarly, Theorem 1.5(iv) implies that there is no upper bound in terms of step length of Γ for $\psi_{\mathrm{RF}}(\Gamma)$. On the other hand, the step size of Γ is not determined by $\psi_{\mathrm{RF}}(\Gamma)$ as seen in Theorem 1.5(iii).

1.2. Effective conjugacy separability. We now turn our attention to the study of effective conjugacy separability. Lawton–Louder–McReynolds [25] introduced a function $\mathrm{Conj}_{\Gamma,S}(n)$ on the natural numbers that quantifies conjugacy separability. To be precise, the value of $\mathrm{Conj}_{\Gamma,S}(n)$ is the maximum order of the minimal finite quotient needed to separate a pair of nonconjugate elements as one varies over nonconjugate pairs of elements in the n-ball. Since the dependence of $\mathrm{Conj}_{\Gamma,S}(n)$ on S is mild (see Lemma 2.1), we will suppress the generating subset throughout this subsection.

The only previous work on the effective behavior of $\mathrm{Conj}_\Gamma(n)$ is due to Lawton–Louder–McReynolds [25]. They demonstrate that if Γ is a surface group or a finite rank free group, then $\mathrm{Conj}_\Gamma(n) \preceq n^{n^2}$ [25, Cor 1.7]. In this subsection, we initiate the study of the effective behavior of $\mathrm{Conj}_\Gamma(n)$ for the collection of finitely generated nilpotent groups.

Our first result is the calculation of $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$ where $\mathrm{H}_{2m+1}(\mathbb{Z})$ is the $(2m+1)$-dimensional integral Heisenberg group.

Theorem 1.6. $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \approx n^{2m+1}$.

For general nilpotent groups, we establish the following upper bound for $\mathrm{Conj}_\Gamma(n)$.

Theorem 1.7. *Let* Γ *be a finitely generated nilpotent group. Then there exists a* $k \in \mathbb{N}$ *such that*

$$\mathrm{Conj}_\Gamma(n) \preceq n^k.$$

Blackburn [3] was the first to prove conjugacy separability of finitely generated nilpotent groups. Our strategy for proving Theorem 1.7 is to effectivize [3].

For the same class of groups, we have the following lower bound which allows us to characterize virtually abelian groups within the class of finitely generated nilpotent groups. Moreoever, we obtain the first example of a class of groups for which the effective behavior of $\mathrm{F}_{\Gamma}(n)$ and $\mathrm{Conj}_{\Gamma}(n)$ are shown to be dramatically different.

Theorem 1.8. *Let Γ be an infinite, finitely generated nilpotent group.*

(i) *If Γ contains a normal abelian subgroup of index m, then*

$$\log(n) \preceq \mathrm{Conj}_{\Gamma}(n) \preceq (\log(n))^m.$$

(ii) *Suppose that Γ is not virtually abelian. There exists a $\psi_{\mathrm{Lower}}(\Gamma) \in \mathbb{N}$ such that*

$$n^{\psi_{\mathrm{Lower}}(\Gamma)} \preceq \mathrm{Conj}_{\Gamma}(n).$$

Additionally, one can compute $\psi_{\mathrm{Lower}}(\Gamma)$ given a basis for $(\Gamma/T(\Gamma))_c$ where c is the step length of $\Gamma/T(\Gamma)$.

The proof of Theorem 1.8(i) is elementary. We prove Theorem 1.8(ii) by finding an infinite sequence of nonconjugate elements $\{\gamma_i, \eta_i\}$ such that the order of the minimal finite group that separates the conjugacy classes of γ_i and η_i is bounded below by $C n_i^{\psi_{\mathrm{Lower}}(\Gamma)}$ for some $C \in \mathbb{N}$ where $\|\gamma_i\|_S, \|\eta_i\|_S \approx n_i$ for some finite generating subset S using tools from §3 and §5.

We have the following theorem which is similar in nature to Theorem 1.3.

Theorem 1.9. *Let Γ and Δ be infinite, finitely generated nilpotent groups of step size greater than or equal to 2, and suppose that $\Gamma/T(\Gamma)$ and $\Delta/T(\Delta)$ have isomorphic Mal'tsev completions. Then*

$$n^{\psi_{\mathrm{Lower}}(\Gamma)} \preceq \mathrm{Conj}_{\Delta}(n) \quad \text{and} \quad n^{\psi_{\mathrm{Lower}}(\Delta)} \preceq \mathrm{Conj}_{\Gamma}(n).$$

We apply Theorem 1.8 to construct nilpotent groups that help demonstrate the various effective behaviors that the growth of conjugacy separability may exhibit.

Theorem 1.10. *For natural numbers $c > 1$ and $k \geq 1$, there exists an irreducible, torsion-free, finitely generated nilpotent group Γ of step length c such that*

$$n^k \preceq \mathrm{Conj}_{\Gamma}(n).$$

Theorem 1.10 implies that the conjugacy separability function does not depend of the step length of the nilpotent group. We consider central products of filiform nilpotent groups for Theorem 1.10.

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2. Background

We will assume the reader is familiar with finitely generated groups, Lie groups, and Lie algebras.

2.1. Notation and conventions. We let $\mathrm{lcm}\{r_1, \dots, r_m\}$ be the lowest common multiple of $\{r_1, \dots, r_m\} \subseteq \mathbb{Z}$ with the convention that $\mathrm{lcm}(a) = |a|$ and $\mathrm{lcm}(a, 0) = 0$. We let $\gcd(r_1, \dots, r_m)$ be the greatest common multiple of $\{r_1, \dots, r_m\} \subseteq \mathbb{Z}$ with the convention that $\gcd(a, 0) = |a|$.

We denote $\|\gamma\|_S$ as the word length of γ with respect to the finite generating subset S and denote the identity of Γ as 1. We denote the order of γ as an element of Γ as $\mathrm{Ord}_\Gamma(\gamma)$ and denote the cardinality of a group Γ as $|\Gamma|$. We write $\gamma \sim \eta$ when there exists an element $g \in \Gamma$ such that $g^{-1}\,\gamma\,g = \eta$. For a normal subgroup $\Delta \trianglelefteq \Gamma$, we set $\pi_\Delta : \Gamma \to \Gamma/\Delta$ to be the natural projection and write $\bar{\gamma} = \pi_\Delta(\gamma)$ when Δ is clear from context. For a subset $X \subseteq \Gamma$, we denote $\langle X \rangle$ to be the subgroup generated by X. For any group Γ, we let $\Gamma^\bullet = \Gamma \backslash \{1\}$.

We define the commutator of γ and η as $[\gamma, \eta] = \gamma^{-1}\,\eta^{-1}\,\gamma\,\eta$. We denote the m-fold commutator of not necessarily distinct elements $\{\gamma_i\}_{i=1}^m \subseteq \Gamma$ as $[\gamma_1, \dots, \gamma_m]$ with the convention that

$$[\gamma_1, \dots, \gamma_m] = [[\gamma_1, \dots, \gamma_{m-1}], \gamma_m].$$

We denote the center of Γ as $Z(\Gamma)$ and the centralizer of γ in Γ as $C_\Gamma(\gamma)$. We define Γ_i to be the i-th term of the lower central series and $Z^i(\Gamma)$ to be the i-th term of the upper central series. For $\gamma \in \Gamma^\bullet$, we denote $\mathrm{Height}(\gamma)$ as the minimal $j \in \mathbb{N}$ such that $\pi_{Z^{j-1}(\Gamma)}(\gamma) \neq 1$.

We define the abelianization of Γ as Γ_{ab} with the associated projection given by $\pi_{\mathrm{ab}} = \pi_{[\Gamma,\Gamma]}$. For $m \in \mathbb{N}$, we define $\Gamma^m \cong \langle \gamma^m \mid \gamma \in \Gamma \rangle$.

When given a basis $X = \{X_i\}_{i=1}^{\dim_\mathbb{R}(\mathfrak{g})}$ for $\mathfrak{g}$, we denote

$$\left\| \sum_{i=1}^{\dim_\mathbb{R}(\mathfrak{g})} \alpha_i\,X_i \right\|_X = \sum_{i=1}^{\dim_\mathbb{R}(\mathfrak{g})} |\alpha_i|.$$

For a Lie algebra $\mathfrak{g}$ with a Lie ideal $\mathfrak{h}$, we define $\pi_\mathfrak{h} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ to be the natural Lie projection.

For a $\mathbb{R}$-Lie algebra $\mathfrak{g}$, we denote $Z(\mathfrak{g})$ to the center of $\mathfrak{g}$, $\mathfrak{g}_i$ to be the i-th term of the lower central series, and $Z^i(\mathfrak{g})$ to be the i-th term of the upper central series.

For $A \in \mathfrak{g}$, we define the map $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$ to be given by

$$\mathrm{ad}_A(B) = [A, B].$$

Denote the m-fold Lie bracket of not necessarily distinct elements $\{A_i\}_{i=1}^m \subseteq \mathfrak{g}$ as $[A_1, \dots, A_m]$ with the convention that

$$[A_1, \dots, A_m] = [[A_1, \dots, A_{m-1}], A_m].$$

2.2. Finitely generated groups and separability.

2.2.1. Residually finite groups. Following [4] (see also [35]), we define the depth function $\mathrm{D}_\Gamma : \Gamma^\bullet \to \mathbb{N} \cup \{\infty\}$ of the finitely generated group Γ to be given by

$$\mathrm{D}_\Gamma(\gamma) \overset{\text{def}}{=} \min \{|Q| \mid \varphi : \Gamma \to Q, |Q| < \infty, \text{ and } \varphi(\gamma) \neq 1\}.$$

We define $\mathrm{F}_{\Gamma,S} : \mathbb{N} \to \mathbb{N}$ by

$$\mathrm{F}_{\Gamma,S}(n) \overset{\text{def}}{=} \max \{\mathrm{D}_\Gamma(\gamma) \mid \|\gamma\|_S \leq n \text{ and } \gamma \neq 1\}.$$

When Γ is a residually finite group, then $\mathrm{F}_{\Gamma,S}(n) < \infty$ for all $n \in \mathbb{N}$. For any two finite generating subsets S_1 and S_2, we have that

$$\mathrm{F}_{\Gamma,S_1}(n) \approx \mathrm{F}_{\Gamma,S_2}(n)$$

(see [4, Lem 1.1]). Thus, we will suppress the choice of finite generating subset.

2.2.2. Conjugacy separable groups. Following [25], we define the conjugacy depth function of Γ, denoted $\mathrm{CD}_\Gamma : (\Gamma \times \Gamma) \setminus \{(\gamma, \eta) \mid \gamma \sim \eta\} \to \mathbb{N} \cup \{\infty\}$, to be given by

$$\mathrm{CD}_\Gamma(\gamma, \eta) \overset{\text{def}}{=} \min \{|Q| \mid \varphi : \Gamma \to Q, |Q| < \infty, \text{ and } \varphi(\gamma) \nsim \varphi(\eta)\}.$$

We define $\mathrm{Conj}_{\Gamma,S}(n) : \mathbb{N} \to \mathbb{N}$ as

$$\mathrm{Conj}_{\Gamma,S}(n) \overset{\text{def}}{=} \max \{\mathrm{CD}_\Gamma(\gamma, \eta) \mid \gamma \nsim \eta \text{ and } \|\gamma\|_S, \|\eta\|_S \leq n\}.$$

When Γ is a conjugacy separable group, then $\mathrm{Conj}_{\Gamma,S}(n) < \infty$ for all $n \in \mathbb{N}$.

Lemma 2.1. *If S_1, S_2 are two finite generating subsets of Γ, then*

$$\mathrm{Conj}_{\Gamma,S_1}(n) \approx \mathrm{Conj}_{\Gamma,S_2}(n).$$

The proof is similar to [4, Lem 1.1] (see also [25, Lem 2.1]). As before, we will suppress the choice of finite generating subset.

[25, Lem 2.1] implies that the order of the minimal finite group required to separate a nonidentity element $\gamma \in \Gamma$ from the identity is bounded above by the order of the minimal finite group required to separate the conjugacy class of γ from the identity. Thus, $\mathrm{F}_\Gamma(n) \preceq \mathrm{Conj}_\Gamma(n)$ for all conjugacy separable groups. In particular, if Γ is conjugacy separable, then Γ is residually finite.

2.3. Nilpotent groups and nilpotent Lie groups. See [13, 17, 23, 36] for a more thorough account of the material in this subsection. Let Γ be a nontrivial, finitely generated group. The i-th term of the *lower central series* is defined by $\Gamma_1 \overset{\text{def}}{=} \Gamma$, and for $i > 1$, we let $\Gamma_i \overset{\text{def}}{=} [\Gamma_{i-1}, \Gamma]$. The i-term of the *upper central series* is defined by $Z^0(\Gamma) \overset{\text{def}}{=} \{1\}$ and $Z^i(\Gamma) \overset{\text{def}}{=} \pi^{-1}_{Z^{i-1}(\Gamma)}(Z(\Gamma / Z^{i-1}(\Gamma)))$ for $i > 1$.

Definition 2.2. We say that Γ is a *nilpotent group of step size* c if c is the minimal natural number such that $\Gamma_{c+1} = \{1\}$, or equivalently, $Z^c(\Gamma) = \Gamma$. If the step size is unspecified, we simply say that Γ is a nilpotent group. When given a nilpotent group Γ, we denote its step length as $c(\Gamma)$.

For a finitely generated nilpotent group Γ, the set of torsion elements of Γ, denoted as $T(\Gamma)$, is a finite, characteristic subgroup. Moreover, when $|\Gamma| = \infty$, then $\Gamma/T(\Gamma)$ is torsion-free.

Let $\mathfrak{g}$ be a nontrivial, finite dimensional $\mathbb{R}$-Lie algebra. The i-th term of the *lower central series* of $\mathfrak{g}$ is defined by $\mathfrak{g}_1 \stackrel{\text{def}}{=} \mathfrak{g}$, and for $i > 1$, we let $\mathfrak{g}_i \stackrel{\text{def}}{=} [\mathfrak{g}_{i-1}, \mathfrak{g}]$. We define the i-th term of the *upper central series* as $Z^0(\mathfrak{g}) \stackrel{\text{def}}{=} \{0\}$ and $Z^i(\mathfrak{g}) \stackrel{\text{def}}{=} \pi^{-1}_{Z^{i-1}(\mathfrak{g})}(Z(\mathfrak{g}/Z^{i-1}(\mathfrak{g})))$ for $i > 1$.

Definition 2.3. We say that $\mathfrak{g}$ is a *nilpotent Lie algebra* of step length c if c is the minimal natural number satisfying $Z^c(\mathfrak{g}) = \mathfrak{g}$, or equivalently, $\mathfrak{g}_{c+1} = \{0\}$. If the step size is unspecified, we simply say that $\mathfrak{g}$ is a nilpotent Lie algebra.

For a connected, simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, the exponential map, written as $\exp : \mathfrak{g} \to G$, is a diffeomorphism [23, Thm 1.127] whose inverse is formally denoted as Log. The Baker–Campbell–Hausdorff formula [13, (1.3)] implies that every $A, B \in \mathfrak{g}$ satisfies

$$(1) \qquad A * B \stackrel{\text{def}}{=} \text{Log}(\exp A \cdot \exp B) \stackrel{\text{def}}{=} A + B + \frac{1}{2}[A, B] + \sum_{m=3}^{\infty} CB_m(A, B)$$

where $CB_m(A, B)$ is a rational linear combination of m-fold Lie brackets of A and B. By assumption, $CB_m(A, B) = 0$ for $m > c(G)$. For $\{A_i\}_{i=1}^m$ in $\mathfrak{g}$, we may more generally write

$$(2) \qquad A_1 * \cdots * A_m = \text{Log}(\exp A_1 \cdots \exp A_m) = \sum_{i=1}^{c(G)} CB_i(A_1, \ldots, A_m)$$

where $CB_i(A_1, \ldots, A_m)$ is a rational linear combination of i-fold Lie brackets of the elements $\{A_{j_t}\}_{t=1}^{\ell} \subseteq \{A_i\}_{i=1}^m$ via repeated applications of the Baker–Campbell–Hausdorff formula.

We define the adjoint representation of G, denoted $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$, as $\text{Ad}(g)(X) = (d\Psi_g)_1(X)$ where $\Psi_g(x) = g\,x\,g^{-1}$. By [23, 1.92], we may write for $\gamma \in \Gamma$ and $A \in \mathfrak{g}$

$$(3) \qquad \text{Ad}(\gamma)(A) = A + \frac{1}{2}[\text{Log}(\gamma), A] + \sum_{i=3}^{c} \frac{(\text{ad}_{\text{Log}(\gamma)})^i(A)}{i!}.$$

By [30], a connected, simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$ admits a cocompact lattice Γ if and only if $\mathfrak{g}$ admits a basis with rational structure constants (see [29, Thm 7] for more details). We say G is $\mathbb{Q}$-*defined* if it admits a cocompact lattice. For any torsion-free, finitely generated nilpotent group Γ, [29, Thm 6] implies that there exists a $\mathbb{Q}$-defined group unique up to isomorphism in which Γ embeds as a cocompact lattice.

Definition 2.4. We call this $\mathbb{Q}$-defined group the *Mal'tsev completion* of Γ. When given a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group G, the tangent space at the identity with the Lie bracket of vector fields is a finite dimensional nilpotent $\mathbb{R}$-Lie algebra.

2.4. Polycyclic groups. See [20, 33, 36] for the material contained in the following subsection.

Definition 2.5. A group Γ is *polycyclic* if there exists an ascending chain of subgroups $\{\Delta_i\}_{i=1}^m$ such that Δ_1 is cyclic, $\Delta_i \trianglelefteq \Delta_{i+1}$, and Δ_{i+1}/Δ_i is cyclic for all i. We call $\{\Delta_i\}_{i=1}^m$ a *cyclic series* for Γ. We say $\{\xi_i\}_{i=1}^m$ is a *compatible generating subset* with respect to the cyclic series $\{\Delta_i\}_{i=1}^m$ if $\langle \xi_1 \rangle = \Delta_1$ and $\langle \xi_{i+1}, \Delta_i \rangle = \Delta_{i+1}$ for all $i > 1$. We define the *Hirsch length* of Γ, denoted as $h(\Gamma)$, as the number of indices i such that $|\Delta_{i+1} : \Delta_i| = \infty$.

For a general polycyclic group, there may be infinitely many different cyclic series of arbitrary length (see [20, Ex 8.2]). However, the Hirsch length of Γ is independent of the cyclic series. With respect to the compatible generating subset $\{\xi_i\}_{i=1}^m$, [20, Lem 8.3] implies that we may represent every element $\gamma \in \Gamma$ uniquely as $\gamma = \prod_{i=1}^m \xi_i^{\alpha_i}$ where $\alpha_i \in \mathbb{Z}$ if $|\Delta_{i+1} : \Delta_i| = \infty$ and $0 \leq \alpha_i < r_i$ if $|\Delta_{i+1} : \Delta_i| = r_i$. If $|\Gamma| < \infty$, then the second paragraph after [20, Defn 8.2] implies that $|\Gamma| = \prod_{i=1}^m r_i$.

Definition 2.6. We call the collection of such m-tuples a *Mal'tsev basis* for Γ with respect to the compatible generating subset $\{\xi_i\}_{i=1}^m$. When $\gamma = \prod_{i=1}^m \xi_i^{\alpha_i}$, we call $(\alpha_i)_{i=1}^m$ the *Mal'tsev coordinates* of γ.

For a finitely generated nilpotent group Γ, we may refine the upper central series to obtain a cyclic series and a compatible generating subset. In particular, we will demonstrate that every finitely generated nilpotent group is polycyclic. First, assume that Γ is abelian. We may write $\Gamma \cong \mathbb{Z}^m \oplus T(\Gamma)$, and we let $\{\xi_i\}_{i=1}^{h(\Gamma)}$ be a free basis for $\mathbb{Z}^m$. Since $T(\Gamma)$ is a finite abelian group, there exists an isomorphism $\varphi : T(\Gamma) \to \oplus_{i=1}^{\ell} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$. If x_i generates $\mathbb{Z}/p_i^{k_i}\mathbb{Z}$ in $\oplus_{i=1}^{\ell} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$, we then set $\xi_i = \varphi^{-1}(x_{i-h(\Gamma)})$ for $h(\Gamma)+1 \leq i \leq h(\Gamma)+\ell$. Thus, the groups $\{\Delta_i\}_{i=1}^{h(\Gamma)+\ell}$ given by $\Delta_i = \langle \xi_t \rangle_{t=1}^{i}$ form a cyclic series for Γ with a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)+\ell}$.

We now assume that $c(\Gamma) > 1$. There exists a generating basis $\{z_i\}_{i=1}^{h(\Gamma)}$ for $Z(\Gamma)$ and integers $\{t_i\}_{i=1}^{h(\Gamma)}$ such that $\{z_i^{t_i}\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ is a basis for $\Gamma_{c(\Gamma)}$, and for each i, there exist $x_i \in \Gamma_{c(\Gamma)-1}$ and $y_i \in \Gamma$ such that $z_i^{t_i} = [x_i, y_i]$. We may choose a cyclic series $\{H_i\}_{i=1}^{h(\Gamma)}$ for $Z(\Gamma)$ such that $H_i = \langle z_s \rangle_{s=1}^{i}$. Induction implies that there exists a cyclic series $\{\Lambda_i\}_{i=1}^{k}$ and a compatible generating subset $\{\lambda_i\}_{i=1}^{k}$ for $\Gamma/Z(\Gamma)$. For $1 \leq i \leq \ell$, we set $\Delta_i = H_i$, and for $\ell+1 \leq i \leq \ell+k$, we set $\Delta_i = \pi^{-1}_{Z(\Gamma)}(\Lambda_{i-\ell})$. For $1 \leq i \leq \ell$, we set $\xi_i = z_i$. For $\ell+1 \leq i \leq \ell+k$, we choose a ξ_i such that $\pi_{Z(\Gamma)}(\xi_i) = \lambda_{i-\ell}$. It then follows that $\{\Delta_i\}_{i=1}^{\ell+k}$ is a cyclic series for Γ with a compatible generating subset $\{\xi_i\}_{i=1}^{\ell+k}$. Moreover, the given construction implies that

$$h(\Gamma) = \sum_{i=1}^{c(\Gamma)} \operatorname{rank}_{\mathbb{Z}} \left(Z^i(\Gamma)/Z^{i-1}(\Gamma)\right).$$

Whenever Γ is a finitely generated nilpotent group, we choose the cyclic series and compatible generating subset this way.

Another way to calculate the Hirsch length of a finitely generated nilpotent group is to use successive quotients of the lower central series. In particular, we have that

$$h(\Gamma) = \sum_{i=1}^{c(\Gamma)} \operatorname{rank}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i+1}).$$

Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. [17, Thm 6.5] implies that multiplication of $\gamma, \eta \in \Gamma$ can be expressed as polynomials in terms of the Mal'tsev basis associated to the cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and the compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Specifically, we may write

$$\gamma\,\eta = \left(\prod_{i=1}^{h(\Gamma)} \xi_i^{a_i}\right) \cdot \left(\prod_{j=1}^{h(\Gamma)} \xi_j^{b_j}\right) = \prod_{s=1}^{h(\Gamma)} \xi_s^{d_s}$$

where each d_s can be expressed as a polynomial in the Mal'stev coordinates of γ and η, respectively. Similarly, we may write

$$\gamma^\ell = \left(\prod_{i=1}^{h(\Gamma)} \xi_i^{a_i}\right)^\ell = \prod_{j=1}^{h(\Gamma)} \xi_j^{e_j}$$

where each e_j can be expressed as a polynomial in the Mal'tsev coordinates of γ and the integer ℓ.

The polynomials that define the group product and group power operation of Γ with respect to the given cyclic series and compatible generating subset have unique extensions to $\mathbb{R}^{h(\Gamma)}$. That implies the Mal'tsev completion of Γ, denoted G, is diffeomorphic to $\mathbb{R}^{h(\Gamma)}$ (see [17, Thm 6.5], [23, Cor 1.126]). Consequently, the dimension and step length of G are equal to the Hirsch length and step length of Γ, respectively. Thus, we may write $h(\Gamma) = \dim(G)$. We may also identify Γ with its image in G which is the set $\mathbb{Z}^{h(\Gamma)}$.

The following definition will be of use for the last lemma of this subsection.

Definition 2.7. Let Γ be a torsion-free, finitely generated nilpotent group, and let $\Delta \leq \Gamma$ be a subgroup. We define the *isolator of* Δ *in* Γ as the subset given by

$$\sqrt[\Gamma]{\Delta} = \{\gamma \in \Gamma \mid \text{ there exists a } k \in \mathbb{N} \text{ such that } \gamma^k \in \Delta\} \cup \{1\}.$$

By the paragraph proceeding exercise 8 of [36, Ch 8] and [17, Thm 4.5], $\sqrt[\Gamma]{\Delta}$ is a subgroup such that $|\sqrt[\Gamma]{\Delta} : \Delta| < \infty$ when Γ is a torsion-free, finitely generated nilpotent group. If Γ is abelian, then we may write

$$\Gamma = (\Gamma/\sqrt[\Gamma]{\Delta}) \oplus \sqrt[\Gamma]{\Delta}.$$

When $\Delta \trianglelefteq \Gamma$, we have that $\Gamma/\sqrt[\Gamma]{\Delta}$ is torsion-free.

We finish this section with the following result. When given an infinite, finitely generated nilpotent group Γ, the following lemma relates the word length of an element γ in Γ to the Mal'tsev coordinates of γ with respect to a cyclic series and a compatible generating subset.

Lemma 2.8. *Let Γ be an infinite, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^k$ and a compatible generating subset $\{\xi_i\}_{i=1}^k$. Let $\gamma \in \Gamma$ such that $\|\gamma\|_S \leq n$. There exists some $C \in \mathbb{N}$ such that $|\alpha_i| \leq C\, n^{c(\Gamma)}$ for all i, where (α_i) are the Mal'tsev coordinates of γ.*

Proof. We proceed by induction on step length, and observe that the base case of abelian groups is clear. Now suppose $c(\Gamma) > 1$ and that $\|\gamma\|_S \leq n$. Since $\|\pi_{\Gamma_i}(\gamma)\|_{\pi_{\Gamma_i}(S)} \leq n$, the inductive hypothesis implies that there exists a constant $C_0 > 0$ such that $|\alpha_i| \leq C_0\, n^t$ when $\pi_{\Gamma_t}(\xi_i) \neq 1$ and $\pi_{\Gamma_{t-1}}(\xi_i) = 1$. Let k be the length of the cyclic series Δ_i, and let $\mathcal{S} \subset \{1, \dots, k\}$ be the set of indices such that $\xi_{i_s} \notin \Gamma_{c(\Gamma)}$ for $i \in \mathcal{S}$ and $\xi_i \in \Gamma_{c(\Gamma)}$, otherwise.

We will demonstrate that there exists some constant $C_1 > 0$ such that the element

$$\zeta = \left(\pi_{i=1, i\notin \mathcal{S}}^k\, \xi_i^{\alpha_i}\right)^{-1} \quad \text{satisfies} \quad \|\zeta\|_S \leq C_1\, n.$$

Suppose for some $i \notin \mathcal{S}$ that $\pi_{\Gamma_t}(\xi_i) \neq 1$ and where $\pi_{\Gamma_{t-1}}(\xi_i) = 1$. We have by induction that $|\alpha_i| \leq C_0\, n^t$. By [16, 3.B2], we have that $\|\xi_i^{\alpha_i}\|_S \approx |\alpha_i|^{1/t}$. Thus, there exists a constant $C_2 > 0$ such that $\|\xi_i^{\alpha_i}\|_S \leq C_2\, |\alpha_i|^{1/t}$ when $\mathrm{Ord}_\Gamma(\xi_i) = \infty$. Therefore, we may write $\|\xi_i^{\alpha_i}\|_S \leq C_3\, n$ for some $C_3 > 0$. By letting $C_4 = \max\{C_3, |T(\Gamma)|\}$, it follows that $\|\xi_i\|_S \leq C_4\, n$ for all $i \notin \mathcal{S}$. In particular, we may write

$$\|\eta\|_S = \|\gamma\, \zeta\|_S \leq \|\gamma\|_S + \|\zeta\|_S \leq C_5\, n$$

for some constant $C_5 > 0$. By taking $C_1 = C_5$, we have our statement. Thus, we may assume that $\gamma \in \Gamma_{c(\Gamma)}$.

We may write $\gamma = \prod_{i \in \mathcal{S}} \xi_i^{\alpha_i}$. If we let $\lambda_t = \prod_{i \in \mathcal{S}, i \neq t} \xi_i^{-\alpha_i}$, it is evident that $\|\lambda_t\|_S \leq n$. Thus, we may write

$$\|\xi_t^{\alpha_t}\|_S = \|\gamma\, \lambda_t\|_S \leq \|\gamma\|_S + \|\lambda_t\|_S \leq 4\, C_6\, n$$

for some $C_6 > 0$. Thus, we need only consider when $\gamma = \xi_i^{\alpha_i}$ for $\xi_i \in \Gamma_{c(\Gamma)}$. Since $\xi_i \in \Gamma_{c(\Gamma)}$, [16, 3.B2] implies that $|\alpha_i| \leq C_7\, n^{c(\Gamma)}$ for some $C_7 \in \mathbb{N}$. $\square$

Part II. Technical tools

3. Admissible quotients

In the following subsection, we define what an admissible quotient with respect to a central, nontrivial element is, what a maximal admissible quotient is, and define the constants $\psi_{\mathrm{RF}}(\Gamma)$ and $\psi_{\mathrm{Lower}}(\Gamma)$ for an infinite, finitely generated nilpotent group Γ.

3.1. Existence of admissible quotients. The following proposition will be useful throughout this article.

Proposition 3.1. *Let Γ be a torsion-free, finitely generated nilpotent group, and suppose that γ is a central, nontrivial element. There exists a normal subgroup Λ in Γ such that Γ/Λ is an irreducible, torsion-free, finitely generated nilpotent group such that $Z(\Gamma/\Lambda) \cong \mathbb{Z}$ and where $\langle \pi_\Lambda(\gamma) \rangle$ is a finite index subgroup of $Z(\Gamma/\Lambda)$. If γ is primitive, then $Z(\Gamma/\Lambda) \cong \langle \pi_\Lambda(\gamma) \rangle$.*

Proof. We construct Λ by induction on Hirsch length, and since the base case is trivial, we may assume that $h(\Gamma) > 1$. If $Z(\Gamma) \cong \mathbb{Z}$, then the proposition is now evident by letting $\Lambda = \{1\}$.

Now assume that $h(Z(\Gamma)) \geq 2$. There exists a basis $\{z_i\}_{i=1}^{h(Z(\Gamma))}$ for $Z(\Gamma)$ such that $z_1^k = \gamma$ for some $k \in \mathbb{Z}^\bullet$. Letting $K = \langle z_i \rangle_{i=2}^{h(Z(\Gamma))}$, we note that $K \trianglelefteq \Gamma$ and $\pi_K(\gamma) \neq 1$. Additionally, it follows that Γ/K is a torsion-free, finitely generated nilpotent group. If $h(Z(\Gamma/K)) = 1$, then our proposition is evident by defining $\Lambda = K$.

Now suppose that $h(Z(\Gamma/K)) \geq 2$. Since $h(\Gamma/K) < h(\Gamma)$, the inductive hypothesis implies that there exists a subgroup Λ_1 such that $\Lambda_1 \trianglelefteq \Gamma/K$ and where $(\Gamma/K)/\Lambda_1$ is a torsion-free, finitely generated nilpotent group. Letting $\rho : \Gamma/K \to (\Gamma/K)/\Lambda_1$ be the natural projection, induction additionally implies that $\langle \rho(\pi_K(\gamma)) \rangle$ is a finite index subgroup of $Z((\Gamma/K)/\Lambda_1)$. Taking $\Lambda_2 = \pi_K^{-1}(\Lambda_1)$, we note that $\Lambda_2/K \cong \Lambda_1$. Thus, the third isomorphism theorem implies that $(\Gamma/K)/(\Lambda_2/K) \cong \Gamma/\Lambda_2$. Hence, Γ/Λ_2 is a torsion-free, finitely generated nilpotent group, and by construction, $\langle \pi_{\Lambda_2}(\gamma) \rangle$ is a finite index subgroup of $Z(\Gamma/\Lambda_2)$.

Letting Λ satisfy the hypothesis of the proposition for γ, we now demonstrate that Γ/Λ is irreducible. Suppose for a contradiction that there exists a pair of nontrivial, finitely generated nilpotent groups Δ_1 and Δ_2 such that $\Gamma/\Lambda \cong \Delta_1 \times \Delta_2$. Since Γ/Λ is torsion-free, Δ_1 and Δ_2 are torsion-free. Thus, $Z(\Delta_1)$ and $Z(\Delta_2)$ are torsion-free, finitely generated abelian groups. Hence, $\mathbb{Z}^2$ is isomorphic to a subgroup of $Z(\Gamma/\Lambda)$. Subsequently, $h(Z(\Gamma/\Lambda)) \geq 2$ which is a contradiction. Thus, either $\Delta_1 \cong \{1\}$ or $\Delta_2 \cong \{1\}$, and subsequently, Γ/Λ is irreducible. $\square$

As a natural corollary of the proof of Proposition 3.1, we have the following. Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. For a central element $\xi_{i_0} \in \{\xi_i\}_{i=1}^{h(\Gamma)}$, the next proposition demonstrates that there exists a normal subgroup $\Lambda \trianglelefteq \Gamma$ such that Γ/Λ satisfies Proposition 3.1 for ξ_{i_0}. Moreover, Λ is generated by a subset $\left\{\xi_{i_j}\right\}_{j=1}^{h(\Lambda)}$ of the compatible generating subset.

Corollary 3.2. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let ξ_{i_0} be a central element of $\{\xi_i\}_{i=1}^{h(\Gamma)}$. There exists a normal subgroup $\Lambda \trianglelefteq \Gamma$ such that Γ/Λ satisfies the conditions of Proposition 3.1 for ξ_{i_0}. Moreover, there exists a subset, possibly empty, $\left\{\xi_{i_j}\right\}_{i=1}^{h(\Gamma)}$ of the compatible generating subset satisfying the following. The subgroups $W_t = \left\langle \xi_{i_j} \right\rangle_{j=1}^{t}$ form a cyclic series for Λ with a compatible generating subset $\left\{\xi_{i_j}\right\}_{j=1}^{h(\Lambda)}$.*

Definition 3.3. Let $\gamma \in \Gamma$ be a central, nontrivial element, and let $\mathcal{J}$ be the set of subgroups of Γ that satisfy Proposition 3.1 for γ. Since the set $\{h(\Gamma/\Lambda) \mid \Lambda \in \mathcal{J}\}$ is bounded below by 1, there exists an $\Omega \in \mathcal{J}$ such that

$$h(\Gamma/\Omega) = \min\{h(\Gamma/\Lambda) \mid \Lambda \in \mathcal{J}\}.$$

We say Γ/Ω is an *admissible quotient of* Γ *with respect to* γ .

For a primitive element $\gamma \in (Z(\Gamma))^\bullet$, we let Γ/Λ_1 and Γ/Λ_2 be two different admissible quotients of Γ with respect to γ. In general, $\Gamma/\Lambda_1 \ncong \Gamma/\Lambda_2$. On the other hand, we have, by definition, that $h(\Gamma/\Lambda_1) = h(\Gamma/\Lambda_2)$. Subsequently, the Hirsch length of an admissible quotient with respect to γ is a natural invariant of Γ associated to γ. Such a quotient corresponds to a torsion-free quotient of Γ of minimal Hirsch length such that γ has a nontrivial image that generates a finite index subgroup of the center. That will be useful in finding the smallest finite quotient in which γ has a nontrivial image.

Definition 3.4. Let Γ be a non-abelian, torsion-free, finitely generated nilpotent group. For each element $\gamma \in (Z(\Gamma))^\bullet$, we let Γ/Λ_γ be an admissible quotient of Γ with respect to γ. Let $\mathcal{J}$ be the set of $\gamma \in (Z(\Gamma))^\bullet$ such that there exists a $k \in \mathbb{Z}^\bullet$ such that $\gamma^k = [a, b]$ where $a \in \Gamma_{c(\Gamma)-1}$ and $b \in \Gamma$. Observe that the set $\{h(\Gamma/\Lambda_\gamma) \mid \gamma \in \mathcal{J})\}$ is bounded above by $h(\Gamma)$. Thus, there exists an $\eta \in \mathcal{J}$ such that

$$h(\Gamma/\Lambda_\eta) = \max\left\{h(\Gamma/\Lambda_\gamma) \mid \gamma \in \mathcal{J}\right\}.$$

We say that Γ/Λ_η is a *maximal admissible quotient of* Γ. When Γ is a torsion-free, finitely generated abelian group, we take any admissible quotient with respect to any central element and denote it as a maximal admissible quotient.

For an infinite, finitely generated nilpotent group Γ, we now define the constants $\psi_{\mathrm{RF}}(\Gamma)$ and $\psi_{\mathrm{Lower}}(\Gamma)$.

Definition 3.5. Let Γ be an infinite, finitely generated nilpotent group. We let $(\Gamma/T(\Gamma))/\Lambda$ be a maximal admissible quotient of $\Gamma/T(\Gamma)$. We then set

$$\psi_{\mathrm{RF}}(\Gamma) = h((\Gamma/T(\Gamma))/\Lambda).$$

Assuming that Γ is not virtually abelian, we define

$$\psi_{\mathrm{Lower}}(\Gamma) = \psi_{\mathrm{RF}}(\Gamma)\left(c(\Gamma/T(\Gamma)) - 1\right).$$

Suppose that Γ/Λ_1 and Γ/Λ_2 are two maximal admissible quotients of Γ when Γ is torsion-free. In general, $\Gamma/\Lambda_1 \not\cong \Gamma/\Lambda_2$. However,

$$h(\Gamma/\Lambda_1) = h(\Gamma/\Lambda_2) = \psi_{\mathrm{RF}}(\Gamma)$$

by definition; hence, $\psi_{\mathrm{RF}}(\Gamma)$ is a well defined invariant of Γ. Similarly, we have that $\psi_{\mathrm{Lower}}(\Gamma)$ is a well defined invariant of finitely generated nilpotent groups that are not virtually abelian.

A natural observation is that if $h(Z(\Gamma/T(\Gamma))) = 1$, then

$$\psi_{\mathrm{RF}}(\Gamma) = h(\Gamma).$$

Additionally, if Γ is an infinite, finitely generated abelian group, then

$$\psi_{\mathrm{RF}}(\Gamma) = 1.$$

Finally, if Γ is a finitely generated nilpotent group that is not virtually abelian where $h(Z(\Gamma/T(\Gamma))) = 1$, then

$$\psi_{\mathrm{Lower}}(\Gamma) = h(\Gamma)(c(\Gamma/T(\Gamma)) - 1).$$

Let Γ be a torsion-free, finitely generated nilpotent group with a primitive element $\gamma \in Z(\Gamma)^{\bullet}$, and let Γ/Λ be an admissible quotient of Γ with respect to γ. The next proposition demonstrates that we may choose a cyclic series and a compatible generating subset such that a subset of the compatible generating subset generates Λ.

Proposition 3.6. *Let Γ be a torsion-free, finitely generated nilpotent group, and let γ be a primitive, central, nontrivial element. Let Γ/Λ be an admissible quotient of Γ with respect to γ. Then there exists a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ such that Γ/Λ is an admissible quotient of Γ with respect to ξ_1 where $\gamma = \xi_1$. Moreover, there exists a subset, possibly empty, $\left\{\xi_{i_j}\right\}_{j=1}^{h(\Lambda)}$ of the compatible generating subset satisfying the following. The subgroups $W_t = \left\langle \xi_{i_j}\right\rangle_{j=1}^{t}$ form a cyclic series for Λ with a compatible generating subset $\left\{\xi_{i_j}\right\}_{j=1}^{h(\Lambda)}$.*

Proof. We proceed by induction on $h(\Gamma)$, and note that the base case of $h(\Gamma) = 1$ is evident. Thus, we may assume that $h(\Gamma) > 1$. If $h(Z(\Gamma)) = 1$, then $\Lambda \cong \{1\}$; hence, we may take any cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ where $\xi_1 = \gamma$. Therefore, we may assume that $h(Z(\Gamma)) > 1$.

There exists a generating basis $\{z_i\}_{i=1}^{h(Z(\Gamma))}$ for $Z(\Gamma)$ such that $z_1 = \gamma$. Letting $K = \langle z_i \rangle_{i=2}^{h(Z(\Gamma))}$, we note that $K \leq \Lambda$. Observe that $(\Gamma/K)/(\Lambda/K)$ is an admissible quotient of Γ/K with respect to $\pi_K(\gamma)$ for the group Γ/K. Induction implies that there exists a cyclic series $\{\Delta_i/K\}_{i=1}^{h(\Gamma/K)}$ and a compatible generating subset $\{\pi_K(\xi_i)\}_{i=1}^{h(\Gamma/K)}$ such that there exists a subset $\left\{\pi_K(\xi_{i_j})\right\}_{j=1}^{h(\Gamma/K)}$ satisfying the following. The subgroups $\{W_t/K\}$ given by $\left\langle \pi_K(\xi_{i_j})\right\rangle_{j=1}^{t}$ form a cyclic series for Λ/K with a compatible generating subset $\left\{\pi_K(\xi_{i_j})\right\}_{j=1}^{h(\Lambda/K)}$. We let $H_i = \langle z_s \rangle_{s=1}^{i}$ for $1 \leq i \leq h(Z(\Gamma))$ and for $i > h(Z(\Gamma))$, we let $H_i = \left\langle \{K\} \cup \{\xi_t\}_{t=1}^{i-h(K)} \right\rangle$. We also take $\eta_i = z_i$ for $1 \leq i \leq h(Z(\Gamma))$ and for $i > h(Z(\Gamma))$, we take $\eta_i = \xi_{i-h(Z(\Gamma))}$. Thus, $\{H_i\}_{i=1}^{h(\Gamma)}$ is cyclic series for Γ with a compatible generating subset $\{\eta_i\}_{i=1}^{h(\Gamma)}$.

Consider the subset $\left\{\eta_{i_j}\right\}_{j=1}^{h(\Lambda)}$ where $\eta_{i_j} = z_{j+1}$ for $1 \leq j \leq h(K)$ and where $\eta_{i_j} = \xi_{i_{j-h(K)}}$ for $j > h(K)$. Thus, one can see that $\left\{\eta_{i_j}\right\}_{j=1}^{h(\Lambda)}$ is the required subset. $\square$

For the next two propositions, we establish some notation. Let Γ be a torsion-free, finitely generated nilpotent group. For each primitive element $\gamma \in Z(\Gamma)^{\bullet}$, we let Γ/Λ_γ be an admissible quotient with respect to γ.

We demonstrate that we may calculate $\psi_{\mathrm{RF}}(\Gamma)$ for Γ when given a generating basis for $(\Gamma/T(\Gamma))_{c(\Gamma/T(\Gamma))}$.

Proposition 3.7. *Let Γ be a torsion-free, finitely generated nilpotent group, and let $\{z_i\}_{i=1}^{h(Z(\Gamma))}$ be a basis of $Z(\Gamma)$. Moreover, assume there exist integers $\{t_i\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ such that $\left\{z_i^{t_i}\right\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ is a basis of $\Gamma_{c(\Gamma)}$ and that there exist $a_i \in \Gamma_{c(\Gamma)-1}$ and $b_i \in \Gamma$ such that $z_i^{t_i} = [a_i, b_i]$. For each $\gamma \in \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, there exists an $i_0 \in \left\{1, \ldots, h(\Gamma_{c(\Gamma)})\right\}$ such that $\Gamma/\Lambda_{z_{i_0}}$ is an admissible quotient with respect to γ. More generally, if $\{z_i\}_{i=1}^{h(Z(\Gamma))}$ is any basis of $Z(\Gamma)$ with $\gamma \in (Z(\Gamma))^{\bullet}$, then there exists an i_0 such that $\Gamma/\Lambda_{z_{i_0}}$ is an admissible quotient of Γ with respect to γ.*

Proof. Letting $M = \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, we may write $\gamma = \prod_{i=1}^{h(M)} z_i^{\alpha_i}$. There exist indices $1 \leq i_1 < \cdots < i_\ell \leq h(M)$ such that $\alpha_{i_j} \neq 0$ for $1 \leq j \leq \ell$ and $\alpha_i = 0$, otherwise. We observe that $\Gamma/\Lambda_{z_{i_t}}$ satisfies the conditions of Proposition 3.1 for γ for each $1 \leq t \leq \ell$. Therefore,

$$h(\Gamma/\Lambda_\gamma) \leq \min\{h(\Gamma/\Lambda_{z_{i_t}}) \mid 1 \leq t \leq \ell\}.$$

Since $\pi_{\Lambda_\gamma}(\gamma) \neq 1$, there exists i_{t_0} such that $\pi_{\Lambda_\gamma}(z_{i_{t_0}}) \neq 1$. Thus, Γ/Λ_γ satisfies the conditions of Proposition 3.1 for $z_{i_{t_0}}$. Thus, $h(\Gamma/\Lambda_{z_{i_{t_0}}}) \leq h(\Gamma/\Lambda_\gamma)$.

In particular,
$$\min\{h(\Gamma/\Lambda_{z_{i_t}}) \mid 1 \leq t \leq \ell\} \leq h(\Gamma/\Lambda_\gamma).$$
Therefore,
$$h(\Gamma/\Lambda_\gamma) = \min\{h(\Gamma/\Lambda_{z_{i_t}}) \mid 1 \leq i \leq \ell\}.$$
The last statement follows using similar reasoning. □

The following proposition demonstrates that $\psi_{\mathrm{RF}}(\Gamma)$ can always be realized as the Hirsch length of an admissible quotient with respect to a central element of a fixed basis of $\Gamma_{c(\Gamma)}$.

Proposition 3.8. *Let Γ be a torsion-free, finitely generated nilpotent group with a basis $\{z_i\}_{i=1}^{h(Z(\Gamma))}$ for $Z(\Gamma)$. Moreover, assume there exist integers $\{t_i\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ such that $\left\{z_i^{t_i}\right\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ is a basis of $\Gamma_{c(\Gamma)}$ and that there exist elements $a_i \in \Gamma_{c(\Gamma)-1}$ and $b_i \in \Gamma$ such that $z_i^{t_i} = [a_i, b_i]$. There exists an $i_0 \in \left\{1, \ldots, h(\Gamma_{c(\Gamma)})\right\}$ such that $\psi_{\mathrm{RF}}(\Gamma) = h(\Gamma/\Lambda_{z_{i_0}})$. Hence,*
$$\psi_{\mathrm{RF}}(\Gamma) = max\left\{h(\Gamma/\Lambda_{z_i}) \mid 1 \leq i \leq h(\Gamma_{c(\Gamma)})\right\}.$$
More generally, if $\{z_i\}_{i=1}^{h(Z(\Gamma))}$ is any basis of $Z(\Gamma)$, then
$$\psi_{\mathrm{RF}}(\Gamma) = max\{h(\Gamma/\Lambda_{z_i}) \mid 1 \leq i \leq h(Z(\Gamma))\}.$$

Proof. Let $\mathcal{J}$ be the set of central, nontrivial elements γ such that there exists a $k \in \mathbb{Z}^\bullet$ where γ^k is a $c(\Gamma)$-fold commutator bracket. Given that the set $\{h(\Gamma/\Lambda_\gamma) \mid \gamma \in \mathcal{J}\}$ is bounded above by $h(\Gamma)$, there exists a nontrivial element $\eta \in \mathcal{J}$ such that
$$h(\Gamma/\Lambda_\eta) = \max\{h(\Gamma/\Lambda_\gamma) \mid \gamma \in \mathcal{J}\}.$$
Proposition 3.7 implies that there exists an $i_0 \in \{1, \ldots, h(Z(\Gamma))\}$ such that $h(\Gamma/\Lambda_\eta) = h(\Gamma/\Lambda_{z_{i_0}})$. By the definition of $\psi_{\mathrm{RF}}(\Gamma)$, it follows that
$$\psi_{\mathrm{RF}}(\Gamma) = \max\{h(\Gamma/\Lambda_{z_i}) \mid 1 \leq i \leq h(\Gamma_{c(\Gamma)})\}.$$
The last statement follows using similar reasoning. □

3.2. Properties of admissible quotients. We demonstrate conditions for an admissible quotient of Γ with respect to some primitive, central, nontrivial element to have the same step length as Γ.

Proposition 3.9. *Let Γ be a torsion-free, finitely generated nilpotent group. If we let $\gamma \in \left(\sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}\right)^\bullet$ be a primitive element with an admissible quotient Γ/Λ with respect to γ, then $c(\Gamma/\Lambda) = c(\Gamma)$. In particular, if Γ/Λ is a maximal admissible quotient of Γ, then $c(\Gamma/\Lambda) = c(\Gamma)$. If $c(\Gamma) > 1$, then $h(\Gamma/\Lambda) \geq 3$.*

Proof. By definition, there exists a $k \in \mathbb{Z}^\bullet$ such that $\gamma^k \in \Gamma_{c(\Gamma)}$. Suppose for a contradiction that $c(\Gamma/\Lambda) < c(\Gamma)$. We then have that $\Gamma_{c(\Gamma)} \leq \ker(\pi_\Lambda)$, and hence, $\pi_\Lambda(\gamma^k) = 1$. Since Γ/Λ is torsion-free, it follows that $\pi_\Lambda(\gamma) = 1$. That contradicts the construction of Γ/Λ, and thus, $c(\Gamma/\Lambda) = c(\Gamma)$.

Since every irreducible, torsion-free, finitely generated nilpotent group Γ such that $c(\Gamma) \geq 2$ contains a subgroup isomorphic to the 3-dimensional integral Heisenberg group, we have that $h(\Gamma/\Lambda) \geq 3$. □

The following proposition relates the value $\psi_{\mathrm{RF}}(\Gamma)$ to the value $\psi_{\mathrm{RF}}(\Lambda)$ when Λ is a torsion-free quotient of Γ of lower step length.

Proposition 3.10. *Let Γ be a torsion-free, finitely generated nilpotent group. If $M = \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, then $\psi_{\mathrm{RF}}(\Gamma) \geq \psi_{\mathrm{RF}}(\Gamma/M)$.*

Proof. There exist elements $\{z_i\}_{i=1}^{h(Z(\Gamma/M))}$ and integers $\{t_i\}_{i=1}^{h(N)}$ satisfying the following. The set $\{\pi_M(z_i)\}_{i=1}^{h(Z(\Gamma/M))}$ generates $Z(\Gamma/M)$ and that there exist $a_i \in (\Gamma/M)_{c(\Gamma)-2}$ and $b_i \in \Gamma/M$ such that $\pi_M([a_i, b_i]) = \pi_M(z_i^{t_i})$. Finally, the set $\left\langle \left\{\pi_M(z_i)^{t_i}\right\}_{i=1}^{h(N)} \right\rangle$ generates $\sqrt[Z(\Gamma/M)]{(\Gamma/M)_{c(\Gamma)-1}}$.

There exist $\gamma_i \in \Gamma$ such that the elements $\{[z_i, \gamma_i]\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ generate $\Gamma_{c(\Gamma)}$. Finally, there exist elements $\{y_i\}_{i=1}^{h(M)}$ in $Z(\Gamma)$ and integers $\{s_i\}_{i=1}^{h(M)}$ such that $y_i^{s_i} = [z_i, \gamma_i]$. For each $i \in \{1, \ldots, h(M)\}$, we let Γ/Λ_i be an admissible quotient with respect to y_i.

Let $(\Gamma/M)/\Omega_i$ be an admissible quotient of Γ/M with respect to $\pi_M(z_i)$. It is evident that $\Lambda_i \leq \pi_M^{-1}(\Omega_i)$. Thus, it follows that

$$h(\Gamma/\Lambda_i) \geq h(\Gamma/\pi_M^{-1}(\Omega_i)) = h((\Gamma/M)/\Omega_i).$$

Proposition 3.8 implies that $\psi_{\mathrm{RF}}(\Gamma) \geq h((\Gamma/M)/\Omega_i)$. Applying Proposition 3.8 again, we have that $\psi_{\mathrm{RF}}(\Gamma) \geq \psi_{\mathrm{RF}}(\Gamma/M)$. □

This last proposition demonstrates that the definition of $\psi_{\mathrm{RF}}(\Gamma)$ is the maximum value over all possible Hirsch lengths of admissible quotients with respect to primitive, central, nontrivial elements of Γ.

Proposition 3.11. *Let Γ be a torsion-free, finitely generated nilpotent group. For each primitive element $\gamma \in (Z(\Gamma))^{\bullet}$, we let Γ/Λ_γ be an admissible quotient with respect to γ. Then*

$$\psi_{\mathrm{RF}}(\Gamma) = \mathit{max}\left\{h(\Gamma/\Lambda_\gamma) \mid \gamma \in (Z(\Gamma))^{\bullet}\right\}.$$

Proof. Suppose that Γ is abelian. We then have that $h(\Gamma/\Lambda_\gamma) = 1$ for all primitive elements. Therefore, we have our statement, and thus, we may assume that $c(\Gamma) > 1$.

Let $M = \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, and let $\gamma \in (Z(\Gamma))^{\bullet}$. There exists a basis $\{z_i\}^{h(Z(\Gamma))}$ for $Z(\Gamma)$ and integers $\{t_i\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ such that $\{z_i^{t_i}\}_{i=1}^{h(\Gamma_{c(\Gamma)})}$ is a basis for $\Gamma_{c(\Gamma)}$. Moreover, there exist $a_i \in \Gamma_{c(\Gamma)-1}$ and $b_i \in \Gamma$ such that $z_i^{t_i} = [a_i, b_i]$. If $\gamma \in M$, then by definition of $\psi_{\mathrm{RF}}(\Gamma)$ and Proposition 3.8, we have that $h(\Gamma/\Lambda_\gamma) \leq \psi_{\mathrm{RF}}(\Gamma)$. Thus, we may assume that $\gamma \notin M$.

Since $\gamma \notin M$, $\pi_M(\gamma) \neq 1$. Hence, it is evident that $(\Gamma/M)/\pi_M(\Lambda_\gamma)$ satisfies Proposition 3.1 for $\pi_M(\gamma)$. Thus, if $(\Gamma/M)/\Omega$ is an admissible quotient with respect to $\pi_M(\gamma)$, we note that $\Gamma/\pi_K^{-1}(\Omega)$ satisfies Proposition 3.1 for γ. Thus, by definition,

$$h(\Gamma/\Lambda_\gamma) \leq h(\Gamma/\pi_K^{-1}(\Omega)) \leq h((\Gamma/M)/\Omega) \leq \psi_{\mathrm{RF}}(\Gamma/M).$$

Proposition 3.10 implies that $\psi_{\mathrm{RF}}(\Gamma/M) \leq \psi_{\mathrm{RF}}(\Gamma)$. Thus,

$$h(Z(\Gamma/\Lambda_\gamma)) \leq \psi_{\mathrm{RF}}(\Gamma). \quad \square$$

Definition 3.12. Let Γ be a torsion-free, finitely generated nilpotent group with a maximal admissible quotient Γ/Λ. Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series and $\{\xi_i\}_{i=1}^{h(\Gamma)}$ be a compatible generating subset that together satisfy Proposition 3.6 for Λ. We take the Mal'tsev completion G to be constructed as defined in §2.4 with Lie algebra $\mathfrak{g}$. We observe that the vectors $\mathrm{Log}(\xi_i)$ span $\mathfrak{g}$. We call the subset $\{\mathrm{Log}(\xi_i)\}_{i=1}^{h(\Gamma)}$ an *induced basis* for $\mathfrak{g}$.

4. Commutator geometry and lower bounds for residual finiteness

The following definitions and propositions will be important in the construction of the lower bounds found in the proof of Theorem 1.1.

4.1. Finite index subgroups and cyclic series. The following proposition tells us how to view finite index subgroups in light of a cyclic series and a compatible generating subset.

Proposition 4.1. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let $K \leq \Gamma$ be a finite index subgroup. Then there exist natural numbers $\{t_i\}_{i=1}^{h(\Gamma)}$ satisfying the following. The subgroups $\{H_i\}_{i=1}^{h(\Gamma)}$ given by*

$$H_i = \left\langle \xi_s^{t_s} \right\rangle_{s=1}^{i}$$

form a cyclic series for K with a compatible generating subset $\{\xi_i^{t_i}\}_{i=1}^{h(\Gamma)}$.

Proof. We proceed by induction on Hirsch length. For the base case, we have that $\Gamma \cong \mathbb{Z}$ and that $K \cong t\mathbb{Z}$ for some $t \geq 1$. Now the statement of the proposition is evident by choosing $H_1 = K$ and the compatible generating subset is given by $\{t\}$.

Thus, we may assume $h(\Gamma) > 1$. Observing that $\Delta_{h(\Gamma)-1} \cap K$ is a finite index subgroup of $\Delta_{h(\Gamma)-1}$ and that $h(\Delta_{h(\Gamma)-1}) = h(\Gamma) - 1$, the inductive hypothesis implies that there exist natural numbers $\{t_i\}_{i=1}^{h(\Gamma)}$ satisfying the following. The groups $\{H_i\}_{i=1}^{h(\Gamma)-1}$ given by $H_i = \left\langle \xi_s^{t_s} \right\rangle_{s=1}^{i}$ form a cyclic series for $\Delta_{h(\Gamma)-1} \cap K$ with a compatible generating subset $\{\xi_i^{t_i}\}_{i=1}^{h(\Gamma)-1}$. We also have that $\pi_{\Delta_{h(\Gamma)-1}}(K)$ is a finite index subgroup of $\Gamma/\Delta_{h(\Gamma)-1}$. Thus, there exists a $t_{h(\Gamma)} \in \mathbb{N}$ such that

$$K/\Delta_{h(\Gamma)-1} \cong \left\langle \pi_{\Delta_{h(\Gamma)-1}}\left(\xi_{h(\Gamma)}^{t_{h(\Gamma)}}\right) \right\rangle.$$

If we set $H_{h(\Gamma)} \cong \left\langle H_{h(\Gamma)-1}, \xi_{h(\Gamma)}^{t_{h(\Gamma)}} \right\rangle$, then the groups $\{H_i\}_{i=1}^{h(\Gamma)}$ form a cyclic series for K with a compatible generating subset $\{\xi_i^{t_i}\}_{i=1}^{h(\Gamma)}$. □

We now apply Proposition 4.1 to give a description of the subgroups of Γ of the form Γ^m for $m \in \mathbb{N}$.

Corollary 4.2. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let $m \in \mathbb{N}$. The subgroups $H_i = \langle \xi_s^m \rangle_{s=1}^{i}$ form a cyclic series for Γ^m with a compatible generating subset $\{\xi_i^m\}_{i=1}^{h(\Gamma)}$. In particular,*

$$|\Gamma/\Gamma^m| = m^{h(\Gamma)}.$$

Proof. Proposition 4.1 implies that there exist natural numbers $\{t_i\}_{i=1}^{h(\Gamma)}$ such that the subgroups $\{H_i\}_{i=1}^{h(\Gamma)}$ given by $H_i = \langle \xi_s^{t_s} \rangle_{s=1}^{i}$ form a cyclic series for Γ^m with a compatible generating subset $\{\xi_i^{t_i}\}_{i=1}^{h(\Gamma)}$. We observe that $(\Gamma/\Delta_1)^m \cong (\Gamma^m/\Delta_1)$. It is also evident that the series $\{H_i/\Delta_1\}_{i=2}^{h(\Gamma)}$ is a cyclic series for $(\Gamma/\Delta_1)^m$ with a compatible generating subset $\{\pi_{\Delta_1}(\xi_i^{t_i})\}_{i=2}^{h(\Gamma)}$. Thus, the inductive hypothesis implies that $t_i = m$ for all $2 \leq i \leq h(\Gamma)$. To finish, we observe that $\Gamma^m \cap \Delta_1 \cong \Delta_1^m$. Thus, $t_1 = m$. □

Let Γ be a torsion-free, finitely generated nilpotent group, and let $K \leq \Gamma$ be a finite index subgroup. The following proposition allows us to understand how K intersects a fixed admissible quotient of Γ with respect to a primitive, central, nontrivial element.

Proposition 4.3. *Let Γ be a torsion-free, finitely generated nilpotent group with a maximal admissible quotient Γ/Λ. Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series and $\{\xi_i\}_{i=1}^{h(\Gamma)}$ be a compatible generating subset that together satisfy Proposition 3.6 for Λ. Let K be a finite index subgroup of Γ. There exist indices $1 \leq i_1 < i_2 < \cdots < i_\ell \leq h(\Gamma)$ with natural numbers $\{t_s\}_{s=1}^{\ell}$ such that the subgroups $H_s = \left\langle \xi_{i_j}^{t_j} \right\rangle_{j=1}^{s}$ form a cyclic series for $K \cap \Lambda$ with compatible generating subset $\{\xi_{i_s}^{t_s}\}_{s=1}^{\ell}$.*

Proof. We proceed by induction on Hirsch length, and since the base case is clear, we may assume that $h(\Gamma) > 1$. By assumption, the cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ together satisfy the conditions of Proposition 3.6 for Λ. Thus, there exists indices $1 \leq i_1 < i_2 < \cdots < i_\ell$ such that the subgroups $M_j = \langle \xi_{i_s} \rangle_{s=1}^{j}$ form a cyclic series for Λ with a compatible generating subset $\{\xi_{i_s}\}_{s=1}^{h(\Lambda)}$. Applying Proposition 4.1 to the torsion-free, finitely generated nilpotent group Γ, cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$, and compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, we have that there exist natural numbers $\{t_i\}_{i=1}^{h(\Gamma)}$ such that the subgroups given by $W_i = \langle \xi_s^{t_s} \rangle_{s=1}^{i}$ form a cyclic series for K with a compatible generating subset $\{\xi_i^{t_i}\}_{i=1}^{h(\Gamma)}$. Since $K \cap \Lambda$ is a finite index subgroup of Λ, the subgroups given by $H_s = \left\langle \xi_{i_j}^{t_{i_j}} \right\rangle_{j=1}^{s}$ form the desired cyclic series for $K \cap \Lambda$ with a compatible generating subset $\{\xi_{i_s}^{t_{i_s}}\}_{s=1}^{\ell}$. Therefore, $\{t_{i_s}\}_{s=1}^{\ell}$ are the desired integers. □

4.2. Reduction of complexity for residual finiteness. We first demonstrate that we may assume that Γ is torsion-free when calculating $\mathrm{F}_\Gamma(n)$.

Proposition 4.4. *Let Γ be an infinite, finitely generated nilpotent group. Then*

$$\mathrm{F}_\Gamma(n) \approx \mathrm{F}_{\Gamma/T(\Gamma)}(n).$$

Proof. We proceed by induction on $|T(\Gamma)|$, and observe that the base case is evident. Thus, we may assume that $|T(\Gamma)| > 1$. Note that

$$\pi_{Z(T(\Gamma))} : \Gamma \to \Gamma/Z(T(\Gamma))$$

is surjective and that $\ker(\pi_{Z(T(\Gamma))}) = Z(T(\Gamma))$ is a finite central subgroup. Since finitely generated nilpotent groups are linear, [7, Lem 2.4] implies that $\mathrm{F}_\Gamma(n) \approx \mathrm{F}_{\Gamma/T(Z(\Gamma))}(n)$. Since

$$(\Gamma/Z(T(\Gamma)))/T(\Gamma/Z(T(\Gamma))) \cong \Gamma/T(\Gamma),$$

the inductive hypothesis implies that $\mathrm{F}_\Gamma(n) \approx \mathrm{F}_{\Gamma/T(\Gamma)}(n)$. □

For a torsion-free, finitely generated nilpotent group Γ, the following proposition implies that we may pass to a maximal admissible quotient of Γ when computing the lower bounds of $\mathrm{F}_\Gamma(n)$.

Proposition 4.5. *Let Γ be a torsion-free, finitely generated nilpotent group with a maximal admissible quotient Γ/Λ. Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series and $\{\xi_i\}_{i=1}^{h(\Gamma)}$ be a compatible generating subset that together satisfy Proposition 3.6 for Λ. If $\varphi : \Gamma \to Q$ is a surjective homomorphism to a finite group, then $\varphi(\xi_1^m) \neq 1$ if and only if $\pi_{\varphi(\Lambda)}(\varphi(\xi_1^m)) \neq 1$ where $m \in \mathbb{N}$.*

Proof. If $\Lambda \cong \{1\}$, then there is nothing to prove. Thus, we may assume that $\Lambda \ncong \{1\}$. Proposition 3.6 implies that $\xi_1 \notin \Lambda$ and that there exists a collection of elements of the Mal'tsev basis $\{\xi_{i_s}\}_{s=1}^{\ell}$ such that $\Lambda \cong \langle \xi_{i_s} \rangle_{s=1}^{h(\Lambda)}$. Moreover, we have that $H_s = \langle \xi_{i_t} \rangle_{t=1}^{s}$ is cyclic series for Λ with compatible generating subset $\{\xi_{i_s}\}_{s=1}^{\ell}$. Proposition 4.3 implies that there exist natural numbers $\{t_s\}_{s=1}^{\ell}$ such that the series of subgroups $\{W_s\}_{s=1}^{h(\Lambda)}$ given by

$$W_s = \left\langle \xi_{i_j}^{t_j} \right\rangle_{j=1}^{s}$$

forms a cyclic series for $\ker(\varphi) \cap \Lambda$ with a compatible generating subset $\{\xi_{i_s}^{t_s}\}_{s=1}^{h(\Lambda)}$.

Since the backwards direction is clear, we proceed with forward direction. To be more specific, we demonstrate that if $\varphi(\xi_1^m) \neq 1$, then $\pi_{\varphi(\Lambda)}(\xi_1^m) \neq 1$.

We proceed by induction on $|\varphi(\Lambda)|$, and observe that the base case is clear. Thus, we may assume that $|\varphi(\Lambda)| > 1$. In order to apply the inductive hypothesis, we find a nontrivial, normal subgroup $M \leq Z(Q)$ such that $\varphi(\xi_1^m) \notin M$.

We first observe that if $\varphi(\xi_{i_0}) \neq 1$ for some $i_0 \in \{2, \ldots, h(Z(\Gamma))\}$, we may set $M = \langle \varphi(\xi_{i_0}) \rangle$. It is straightforward to see that $M \ncong \{1\}$ and that $\varphi(\xi_1^m) \notin M$. Thus, we may assume that $\xi_i \in \ker(\varphi)$ for $i \in \{2, \ldots, h(Z(\Gamma))\}$.

In this next paragraph, we prove that there exists an element of the compatible generating subset, say ξ_{i_0}, such that $\xi_{i_0} \in \Lambda$, $\xi_{i_0} \notin \ker(\varphi)$, and $\varphi(\xi_{i_0}) \in Z(Q)$. To that end, we note that if $|t_{i_s}| = 1$, then $\xi_{i_s} \in \ker(\varphi)$. Since $|\varphi(\Lambda)| > 1$, the set $E = \{\xi_{i_s} \mid |t_{i_s}| \neq 1\}$ is non empty. Given that E is a finite set, there exists a $\xi_{i_{s_0}} \in E$ such that

$$\text{Height}(\xi_{i_{s_0}}) = \min\{\text{Height}(\xi_{i_s}) \mid \xi_{i_s} \in E\}.$$

We claim that $\varphi(\xi_{i_{s_0}})$ is central in Q, and since we are assuming that $\varphi(\xi_i) = 1$ for $i \in \{2, \ldots, h(Z(\Gamma))\}$, we may assume that $\text{Height}(\xi_{i_{s_0}}) > 1$. Since $\text{Height}([\xi_{i_{s_0}}, \xi_t]) < \text{Height}(\xi_{i_{s_0}})$ for any ξ_t and that $\varphi(\Lambda) \trianglelefteq Q$, it follows that $[\varphi(\xi_{i_{s_0}}), \varphi(\xi_t)] \in \varphi(\Lambda)$. Thus, $[\varphi(\xi_{i_{s_0}}), \varphi(\xi_t)]$ is a product of $\varphi(\xi_{i_{s_j}})$ where $\text{Height}(\xi_{i_{s_j}}) < \text{Height}(\xi_{i_{s_0}})$. Since $\xi_{i_{s_j}} \in \varphi(\Lambda)$ and $\text{Height}(\xi_{i_{s_j}}) < \text{Height}(\xi_{i_{s_0}})$, the definition of E and the choice of ξ_{i_0} imply that $t_{i_{s_j}} = 1$. Thus, $\xi_{i_{s_j}} \in \ker(\varphi)$, and subsequently, $\varphi(\xi_{i_{s_j}}) = 1$. Hence, $[\varphi(\xi_{i_{s_0}}), \varphi(\xi_t)] = 1$, and thus, $\varphi(\xi_{i_0}) \in (Z(Q))^{\bullet}$.

Since $\varphi(\xi_{i_{s_0}})$ is central in Q, the group $M = \langle \varphi(\xi_{i_0}) \rangle$ is a normal subgroup of Q. By selection, $\varphi(\xi_1^m) \notin M$, and since $|\pi_M(\varphi(\Lambda))| < |\varphi(\Lambda)|$, we may apply the inductive hypothesis to the surjective homomorphism

$$\pi_M \circ \varphi : \Gamma \to Q/M.$$

Letting $N = \pi_M \circ \varphi(\Lambda)$, we have that $\pi_N(\pi_M(\varphi(\xi_1^m))) \neq 1$. Thus,

$$\pi_{\varphi(\Lambda)}(\xi_1^m) \neq 1. \qquad \square$$

As a natural consequence of the techniques used in the proof of the above proposition, we have the following corollary.

Corollary 4.6. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let ξ_{t_0} be a central element of $\{\xi_i\}_{i=1}^{h(\Gamma)}$. If Λ satisfies the conditions of Corollary 3.2 for ξ_{t_0}, then $\varphi(\xi_{t_0}) \neq 1$ if and only if $\pi_{\varphi(\Lambda)} \circ \varphi(\xi_{t_0}) \neq 1$.*

4.3. Rank and step estimates.

Definition 4.7. Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let $\vec{a} = (a_i)_{i=1}^{\ell}$ where $1 \leq a_i \leq h(\Gamma)$ for all i. We write

$$[\xi_{\vec{a}}] = [\xi_{a_1}, \ldots, \xi_{a_\ell}].$$

We call $[\xi_{\vec{a}}]$ a *simple commutator of weight ℓ with respect to $\vec{a}$*. Let

$$W_k(\Gamma, \Delta, \xi)$$

be the set of nontrivial simple commutators of weight k. Since Γ is a nilpotent group, $W_{c(\Gamma)+1}$ is empty. Thus, the set of nontrivial simple commutators of any weight, denoted as $W(\Gamma, \Delta, \xi)$, is finite.

When considering a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$, we need to ensure that the step length of Q is equal to the step length of Γ. We do that by assuming that $\varphi([\xi_{\vec{a}}]) \neq 1$ for all $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \cap Z(\Gamma)$.

Proposition 4.8. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let $\varphi : \Gamma \to Q$ be a surjective homomorphism to a finite group such that if $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \cap Z(\Gamma)$, then $\varphi([\xi_{\vec{a}}]) \neq 1$. Then $\varphi([\xi_{\vec{a}}]) \neq 1$ for all $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi)$. Lastly, $c(\Gamma) = c(Q)$.*

Proof. We first demonstrate that $\varphi([\xi_{\vec{a}}]) \neq 1$ for all $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi)$ by induction on $\mathrm{Height}([\xi_{\vec{a}}])$. Observe that if $[\xi_{\vec{a}}] \in W_k(\Gamma, \Delta, \xi)$, then $\mathrm{Height}([\xi_{\vec{a}}]) \leq c(\Gamma) - k + 1$. Thus, if $[\xi_{\vec{a}}] \in W_{c(\Gamma)}(\Gamma, \Delta, \xi)$, then $\mathrm{Height}([\xi_{\vec{a}}]) = 1$. Hence, the base case follows from assumption.

Now consider $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi)$ where $\mathrm{Height}([\xi_{\vec{a}}]) = \ell > 1$. If $[\xi_{\vec{a}}] \in Z(\Gamma)$, then the assumptions of the proposition imply that $\varphi([\xi_{\vec{a}}]) \neq 1$. Thus, we may assume there exists an element ξ_{i_0} of the Mal'tsev basis such that $[[\xi_{\vec{a}}], \xi_{i_0}] \neq 1$. The induction hypothesis implies that $\varphi([[\xi_{\vec{a}}], \xi_{i_0}]) \neq 1$ since $[[\xi_{\vec{a}}], \xi_{i_0}]$ is a simple commutator of $\mathrm{Height}([[\xi_{\vec{a}}], \xi_{i_0}]) \leq \ell - 1$. Thus, $\varphi([\xi_{\vec{a}}]) \neq 1$. Therefore, for each $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi)$, it follows that $\varphi([\xi_{\vec{a}}]) \neq 1$.

If $c(Q) < c(\Gamma)$, then φ factors through $\Gamma / \Gamma_{c(\Gamma)}$, and thus,

$$W_{c(\Gamma)}(\Gamma, \Delta, \xi) \subseteq \ker(\varphi).$$

Since

$$W_{c(\Gamma)}(\Gamma, \Delta, \xi) \subseteq W(\Gamma, \Delta, \xi) \cap Z(\Gamma),$$

we have a contradiction. Hence, $c(Q) = c(\Gamma)$. $\square$

The following definition will be important in the proofs of Theorem 1.1 and Theorem 1.8.

Definition 4.9. Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. For $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi)$, we may write $[\xi_{\vec{a}}] = \prod_{i=1}^{h(\Gamma)} \xi_i^{\delta_{\vec{a},i}}$. Let

$$\mathrm{B}(\Gamma, \Delta, \xi) = \mathrm{lcm}\Big\{ |\delta_{\vec{a},i}| \;\Big|\; 1 \leq i \leq h(\Gamma), \delta_{\vec{a},i} \neq 0 \text{ and } [\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \Big\}.$$

Suppose that Γ is a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. For a surjective homomorphism to a finite p-group $\varphi : \Gamma \to Q$, the following proposition gives conditions so that $|Q| \geq p^{h(\Gamma)}$. To be more specific, if φ is an injective map when restricted to the set of central simple commutators, is an injective map when restricted to central elements of a fixed compatible generating subset, and $p > \mathrm{B}(\Gamma, \Delta, \xi)$, then φ is an injection when restricted to that same compatible generating subset.

Proposition 4.10. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let $\varphi : \Gamma \to Q$ be a surjective homomorphism to a finite p-group where $p > \mathrm{B}(\Gamma, \Delta, \xi)$. Suppose that $\varphi([\xi_{\vec{a}}]) \neq 1$ for all $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \cap Z(\Gamma)$. Also, suppose that $\varphi(\xi_i) \neq 1$ for $\xi_i \in Z(\Gamma)$ and $\varphi(\xi_i) \neq \varphi(\xi_j)$ for $\xi_i, \xi_j \in Z(\Gamma)$ where $i \neq j$. Then $\varphi(\xi_t) \neq 1$ for $1 \leq t \leq h(\Gamma)$ and $\varphi(\xi_i) \neq \varphi(\xi_j)$ for $1 \leq i < j \leq h(\Gamma)$. Finally, $|Q| \geq p^{h(\Gamma)}$.*

Proof. Let $\xi_t \notin Z(\Gamma)$. By selection, there exists a ξ_s such that $[\xi_t, \xi_s] \neq 1$. Since $[\xi_t, \xi_s]$ is a simple commutator of weight 2, we have that $\varphi([\xi_t, \xi_s]) \neq 1$ by Proposition 4.8. Thus, $\varphi(\xi_t) \neq 1$.

We now demonstrate that $\varphi(\xi_i) \neq \varphi(\xi_j)$ for all $1 \leq i < j \leq h(\Gamma)$ by induction on $h(\Gamma)$. If $\xi_i, \xi_j \in Z(\Gamma)$, then by assumption, $\varphi(\xi_i) \neq \varphi(\xi_j)$. Now suppose that $\xi_i \in Z(\Gamma)$ and that $\xi_j \notin Z(\Gamma)$. Then there exists a ξ_s such that $[\xi_j, \xi_s] \neq 1$, and subsequently, the above paragraph implies that $\varphi([\xi_j, \xi_s]) \neq 1$. In particular, $\varphi(\xi_j) \notin Z(Q)$, and thus, $\varphi(\xi_i) \neq \varphi(\xi_j)$.

We now may assume that $\xi_i, \xi_j \notin Z(\Gamma)$. Proposition 3.2 implies that there exists a normal subgroup $\Lambda/K \trianglelefteq \Gamma/K$ such that Λ/K satisfies the conditions of Proposition 3.1 for $\pi_K(\xi_i)$ where $K = \langle \xi_s \rangle_{s=1}^{i-1}$. Moreover, $\Lambda \cong \langle \xi_{i_\ell} \rangle_{\ell=1}^{h(\Lambda)}$ where the subgroups given by $W_t = \langle \xi_{i_\ell} \rangle_{\ell=1}^{t}$ form a cyclic series for Λ with a compatible generating subset $\{\xi_{i_\ell}\}_{\ell=1}^{h(\Lambda)}$. Thus, $\{\pi_\Lambda(\Delta_s)\}_{s=1, s \notin \mathcal{S}}^{h(\Gamma)}$ is a cyclic series for Γ/Λ with a compatible generating subset $\{\pi_\Lambda(\xi_s)\}_{s=1, s \notin \mathcal{S}}^{h(\Gamma)}$. That implies $\pi_\Lambda(W(\Gamma, \Delta, \xi)) = W(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi))$. For simplicity, we indicate elements of $W(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi))$ as $[\pi_\Lambda(\xi_{\vec{a}})]$.

Corollary 4.6 implies that $\pi_{\varphi(\Lambda)} \circ \varphi(\pi_\Lambda(\xi_i)) \neq 1$. Thus, we proceed based on whether $\varphi(\xi_j) \in \varphi(\Lambda)$ or not. If $\varphi(\xi_j) \in \varphi(\Lambda)$, then $\varphi(\xi_i) \neq \varphi(\xi_j)$ since $\varphi(\xi_i) \notin \varphi(\Lambda)$. Thus, we may assume that $\varphi(\xi_i), \varphi(\xi_j) \notin \varphi(\Lambda)$.

Since we have a homomorphism

$$\pi_{\varphi(\Lambda)} \circ \varphi : \Gamma \to \Gamma/\varphi(\Lambda)$$

where $\Lambda \leq \ker(\pi_{\varphi(\Lambda)} \circ \varphi)$, we obtain an induced homomorphism

$$\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi} : \Gamma/\Lambda \to Q/\varphi(\Lambda).$$

We now demonstrate the hypotheses of our proposition hold for the homomorphism $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi} : \Gamma/\Lambda \to Q/\varphi(\Lambda)$. Since $\pi_{\varphi(\Lambda)} \circ \varphi(\pi_\Lambda(\xi_i)) \neq 1$, it follows that $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_K(\xi_i)) \neq 1$. Thus, we have that $Z(\Gamma/\Lambda) \cong \langle \pi_\Lambda(\xi_i) \rangle$.

In particular, $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}$ is injective when restricted to the subset of central elements of the compatible generating subset for Γ/Λ given by $\{\pi_\Lambda(\xi_s)\}$. For each $[\pi_\Lambda(\xi_{\vec{a}})] \in W(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi))$, there exists a $[\xi_{\vec{b}}] \in W(\Gamma, \Delta, \xi)$ such that $\pi_\Lambda([\xi_{\vec{b}}]) = [\pi_\Lambda(\xi_{\vec{a}})]$. We may write

$$\pi_\Lambda([\xi_{\vec{b}}]) = \pi_\Lambda\left(\prod_{s=1}^{h(\Gamma)} \xi_i^{\delta_{\vec{b},s}}\right) = \prod_{s=1}^{h(\Gamma)} \pi_\Lambda(\xi_s)^{\delta_{\vec{b},i}} = \prod_{s=1, s\neq \mathcal{S}}^{h(\Gamma)} \pi_\Lambda(\xi_s)^{\delta_{\vec{a},s}} = [\pi_\Lambda(\xi_{\vec{a}})].$$

By construction, $\delta_{\vec{b},i} = \delta_{\vec{a},i}$ for $1 \leq i \leq h(\Gamma)$ and $i \notin \mathcal{S}$. By the definition of lcm, we have that

$$\mathrm{B}(\Gamma, \Delta, \xi) \geq \mathrm{B}(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi)).$$

Additionally, for

$$[\pi_\Lambda(\xi_{\vec{a}})] \in W(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi)) \cap Z(\Gamma/\Lambda)$$

we have that $[\xi_{\vec{a}}] = \xi_i^{\delta_{\vec{a},i}}$. Since $p > \mathrm{B}(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi))$, we have that

$$\mathrm{Ord}_{Q/\varphi(\Lambda)}(\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_\Lambda(\xi_i))) > p.$$

Thus, we have $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}([\pi_\Lambda(\xi_{\vec{a}})]) \neq 1$ for all

$$[\pi_\Lambda(\xi_{\vec{a}})] \in W(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi)) \cap Z(\Gamma/\Lambda).$$

By the proof of the first statement, we have that $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_K(\xi_k)) \neq 1$ for $1 \leq k \leq h(\Gamma)$ where $k \notin \mathcal{S}$. We also have that $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}([\xi_{\vec{a}}]) \neq 1$ for all

$$[\pi_\Lambda(\xi_{\vec{a}})] \in W(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi)).$$

By construction, there exists a ξ_{s_0} such that $[\pi_\Lambda(\xi_j), \pi_\Lambda(\xi_{s_0})] \neq 1$. Proposition 4.8 implies that $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}([\pi_\Lambda(\xi_j), \pi_\Lambda(\xi_{s_0})]) \neq 1$. Thus, it follows that

$$\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_\Lambda(\xi_j)) \notin Z(Q/\varphi(\Lambda))$$

whereas $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_\Lambda(\xi_i)) \in Z(Q/\varphi(\Lambda))$. Thus,

$$\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_\Lambda(\xi_i)) \neq \widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_\Lambda(\xi_j)).$$

Given that $\widetilde{\pi_{\varphi(\Lambda)} \circ \varphi}(\pi_\Lambda(g)) = \pi_{\varphi(\Lambda)} \circ \varphi(g)$ for all $g \in \Gamma$, we have that

$$\pi_{\varphi(A)} \circ \varphi(\xi_i) \neq \pi_{\varphi(A)} \circ \varphi(\xi_j).$$

In particular, $\varphi(\xi_i) \neq \varphi(\xi_j)$.

Subsequently $\{\varphi(\xi_i)\}_{i=1}^{h(\Gamma)}$ is a generating subset of Q where $\mathrm{Ord}_Q(\varphi(\xi_i)) \geq p$ for all i. [17, Thm 1.10] implies that $|Q|$ divides nonzero some power of $p^{h(\Gamma)}$. Hence, $|Q| \geq p^{h(\Gamma)}$. □

Proposition 4.11. *Let Γ be a torsion-free, finitely generated nilpotent group such that $h(Z(\Gamma)) = 1$ with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Suppose that $\varphi : \Gamma \to Q$ is a surjective homomorphism to a finite p-group such that $p > \mathrm{B}(\Gamma, \Delta, \xi)$, and suppose that $\varphi(\xi_1) \neq 1$. Then $c(\Gamma) = c(Q)$, $Z(Q) = \langle \varphi(\xi_1) \rangle$, and $|Q| \geq p^{h(\Gamma)}$.*

Proof. Since $\varphi(\xi_1) \neq 1$ and Q is p-group, we have that $\mathrm{Ord}_Q(\varphi(\xi_1)) \geq p$. We claim that if $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \cap Z(\Gamma)$, then $\varphi([\xi_{\vec{a}}]) \neq 1$. Suppose for a contradiction that $\varphi([\xi_{\vec{a}}]) = 1$ for some $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \cap Z(\Gamma)$. Since $\varphi(\xi_1^{\mathrm{B}(\Gamma,\Delta,\xi)})$ is a power of $\varphi([\xi_{\vec{a}}])$ by definition, we have that $\varphi(\xi_1^{\mathrm{B}(\Gamma,\Delta,\xi)}) = 1$. Thus, $\varphi(\xi_1)$ has order strictly less than p which is a contradiction.

Since $\varphi([\xi_{\vec{a}}]) \neq 1$ for all $[\xi_{\vec{a}}] \in W(\Gamma, \Delta, \xi) \cap Z(\Gamma)$, Proposition 4.8 implies that $c(\Gamma) = c(Q)$. On the other hand, Proposition 4.10 implies that $\varphi(\xi_i) \neq 1$ for all $1 \leq i \leq h(\Gamma)$ and $\varphi(\xi_{j_1}) \neq \varphi(\xi_{j_2})$ for all $1 \leq j_1 < j_2 \leq h(\Gamma)$. Thus, $\{\varphi(\Delta_i)\}_{i=1}^{h(\Gamma)}$ is a cyclic series for Q and $\{\varphi(\xi_i)\}_{i=1}^{h(\Gamma)}$ is a compatible generating subset for Q. Since Q is a p-group, we have that $|\varphi(\Delta_i) : \varphi(\Delta_{i-1})| \geq p$ for each $1 \leq i \leq h(\Gamma)$ with the convention that $\Delta_0 = \{1\}$. Hence, the second paragraph after [20, Defn 8.2] implies that

$$|Q| = \prod_{i=1}^{h(\Gamma)} |\Delta_i : \Delta_{i-1}| \geq p^{h(\Gamma)}.$$

We finish by demonstrating $Z(Q) = \langle \varphi(\xi_1) \rangle$. Since $\{\varphi(\Delta_i)\}_{i=1}^{h(\Gamma)}$ is an ascending central series that is a refinement of the upper central series, there exists an i_0 such that $\varphi(\Delta_{i_0}) = Z(Q)$. For $t > 1$, there exists a $j \neq t$ such that $[\xi_t, \xi_j] \neq 1$. Since $[\xi_t, \xi_j]$ is a simple commutator of weight 2, Proposition 4.8 implies that $\varphi([\xi_t, \xi_j]) \neq 1$. Given that

$$\varphi([\xi_t, \xi_j]) = [\varphi(\xi_t), \varphi(\xi_j)],$$

it follows that $\varphi(\xi_t) \notin Z(Q)$. Thus, we have that $\varphi(\Delta_t) \gneq Z(Q)$ for all $t > 1$. Hence, $Z(Q) = \langle \varphi(\xi_1) \rangle$. □

Let Γ be a torsion-free, finitely generated nilpotent group with a maximal admissible quotient Γ/Λ. Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series and $\{\xi_i\}_{i=1}^{h(\Gamma)}$ be a compatible generating subset that together satisfy Proposition 3.6 for Λ. Suppose that $\varphi : \Gamma \to Q$ is a surjective homomorphism to a finite group and $m \in \mathbb{Z}^{\bullet}$. The following proposition gives conditions such that Q has no nontrivial quotients in which $\varphi(\xi_1^m) \neq 1$.

Proposition 4.12. *Let Γ be a torsion-free, finitely generated nilpotent group with a maximal admissible quotient Γ/Λ. Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series and $\{\xi_i\}_{i=1}^{h(\Gamma)}$ be a compatible generating subset that together satisfy Proposition 3.6 for Λ. Suppose that $\varphi : \Gamma \to Q$ is a surjective homomorphism to a finite p-group where $\varphi(\Lambda) \cong \{1\}$, $p > B(\Gamma/\Lambda, \pi_\Lambda(\Delta), \pi_\Lambda(\xi))$, and $|Q| \leq p^{\psi_{\mathrm{RF}}(\Gamma)}$. If $\varphi(\xi_1^m) \neq 1$ for some $m \in \mathbb{Z}$, then $|Q| = p^{\psi_{\mathrm{RF}}(\Gamma)}$. Additionally, if N is a proper quotient of Q, then $\rho(\varphi(\xi_1^m)) = 1$ where $\rho : Q \to N$ is the natural projection. Finally, $Z(Q) \cong \mathbb{Z}/p\mathbb{Z}$.*

Proof. Let us first demonstrate that $|Q| = p^{\psi_{\mathrm{RF}}(\Gamma)}$. Since $\Lambda \leq \ker(\varphi)$, we have an induced homomorphism $\widetilde{\varphi} : \Gamma/\Lambda \to Q$ such that $\widetilde{\varphi} \circ \pi_\Lambda = \varphi$. Hence, Proposition 4.11 implies that $\varphi(Z(\Gamma/\Lambda)) \cong Z(Q)$ and $|Q| \geq p^{\psi_{\mathrm{RF}}(\Gamma)}$. Thus, $|Q| = p^{\psi_{\mathrm{RF}}(\Gamma)}$.

We now demonstrate that $Z(Q) \cong \mathbb{Z}/p\mathbb{Z}$. Since $\{\varphi(\pi_\Lambda(\Delta_i)) \mid \xi \notin \Lambda\}$ is a cyclic series for Q with a compatible generating $\{\varphi(\xi_i) \mid \xi_i \notin \Lambda\}$, it follows that

$$|Q| = \prod_{\xi_i \notin \Lambda} \mathrm{Ord}_Q(\varphi(\xi_i))$$

(see the second paragraph after [20, Defn 8.2]). Thus, we must have that $\mathrm{Ord}_Q(\varphi(\xi_i)) \leq p$. Since $\mathrm{Ord}_Q(\varphi(\xi_1)) \geq p$, we have that $\mathrm{Ord}_Q(\varphi(\xi_1)) = p$. Since $Z(Q) \cong \langle \varphi(\xi_1) \rangle$, it follows that $Z(Q) \cong \mathbb{Z}/p\mathbb{Z}$.

Since $Z(Q) \cong \mathbb{Z}/p\mathbb{Z}$, there are no proper, nontrivial, normal subgroups of $Z(Q)$. Given that $\ker(\rho) \trianglelefteq Q$, we have that $Z(Q) \cap \ker(\rho) = Z(Q)$; hence, $\rho(\varphi(\xi_i^m)) = 1$ because $\varphi(\xi_1^m) \in Z(Q) \leq \ker(\rho)$. □

5. Some examples of precise residual finiteness

To demonstrate the techniques used in the proof of Theorem 1.1, we make a precise calculation of $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$ where $\mathrm{H}_{2m+1}(\mathbb{Z})$ is the $(2m+1)$-dimensional integral Heisenberg group.

5.1. Basics facts about the integral Heisenberg group. We start by introducing basic facts about the $(2m+1)$-dimensional integral Heisenberg group which will be useful in the calculation of $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$ and in Section 9. We may write

$$\mathrm{H}_{2m+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & \vec{x} & z \\ \vec{0} & \mathbf{I}_m & \vec{y} \\ 0 & \vec{0} & 1 \end{pmatrix} \middle| \, z \in \mathbb{Z}, \;\; \vec{x}, \vec{y}^T \in \mathbb{Z}^m \right\}$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix. If $\gamma \in \mathrm{H}_{2m+1}(\mathbb{Z})$, we write

$$\gamma = \begin{pmatrix} 1 & \vec{x}_\gamma & z_\gamma \\ \vec{0} & \mathbf{I}_m & \vec{y}_\gamma \\ 0 & \vec{0} & 1 \end{pmatrix}$$

where $\vec{x}_\gamma = [x_{\gamma,1}, \ldots, x_{\gamma,m}]$ and $\vec{y}_\gamma^T = [y_{\gamma,1}, \ldots, y_{\gamma,m}]$.

We let $E = \{\vec{e}_i\}_{i=1}^m$ be the standard basis of $\mathbb{Z}^m$ and then choose a generating subset for $\mathrm{H}_{2m+1}(\mathbb{Z})$ given by $S = \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m, \lambda\}$ where

$$\alpha_i = \begin{pmatrix} 1 & \vec{e}_i & 0 \\ \vec{0} & \mathbf{I}_m & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix}, \quad \beta_i = \begin{pmatrix} 1 & \vec{0} & 0 \\ \vec{0} & \mathbf{I}_m & \vec{e}_i^T \\ 0 & \vec{0} & 1 \end{pmatrix}, \quad \text{and } \lambda = \begin{pmatrix} 1 & \vec{0} & 1 \\ \vec{0} & \mathbf{I}_m & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix}.$$

Thus, if $\gamma \in B_{\mathrm{H}_{2m+1}(\mathbb{Z}),S}(n)$, then $\vec{x}_\gamma, \vec{x}_\eta, \vec{y}_\gamma^T, \vec{y}_\eta^T \in B_{\mathbb{Z}^m,E}(C_0\, n)$ and $|z_\gamma| \leq C_0\, n^2$ for some $C_0 \in \mathbb{N}$ [16, 3.B2]. We obtain a finite presentation for $\mathrm{H}_{2m+1}(\mathbb{Z})$ written as

$$\mathrm{H}_{2m+1}(\mathbb{Z}) = \langle \kappa, \mu_i, \nu_j \text{ for } 1 \leq i, j \leq m \mid [\mu_t, \nu_t] = \kappa \text{ for } 1 \leq t \leq m \rangle \tag{4}$$

with all other commutators being trivial.

Finally, we let $\Delta_1 = \langle \kappa \rangle$, $\Delta_i = \langle \{\kappa\} \cup \{\mu_s\}\rangle_{s=1}^{t-1}$ for $2 \le i \le m+1$, and $\Delta_i = \left\langle \{\Delta_{m+1}\}, \{\nu_t\}_{t=1}^{i-m-1} \right\rangle$ for $m+2 \le i \le 2m+1$. One can see that $\{\Delta_i\}_{i=1}^{2m+1}$ is a cyclic series for $\mathrm{H}_{2m+1}(\mathbb{Z})$ and that S is a compatible generating subset.

5.2. Residual finiteness of $\mathbf{H_{2m+1}}(\mathbb{Z})$. The upper and lower bounds for $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$ require different strategies, so we approach them separately. We start with the upper bound as it is more straightforward.

Proposition 5.1. $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \preceq (\log(n))^{2m+1}$.

Proof. For $\|\gamma\|_S \le n$, we will construct a surjective homomorphism to a finite group $\varphi : \mathrm{H}_{2m+1}(\mathbb{Z}) \to Q$ such that $\varphi(\gamma) \neq 1$ and where

$$|Q| \le C_0 (\log(C_0\, n))^{2m+1}$$

for some $C_0 > 0$.

Via the Mal'tsev basis, we may write

$$\gamma = \kappa^{\alpha} \left(\prod_{i=1}^{m} \mu_i^{\beta_i} \right) \left(\prod_{j=1}^{m} \nu_j^{\lambda_j} \right).$$

We proceed based on whether $\pi_{\mathrm{ab}}(\gamma)$ is trivial or not.

Suppose that $\pi_{\mathrm{ab}}(\gamma) \neq 1$. Since $\gamma \neq 1$, either $\beta_{i_0} \neq 0$ for some i_0, or $\lambda_{j_0} \neq 0$ for some j_0. Without loss of generality, we may assume that there exists some i_0 such that $\beta_{i_0} \neq 0$. The Prime Number Theorem [38, 1.2] implies that there exists a prime p such that $p \nmid |\beta_{i_0}|$ and where

$$p \le C_2 \log(C_2\, |\beta_{i_0}|) \le C_2 \log(C_1\, C_2\, n^2).$$

Consider the homomorphism $\rho : \mathrm{H}_{2m+1}(\mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z}$ given by

$$\kappa^{\alpha} \left(\prod_{i=1}^{m} \mu_i^{\beta_i} \right) \left(\prod_{j=1}^{m} \nu_j^{\lambda_j} \right) \longrightarrow (\beta_1, \ldots, \beta_m, \lambda_1, \ldots, \lambda_m) \longrightarrow \beta_{i_0} \;(\bmod\; p\,).$$

Here, the first arrow is the abelianization homomorphism and the second arrow is the natural projection from $\mathbb{Z}^{2m}$ to $\mathbb{Z}/p\mathbb{Z}$. By construction, $\rho(\gamma) \neq 1$ and

$$|\mathbb{Z}/p\mathbb{Z}| \le C_1\, C_2 \log(C_1\, C_2\, n^2).$$

Thus, for some $C_3 > 0$, we have that

$$\mathrm{D}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma) \le C_3 \log(C_3\, n).$$

Now suppose that $\pi_{\mathrm{ab}}(\gamma) = 1$. That implies $\beta_i, \lambda_j = 0$ for all i, j. As before, the Prime Number Theorem [38, 1.2] implies that there exists a prime p such that $p \nmid |\alpha|$ and $p \le C_4\, \log(C_4\, n)$ for some $C_4 \in \mathbb{N}$. We have that $\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^p}(\gamma) = \pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^p}(\kappa^{\alpha}) \neq 1$. Corollary 4.2 implies that

$$|\,\mathrm{H}_{2m+1}(\mathbb{Z})/(\mathrm{H}_{2m+1}(\mathbb{Z}))^p| = p^{2m+1}.$$

Hence,

$$|\,\mathrm{H}_{2m+1}(\mathbb{Z})/(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p}| \leq (C_4)^{2m+1}\,(\log(C_4\,n))^{2m+1}.$$

Thus, $\mathrm{D}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma) \leq C_4(\log(C_4\,n))^{2m+1}$, and therefore,

$$\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \preceq (\log(n))^{2m+1}\,. \qquad \square$$

We now proceed with the lower bound calculation of $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$.

Proposition 5.2. $(\log(n))^{2m+1} \preceq \mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$.

Proof. To demonstrate that $(\log(n))^{2m+1} \preceq \mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$, we construct a sequence of elements $\{\gamma_i\}$ such that there exists a constant $C_1 > 0$ where

$$C_1(\log(C_1\,\|\gamma_i\|_S))^{2m+1} \leq \mathrm{D}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma_i)$$

independent of i. The proof of Proposition 5.1 implies that

$$\mathrm{D}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma) \leq C_2\,\log(C_2\,\|\gamma\|_S)$$

for some $C_2 \in \mathbb{N}$, when $\gamma \notin Z(\mathrm{H}_{2m+1}(\mathbb{Z}))$. That implies that we will be looking for central elements.

Let $\{p_i\}$ be an enumeration of the primes, and let

$$\alpha_i = (\mathrm{lcm}\{1, 2, \ldots, p_i - 1\})^{2m+2}.$$

We claim for all i that $\mathrm{D}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\kappa^{\alpha_i}) \approx \log(\|\kappa^{\alpha_i}\|_S))^{2m+1}$. It is clear that $\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}}\,(\kappa)^{\alpha_i} \neq 1$ in $\mathrm{H}_{2m+1}(\mathbb{Z})/(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}$. [16, 3.B2] implies that $\|\kappa^{\alpha_i}\|_S \approx \sqrt{|\alpha_i|}$, and the Prime Number Theorem [38, 1.2] implies that $\log(|\alpha_i|) \approx p_i$. Subsequently, $\log(\|\kappa^{\alpha_i}\|_S) \approx p_i$, and thus,

$$(\log(\|\kappa^{\alpha_i}\|_S))^{2m+1} \approx p_i^{2m+1}.$$

Corollary 4.2 implies that

$$|\,\mathrm{H}_{2m+1}(\mathbb{Z})/\,(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}\,| = p_i^{2m+1};$$

thus, we will establish that

$$\mathrm{D}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\kappa^{\alpha_i}) \approx (\log(\|\kappa^{\alpha_i}\|_S))^{2m+1}$$

by demonstrating that if given a surjective homomorphism to a finite group $\varphi : \mathrm{H}_{2m+1}(\mathbb{Z}) \to Q$ satisfying $|Q| < p_i^{2m+1}$, then $\varphi(\kappa)^{\alpha_i} = 1$.

[17, Thm 2.7] implies that we may assume that $|Q| = q^{\beta}$ where q is a prime. Since $\varphi(\kappa^{\alpha_i}) = 1$ when $\varphi(\kappa) = 1$, we may assume that $\varphi(\kappa) \neq 1$. Give that $[\mu_t, \nu_t] = \kappa$ for all t, it follows that $\varphi(\nu_s), \varphi(\mu_j) \neq 1$ for all s, j and that $|Q| \geq q^{2m+1}$ (see the second paragraph after [20, Defn 8.2]).

Suppose Q is a p_i-group. If $\varphi(\kappa^{\alpha_i}) \neq 1$, then Proposition 4.12 implies that $|Q| = p_i^{2m+1}$ and that there are no proper quotients of Q where the image of $\varphi(\kappa^{\alpha_i})$ does not vanish. In particular, there are no proper quotients of $\mathrm{H}_{2m+1}(\mathbb{Z})/(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}$ where $\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}}(\kappa^{\alpha_i})$ does not vanish. Thus, we may assume that $q \neq p_i$.

If $q > p_i$, then we have that $\mathrm{Ord}_Q(\varphi(\nu_i)), \mathrm{Ord}_Q(\varphi(\mu_j)) \geq p_i$ for all i, j. That implies $|\Delta_i : \Delta_{i-1}| > p_i$. Thus, the second paragraph after [20, Defn

8.2] implies that $|Q| > p_i^{2m+1}$; hence, we may disregard this possibility. We now assume that Q is a q-group where $q < p_i$. If $q^\beta < p$, then $|Q| \mid \alpha_i$. Since the order of an element of a finite group divides the order of the group, we have that $\lambda \mid \alpha_i$ where $\lambda = \mathrm{Ord}_Q(\varphi(\kappa))$. Thus, $\varphi(\kappa^{\alpha_i}) = 1$.

Hence, we may assume that Q is a q-group where $q < p_i$ and $p_i < q^\beta < p_i^{2m+1}$. There exists v such that

$$q^{(2m+1)v} < p_i^{2m+1} < q^{(2m+1)(v+1)}.$$

Thus, we may write

$$\beta = vt + r$$

where $t \leq 2m+1$ and $0 \leq r < t$. By construction, $q^{(2m+1)t+r} \leq \alpha_i$, and since $q < p_i$, it follows that

$$q^{(2m+1)t+r} \mid \alpha_i.$$

Subsequently, $\lambda \mid \alpha_i$ and $\varphi(\kappa^{\alpha_i}) = 1$ as desired. □

Corollary 5.3. *Let* $\mathrm{H}_{2m+1}(\mathbb{Z})$ *be the integral Heisenberg group. Then*

$$\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \approx (\log(n))^{2m+1}.$$

Part III. Residual finiteness

6. Proof of Theorem 1.1

Our goal for Theorem 1.1 is to demonstrate that $\mathrm{F}_\Gamma(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}$. Proposition 4.4 implies that we may assume that Γ is torsion-free. We proceed with the proofs of the upper and lower bounds for $\mathrm{F}_\Gamma(n)$ separately since they require different strategies. We start with the upper bound as its proof is simpler.

For the upper bound, our task is to prove for a nonidentity element $\gamma \in \Gamma$ that there exists a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $\varphi(\gamma) \neq 1$ and where

$$|Q| \leq C_0 \left(\log(C_0 \|\gamma\|_S)\right)^{\psi_{\mathrm{RF}}(\Gamma)}$$

for some $C_0 \in \mathbb{N}$. When $\gamma \notin \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, we pass to the quotient given by $\Gamma / \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$ and then appeal to induction on step length. Otherwise, for $\gamma \in \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, we find an admissible quotient of Γ with respect to some primitive central element in which γ has a nontrivial image.

Proposition 6.1. *Let* Γ *be a torsion-free, finitely generated nilpotent group. Then*

$$\mathrm{F}_\Gamma(n) \preceq (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Proof. Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series with a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Suppose $\gamma \in \Gamma$ such that $\|\gamma\|_S \leq n$. Using the Mal'tsev coordinates of γ, we may write

$$\gamma = \prod_{i=1}^{h(\Gamma)} \xi_i^{\alpha_i}.$$

Lemma 2.8 implies that $|\alpha_i| \leq C_1\, n^{c(\Gamma)}$ for some $C_1 \in \mathbb{N}$ for all i. By induction, we will construct a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $\varphi(\gamma) \neq 1$ and where

$$|Q| \leq C_2\, (\|\gamma\|_S)^{\psi_{\mathrm{RF}}(\Gamma)}$$

for some constant $C_2 > 0$.

When Γ is a torsion-free, finitely generated abelian group, [4, Cor 2.3] implies that there exists a surjective homomorphism $\varphi : \Gamma \to Q$ such that $\varphi(\gamma) \neq 1$ and where

$$|Q| \leq C_3\, \log(C_3\, n)$$

for some constant $C_3 > 0$. Thus, we may assume that Γ is nonabelian.

Letting $M = \sqrt[Z(\Gamma)]{\Gamma_{c(\Gamma)}}$, suppose that $\pi_M(\gamma) \neq 1$. Passing to the group Γ/M, the inductive hypothesis implies that there exists a surjective homomorphism $\varphi : \Gamma/M \to Q$ such that $\varphi(\pi_M(\gamma)) \neq 1$ and where

$$\mathrm{D}_\Gamma(\gamma) \leq C_4\, (\log(C_4\, n))^{\psi_{\mathrm{RF}}(\Gamma/M)}$$

for some $C_4 \in \mathbb{N}$. Proposition 3.10 implies that $\psi_{\mathrm{RF}}(\Gamma/M) \leq \psi_{\mathrm{RF}}(\Gamma)$, and thus,

$$\mathrm{D}_\Gamma(\gamma) \leq C_4\, (\log(C_4\, n))^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Otherwise, we may assume that $\gamma \in M$. Thus, we may write

$$\gamma = \prod_{i=1}^{h(\Gamma_{c(\Gamma)})} \xi_i^{\alpha_i},$$

and since $\gamma \neq 1$, there exists a $1 \leq j \leq h(\Gamma_c)$ such that $\alpha_j \neq 0$. The Prime Number Theorem [38, 1.2] implies that there exists a prime p such that $p \nmid |\alpha_j|$ and $p \leq C_5\, \log(C_5\, |\alpha_j|)$ for some $C_5 \in \mathbb{N}$. If Γ/Λ_j is an admissible quotient with respect to ξ_j, then $\pi_{\Lambda_j \cdot \Gamma^p}(\gamma) \neq 1$. Corollary 4.2 implies that

$$|\Gamma/\Lambda_j \cdot \Gamma^p| \leq C_5^{h(\Gamma/\Lambda_j)} \cdot (\log(C_5\, |\alpha_j|))^{h(\Gamma/\Lambda_j)}.$$

Proposition 3.8 implies that $h(\Gamma/\Lambda_j) \leq \psi_{\mathrm{RF}}(\Gamma)$. Thus, we have that

$$\mathrm{D}_\Gamma(\gamma) \leq C_6\, (\log(C_6\, n))^{\psi_{\mathrm{RF}}(\Gamma)}$$

for some $C_6 \in \mathbb{N}$. Hence,

$$\mathrm{F}_\Gamma(n) \preceq (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}. \qquad \square$$

In order to demonstrate that $(\log(n))^{\psi_{\mathrm{RF}}(\Gamma)} \preceq \mathrm{F}_\Gamma(n)$, we require an infinite sequence of elements $\{\gamma_j\} \subseteq \Gamma$ such that

$$C\left(\log(C\,\|\gamma_j\|_S)\right)^{\psi_{\mathrm{RF}}(\Gamma)} \leq \mathrm{D}_\Gamma(\gamma_j)$$

for some $C \in \mathbb{N}$ independent of j. That entails finding elements that are of high complexity with respect to residual finiteness, i.e., nonidentity elements that have relatively short word length in comparison to the order of the minimal finite group required to separate them from the identity.

Proposition 6.2. *Let Γ be torsion-free, finitely generated nilpotent group. Then*

$$(\log(n))^{\psi_{\mathrm{RF}}(\Gamma)} \preceq \mathrm{F}_\Gamma(n).$$

Proof. Suppose that Γ is a torsion-free, finitely generated abelian group. Then [4, Cor 2.3] implies that $\mathrm{F}_\Gamma(n) \approx \log(n)$ which gives our theorem in this case. Thus, we may assume that Γ is not abelian.

Let Γ/Λ be a maximal admissible quotient of Γ. There exists a $g \in (Z(\Gamma))^\bullet$ such that Γ/Λ is an admissible quotient with respect to g. Moreover, there exists a $k \in \mathbb{Z}^\bullet$, $a \in \Gamma_{c(\Gamma)-1}$, and $b \in \Gamma$ such that $g^k = [a,b]$. If g is not primitive, then there exists a primitive element $x_\Lambda \in Z(\Gamma)$ such that $x_\Lambda^s = g$ for some $s \in \mathbb{N}$. In particular, x_Λ is a primitive, central, nontrivial element such that $x_\Lambda^{s\,k} = [a,b]$.

Let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ be a cyclic series with a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ that together satisfy Proposition 3.6 for Λ such that $\xi_1 = x_\Lambda$. Let

$$\alpha_{j,\Lambda,\Delta,\xi} = (\mathrm{lcm}\{1,2,\ldots,p_{j,\Lambda,\Delta,\xi}-1\})^{\psi_{\mathrm{RF}}(\Gamma)+1}$$

where $\{p_{j,\Lambda,\Delta,\xi}\}$ is an enumeration of primes greater than $\mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})$. Letting $\gamma_{j,\Lambda,\Delta,\xi} = \xi_1^{\alpha_{j,\Lambda,\Delta,\xi}}$, we claim that $\{\gamma_{j,\Lambda,\Delta,\xi}\}_j$ is our desired sequence.

Before continuing, we make some remarks. The value $\mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})$ depends on the maximal admissible quotient Γ/Λ of Γ and the cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ that together satisfy Proposition 3.6 for Λ. To be more specific, if Γ/Ω is another maximal admissible quotient of Γ with cyclic series $\{H_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{g_i\}_{i=1}^{h(\Gamma)}$ that together satisfy Proposition 3.6 for Ω, then, in general, $\Gamma/\Lambda \ncong \Gamma/\Omega$, and subsequently, $\mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi}) \neq \mathrm{B}(\Gamma/\Omega,\bar{H},\bar{g})$. Even when we have a fixed maximal admissible quotient, i.e., $\Lambda \cong \Omega$, we still may run into ambiguity in the value $\mathrm{B}(\Gamma/\Lambda,\Delta,\xi)$. To this end, let $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and $\{H_i\}_{i=1}^{h(\Gamma)}$ be distinct cyclic series which satisfy Proposition 3.6 for Λ with respective compatible generating subsets $\{\xi_i\}_{i=1}^{h(\Gamma)}$ and $\{g_i\}_{i=1}^{h(\Gamma)}$. Then $\mathrm{B}(\Gamma/\Lambda,\bar{K},\bar{\xi}) \neq \mathrm{B}(\Gamma/\Lambda,\bar{H},\bar{g})$, in general. Finally, assuming that we have a fixed cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ with two distinct compatible generating subsets $\{\xi_i\}_{i=1}^{h(\Gamma)}$ and $\{g_i\}_{i=1}^{h(\Gamma)}$, we then have that $\mathrm{B}(\Gamma/\Lambda,\bar{H},\bar{\xi}) \neq \mathrm{B}(\Gamma/\Lambda,\bar{H},\bar{g})$, in general. Thus, the sequence of elements $\{\gamma_{j,\Lambda,\Delta,\xi}\}$ depends on the maximal admissible quotient of Γ and the cyclic series and compatible generating subset that together satisfy Proposition 3.6 for Λ. However, we will demonstrate that the given construction will work for any maximal admissible quotient we take and any cyclic series and compatible generating subset that together satisfy Proposition 3.6 for Λ.

We claim for all j that $\mathrm{D}_\Gamma(\gamma_j) \approx (\log(p_{j,\Lambda,\Delta,\xi}))^{\psi_{\mathrm{RF}}(\Gamma)}$. It is evident that $\pi_{\Lambda\cdot\Gamma^{p_{j,\Lambda,\Delta,\xi}}}(\gamma_{j,\Lambda,\Delta,\xi}) \neq 1$ in $\Gamma/\Lambda\cdot\Gamma^{p_{j,\Lambda,\Delta,\xi}}$, and Proposition 4.2 implies that $|\Gamma/\Lambda\cdot\Gamma^{p_{j,\Lambda,\Delta,\xi}}| = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$. To proceed, we show that if given a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $|Q| < (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$, then $\varphi(\gamma) = 1$.

[17, Thm 2.7] implies that we may assume that $|Q| = q^\beta$ where q is a prime. If $\xi_1 \in \ker(\varphi)$, then $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) = 1$. Thus, we may assume that $\varphi(\xi_1) \neq 1$. Proposition 4.5 implies that $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) \neq 1$ if and only if $\pi_{\varphi(\Lambda)}(\varphi(\gamma_{j,\Lambda,\Delta,\xi})) \neq 1$. Thus, we may restrict our attention to surjective homomorphisms that factor through Γ/Λ, i.e., homomorphisms $\varphi : \Gamma \to Q$ where $\varphi(\Lambda) \cong \{1\}$.

Suppose that $q = p_{j,\Lambda,\Delta,\xi}$. If $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) = 1$, then there is nothing to prove. So we may assume that $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) \neq 1$. Since $|Q| \leq (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$, Proposition 4.12 implies that $|Q| = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$ and that if N is a proper quotient of Q with natural projection given by $\rho : Q \to N$, then $\rho(\varphi(\gamma_{j,\Lambda,\Delta,\xi})) = 1$. We have two natural consequences. There are no proper quotients of $\Gamma/\Lambda\cdot\Gamma^{p_{j,\Lambda,\Delta,\xi}}$ where $\varphi(\gamma_{j,\Lambda,\Delta,\xi})$ has nontrivial image. Additionally, if $\varphi : \Gamma \to Q$ is a surjective homomorphism to a finite $p_{j,\Lambda,\Delta,\xi}$-group where $|Q| < (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$, then $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) = 1$. Thus, we may assume that $q \neq p_{j,\Lambda,\Delta,\xi}$.

Suppose that $q > p_{j,\Lambda,\Delta,\xi}$. Since $\widetilde{\varphi} : \Gamma/\Lambda \to Q$ is a surjective homomorphism to a finite q-group where $q > \mathrm{B}(\Gamma/\Lambda, \bar{\Delta}, \bar{\xi})$, Proposition 4.11 implies that $|Q| > (p_{j,\Lambda,\Delta_i,\xi_i})^{\psi_{\mathrm{RF}}(\Gamma)}$. Hence, we may assume that $q < p_{j,\Lambda,\Delta,\xi}$.

Now suppose that Q is a q-group where $|Q| < p_{j,\Lambda,\Delta,\xi}$. By selection, it follows that $|Q|$ divides $\alpha_{j,\Lambda,\Delta,\xi}$. Since the order of an element divides the order of the group, we have that $\mathrm{Ord}_Q(\varphi(\xi_1))$ divides $\alpha_{j,\Lambda,\Delta,\xi}$. In particular, we have that $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) = 1$.

Now suppose that Q is a q-group where $q < p_{j,\Lambda,\Delta,\xi}$ and $q^\beta > p_{j,\Lambda,\Delta,\xi}$. Thus, there exists a $\nu \in \mathbb{N}$ such that

$$q^{\nu\,\psi_{\mathrm{RF}}(\Gamma)} < (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)} < q^{(\nu+1)\;\psi_{\mathrm{RF}}(\Gamma)}.$$

Subsequently, we may write $\beta = \nu t + r$ where $t \leq \psi_{\mathrm{RF}}(\Gamma)$ and $0 \leq r < \nu$. By construction, $q^{vt+r} \leq \alpha_{j,\Lambda,\Delta,\xi}$, and since $q < p_{j,\Lambda,\Delta,\xi}$, it follows that $q^\beta = q^{vt+r} \mid \alpha_{j,\Lambda,\Delta,\xi}$. Given that the order of any element in a finite group divides the order of the group, it follows that $\mathrm{Ord}_Q(\varphi(\xi_1))$ divides $\alpha_{j,\Lambda,\Delta,\xi}$. Thus, $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) = 1$, and therefore,

$$\mathrm{D}_\Gamma(\gamma_{j,\Lambda,\Delta,\xi}) = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Since $\gamma_{j,\Lambda,\Delta,\xi} \in \Gamma_{c(\Gamma)}$, [16, 3.B2] implies that

$$(\|\gamma_{j,\Lambda,\Delta,\xi}\|_S) \approx (|\alpha_{j,\Lambda,\Delta,\xi}|)^{1/c(\Gamma)},$$

and the Prime Number Theorem [38, 1.2] implies that

$$\log(|\alpha_{j,\Lambda,\Delta,\xi}|) \approx p_{j,\Lambda,\Delta,\xi}.$$

Hence,

$$(\log(\|\gamma_{j,\Lambda,\Delta,\xi}\|_S))^{\psi_{\mathrm{RF}}(\Gamma)} \approx (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Thus, $\mathrm{D}_\Gamma(\gamma_{j,\Lambda,\Delta,\xi}) \approx (\log(\|\gamma_{j,\Lambda,\Delta,\xi}\|_S))^{\psi_{\mathrm{RF}}(\Gamma)}$, and subsequently,

$$(\log(n))^{\psi_{\mathrm{RF}}(\Gamma)} \preceq \mathrm{F}_\Gamma(n).$$

□

We now prove the main result of this section.

Theorem 1.1. *Let Γ be an infinite, finitely generated nilpotent group. Then*

$$\mathrm{F}_\Gamma(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Additionally, one can compute $\psi_{\mathrm{RF}}(\Gamma)$ given a basis for $(\Gamma/T(\Gamma))_{c(\Gamma)/T(\Gamma)}$.

Proof. Let Γ be an infinite, finitely generated nilpotent group. Proposition 4.4 implies that

$$\mathrm{F}_\Gamma(n) \approx \mathrm{F}_{\Gamma/T(\Gamma)}(n).$$

Proposition 6.1 and Proposition 6.2 together imply that

$$\mathrm{F}_{\Gamma/T(\Gamma)}(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Thus,

$$\mathrm{F}_\Gamma(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma)}.$$

The last statement in the theorem follows from Proposition 3.8. □

7. Cyclic series, lattices in nilpotent Lie groups, and Theorem 1.3

Let Γ be a torsion-free, finitely generated nilpotent group. The main task of this section is to demonstrate that the value $h(\Gamma/\Lambda)$ is a well-defined invariant of the Mal'tsev completion of Γ. Thus, we need to establish some properties of cocompact lattices in connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie groups. We start with the following lemma that relates the Hirsch lengths of centers of cocompact lattices within the same connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group.

Lemma 7.1. *Let G be a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group with two cocompact lattices Γ_1 and Γ_2. Then*

$$\dim(Z(G)) = h(Z(\Gamma_1)) = h(Z(\Gamma_2)).$$

Proof. This proof is a straightforward application of [13, Lem 1.2.5]. □

We now introduce the notion of one parameter families of group elements of a Lie group.

Definition 7.2. Let G be a connected, simply connected Lie group. We call a map $f : \mathbb{R} \to G$ an *one parameter family of group elements of* G if f is an injective group homomorphism from the real line with addition.

Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. We let G be the Mal'tsev completion with Lie algebra $\mathfrak{g}$ and induced basis for $\mathfrak{g}$ given by $\{\text{Log}(\xi_i)\}_{i=1}^{h(\Gamma)}$. Via the exponential map and [13, Lem 1.2.5], the maps given by

$$f_{i,\Gamma,\Delta,\xi}(t) = \exp(t \ \text{Log}(\xi_i))$$

are one parameter families of group elements. The discussion below [13, Thm 1.2.4 Pg 9] implies that we may uniquely write each $g \in G$ as

$$g = \prod_{i=1}^{h(\Gamma)} f_{i,\Gamma,\Delta,\xi}(t_i)$$

where $t_i \in \mathbb{R}$ for all i.

Definition 7.3. We say the one parameter families of group elements $f_{i,\Gamma,\Delta,\xi}$ are *associated* to the group Γ, cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$, and compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$.

We characterize when a discrete subgroup of an connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group is a cocompact lattice based on how it intersects a collection of one parameter families of group elements.

Proposition 7.4. *Let G be a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group, and suppose that Γ is a discrete subgroup of G. Suppose there exists a collection of one parameter families of group elements of G, written as $f_i : \mathbb{R} \to G$ for $1 \leq i \leq \dim(G)$, such that G is diffeomorphic to $\prod_{i=1}^{\dim(G)} f_i(\mathbb{R})$. Then Γ is a cocompact lattice in G if and only $\Gamma \cap f_i(\mathbb{R}) \cong \mathbb{Z}$ for all i.*

Proof. Let $\rho : G \to G/\Gamma$ be the natural projection onto the space of cosets. Suppose that there exists an i_0 such that $f_{i_0}(\mathbb{R}) \cap \Gamma \not\cong \mathbb{Z}$. Since Γ is discrete in G, we have that $\Gamma \cap f_{i_0}(\mathbb{R})$ is a discrete subset of $f_{i_0}(\mathbb{R})$. Given that $\Gamma \cap f_{i_0}(\mathbb{R})$ is discrete and not infinite cyclic, we have that $\Gamma \cap f_{i_0}(\mathbb{R}) \cong \{1\}$.

We claim that each element of the sequence $\{f_{i_0}(t)\}_{t \in \mathbb{N}}$ projects to a unique element of G/Γ. To this end, suppose that there exists integers t_0 and t_1 such that $\rho(f_{i_0}(t_0)) = \rho(f_{i_0}(t_1))$. That implies that there exists an element $g \in \Gamma$ such that $f_{i_0}(t_0 - t_1) = g$. In particular, $g \in f_{i_0}(\mathbb{R}) \cap \Gamma$, and thus, $g = 1$. Hence, $f_{i_0}(t_0) = f_{i_0}(t_1)$ which gives our claim.

Thus, $\{\rho(f_{i_0}(t))\}_{t \in \mathbb{N}}$ is an infinite sequence in G/Γ with no convergent subsequence. Hence, Γ is not a cocompact lattice of G

Now suppose that $f_i(\mathbb{R}) \cap \Gamma \cong \mathbb{Z}$ for all i. That implies for each $i \in \{1, \ldots, h(\Gamma)\}$ that there exists a $t_i > 0$ such that

$$\Gamma \cap f_i(\mathbb{R}) \cong \{f_i(n \, t_i) \mid n \in \mathbb{Z}\}.$$

Let

$$E = \prod_{i=1}^{\dim(G)} f_i([0, t_i]).$$

We claim that E is compact and that $\rho(E) \cong G/\Gamma$.

Let $f : \mathbb{R}^{\dim(G)} \to G$ be the continuous map given by

$$f\big((a_i, \ldots, a_{\dim(G)})\big) = \prod_{i=1}^{\dim(G)} f_i(a_i).$$

Since $\prod_{i=1}^{\dim(G)}[0, t_i]$ is a closed and bounded subset of $\mathbb{R}^{\dim(G)}$, the Heine–Borel theorem implies that $\prod_{i=1}^{\dim(G)}[0, t_i]$ is compact. Since f is continuous, E is compact.

We now claim that each coset of Γ in G has a representative in E. Let $g = \prod_{i=1}^{\dim(G)} f_i(\ell_i)$ where $\ell_i \in \mathbb{R}$ for each $i \in \{1, \ldots, \dim(G)\}$. For each i, there exists a $s_i \in \mathbb{Z}$ such that $s_i\, t_i \leq \ell_i \leq (s_i + 1)\, t_i$. Let $k_i = \ell_i - s_i\, t_i$ and write $h \in E$ to be given by $h = \prod_{i=1}^{\dim(G)} f(k_i)$. By construction, $\rho(h) = \rho(g)$, and subsequently, $\rho(g) \in \rho(E)$. Thus, $\rho(E) = \rho(G)$. Since G/Γ is the image of a compact set under a continuous map, G/Γ is compact. [33, Thm 2.1] implies that Γ is a cocompact lattice of G. □

These next two propositions give some structural information needed about the Mal'tsev completion of a torsion-free, finitely generated nilpotent group and some structural information of admissible quotients with respect to some primitive, central, nontrivial element.

Proposition 7.5. *Let Γ be a torsion-free, finitely generated nilpotent group. Let $\gamma \in (Z(\Gamma))^{\bullet}$ be a primitive element, and let Γ/Λ be an admissible quotient with respect to γ. Suppose that G is the Mal'tsev completion of Γ, and let H be the Mal'tsev completion of Λ. Then H is isomorphic to a closed, connected, normal subgroup of G.*

Proof. Proposition 3.6 there exists a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ satisfying the following. There exists a subset $\{\xi_{i_s}\}_{s=1}^{h(\Lambda)} \subseteq \{\xi_i\}_{i=1}^{h(\Gamma)}$ such that if $W_s = \langle \xi_{i_t} \rangle_{t=1}^{s}$, then $\{W_s\}_{s=1}^{h(\Lambda)}$ is a cyclic series for Λ with compatible generating subset $\{\xi_{i_s}\}_{s=1}^{h(\Lambda)}$ where $\xi_1 = \gamma$. Let $\{f_{i,\Gamma,\Delta,\xi}\}_{i=1}^{h(\Gamma)}$ be the one parameter families of group elements of G associated to the torsion-free, finitely generated nilpotent group Γ, cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$.

[13, Thm 1.2.3] implies that we may view H as a connected subgroup of G. We proceed by induction on $h(\Gamma)$ to demonstrate that H is a closed and normal subgroup of G. If $h(\Gamma) = 1$, then $\Gamma = \mathbb{Z}$. It follows that G is isomorphic to $\mathbb{R}$ and that $H \cong \{1\}$. Now our claim is evident.

Now suppose that $h(\Gamma) > 1$. If $h(Z(\Gamma)) = 1$, then we may take $\Lambda = \{1\}$ which implies that $H \cong \{1\}$. Thus, our claims are evident. Now suppose that $h(Z(\Gamma)) > 1$. Let $\Omega = \langle \xi_i \rangle_{i=1}^{h(Z(\Gamma))}$, and let K be the Mal'tsev completion of Ω. [13, Lem 1.2.5] implies that $K \leq Z(G)$. Thus, K is a closed, connected, normal subgroup of G.

We will demonstrate H/K is Mal'tsev completion of $\pi_K(\Lambda)$. We may write $H = \prod_{s=1}^{h(\Lambda)} f_{i_s,\Gamma,\Delta,\xi}(\mathbb{R})$. Since Λ is a cocompact lattice of H, Proposition 7.4 implies that $\Lambda \cap f_{i_s,\Gamma,\Delta,\xi}(\mathbb{R}) \cong \mathbb{Z}$ for all $1 \leq s \leq h(\Lambda)$. By Proposition 7.4 again, we have that $K \cap \Lambda$ is a cocompact lattice of K. [12, Prop 5.1.4] implies that $\pi_K(\Lambda)$ is a cocompact lattice in H/K.

Observe that $\pi_K(\Lambda) \cong \Lambda/\Omega$. We have that Λ/Ω satisfies Proposition 3.1 for $\pi_K(\xi_1)$. Thus, the inductive hypothesis implies that H/K is a closed, normal subgroup of G/K. Since H isomorphic to the pullback of a closed, normal subgroup of G/K, H is a closed, normal subgroup of G. □

Suppose that G is the Mal'tsev completion of Γ, and let Γ/Λ be an admissible quotient with respect to a primitive, central, nontrivial element of Γ. If H is a Mal'tsev completion of Λ, then H intersects any cocompact lattice as a cocompact lattice.

Proposition 7.6. *Let Γ be a torsion-free, finitely generated nilpotent group. Let $\gamma \in (Z(\Gamma))^\bullet$ be a primitive element, and let Γ/Λ be an admissible quotient with respect to γ, G be the Mal'tsev completion of Γ, and H be the Mal'tsev completion of Λ. If $\Omega \leq G$ is another cocompact lattice of G, then $\Omega \cap H$ is a cocompact lattice of H.*

Proof. Proposition 3.6 implies that there exists a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ satisfying the following. There exists a subset $\{\xi_{i_j}\}_{j=1}^{h(\Lambda)}$ such that the groups $\{W_i\}_{i=1}^{h(\Lambda)}$ where $W_i \cong \langle \xi_{i_j} \rangle_{j=1}^{i}$ form a cyclic series for Λ with a compatible generating subset $\{\xi_{i_j}\}_{j=1}^{h(\Lambda)}$. Let $\{f_{i,\Gamma,\Delta,\xi}\}_{i=1}^{h(\Gamma)}$ be the associated one parameter families of group elements of the Mal'tsev completion G of Γ. It follows that G is diffeomorphic to $\prod_{i=1}^{h(\Gamma)} f_{i,\Gamma,\Delta,\xi}(\mathbb{R})$. By construction, $H \cong \prod_{j=1}^{h(\Lambda)} f_{i_j,\Gamma,\Delta,\xi}(\mathbb{R})$. Proposition 7.4 implies that $\Omega \cap f_{i,\Gamma,\Delta,\xi}(\mathbb{R}) \cong \mathbb{Z}$ for all i. In particular, $\Omega \cap f_{i_j,\Gamma,\Delta,\xi}(\mathbb{R}) \cong \mathbb{Z}$ for all j. Proposition 7.4 implies that $\Omega \cap H$ is a cocompact lattice in H as desired. □

The following lemma demonstrates that you can select a cyclic series and a compatible generating subset for a cocompact lattice in a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group by intersecting the lattice with a collection of one parameter families of group elements.

Lemma 7.7. *Let G be a connected, simply connected, $\mathbb{Q}$-defined nilpotent Lie group with a cocompact lattice Γ. Let f_i be a collection of one parameter families of elements of G such that G is diffeomorphic to $\prod_{i=1}^{\dim(G)} f_i(\mathbb{R})$. Let*

$\langle \xi_i \rangle \cong f_i(\mathbb{R}) \cap \Gamma$. *Then the groups given by* $\Delta_s = \langle \xi_t \rangle_{t=1}^{s}$ *form a cyclic series for* Γ *with a compatible generating subset* $\{\xi_i\}_{i=1}^{h(\Gamma)}$.

Proof. We proceed by induction on the $\dim(G)$. Since the statement is clear for the base case, we may assume that $\dim(G) > 1$. If we let

$$H \cong \prod_{i=1}^{\dim(G)-1} f_i(\mathbb{R}),$$

then Proposition 7.4 implies that $\Gamma \cap H$ is a cocompact lattice in H. The inductive hypothesis implies that the elements ξ_i given by

$$\langle \xi_i \rangle \cong f_i(\mathbb{R}) \cap \Gamma$$

satisfy the following. The groups given by $\Delta_s = \langle \xi_i \rangle_{i=1}^{s}$, $1 \leq s \leq \dim(G)-1$, form a cyclic series for $\Gamma \cap H$ with a compatible generating subset $\{\xi_i\}_{i=1}^{\dim(G)-1}$. Since Γ is a cocompact lattice in G, Proposition 7.4 implies that

$$f_{\dim(G)}(\mathbb{R}) \cap \Gamma \cong \mathbb{Z}.$$

Letting $\Delta_{\dim(G)} = \left\langle \Delta_{\dim(G)-1}, \xi_{\dim(G)} \right\rangle$, we have that the groups given by $\{\Delta_i\}_{i=1}^{\dim(G)}$ form a cyclic series for Γ with a compatible generating subset $\{\xi_i\}_{i=1}^{\dim(G)}$. □

Let Γ be a torsion-free, finitely generated nilpotent group. We now demonstrate that the value $\psi_{\mathrm{RF}}(\Gamma)$ is a well-defined invariant of the Mal'tsev completion of Γ.

Proposition 7.8. *Let* G *be a connected, simply connected,* $\mathbb{Q}$*-defined nilpotent Lie group, and suppose that* Γ_1 *and* Γ_2 *are two cocompact lattices of* G*. Then* $\psi_{\mathrm{RF}}(\Gamma_1) = \psi_{\mathrm{RF}}(\Gamma_2)$.

Proof. If $h(Z(\Gamma_1)) = 1$, then Proposition 7.1 implies that $h(Z(\Gamma_2)) = 1$. It then follows from the definition of $\psi_{\mathrm{RF}}(\Gamma_1)$ and $\psi_{\mathrm{RF}}(\Gamma_2)$ that

$$\psi_{\mathrm{RF}}(\Gamma_1) = h(\Gamma) = \psi_{\mathrm{RF}}(\Gamma_2).$$

Therefore, we may assume that $h(Z(\Gamma_1)), h(Z(\Gamma_2)) \geq 2$. In this case, we demonstrate the equality by showing that $\psi_{\mathrm{RF}}(\Gamma_1) \leq \psi_{\mathrm{RF}}(\Gamma_2)$ and $\psi_{\mathrm{RF}}(\Gamma_2) \leq \psi_{\mathrm{RF}}(\Gamma_1)$.

Let G be the Mal'tsev completion of Γ_1. Let $\{\Delta_i\}_{i=1}^{h(\Gamma_1)}$ be a cyclic series for Γ_1 with a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma_1)}$, and let $\{f_{i,\Gamma_1,\Delta,\xi}\}_{i=1}^{h(\Gamma_1)}$ be the associated one parameter families of group elements. We have that G is diffeomorphic to $\prod_{i=1}^{h(\Gamma_1)} f_{i,\Gamma_1,\Delta,\xi}(\mathbb{R})$. Let

$$\{\eta_i\}_{i=1}^{h(\Gamma_2)} \subseteq \Gamma_2 \quad \text{such that} \quad \langle \eta_i \rangle \cong \Gamma_2 \cap f_i(\mathbb{R}).$$

If we let $W_i \cong \langle \eta_j \rangle_{j=1}^{i}$, then Proposition 7.7 implies that $\{W_i\}_{i=1}^{h(\Gamma_2)}$ is a cyclic series for Γ_2 with a compatible generating subset $\{\eta_i\}_{i=1}^{h(\Gamma_2)}$. Let ξ_i be a central element of the compatible generating subset of Γ_1, and let Γ_1/Λ be an admissible quotient with respect to ξ_i. Let H be the Mal'tsev completion of Λ. Since $\pi_\Lambda(\xi_i) \cong Z(\Gamma_1/H)$, it is evident that $\langle \pi_H(\eta_i) \rangle \cong Z(\Gamma_2/H)$. In particular, $\pi_H(\eta_i) \neq 1$. Proposition 7.6 implies that $H \cap \Omega$ is a compact lattice of H and Proposition 7.5 implies that H is a closed, connected, normal subgroup of G. Thus, [12, Prop 5.1.4] implies that $\pi_H(\Omega)$ is a compact lattice in G/H. Proposition 7.1 implies that $h(\Gamma_2/\Lambda) = h(\pi_H(\Gamma_2))$; thus, it follows that $\pi_H(\Gamma_2)$ satisfies the conditions of Proposition 3.1 for η_i. If we let Γ_2/Ω be an admissible quotient with respect to η_i, it follows that

$$h(\Gamma/\Omega) \leq \pi_H(\Gamma_2) \leq h(\Gamma/\Lambda),$$

By Proposition 3.8, $h(\Gamma_2/\Omega) \leq \psi_{\mathrm{RF}}(\Gamma_1)$. [13, Lem 1.2.5] implies that $\eta_i \in Z(\Gamma_2)$, and thus, the above inequality holds for each central element of the compatible generating subset of Γ_2 in $Z(\Gamma_2)$. Therefore, Proposition 3.8 implies that $\psi_{\mathrm{RF}}(\Gamma_2) \leq \psi_{\mathrm{RF}}(\Gamma_1)$. By interchanging Γ_1 and Γ_2, we have that $\psi_{\mathrm{RF}}(\Gamma_1) \leq \psi_{\mathrm{RF}}(\Gamma_2)$. □

We now come to the main result of this section.

Theorem 1.3. *Suppose that Γ_1 and Γ_2 are two infinite, finitely generated nilpotent groups such that $\Gamma_1/T(\Gamma_1)$ and $\Gamma_2/T(\Gamma_2)$ have isomorphic Mal'tsev completions. Then $\mathrm{F}_{\Gamma_1}(n) \approx \mathrm{F}_{\Gamma_2}(n)$.*

Proof. Suppose that Γ_1 and Γ_2 are two infinite, finitely generated nilpotent groups such that $\Gamma_1/T(\Gamma_1)$ and $\Gamma_2/T(\Gamma_2)$ have isomorphic Mal'tsev completions. Proposition 4.4 implies that

$$\mathrm{F}_{\Gamma_1}(n) \approx \mathrm{F}_{\Gamma_1/T(\Gamma_1)}(n) \quad \text{and} \quad \mathrm{F}_{\Gamma_2}(n) \approx \mathrm{F}_{\Gamma_2/T(\Gamma_2)}(n).$$

We also have that Theorem 1.1 implies

$$\mathrm{F}_{\Gamma_1/T(\Gamma_1)}(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma_1/T(\Gamma_1))},$$
$$\mathrm{F}_{\Gamma_2/T(\Gamma_2)}(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\Gamma_2/T(\Gamma_2))}.$$

Proposition 7.8 implies that $\psi_{\mathrm{RF}}(\Gamma_1/T(\Gamma_1)) = \psi_{\mathrm{RF}}(\Gamma_2/T(\Gamma_2))$. Thus,

$$\mathrm{F}_{\Gamma_1}(n) \approx \mathrm{F}_{\Gamma_2}(n).$$ □

8. Some examples and the proof of Theorem 1.5

8.1. Free nilpotent groups and Theorem 1.5(i).

Definition 8.1. Let $\mathrm{F}(X)$ be the free group of rank m generated by X. We define $\mathrm{N}(X, c, m) = \mathrm{F}(X)/(\mathrm{F}(X))_{c+1}$ as *the free nilpotent group of step size c and rank m on the set X.*

Following [22, Sec 2.7], we construct a cyclic series for $\mathrm{N}(X, c, m)$ and a compatible generating subset using iterated commutators in the set X.

Definition 8.2. We formally call elements of X *basic commutators of weight* 1 *of* $\mathrm{N}(X,c,m)$, and we choose an arbitrary linear order for weight 1 basic commutators. If γ_1 and γ_2 are basic commutators of weight i_1 and i_2, respectively, then $[\gamma_1,\gamma_2]$ is a *basic commutator of weight* i_1+i_2 *of* $\mathrm{N}(X,c,m)$ if $\gamma_1>\gamma_2$. If, in addition, we can write $\gamma_1=[\gamma_{1,1},\gamma_{1,2}]$ where $\gamma_{1,1}$ and $\gamma_{1,2}$ are basic commutators, then we also assume that $\gamma_{1,2}\leq\gamma_2$.

Basic commutators of higher weight are greater with respect to the linear order than basic commutators of lower weight. Moreover, we choose an arbitrary linear order for commutators of the same weight.

For $x_{i_0}\in X$, we say that a 1-fold commutator γ contains x_{i_0} if $\gamma=x_{i_0}$. Inductively, we say that a j-fold commutator $[\gamma_1,\gamma_2]$ contains x_{i_0} if either γ_1 contains x_{i_0} or γ_2 contains x_{i_0}.

Note that any basic commutator of weight greater or equal to 2 must contain two distinct commutators of weight 1 but not necessarily more than 2. Additionally, if γ is a basic commutator of weight k, then γ can contain at most k distinct basic commutators of weight 1.

It is well known that the number of basic commutators of $\mathrm{N}(X,c,m)$ is equal to the Hirsch length of $\mathrm{N}(X,c,m)$. Letting μ be the Möbius function, we may write

$$h(\mathrm{N}(X,c,m))=\sum_{r=1}^{c}\left(\frac{1}{r}\sum_{d|r}\mu(d)m^{\frac{r}{d}}\right).$$

We label the basic commutators as $\{\xi_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ with respect to the given linear order.

Definition 8.3. One can see that the subgroup series $\{\Delta_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ where $\Delta_i=\langle\xi_t\rangle_{t=1}^{i}$ is a cyclic series for $\mathrm{N}(X,c,m)$, and [22, Cor 2.7.3] implies that $\{\xi_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ is a compatible generating subset. We call $\{\Delta_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ the *cyclic series of basic commutators for* $\mathrm{N}(X,c,m)$ and $\{\xi_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ *the compatible generating subset of basic commutators for* $\mathrm{N}(X,c,m)$.

Proposition 8.4. *Let* $\mathrm{N}(X,c,m)$ *be the free nilpotent group of step size* c *and rank* m *on the set* $X=\{x_i\}_{i=1}^{m}$. *Let* γ *be a basic commutator of weight* c *in the set* X *that contains only elements of* $Y\subsetneq X$ *where* $Y\neq\emptyset$. *There exists a normal subgroup* Ω *such that* $\mathrm{N}(X,c,m)/\Omega$ *is torsion-free where* $\langle\pi_\Omega(\gamma)\rangle\cong Z(\mathrm{N}(X,c,m)/\Omega)$. *Additionally, if* η *is a* j*-fold commutator that contains elements of* $X\backslash Y$, *then* $\pi_\Omega(\eta)=1$.

Proof. Let $\{\Delta_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ be the cyclic series of basic commutators and $\{\xi_i\}_{i=1}^{h(\mathrm{N}(X,c,m))}$ be the compatible generating subset of basic commutators. By assumption, there exists an $i_0\in\{1,\ldots,h(Z(\mathrm{N}(X,c,m)))\}$ such that $\xi_{i_0}=\gamma$.

We will demonstrate that there exists a normal descending series $\{K_t\}_{t=1}^{c}$ such that $\mathrm{N}(X,c,m)/K_t$ is torsion-free for each t, $\pi_{K_t}(\xi_1)\neq 1$ for each t, and if η is a i-fold commutator that contains only elements of $X\backslash Y$ where $i \geq t$, then $\pi_{K_t}(\eta) = 1$. We will also have that K_t is generated by basic commutators of weight greater than or equal to t, and finally, we will have that $\langle \pi_{K_1}(\xi_{i_0}) \rangle \cong Z(\mathrm{N}(X, c, m)/K_1)$. We proceed by induction on t.

Consider the subgroup given by $K_c = \langle \xi_i \rangle_{i=1, i \neq i_0}^{h(Z(\mathrm{N}(X,c,m)))}$. Observe that if η is a c-fold commutator such that η contains only elements of $X\backslash Y$, then it follows by selection that $\pi_{K_c}(\eta) = 1$. Thus, we have the base case.

Thus, we may assume that the subgroup K_t has been constructed for $t < c$, and let η be a $(t-1)$-fold commutator bracket that contains elements of $X\backslash Y$. It then follows that $[\eta, x_i]$ contains elements of $X\backslash Y$. Thus, $\pi_{K_t}([\eta, x_i]) = 1$ by assumption. Since that is true for all $1 \leq i \leq m$, we have that $\pi_{K_t}(\eta) \in Z(\mathrm{N}(X, c, m)/K_t)$. Let W be the set of basic commutator brackets ξ_i such that $\pi_{K_t}(\xi_i)$ is central and where $\pi_{K_t}(\xi_i) \neq \pi_{K_t}(\xi_{i_0})$. By construction, $\pi_{K_t}(\xi_1) \notin \langle \pi_{K_t}(W) \rangle$ and if η is a ℓ-fold commutator bracket that contains elements of $X\backslash Y$ where $\ell \geq t-1$, then $\pi_{K_t}(\eta) \in \langle \pi_{K_t}(W) \rangle$. We set $K_{t-1} \cong \langle K_t, W \rangle$, and suppose that η is a ℓ-fold commutator that contains elements of X/Y and where $\ell \geq t-1$. By construction, we have that $\pi_{K_{t-1}}(\eta) = 1$. Since $K_{t-1} \cong \pi_{K_t}^{-1}(\langle \pi_{K_t}(W) \rangle)$, we have that K_{t-1} is a normal subgroup of $\mathrm{N}(X, c, m)$ and $K_t \leq K_{t-1}$. Finally, it is evident that $\mathrm{N}(X, c, m)/K_t$ is torsion-free. Hence, induction gives the construction of K_t for all i.

We now demonstrate that $Z(\mathrm{N}(X, c, m)/K_1) \cong \langle \pi_{K_1}(\xi_1) \rangle$ by first showing that $\xi_1 \notin K_i$ for all i. We proceed by induction, and note that the base case follows from the definition of K_c. Now assume that $\xi_1 \notin K_t$ for the inductive hypothesis. By definition, $K_{t-1} = \langle K_t, W \rangle$ where W is the set of basic commutator brackets ξ_i such that $\pi_{K_t}(\xi_i)$ is central in $\mathrm{N}(X, c, m)/K_t$ and where $\pi_{K_t}(\xi_i) \neq \pi_{K_t}(\xi_{i_0})$. By the construction of the upper central series, we have that the subset $\{\pi_{K_t}(\xi_t), \pi_{K_t}(W)\}$ is a free basis of $Z(\mathrm{N}(X, c, m)/K_t)$. Suppose for a contradiction that $\xi_{i_0} \in K_{t-1}$. Given that $K_t \trianglelefteq K_{t-1}$, we have that $\xi_{i_0} \in W \bmod K_t$ which contradicts the fact that $\{\pi_{K_t}(\xi_t), \pi_{K_t}(W)\}$ is a free basis of $Z(\mathrm{N}(X, c, m)/K_t)$. Thus, induction implies that $\xi_{i_0} \notin K_1$. In particular, $\pi_{K_1}(\xi_{i_0}) \subseteq (Z(\mathrm{N}(X, c, m)/K_1))^{\bullet}$. The construction of K_1 mirrors the techniques used in the proof of Proposition 3.1. Thus, $h(Z(\mathrm{N}(X, c, m)/K_1)) = 1$, and therefore, $Z(\mathrm{N}(X, c, m)/K_1) \cong \langle \pi_{K_1}(\xi_{i_0}) \rangle$. By taking $\Omega = K_1$, we have our proposition. $\square$

We now come to the main result of this subsection.

Theorem 1.5(i). *For each $c \in \mathbb{N}$, there exists a $m(c) \in \mathbb{N}$ satisfying the following. For each $\ell \in \mathbb{N}$, there exists an irreducible, torsion-free, finitely generated nilpotent group Γ of step length c and $h(\Gamma) \geq \ell$ such that*

$$\mathrm{F}_\Gamma(n) \preceq (\log(n))^{m(c)}.$$

Proof. Let $c \geq 1$, $\ell \geq 2$, and $X_\ell = \{x_i\}_{i=1}^{\ell}$. Let $\mathrm{N}(X_\ell, c, \ell)$ to be the free nilpotent group of step size c and rank ℓ on the set X_ℓ. Theorem 1.1 implies that there exists a $\psi_{\mathrm{RF}}(\mathrm{N}(X_\ell, c, \ell)) \in \mathbb{N}$ such that

$$\mathrm{F}_{\mathrm{N}(X_\ell,c,\ell)}(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(\mathrm{N}(X_\ell,c,\ell))}.$$

Before we continue, we make an observation. One can see that we may take the groups given by $\mathrm{N}(X_c, c, c) \times \mathbb{Z}^\ell$ to satisfy a weak version of our theorem. However, $\mathrm{N}(X_c, c, c) \times \mathbb{Z}^\ell$ is not irreducible. Thus, we will find an irreducible, torsion-free quotient of $N(X_\ell, c, \ell)$ which achieves the desired end.

We will demonstrate that

$$(\log(n))^{\psi_{\mathrm{RF}}(\mathrm{N}(X_\ell,c,\ell))} \preceq (\log(n))^{\psi_{\mathrm{RF}}(\mathrm{N}(X_c,c,c))}$$

for each $\ell > c$, and since $\mathrm{N}(X_\ell, c, \ell)$ is a nilpotent group of step size c and Hirsch length greater than ℓ, we will have our desired result.

We let $\{\Delta_i\}_{i=1}^{h(\mathrm{N}(X_\ell,c,\ell))}$ be the cyclic series of basic commutators and $\{\xi_i\}_{i=1}^{h(\mathrm{N}(X_\ell,c,\ell))}$ be the compatible generating subset of basic commutators for $\mathrm{N}(X_\ell, c, \ell)$. For each $\xi_i \in Z(\mathrm{N}(X_\ell, c, \ell))$, let $\mathrm{N}(X_\ell, c, \ell)/\Lambda_i$ be an admissible quotient with respect to ξ_i. Proposition 3.8 implies that there exists an $i_0 \in \{1, \ldots, h(Z(\mathrm{N}(X_\ell, c, \ell)))\}$ such that $h(\Gamma/\Lambda_{i_0}) = \psi_{\mathrm{RF}}(\mathrm{N}(X_\ell, c, \ell))$.

For each $\xi_i \in Z(\mathrm{N}(X_\ell, c, \ell))$, there exists a subset $Y_i \subseteq X$ such that ξ_i is a basic commutator of weight c that contains only elements of Y_i. Proposition 8.4 implies that there exists a subgroup Ω_i such that $\mathrm{N}(X_\ell, c, \ell)/\Omega_i$ satisfies Proposition 3.1 with respect to ξ_i. Moreover, elements of $X\backslash Y_i$ are contained in Ω_i.

There is a natural surjective homomorphism $\rho_i : \mathrm{N}(X_\ell, c, \ell) \to \mathrm{N}(Y_i, c, |Y_i|)$ given by sending elements of $X\backslash Y_i$ to the identity. Thus, we have an induced homomorphism $\varphi : \mathrm{N}(Y_i, c, |Y_i|) \to \mathrm{N}(X_\ell, c, \ell)/\Omega_i$ such that $\pi_{\Omega_i} = \varphi \circ \rho_i$. In particular, $\mathrm{N}(X_\ell, c, \ell)/\Omega_i \cong \mathrm{N}(Y_i, c, |Y_i|)/\rho_i(\Omega_i)$. Therefore, $\mathrm{N}(X_\ell, c, \ell)/\Omega_i$ satisfies the conditions of Proposition 3.1 for $\rho_i(\xi_i)$. Proposition 3.8 implies that

$$h(\mathrm{N}(X_\ell, c, \ell)/\Lambda_i) \leq \psi_{\mathrm{RF}}(\mathrm{N}(Y_i, c, |Y_i|)).$$

Since $\mathrm{N}(X_\ell, c, \ell)$ has step size c, we have that $|Y_i| \leq c$ for any $\xi_i \in Z(\mathrm{N}(X_\ell, c, \ell))$. Additionally, we have that $\mathrm{N}(Y_i, c, |Y_i|) \cong \mathrm{N}(X_j, c, j)$ when $|Y_i| = j$. In particular,

$$\psi_{\mathrm{RF}}(\mathrm{N}(Y_i, c, |Y_i|)) = \psi_{\mathrm{RF}}(\mathrm{N}(X_j, c, j)).$$

By setting

$$m(c) = \max\{\psi_{\mathrm{RF}}(\mathrm{N}(X_j, c, j)) \mid 1 \leq j \leq c\},$$

Proposition 3.8 implies that $\mathrm{F}_{\mathrm{N}(X_\ell,c,\ell)}(n) \preceq (\log(n))^{m(c)}$. □

8.2. Central products and applications. The examples we contruct for Theorem 1.5(ii), (iii) and (iv) arise as iterated central products of torsion-free, finitely generated nilpotent groups whose centers have Hirsch length 1. In the given context, Corollary 1.2 allows us to compute the precise residually finiteness function in terms of the Hirsch length of the torsion-free, finitely generated nilpotent groups of whom we take the central product of.

Definition 8.5. Let Γ and Δ be finitely generated groups, and let

$$\theta : Z(\Gamma) \to Z(\Delta)$$

be an isomorphism. We define the *central product of* Γ *and* Δ *with respect to* θ as

$$\Gamma \circ_\theta \Delta = (\Gamma \times \Delta)/K \quad \text{where} \quad K = \{(z, \theta(z)^{-1}) \mid z \in Z(\Gamma)\}.$$

We define the central product of the groups $\{\Gamma_i\}_{i=1}^{\ell}$ with respect to the automorphisms $\theta_i : Z(G_i) \to Z(G_{i+1})$ for $1 \leq i \leq \ell - 1$ inductively. Assuming that $(\Gamma_i \circ_{\theta_i})_{i=1}^{\ell}$ has already been defined, we define $(\Gamma_i \circ_{\theta_i})_{i=1}^{\ell}$ as the central product of $(\Gamma_i \circ_{\theta_i})_{i=1}^{\ell-1}$ and Γ_ℓ with respect to the induced isomorphism $\bar{\theta}_{\ell-1} : Z((\Gamma_i \circ_{\theta_i})_{i=1}^{\ell-1}) \to Z(\Gamma_\ell)$. When $\Gamma = \Gamma_i$ and $\theta = \theta_i$ for all i, we write the central product as $(\Gamma \circ_\theta)_{i=1}^{\ell}$.

Suppose that $\Gamma \circ_\theta \Delta$ is a central product of two nilpotent groups. Since products and quotients of nilpotent groups are nilpotent, it follows that $\Gamma \circ_\theta \Delta$ is a nilpotent group. However, the isomorphism type of $\Gamma \circ_\theta \Delta$ is dependent on θ.

Proposition 8.6. *Let* $\{\Gamma_i\}_{i=1}^{\ell}$ *be a collection of torsion-free, finitely generated nilpotent groups where* $h(Z(\Gamma_i)) = 1$ *for all* i. *Let* $Z(\Gamma_i) = \langle z_i \rangle$, *and let* $\theta_i : Z(\Gamma_i) \to Z(\Gamma_{i+1})$ *be the isomorphism given by* $\theta(z_i) = z_{i+1}$ *for* $1 \leq i \leq \ell - 1$. *Then*

$$h((\Gamma_i \circ_{\theta_i})_{i=1}^{\ell}) = \sum_{i=1}^{\ell} h(\Gamma_i) - \ell + 1 \quad \textit{and} \quad h(Z(\Gamma_i \circ_{\theta_i})_{i-1}^{\ell})) = 1.$$

Proof. We may assume that $\ell = 2$. First note that if Γ is a torsion-free, finitely generated nilpotent group with a normal subgroup $\Delta \trianglelefteq \Gamma$ such that Γ/Δ is torsion-free, then $h(\Gamma) = h(\Delta) + h(\Gamma/\Delta)$. Observe that

$$\Gamma_1 \circ_\theta \Gamma_2 / Z(\Gamma_1 \circ_\theta \Gamma_2) \cong \Gamma_1/Z(\Gamma_1) \times \Gamma_2/Z(\Gamma_2).$$

It is evident that $h(Z(\Gamma_1 \circ_\theta \Gamma_2)) = 1$, and thus, we may write

$$h(\Gamma_1/Z(\Gamma_1)) + h(\Gamma_2/Z(\Gamma_2)) + 1 = h(\Gamma_1 \circ_\theta \Gamma_2).$$

Therefore,

$$h(\Gamma_1 \circ_\theta \Gamma_2) = h(\Gamma_1) - 1 + h(\Gamma_2) - 1 + 1 = h(\Gamma_1) + h(\Gamma_2) - 1. \quad \square$$

Definition 8.7. For $\ell \geq 3$, we define Λ_ℓ to be the torsion-free, finitely generated nilpotent group generated by the set $S_\ell = \{x_i\}_{i=1}^{\ell}$ with relations consisting of commutator brackets of the form $[x_1, x_i] = x_{i+1}$ for $2 \leq i \leq \ell - 1$ and all other commutators being trivial.

Λ_ℓ is an example of a *Filiform nilpotent group*. It has Hirsch length ℓ and has step length $\ell - 1$. Defining $\Delta_i = \langle x_s \rangle_{s=m-i+1}^{\ell}$, it follows that $\{\Delta_i\}_{i=1}^{\ell}$ is a cyclic series for Λ_ℓ and $\{\xi_i\}_{i=1}^{\ell}$ is a compatible generating subset where $\xi_i = x_{\ell-i+1}$. Additionally, $h(Z(\Lambda_\ell)) = 1$.

Theorem 1.5(ii), (iii), and (iv).

(ii) *Suppose $\ell \neq 2$. Then there exists an irreducible, torsion-free, finitely generated nilpotent group Γ_ℓ such that $\mathrm{F}_\Gamma(n) \approx (\log(n))^{\ell}$.*

(iii) *Suppose $2 \leq c_1 < c_2$ are natural numbers. For each $\ell \in \mathbb{N}$, there exist irreducible, torsion-free, finitely generated nilpotent groups Γ_ℓ and Δ_ℓ of step lengths c_1 and c_2, respectively, such that*

$$\mathrm{F}_{\Gamma_\ell}(n) \approx \mathrm{F}_{\Delta_\ell}(n) \approx (\log(n))^{\ell \ \mathrm{lcm}(c_1+1, c_2+1)}.$$

(iv) *For natural numbers $c > 1$ and $m \geq 1$, there exists an irreducible, torsion-free, finitely generated nilpotent group Γ of step length c such that $(\log(n))^m \preceq \mathrm{F}_\Gamma(n)$.*

Proof. Assume that $\ell \geq 3$. By construction, Λ_ℓ is a torsion-free, finitely generated nilpotent group of Hirsch length ℓ such that $h(Z(\Gamma_\ell)) = 1$. Corollary 1.2 implies that

$$\mathrm{F}_{\Lambda_\ell}(n) \approx (\log(n))^{\ell}$$

and since $\mathrm{F}_{\mathbb{Z}}(n) \approx \log(n)$, we have Theorem 1.5(ii).

For $2 \leq c_1 < c_2$ and $\ell \geq 1$, there exist natural numbers j_ℓ and ι_ℓ satisfying

$$(j_\ell - 1)\,(c_1 + 1) = \ell \ \mathrm{lcm}(c_1 + 1, c_2 + 1)$$

and

$$(\iota_\ell - 1)\,(c_2 + 1) = \ell \ \mathrm{lcm}(c_1 + 1, c_2 + 1),$$

respectively. Let

$$\Gamma_\ell = (\Lambda_i \circ_{\theta_\Gamma})_{i=1}^{j_\ell} \quad \text{and} \quad \Delta_\ell = (\Lambda_i \circ_{\theta_\Delta})_{i=1}^{\iota_\ell}$$

where

$$\theta_\Gamma : Z(\Lambda_{c_1+1}) \to Z(\Lambda_{c_1+1}) \quad \text{and} \quad \theta_\Delta : Z(\Lambda_{c_2+1}) \to Z(\Lambda_{c_2+1})$$

are the identity isomorphisms, respectively. Proposition 8.6 implies that $h(\Gamma_\ell) = h(\Delta_\ell)$ and $h(Z(\Gamma_\ell)) = h(Z(\Delta_\ell)) = 1$, and thus, Corollary 1.2 implies that

$$\mathrm{F}_{\Gamma_\ell}(n) \approx \mathrm{F}_{\Delta_\ell}(n).$$

Lastly, let $c > 1$ and $m \geq 1$, and consider the group

$$\Gamma_{c\,m} = (\Lambda_{c+1} \circ_\theta)_{i=1}^{c\,m}$$

with finite generating subset S_{cm}. Proposition 8.6 implies that

$$h(\Gamma_{cm}) = c\,m^2 + c\,m - 1,$$

and since $c\,m^2 + c\,m - 1 \geq m$, Corollary 1.2 implies that

$$(\log(n))^m \preceq \mathrm{F}_{\Gamma_{c\,m}}(n)$$

as desired. ∎

Part IV. Conjugacy separability

9. A review of Blackburn and a proof of Theorem 1.6

We start with a review of Blackburn's proof of conjugacy separability for infinite, finitely generated nilpotent groups. This section provides motivation for estimates in the following sections and how one obtains an upper bound for $\mathrm{Conj}_\Gamma(n)$.

Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let $\gamma, \eta \in \Gamma$ be elements such that $\gamma \nsim \eta$. In order to construct a surjective homomorphism to a finite group that separates the conjugacy classes of γ and η, we proceed by induction on $h(\Gamma)$. Since the base case is evident, we may assume that $h(\Gamma) > 1$. When $\pi_{\Delta_1}(\gamma) \nsim \pi_{\Delta_1}(\eta)$, induction implies that there exists a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $\varphi(\gamma) \nsim \varphi(\eta)$. Otherwise, by passing to a conjugate element, we may assume that $\eta = \gamma\, \xi_1^t$ for some $t \in \mathbb{Z}^\bullet$. The following integer is of particular importance.

Definition 9.1. Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let $\gamma \in \Gamma$. If we let $\varphi : \pi_{\Delta_1}^{-1}(C_{\Gamma/\Delta_1}(\bar{\gamma})) \to \Delta_1$ be given by $\varphi(\eta) = [\gamma, \eta]$, we define $\tau(\Gamma, \Delta, \xi, \gamma) \in \mathbb{N}$ so that $\left\langle \xi_1^{\tau(\Gamma,\Delta,\xi,\gamma)} \right\rangle \cong \mathrm{Im}(\varphi)$.

Since we are trying to separate the conjugacy classes of γ and $\gamma\, \xi_1^t$, we choose a prime power p^α such that $p^\alpha \mid \tau(\Gamma, \Delta, \xi, \gamma)$ and $p^\alpha \nmid t$. We then find a $w \in \mathbb{N}$ such that if $\beta \geq w$, then for each $\gamma \in \Gamma^{p^\beta}$ there exists an element $\eta \in \Gamma$ satisfying $\eta^{p^{\beta-w}} = \gamma$ (see [3, Lem 2]).

Consider the following definition (see [3, Lem 3]).

Definition 9.2. Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let $\gamma \in \Gamma$. We define $e(\Gamma, \Delta, \xi, \gamma) \in \mathbb{N}$ to be the smallest natural number such that if $\lambda \geq e(\Gamma, \Delta, \xi, \gamma)$, then

$$C_{\Gamma/\Gamma^{p^\lambda}}(\bar{\gamma}) \subseteq \pi_{\Gamma^{p^\lambda}}\left(C_\Gamma(\gamma) \cdot \Gamma^{p^{\lambda - e(\Gamma,\Delta,\xi,\gamma)}}\right).$$

We set $\omega = \alpha + w + e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma})$. Blackburn then proves that

$$\pi_{\Gamma^{p^\omega}}(\gamma) \nsim \pi_{\Gamma^{p^\omega}}(\eta)$$

(see §12 and [3]). However, as a consequence of the choice of a cyclic series and a compatible generating subset, it becomes evident that the integer w is unnecessary. When Γ has torsion elements, Blackburn inducts on $|T(\Gamma)|$. Thus, it suffices to bound $p^{e(\Gamma,\Delta,\xi,\gamma)}$ and $\tau(\Gamma, \Delta, \xi, \gamma)$ in terms of $\|\gamma\|_S$ and $\|\eta\|_S$. Following Blackburn's method, we calculate the upper bound for $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$. We then demonstrate that the given upper bound is sharp.

Before starting, we make the following observations for $\mathrm{H}_{2m+1}(\mathbb{Z})$. Using the cyclic series and a compatible generating subset given in Subsection 5.1, we have that

$$\tau(\gamma) = \tau(\mathrm{H}_{2m+1}(\mathbb{Z}), \Delta, \xi, \gamma) = \gcd\{x_{\gamma,i}, y_{\gamma,j} | 1 \leq i, j \leq m\}.$$

Thus, Proposition 2.8 implies that $\tau(\gamma) \leq C_0 \|\gamma\|_S$ for some $C_0 \in \mathbb{N}$. Moreover, via Subsection 5.1 we may write the conjugacy class of γ as

$$\left\{ \begin{pmatrix} 1 & \vec{x}_\gamma & \tau(\gamma)\,\beta + z_\gamma \\ \vec{0} & \mathbf{I}_m & \vec{y}_\gamma \\ 0 & \vec{0} & 1 \end{pmatrix} \middle| \beta \in \mathbb{Z} \right\}. \tag{5}$$

The following proposition gives the upper bound for $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$.

Proposition 9.3. $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \preceq n^{2m+1}$.

Proof. Let $\gamma, \eta \in \Gamma$ such that $\|\gamma\|_S, \|\eta\|_S \leq n$ and $\gamma \nsim \eta$. We need to construct a surjective homomorphism to a finite group $\varphi : \mathrm{H_{2m+1}}(\mathbb{Z}) \to Q$ such that $\varphi(\gamma) \nsim \varphi(\eta)$ and where $|Q| \leq C\, n^{2m+1}$ for some $C \in \mathbb{N}$. We proceed based on whether γ and η have equal images in $(\mathrm{H}_{2m+1}(\mathbb{Z}))_{\mathrm{ab}}$. To this end, assume that $\pi_{\mathrm{ab}}(\gamma\,\eta^{-1}) \neq 1$. Corollary 1.4 (see also [4, Cor 2.3]) implies that there exists a surjective homomorphism to a finite group $\varphi : \mathbb{Z}^{2m} \to Q$ such that $\varphi(\pi_{\mathrm{ab}}(\gamma\,\eta^{-1})) \neq 1$ and $|Q| \leq C_1\, \log(C_1\, n)$ for some $C_1 \in \mathbb{N}$. Since the images of γ and η are nonequal, central elements in Q, it follows that $\varphi(\pi_{\mathrm{ab}}(\gamma)) \nsim \varphi(\pi_{\mathrm{ab}}(\eta))$, and thus,

$$\mathrm{CD}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma, \eta) \leq C_1 \log(C_1\, n).$$

Thus, we may assume that $\pi_{\mathrm{ab}}(\gamma) = \pi_{\mathrm{ab}}(\eta)$. In particular, we may write $\eta = \gamma\, \lambda^t$, and Proposition 2.8 implies that $|t| \leq C_0\, n^2$. Let p^ω be a prime power that divides $\tau(\gamma)$ but does not divide t. We claim that

$$\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p^\omega}}(\gamma) \nsim \pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p^\omega}}(\gamma\, \lambda^t),$$

and for a contradiction, suppose otherwise. That implies there exists an element $x \in \mathrm{H}_{2m+1}(\mathbb{Z})$ such that

$$\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p^\omega}}([\gamma, x]) = \pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p^\omega}}(\lambda^t).$$

Equation (5) implies that

$$z_\eta \in \{\ell_\gamma\, \beta + z_\gamma \mid \beta \in \mathbb{Z}\} \pmod{p^\omega}.$$

Therefore, there exist $a, b \in \mathbb{Z}$ such that $t = a\, \tau(\gamma) + b\, p^\omega$. Thus, $p^\omega \mid t$, a contradiction. Hence,

$$\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p^\omega}}(\gamma) \nsim \pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p^\omega}}(\gamma\, \lambda^t).$$

When $\tau(\gamma) \neq 0$, we have that $p^\omega \leq \tau(\gamma) \leq C_0\, n$. Hence,

$$\mathrm{CD}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma, \eta) \leq C_0^{2m+1}\, n^{2m+1}.$$

When $\tau(\gamma) = 0$, the Prime Number Theorem [38, 1.2] implies that there exists a prime p such that $p \nmid t$ where $p \leq C_2 \log(C_2 |t|)$ for some $C_2 \in \mathbb{N}$. Hence, $p \leq C_3 \log(C_3 n)$ for some $C_3 \in \mathbb{N}$, and thus,

$$\mathrm{CD}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma, \eta) \leq C_3 (\log(C_3 n))^{2m+1}.$$

Therefore, $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \preceq n^{2m+1}$. □

The following proposition gives the lower bound of $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$.

Proposition 9.4. $n^{2m+1} \preceq \mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$.

Proof. We will construct a sequence of nonconjugate pairs $\{\gamma_i, \eta_i\}$ such that

$$\mathrm{CD}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma_i, \eta_i) = n_i^{2m+1}$$

where $\|\gamma_i\|, \|\eta_i\| \approx n_i$ for all i. Let $\{p_i\}$ be an enumeration of the primes. Writing $p_i \cdot e_1$ as the scalar product, we consider the following pair of elements:

$$\gamma_i = \begin{pmatrix} 1 & p_i \cdot \vec{e}_1 & 1 \\ \vec{0} & \mathbf{I}_m & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix} \quad \text{and} \quad \eta_i = \begin{pmatrix} 1 & p_i \cdot \vec{e}_1 & 2 \\ \vec{0} & \mathbf{I}_m & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix}.$$

Equation (5) implies that we may write the conjugacy class of γ_i as

$$\left\{ \begin{pmatrix} 1 & p_i \cdot \vec{e}_1 & tp_i + 1 \\ \vec{0} & \mathbf{I}_m & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix} \middle| \, t \in \mathbb{Z} \right\}. \tag{6}$$

Since $\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}}(\gamma_i)$ and $\pi_{(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}}(\eta_i)$ are nonequal, central elements of $\mathrm{H}_{2m+1}(\mathbb{Z})/(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}$, it follows that $\gamma_i \nsim \eta_i$ for all i. Moreover, we have that $\|\gamma_i\|_S, \|\eta_i\|_S \approx p_i$. Given that

$$|\,\mathrm{H}_{2m+1}(\mathbb{Z})/(\mathrm{H}_{2m+1}(\mathbb{Z}))^{p_i}| = p_i^{2m+1},$$

we claim that

$$\mathrm{CD}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(\gamma_i, \eta_i) = p_i^{2m+1}.$$

In order to demonstrate our claim, we show if given a surjective homomorphism to a finite group $\varphi : \mathrm{H}_{2m+1}(\mathbb{Z}) \to Q$ such that $|Q| < p_i^{2m+1}$, then $\varphi(\gamma_i) \sim \varphi(\eta_i)$. [17, Thm 2.7] implies that we may assume that $|Q| = q^{\mu}$. Since $\varphi(\gamma_i) = \varphi(\eta_i)$ when $\varphi(\lambda) = 1$, we may assume that $\varphi(\lambda) \neq 1$.

Suppose first that $q = p_i$. We demonstrate that if Q is a group where $\varphi(\gamma_i) \nsim \varphi(\eta_i)$, then there exists no proper quotient of Q such that the images of $\varphi(\gamma_i)$ and $\varphi(\eta_i)$ are nonconjugate. Since $\mathrm{B}(\mathrm{H}_{2m+1}(\mathbb{Z}), \Delta, \xi) = 1$, Proposition 4.11 implies that $|Q| = p_i^{2m+1}$. Since every admissible quotient with respect to any primitive, central, nontrivial element is isomorphic to the trivial subgroup, Proposition 4.12 implies that there exist no proper quotients of Q such that the image of $\varphi(\lambda^2)$ is nontrivial. Thus, if N is a proper quotient of Q with natural projection $\rho : Q \to N$, then

$$\ker(\rho) \cap Z(Q) \cong Z(Q)$$

since $Z(Q) \cong \mathbb{Z}/p_j\mathbb{Z}$ by Proposition 4.12. Thus, $\rho(\varphi(\gamma_i)) = \rho(\varphi(\eta_i))$; hence, $\rho(\varphi(\gamma_i)) \sim \rho(\varphi(\eta_i))$. In particular, if Q is a p_i-group where $|Q| < p_i^{2m+1}$, then $\varphi(\gamma_i) \sim \varphi(\eta_i)$. Thus, we may assume that $q \neq p_i$.

If $q > p_i$, then Proposition 4.11 implies that $p_i^{2m+1} > q^\mu$. Thus, we may assume that $q < p_i$. Since Proposition 4.11 implies that $\mathbb{Z}/q^\nu\mathbb{Z} \cong Z(Q)$, Equation (6) implies that if $1 \equiv p\, t \pmod{q^\nu \mathbb{Z}}$ for some $t \in \mathbb{Z}$, then $\varphi(\gamma_p) \sim \varphi(\eta_p)$. The smallest q^ν where this fails is $q^\nu = p_i$ since the image of p_i is a unit in $\mathbb{Z}/q^\nu\mathbb{Z}$ if and only if $\gcd(p_i, q^\nu) = 1$. Therefore, $\varphi(\gamma_i) \sim \varphi(\eta_i)$ when $q^\mu < p_i$. Hence, $n^{2m+1} \preceq \mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$. □

Taking these propositions together, we obtain the main result of this section.

Theorem 1.6. $\mathrm{Conj}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \approx n^{2m+1}$.

Proof. Proposition 9.3 implies that $\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \preceq n^{2m+1}$, and Proposition 9.4 implies that $n^{2m+1} \preceq \mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n)$. Therefore,

$$\mathrm{F}_{\mathrm{H}_{2m+1}(\mathbb{Z})}(n) \approx n^{2m+1}.$$ □

The following corollary will be useful for the proof of Theorem 1.8.

Corollary 9.5. *Let $\mathrm{H}_3(\mathbb{Z})$ be the 3-dimensional Heisenberg group with the presentation given by $\langle \kappa, \mu, \nu \mid [\mu, \nu] = \kappa, \kappa \text{ central} \rangle$, and let p be a prime. Suppose $\varphi : \mathrm{H}_3(\mathbb{Z}) \to Q$ is a surjective homomorphism to a finite group such that Q is a q-group where q is a prime distinct from p and where $\varphi(\kappa) \neq 1$. Then*

$$\varphi(\mu^p\, \kappa) \sim \varphi(\mu^p\, \kappa^2).$$

Proof. We may write the conjugacy class of $\mu^p\, \kappa$ as

$$\{\mu^p\, \kappa^{t\, p+1} \mid t \in \mathbb{Z}\}.$$

Proposition 4.11 implies that $Z(Q) \cong \langle \varphi(\kappa) \rangle$. Hence, $Z(Q) \cong \mathbb{Z}/m\mathbb{Z}$ where $m = \mathrm{Ord}_Q(\varphi(\kappa))$. Since Q is a q-group, it follows that $m = q^\beta$ for some $\beta \in \mathbb{Z}$. Given that $\gcd(p, q^\beta) = 1$, there exists integers r, s such that $rp + sq^\beta = 1$. We have that

$$\mu^p\, \kappa^{r\, p+1} \sim \mu^p\, \kappa.$$

We may write

$$\varphi(\mu^p\, \kappa^{r\, p+1}) = \varphi(\mu^p\, \kappa^{1-s\, q^\beta+1}) = \varphi(\mu^p\, \kappa^2).$$

Therefore, $\varphi(\mu^p\, \kappa) \sim \varphi(\mu^p\, \kappa^2)$ as desired. □

10. Relating complexity in groups and Lie algebras

Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$, and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let G be the Mal'tsev completion of Γ with Lie algebra $\mathfrak{g}$. The overall goal of this section is to provide a bound of $\|\mathrm{Log}(\gamma)\|_{\mathrm{Log}(S)}$ in terms of $\|\gamma\|_S$ where $\mathrm{Log}(S)$ gives a norm for the additive structure of $\mathfrak{g}$.

Proposition 10.1. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let G be the Mal'tsev completion of Γ with Lie algebra $\mathfrak{g}$. Let $\gamma \in \Gamma$. Then there exists a constant $C \in \mathbb{N}$ such that*

$$\| \operatorname{Log}(\gamma)\|_{\operatorname{Log}(S)} \leq C\, (\|\gamma\|_S)^{(c(\Gamma))^2}.$$

Proof. Using the Mal'tsev coordinates of γ, we may write

$$\gamma = \prod_{i=1}^{h(\Gamma)} \xi_i^{\alpha_i}.$$

Lemma 2.8 implies that there exists $C_1 \in \mathbb{N}$ such that $|\alpha_i| \leq C_1(\|\gamma\|_S)^{c(\Gamma)}$ for all i. A straightforward application of the Baker–Campbell–Hausdorff formula (2) implies that $\operatorname{Log}(\xi_i^{\alpha_i}) = \alpha_i \operatorname{Log}(\xi_i)$. Writing $A_i = \alpha_i \operatorname{Log}(\xi_i)$, it follows that

$$\|A_i\|_{\operatorname{Log}(S)} \leq C_1(\|\gamma\|_S)^{c(\Gamma)}.$$

Equation (2) implies that we may write

$$\| \operatorname{Log}(\gamma)\|_{\operatorname{Log}(S)} \leq \sum_{i=1}^{c(\Gamma)} \|CB_i(A_1, \ldots, A_{h(\Gamma)})\|_{\operatorname{Log}(S)}$$

where $CB_i(A_1, \ldots, A_{h(\Gamma)})$ is a rational linear combination of i-fold Lie brackets of $\{A_{j_s}\}_{s=1}^{t} \subseteq \{A_i\}_{i=1}^{h(\Gamma)}$. Let $\{A_{j_s}\}_{s=1}^{t} \subset \{A_i\}_{i=1}^{h(\Gamma)}$ where $[A_{j_1}, \ldots A_{j_t}] \neq 0$. Via induction on the length of the iterated Lie bracket, one can see that there exists a constant $C_t \in \mathbb{N}$ such that

$$[A_{j_1}, \ldots, A_{j_t}] \leq C_t \prod_{s=1}^{t} \|A_{j_s}\|_{\operatorname{Log}(S)} \leq C_t\, C_1\, (\|\gamma\|_S)^{t\, c(\Gamma)}.$$

By maximizing over all possible t-fold Lie brackets of elements of $\{A_i\}_{i=1}^{h(\Gamma)}$, there exists a constant $D_i \in \mathbb{N}$ such that

$$\|CB_i(A_1, \ldots, A_{h(\Gamma)})\|_{\operatorname{Log}(S)} \leq D_i\, (\|\gamma\|_S)^{t\, c(\Gamma)}.$$

Hence,

$$\| \operatorname{Log}(\gamma)\|_{\operatorname{Log}(S)} \leq C\, (\|\gamma\|_S)^{(c(\Gamma))^2}$$

for some $C \in \mathbb{N}$. □

An immediate application of Proposition 10.1 is that the adjoint representation of Γ has matrix coefficients bounded by a polynomial in terms of word length.

Proposition 10.2. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating set $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let G be the Mal'tsev completion of Γ with Lie algebra $\mathfrak{g}$. Let $\gamma \in \Gamma$, and let $(\mu_{i,j})$ be the matrix representative of $\operatorname{Ad}(\gamma)$ with respect to the basis $\{\operatorname{Log}(\xi)\}_{i=1}^{h(\Gamma)}$. Then $|\mu_{i,j}| \leq C\, (\|\gamma\|_S)^{(c(\Gamma))^3}$ for some $C \in \mathbb{N}$.*

Proof. Proposition 10.1 implies that there exists a constant $C_1 \in \mathbb{N}$ such that

$$\| \operatorname{Log}(\gamma)\|_{\operatorname{Log}(S)} \leq C_1 \left(\|\gamma\|_S\right)^{(c(\Gamma))^2}.$$

Via induction on the length of the Lie bracket and Equation (3), we have that

$$\| \operatorname{Ad}(\gamma)(v_i)\|_{\operatorname{Log}(S)} \leq C_2(\|\gamma\|_S)^{(c(\Gamma))^3}$$

for some $C_2 \in \mathbb{N}$. □

11. Preliminary estimates for Theorem 1.7

Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let γ be a nontrivial element of Γ, and let p be some prime. In the following section, we demonstrate the construction of the integer $e(\Gamma, \Delta, \xi, \gamma)$ and give an upper bound for $p^{e(\Gamma,\Delta,\xi,\gamma)}$ in terms of $\|\gamma\|_S$ independent of the prime p. We first provide a bound for $\tau(\Gamma, \Delta, \xi, \gamma)$ in terms of $\|\gamma\|_S$.

Proposition 11.1. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. There exists $k, C \in \mathbb{N}$ such that*

$$|\tau(\Gamma, \Delta, \xi, \gamma)| \leq C \left(\|\gamma\|_S\right)^k.$$

Proof. Let G be the Mal'tsev completion of Γ with Lie algebra $\mathfrak{g}$. Consider the smooth map $\Phi : G \to G$ given by $\Phi(g) = [\gamma, g]$. Suppose $\eta \in \Gamma$ satisfies $\Phi(\eta) = \xi_1^{\tau(\Gamma,\Delta,\xi,\gamma)}$. The commutative diagram (1.2) on [13, Pg 7] implies that we may write

$$(I - \operatorname{Ad}(\gamma^{-1}))(\operatorname{Log}(\eta)) = \operatorname{Log}(\xi_1^{\tau(\Gamma,\Delta,\xi,\gamma)})$$

where $(d\Phi_\gamma)_1 = I - \operatorname{Ad}(\gamma^{-1})$. Proposition 10.2 implies that $I - \operatorname{Ad}(\gamma^{-1})$ is a strictly upper triangular matrix whose coefficients are bounded by $C \left(\|\gamma\|_S\right)^{(c(\Gamma))^3}$ for some $C \in \mathbb{N}$. Since it is evident that

$$\operatorname{Log}(\xi_1^{\tau(\Gamma,\Delta,\xi,\gamma)}) = \tau(\Gamma, \Delta, \xi, \gamma) \operatorname{Log}(\xi_1),$$

backwards substitution gives our result. □

The first statement of the following proposition is originally found in [3, Lem 3]. We reproduce its proof so that we may provide estimates for the value $p^{e(\Gamma,\Delta,\xi,\gamma)}$ in terms of $\|\gamma\|_S$ where p is an arbitrary prime.

Proposition 11.2. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Let p be a prime number, and let $\gamma \in \Gamma$. Then there exists $e(\Gamma, \Delta, \xi, \gamma) \in \mathbb{N}$ such that if $\alpha \geq e(\Gamma, \Delta, \xi, \gamma)$, then*

$$C_{\Gamma/\Gamma^{p^\alpha}}(\bar{\gamma}) \subseteq \pi_{\Gamma^{p^\alpha}} \left(C_\Gamma(\gamma) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)}} \right).$$

Moreover, $p^{e(\Gamma,\Delta,\xi,\gamma)} \leq C(\|\gamma\|_S)^k$ for some $C \in \mathbb{N}$ and $k \in \mathbb{N}$.

Proof. We proceed by induction on Hirsch length, and given that the statement is clear for $\mathbb{Z}$ by setting $e(\Gamma, \Delta, \xi, \gamma) = 0$ for all γ, we may assume that $h(\Gamma) > 1$.

We construct $e(\Gamma, \Delta, \xi, \gamma)$ based on the value of $\tau(\Gamma, \Delta, \xi, \gamma)$ (see Definition 9.1). By induction, we may assume that we have already constructed $e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma})$. When $\tau(\Gamma, \Delta, \xi, \gamma) = 0$, we set

$$e(\Gamma, \Delta, \xi, \gamma) = e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma}).$$

Suppose $\alpha \geq e(\Gamma, \Delta, \xi, \gamma)$ and that $\bar{\eta} \in C_{\Gamma/\Gamma^{p^\alpha}}(\bar{\gamma})$ for some $\eta \in \Gamma$. By selection, $\bar{\eta} \in C_{\Gamma/\Gamma^{p^\alpha}\cdot\Delta_1}(\bar{\gamma})$. Thus, we may write

$$\eta \in \pi_{\Delta_1}^{-1}(C_{\Gamma/\Delta_1}(\bar{\gamma})) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)}}.$$

Since $\pi_{\Delta_1}^{-1}(C_{\Gamma/\Delta_1}(\bar{\gamma})) = C_\Gamma(\gamma)$, it follows that

$$\bar{\eta} \in \pi_{\Gamma^{p^\alpha}} \left(C_\Gamma(\gamma) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)}} \right),$$

Thus,

$$C_{\Gamma/\Gamma^{p^\alpha}}(\bar{\gamma}) \subseteq \pi_{\Gamma^{p^\alpha}} \left(C_\Gamma(\gamma) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)}} \right).$$

When $\tau(\Gamma, \Delta, \xi, \gamma) \neq 0$, we let β be the largest power of p such that $p^\beta \mid \tau(\Gamma, \Delta, \xi, \gamma)$ and set

$$e(\Gamma, \Delta, \xi, \gamma) = e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma}) + \beta.$$

Let $\alpha \geq e(\Gamma, \Delta, \xi, \gamma)$, and let $\eta \in \Gamma$ satisfy $\bar{\eta} \in C_{\Gamma/\Gamma^{p^\alpha}}(\bar{\gamma})$. Thus, $\bar{\eta} \in C_{\Gamma/\Gamma^{p^\alpha}\cdot\Delta_1}(\bar{\gamma})$, and subsequently, induction implies that

$$\bar{\eta} \in \pi_{\Gamma^{p^\alpha}\cdot\Delta_1} \left(C_{\Gamma/\Delta_1}(\bar{\gamma}) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)+\beta}} \right).$$

Thus, we may write $\eta = \mu\, \epsilon^a\, \lambda$ where

$$\mu \in C_\Gamma(\gamma), \quad \lambda \in \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)+\beta}}, \quad \text{and} \quad \varphi_\gamma(\epsilon) = \xi_1^{\tau(\Gamma,\Delta,\xi,\gamma)}.$$

Hence, we have that

$$[\gamma, \eta] = [\gamma, \varepsilon^a] \in \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)+\beta}}.$$

Since $[\gamma, \epsilon^a] \in \Gamma^{p^{\alpha - e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma})}}$ and $[\gamma, \epsilon^a] \in \Delta_1$, we have that

$$p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)+\beta} \mid a\, \tau(\Gamma, \Delta, \xi, \gamma).$$

By definition of p^β, it follows that $p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)} \mid a$, and thus,

$$\bar{\eta} \in \pi_{\Gamma^{p^\alpha}} \left(C_\Gamma(\gamma) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)}} \right).$$

Hence,

$$C_{\Gamma/\Gamma^{p^\alpha}}(\bar{\gamma}) \subseteq \pi_{\Gamma^{p^\alpha}} \left(C_\Gamma(\gamma) \cdot \Gamma^{p^{\alpha - e(\Gamma,\Delta,\xi,\gamma)}} \right).$$

We proceed by induction on Hirsch length to demonstrate the upper bound, and since the base case is clear, we may assume that $h(\Gamma) > 1$. Let $\gamma \in \Gamma$, and suppose that $\tau(\Gamma, \Delta, \xi, \gamma) = 0$. By construction,

$$e(\Gamma, \Delta, \xi, \gamma) = e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma}),$$

and thus, induction implies that there exist $C_1, k_1 \in \mathbb{N}$ such that

$$p^{e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma})} \leq C_1 \left(\|\bar{\gamma}\|_{\bar{S}}\right)^{k_1}.$$

When $\tau(\Gamma, \Delta, \xi, \gamma) \neq 0$, it follows that $e(\Gamma, \Delta, \xi, \gamma) = e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma}) + \beta$ where β is the largest power of p that divides $\tau(\Gamma, \Delta, \xi, \gamma)$. Proposition 11.1 implies that there exist $k_2, C_2 \in \mathbb{N}$ such that $p^\beta \leq C_2 \left(\|\gamma\|_S\right)^{k_2}$. Consequently,

$$p^{e(\Gamma, \Delta, \xi, \gamma)} \leq C_1 \, C_2 \left(\|\gamma\|_S\right)^{k_1 + k_2}. \qquad \square$$

We finish with the following technical result.

Proposition 11.3. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$, and let $\gamma, \eta \in \Gamma$. Suppose that $\gamma \nsim \eta$, but $\pi_{\Delta_1}(\gamma) \sim \pi_{\Delta_1}(\eta)$. Then there exists an element $g \in \Gamma$ such that $g^{-1}\, \eta \, g = \gamma \, \xi_1^t$ where*

$$\|t\| \leq C \, max\{\|\gamma\|_S, \|\eta\|_S\}^k$$

for some constant $C > 0$ and $k \in \mathbb{N}$.

Proof. Let G be the Mal'tsev completion of Γ with Lie algebra $\mathfrak{g}$. Consider the smooth map $\Phi : G \to G$ given by $\Phi(x) = [\eta, x]$. By assumption, there exists an element $g \in \Gamma$ such that $g^{-1}\, \eta \, g \equiv \gamma \mod \Delta_1$, and thus, the commutative diagram (1.2) on [13, Pg 7] implies that we may write

$$\left(I - \text{Ad}(\eta^{-1})\right)(\text{Log}(g)) = \text{Log}(\eta^{-1}\, \gamma \, \xi_1^t)$$

for some $t \in \mathbb{Z}^\bullet$. Since ξ_1^t is central, Equation 2 implies that we may write

$$\left(I - \text{Ad}(\eta^{-1})\right)(\text{Log}(g)) = \text{Log}(\eta^{-1}\, \gamma) + \text{Log}(\xi_1^t).$$

Proposition 10.2 implies that $I - \text{Ad}(\eta^{-1})$ is a strictly upper triangular matrix whose coefficients are bounded by $C_1 \left(\|\eta\|_S\right)^{(c(\Gamma))^3}$ for some $C_1 \in \mathbb{N}$. Lemma 10.1 implies that we may write $\text{Log}(\eta^{-1}\gamma) = \sum_{i=1}^{h(\Gamma)} \alpha_i \, \text{Log}(\xi_i)$ where $|\alpha_i| \leq C_2 \left(\|\eta^{-1}\, \gamma\|_S\right)^{(c(\Gamma))^2}$ for some $C_2 \in \mathbb{N}$. Thus, we may write

$$\left(I - \text{Ad}(\eta^{-1})\right)(\text{Log}(g)) = (t + \alpha_1)\, \text{Log}(\xi_1) + \sum_{i=2}^{h(\Gamma)} \alpha_i \, \text{Log}(\xi_i).$$

Thus, backwards substitution gives our result. $\square$

12. Proof of Theorem 1.7

Let Γ be an infinite, finitely generated nilpotent group. In order to demonstrate that there exists a $k_1 \in \mathbb{N}$ such that $\text{Conj}_\Gamma(n) \preceq n^{k_1}$, we need to show for any elements $\gamma, \eta \in \Gamma$ where $\gamma \nsim \eta$ and $\|\gamma\|_S, \|\eta\|_S \leq n$ that there exists a prime power $p^\omega \leq C\, n^{k_2}$ such that $\pi_{\Gamma^{p^\omega}}(\gamma) \nsim \pi_{\Gamma^{p^\omega}}(\eta)$ for some $C, k_2 \in \mathbb{N}$. It then follows that $\text{CD}_\Gamma(\gamma, \eta) \leq C^{h(\Gamma)} n^{h(\Gamma)\, k_2}$. We first specialize to torsion-free, finitely generated nilpotent groups.

Proposition 12.1. *Let Γ be a torsion-free, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$. Then there exists a $k \in \mathbb{N}$ such that*

$$\text{Conj}_\Gamma(n) \preceq n^k.$$

Proof. Let $\gamma, \eta \in \Gamma$ such that $\|\gamma\|_S, \|\eta\|_S \leq n$ and where $\gamma \nsim \eta$. We demonstrate that there exists a $k_0 \in \mathbb{N}$ such that $\text{CD}_\Gamma(\gamma, \eta) \leq C_0\, n^{k_0}$ for some $C_0 \in \mathbb{N}$ by induction on $h(\Gamma)$, and since the base case is clear, we may assume that $h(\Gamma) > 1$. If $\pi_{\Delta_1}(\gamma) \nsim \pi_{\Delta_1}(\eta)$, then the inductive hypothesis implies that there exists a surjective homomorphism to a finite group

$$\varphi : \Gamma/\Delta_1 \to Q$$

such that $\varphi(\gamma) \nsim \varphi(\eta)$ and where $|Q| \leq C_1\, n^{k_1}$ for some $C_1, k_1 \in \mathbb{N}$. Thus,

$$\text{CD}_\Gamma(\gamma, \eta) \leq C_1\, n^{k_1}.$$

Thus, Proposition 11.3 implies that there exists an element $\zeta \in \Gamma$ such that $\zeta\, \eta\, \zeta^{-1} = \gamma\, \xi_1^t$ where $|t| \leq C_2\, n^{k_2}$ for some $C_2 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$. Since $\gamma \nsim \gamma \xi_1^t$, there exists a prime power p^α such that $p^\alpha \mid \tau(\Gamma, \Delta, \xi, \gamma)$ but $p^\alpha \nmid t$. We set

$$\omega = \alpha + e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma}),$$

and suppose for a contradiction that there exists an element $x \in \Gamma$ such that $\pi_{\Gamma^{p^\omega}}(x^{-1}\, \gamma\, x) = \pi_{\Gamma^{p^\omega}}(\gamma\, \xi_1)^t$. That implies $\bar{x} \in C_{\Gamma/\Gamma^{p^\omega}\cdot\Delta_1}(\bar{\gamma})$, and thus,

$$\bar{x} \in \pi_{\Gamma^{p^\omega}\cdot\Delta_1}(C_{\Gamma/\Delta_1}(\gamma) \cdot \Gamma^{p^\alpha})$$

by Proposition 11.2. Subsequently, $x = g\, \mu$ for some $g \in \pi_{\Delta_1}^{-1}(C_{\Gamma/\Delta_1}(\bar{\gamma}))$ and $\mu \in \Gamma^{p^\alpha}$. Hence, $\pi_{\Gamma^{p^\omega}}([\gamma, g]) = \pi_{\Gamma^{p^\omega}}(\xi_1)^t$, and since $[\gamma, g] = \xi_1^{q\, \tau(\Gamma,\Delta,\xi,\gamma)}$ for some $q \in \mathbb{Z}$, it follows that

$$\xi_1^{t - q\, \tau(\Gamma,\Delta,\xi,\gamma)} \in \Gamma^{p^{\alpha + e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi} \bar{\gamma})}}.$$

That implies $p^\alpha \mid t$, which is a contradiction. Hence, $\pi_{\Gamma^{p^\omega}}(\gamma) \nsim \pi_{\Gamma^{p^\omega}}(\eta)$.

Proposition 11.2 implies that $p^{e(\Gamma/\Delta_1, \bar{\Delta}, \bar{\xi}, \bar{\gamma})} \leq C_3\, n^{k_3}$ for some $C_3, k_3 \in \mathbb{N}$. When $\tau(\Gamma, \Delta, \xi, \gamma) = 0$, the Prime Number Theorem [38, 1.2] implies that we may take $|p| \leq C_4\, \log(C_4\, n)$ for some $C_4 \in \mathbb{N}$.

Hence,

$$\text{CD}_\Gamma(\gamma, \eta) \leq C_5\, (\log(C_5\, n))^{h(\Gamma)}$$

for some $C_5 \in \mathbb{N}$. When $\tau(\Gamma, \Delta, \xi, \gamma) \neq 0$, Proposition 11.1 implies that $\tau(\Gamma, \Delta, \xi, \gamma) \leq C_6\, n^{k_4}$ for some $C_6, k_4 \in \mathbb{N}$. Thus, $p^{\omega} \leq C_3\, C_6\, n^{k_3+k_4}$. Therefore,

$$\mathrm{CD}_{\Gamma}(\gamma, \eta) \leq (C_3\, C_6)^{h(\Gamma)}\, n^{h(\Gamma)(k_3+k_4)}.$$

Thus, by letting

$$k_5 = \max\{k_1, h(\Gamma)(k_3 + k_4)\},$$

we have

$$\mathrm{Conj}_{\Gamma}(n) \preceq n^{k_5}.$$

□

We now come to the main result of this section.

Theorem 1.7. *Let Γ be a finitely generated nilpotent group. Then there exists a $k \in \mathbb{N}$ such that*

$$\mathrm{Conj}_{\Gamma}(n) \preceq n^{k}.$$

Proof. Let Γ be an infinite, finitely generated nilpotent group Γ with a cyclic series $\{\Delta_i\}_{i=1}^{m}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{m}$. Let k_1 be the natural number from Proposition 12.1 and k_2 be the natural number from Proposition 11.2, both for $\Gamma/T(\Gamma)$. Letting $k_3 = h(\Gamma) \cdot \max\{k_1, k_2\}$, we claim that $\mathrm{Conj}_{\Gamma}(n) \preceq n^{k_3}$. Let $\gamma, \eta \in \Gamma$ satisfy $\gamma \nsim \eta$ and $\|\gamma\|_S, \|\eta\|_S \leq n$. In order to show that $\mathrm{CD}_{\Gamma}(\gamma, \eta) \leq C_0\, n^{k_3}$ where $C_0 \in \mathbb{N}$, we construct a surjective homomorphism to a finite group that distinguishes the conjugacy classes of γ and η via induction on $|T(\Gamma)|$.

Proposition 12.1 implies that we may assume that there exists a subgroup $P \subseteq Z(\Gamma)$ of prime order p. If $\pi_P(\gamma) \nsim \pi_P(\eta)$, then induction implies that there exists a surjective homomorphism to a finite group $\varphi : \Gamma/P \to N$ such that $\varphi(\gamma) \nsim \varphi(\eta)$ and where $|N| \leq C_1\, n^{k_3}$ for some $C_1 \in \mathbb{N}$. Thus,

$$\mathrm{CD}_{\Gamma}(\gamma, \eta) \leq C_1\, n^{k_3}.$$

Otherwise, we may assume that $\eta = \gamma\, \mu$ where $\langle \mu \rangle = P$.

Suppose there exists a subgroup $Q \subseteq Z(\Gamma)$ such that $|Q| = q$ where q is a prime distinct from p. Suppose for a contradiction that there exists an element $x \in \Gamma$ such that $x^{-1}\, \gamma\, x = \gamma\, \mu\, \lambda$ where $Q = \langle \lambda \rangle$. Since $[\gamma, x] \in Z(\Gamma)$ and $\mathrm{Ord}_{\Gamma}(\lambda) = q$, basic commutator properties imply that $[\gamma, x^q] = \mu^q$. Given that $p\, s + q\, r = 1$ for some $r, s \in \mathbb{Z}$, it follows that

$$[\gamma, x^{q\,r}] = \gamma\, \mu^{1-p\,s} = \gamma\, \mu$$

which is a contradiction. Hence, induction implies that there exists a surjective homomorphism to a finite group $\theta : \Gamma/Q \to M$ such that $\theta(\gamma) \nsim \theta(\gamma\, \mu)$ and where $|M| \leq C_2\, n^{k_3}$ for some $C_2 \in \mathbb{N}$. Thus,

$$\mathrm{CD}_{\Gamma}(\gamma, \eta) \leq C_2\, n^{k_3}.$$

We now may assume that $T(\Gamma)$ is a p-group with exponent p^m. We set

$$\omega = m + e(\Gamma/T(\Gamma), \bar{\Delta}, \bar{\xi}, \bar{\gamma}),$$

and suppose for a contradiction that there exists an element $x \in \Gamma$ such that

$$\pi_{\Gamma^{p^\omega}}(x^{-1}\,\gamma\,x) \sim \pi_{\Gamma^{p^\omega}}(\gamma\,\mu).$$

Thus, $\bar{x} \in C_{\Gamma/T(\Gamma)\cdot\Gamma^{p^\omega}}(\bar{\gamma})$; hence,

$$\bar{x} \in \pi_{T(\Gamma)\cdot\Gamma^{p^\omega}}(C_{\Gamma/T(\Gamma)}(\bar{\gamma}) \cdot \Gamma^{p^m})$$

by Proposition 11.2. Therefore, we may write $x = g\,\lambda$ where $\lambda \in \Gamma^{p^m}$ and $g \in \pi^{-1}_{T(\Gamma)}(C_{\Gamma/T(\Gamma)}(\bar{\gamma}))$. Subsequently, $[\gamma, g]\,\mu^{-1} \in \Gamma^{p^m}$. Moreover, since $[\gamma, g] \in T(\Gamma)$ and $T(\Gamma) \cap \Gamma^{p^m} = \{1\}$, it follows that $[\gamma, x] = \mu$ which is a contradiction. Proposition 11.2 implies that $p^{e(\Gamma/T(\Gamma),\bar{\Delta},\bar{\xi},\bar{\gamma})} \leq C_3\,n^{k_2}$ for some $C_3 \in \mathbb{N}$. Thus,

$$\mathrm{CD}_\Gamma(\gamma,\eta) \leq C_3^{h(\Gamma)}\,|T(\Gamma)|\,n^{h(\Gamma)\,k_2},$$

and subsequently,

$$\mathrm{Conj}_\Gamma(n) \preceq n^{k_3}. \qquad \square$$

13. Proofs of Theorem 1.8 and Theorem 1.9

Let Γ be an infinite, finitely generated nilpotent group with a cyclic series $\{\Delta_i\}_{i=1}^m$ and a compatible generating subset $\{\xi_i\}_{i=1}^m$. Since the proofs of Theorem 1.8(i) and Theorem 1.8(ii) require different strategies, we approach them separately. We start with Theorem 1.8(i) since it only requires elementary methods.

We assume that Γ contains an infinite, finitely generated abelian group K of index m. We want to demonstrate that

$$\log(n) \preceq \mathrm{Conj}_\Gamma(n) \preceq (\log(n))^m.$$

Since $\mathrm{F}_\Gamma(n) \preceq \mathrm{Conj}_\Gamma(n)$, Corollary 1.4 (see also [4, Cor 2.3]) implies that $\log(n) \preceq \mathrm{Conj}_\Gamma(n)$. Thus, we need only to demonstrate that

$$\mathrm{Conj}_\Gamma(n) \preceq (\log(n))^m.$$

For any two nonconjugate elements $\gamma, \eta \in \Gamma$ where $\|\gamma\|_S, \|\eta\|_S \leq n$, we want to construct a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $\varphi(\gamma) \nsim \varphi(\eta)$ and where $|Q| \leq C\,(\log(C\,n))^m$ for $C \in \mathbb{N}$.

Theorem 1.8(i). *Suppose that Γ is an infinite, finitely generated nilpotent group. If Γ contains a normal abelian subgroup of index m, then*

$$\log(n) \preceq \mathrm{Conj}_\Gamma(n) \preceq (\log(n))^m.$$

Proof. Let K be a normal abelian subgroup of Γ of index m. Let S_1 be a finite generating subset for K, and let $\{\upsilon_i\}_{i=1}^m$ be a set of coset representatives of K in Γ. We take $S = S_1 \cup \{\upsilon_i\}_{i=1}^m$ as the generating subset for Γ. If $\|\gamma\|_S \leq n$, we may write $\gamma = g_\gamma\,\upsilon_\gamma$ where $\|g_\gamma\|_{S_1} \leq C_1\,n$ for some $C_1 \in \mathbb{N}$ and $\upsilon_\gamma \in \{\upsilon_i\}_{i=1}^m$. Conjugation in Γ induces a map $\varphi : \Gamma/K \to \mathrm{Aut}(K)$ given by $\varphi(\pi_K(\upsilon_i)) = \varphi_i$. Thus, we may write

$$[\gamma]_\Gamma = \left\{\varphi_i(g_\gamma)\,(\upsilon_i^{-1}\,\upsilon_\gamma\,\upsilon_i)\right\}_{i=1}^m.$$

Finally, there exists a constant $C_2 \in \mathbb{N}$ such that if $\|\gamma\|_{S_1} \leq n$, then $\|\varphi_i(\gamma)\|_S \leq C_2\, n$ for all i.

Suppose $\gamma, \eta \in \Gamma$ are two nonconjugate elements such that $\|\gamma\|_S, \|\eta\|_S \leq n$. If $\pi_K(\gamma) \nsim \pi_K(\eta)$, then by taking the homomorphism $\pi_K : \Gamma \to \Gamma/K$, it follows that $\mathrm{CD}_\Gamma(\gamma,\eta) \leq m$. Otherwise, we may assume that $\eta = g_\eta\, \upsilon_\gamma$. By Corollary 1.4 (see also [4, Cor 2.3]), there exists a surjective homomorphism $f_i : \Gamma \to Q_i$ such that

$$f_i(g_\gamma^{-1} \upsilon_\gamma^{-1}\, \varphi_i(g_\eta)\, (\upsilon_i^{-1}\, \upsilon_\eta\, \upsilon_i)) \neq 1$$

and

$$|Q_i| \leq C_3\, \log(2\, C_2\, C_3\, n)$$

for some $C_3 \in \mathbb{N}$. By letting $H = \cap_{i=1}^m \ker(f_i)$, it follows that $\pi_H(\gamma) \nsim \pi_H(\eta)$ and

$$|\Gamma/H| \leq C_3^m\, (\log(2\, C_2\, C_3\, n))^m.$$

Hence, $\mathrm{Conj}_\Gamma(n) \preceq (\log(n))^m$ and thus, $\log(n) \preceq \mathrm{Conj}_\Gamma(n) \preceq (\log(n))^m$. □

For Theorem 1.8(ii), suppose that Γ does not contain an abelian group of finite index. In order to demonstrate that

$$n^{\psi_{\mathrm{RF}}(\Gamma)(c(\Gamma/T(\Gamma))-1)} \preceq \mathrm{Conj}_\Gamma(n),$$

we desire a sequence of nonconjugate pairs $\{\gamma_i, \eta_i\}$ such that

$$\mathrm{CD}_\Gamma(\gamma_i, \eta_i) = n_i^{\psi_{\mathrm{RF}}(\Gamma)(c(\Gamma/T(\Gamma))-1)}$$

where $\|\gamma_i\|_S, \|\eta_i\|_S \approx n_i$ for all i. In particular, we must find nonconjugate elements whose conjugacy classes are difficult to separate, i.e., nonconjugate elements that have relatively short word length in comparison to the order of the minimal finite group required to separate their conjugacy classes.

We first reduce to the calculation of the lower bounds for $\mathrm{Conj}_\Gamma(n)$ to torsion-free, finitely generated nilpotent groups by appealing to the conjugacy separability of two elements within a finite index subgroup.

Proposition 13.1. *Let Γ be an infinite, finitely generated nilpotent group, and let Δ be a subgroup. Suppose there exist $\gamma, \eta \in \Delta$ such that $\gamma \nsim \eta$ in Γ. Then $\mathrm{CD}_\Delta(\gamma,\eta) \leq \mathrm{CD}_\Gamma(\gamma,\eta)$.*

Proof. We first remark that since Γ and Δ are finitely generated nilpotent groups, Theorem 1.7 implies that $\mathrm{CD}_\Gamma(\gamma,\eta) < \infty$ and $\mathrm{CD}_\Delta(\gamma,\eta) < \infty$. Suppose that $\varphi : \Gamma \to Q$ is surjective homomorphism to a finite group such that $|Q| = \mathrm{CD}_\Gamma(\gamma,\eta)$. If we let $\iota : \Delta \to \Gamma$ be the inclusion, then we have a surjective homomorphism $\varphi \circ \iota : \Delta \to \varphi(\Delta)$ to a finite group where $\varphi(\iota(\gamma)) \nsim \varphi(\iota(\eta))$. By definition, $\mathrm{CD}_\Delta(\gamma,\eta) \leq |\varphi(\Delta)| \leq |Q|$. Thus, $\mathrm{CD}_\Delta(\gamma,\eta) \leq \mathrm{CD}_\Gamma(\gamma,\eta)$. □

Theorem 1.8(ii). *Let Γ be an infinite, finitely generated nilpotent group, and suppose that Γ is not virtually abelian. Then*

$$n^{\psi_{\mathrm{Lower}}(\Gamma)} \preceq \mathrm{Conj}_\Gamma(n).$$

Additionally, one can compute $\psi_{\text{Lower}}(\Gamma)$ *given a basis for* $(\Gamma/T(\Gamma))_c$ *where* c *is the step length of* $\Gamma/T(\Gamma)$.

Proof. We first assume that Γ is torsion-free. Let Γ/Λ be a maximal admissible quotient of Γ. There exists an element $g \in (Z(\Gamma))^\bullet$ such that Γ/Λ is an admissible quotient with respect to g. Moreover, there exists a $k \in \mathbb{Z}^\bullet$ such that $g^k = [y, z]$ for some $y \in \Gamma_{c(\Gamma)-1}$ and $z \in \Gamma$. If g is not primitive, then there exists an element $x_\Lambda \in (Z(\Gamma))^\bullet$ such that $x_\Lambda^s = g$ for some $s \in \mathbb{Z}^\bullet$. In particular, $x_\Lambda^t = [y, z]$ where $t = s\,k$.

There exists a cyclic series $\{\Delta_i\}_{i=1}^{h(\Gamma)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Gamma)}$ that together satisfy Proposition 3.6 for Λ where $\xi_1 = x_\Lambda$. There exists $a_{\Lambda,\Delta,\xi} \in \Gamma_{c(\Gamma)-1}$ and $b_{\Lambda,\Delta,\xi} \in \Gamma$ such that $[a_{\Lambda,\Delta,\xi}, b_{\Lambda,\Delta,\xi}] = \xi_1^{t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})}$. Equation 4 implies that

$$H_{\Lambda,\Delta,\xi} = \left\langle a_{\Lambda,\Delta,\xi}, b_{\Lambda,\Delta,\xi}, \xi_1^{t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})} \right\rangle \cong \mathrm{H}_3(\mathbb{Z}).$$

Let $\{p_{j,\Lambda,\Delta,\xi}\}$ be an enumeration of primes greater than $\mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})$. Let

$$\gamma_{j,\Lambda,\Delta,\xi} = (a_{\Lambda,\Delta,\xi})^{p_{j,\Lambda,\Delta,\xi}}\ \xi_1^{t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})}$$

and

$$\eta_{j,\Lambda,\Delta,\xi} = (a_{\Lambda,\Delta,\xi})^{p_{j,\Lambda,\Delta,\xi}}\ \xi_1^{2\,t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})}.$$

Since the images of $\gamma_{j,\Lambda,\Delta,\xi}$ and $\eta_{j,\Lambda,\Delta,\xi}$ are nonequal, central elements of $\Gamma/\Lambda \cdot \Gamma^{p_{j,\Lambda,\Delta,\xi}}$, it follows that $\gamma_{j,\Lambda,\Delta,\xi} \nsim \eta_{j,\Lambda,\Delta,\xi}$ for all j.

We claim that $\gamma_{j,\Lambda,\Delta,\xi}$ and $\eta_{j,\Lambda,\Delta,\xi}$ are our desired nonconjugate elements. In particular, we will demonstrate that

$$\mathrm{CD}_\Gamma(\gamma_{j,\Lambda,\Delta,\xi}, \eta_{j,\Lambda,\Delta,\xi}) \approx ((p_{j,\Lambda,\Delta,\xi})^{1/(c(\Gamma)-1)})^{\psi_{\text{Lower}}(\Gamma)} = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\text{RF}}(\Gamma)}$$

and that

$$\|\gamma_{j,\Lambda,\Delta,\xi}\|_S, \|\eta_{j,\Lambda,\Delta,\xi}\|_S \approx (p_{j,\Lambda,\Delta,\xi})^{1/(c(\Gamma)-1)}.$$

By construction, we have that $\gamma_{j,\Lambda,\Delta,\xi}, \eta_{j,\Lambda,\Delta,\xi} \in \Gamma_{c(\Gamma)-1}$ and

$$\|\gamma_{j,\Lambda,\Delta,\xi}\|_{S'}, \|\eta_{j,\Lambda,\Delta,\xi}\|_{S'} \approx p_{j,\Lambda,\Delta,\xi}$$

where $S' = S \cap \Gamma^2$. [16, 3.B2] implies that

$$\|\gamma_{j,\Lambda,\Delta,\xi}\|_S, \|\eta_{j,\Lambda,\Delta,\xi}\|_S \approx (p_{j,\Lambda,\Delta,\xi})^{1/(c(\Gamma)-1)}.$$

Therefore,

$$(\|\gamma_{j,\Lambda,\Delta,\xi}\|_S)^{\psi_{\text{Lower}}(\Gamma)}, (\|\eta_{j,\Lambda,\Delta,\xi}\|_S)^{\psi_{\text{Lower}}(\Gamma)} \approx (p_{j,\Lambda,\Delta,\xi})^{\psi_{\text{RF}}(\Gamma)}.$$

Hence, we need to demonstrate that if given a surjective homomorphism to a finite group $\varphi : \Gamma \to Q$ such that $|Q| < (p_{j,\Lambda,\Delta,\xi})^{\psi_{\text{RF}}(\Gamma)}$, then $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) \sim \varphi(\eta_{j,\Lambda,\Delta,\xi})$.

[17, Thm 2.7] implies that we may assume that $|Q| = q^\beta$ where q is prime. Since $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) = \varphi(\eta_{j,\Lambda,\Delta,\xi})$ when $\varphi(\xi_1^{t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})}) = 1$, we may assume that $\varphi(\xi_1^{t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})}) \neq 1$. In particular, we have $\pi_{\varphi(\Lambda)} \circ \varphi(\xi_1^{t\ \mathrm{B}(\Gamma/\Lambda,\bar{\Delta},\bar{\xi})}) \neq 1$ by Proposition 4.5.

Now assume that $q = p_{j,\Lambda,\Delta,\xi}$, and suppose that $\varphi(\Lambda) \neq \{1\}$. As before, we have the homomorphism $\widetilde{\rho \circ \varphi} : \Gamma/\Lambda \to Q/\varphi(\Lambda)$. Since $|Q/\varphi(Q)| \leq (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$, Proposition 4.12 implies that $|Q/\varphi(Q)| = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$. Thus, we have that $|Q| > (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$. Hence, we may assume that $\varphi(\Lambda) = \{1\}$.

Now assume that $q = p_{j,\Lambda,\Delta,\xi}$ and $\varphi(\Lambda) \cong \{1\}$. If $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) \sim \varphi(\eta_{j,\Lambda,\Delta,\xi})$, then there is nothing to prove. Thus, we may assume that $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) \nsim \varphi(\eta_{j,\Lambda,\Delta,\xi})$. Proposition 4.12 implies that $|Q| = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$ and that if N is a proper quotient of Q with natural projection $\theta : Q \to N$, then $\ker(\theta) \cap Z(Q) \cong Z(Q)$. Thus, we have that $\theta(\varphi(\gamma_{j,\Lambda,\Delta,\xi})) = \theta(\varphi(\eta_{j,\Lambda,\Delta,\xi}))$ since $\theta(\varphi(\xi_1)) = 1$. In particular, if Q is a $p_{j,\Lambda,\Delta,\xi}$-group where $\varphi(\Lambda) \cong \{1\}$ and $|Q| < (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}$, then $\varphi(\gamma_{j,\Lambda}) \sim \varphi(\eta_{j,\Lambda})$. Hence, we may assume that $q \neq p_{j,\Lambda,\Delta,\xi}$.

Now suppose that Q is a q-group where $q \neq p_{j,\Lambda,\Delta,\xi}$, Corollary 9.5 implies that there exists an element $g \in H_{\Lambda,\Delta,\xi}$ such that $\varphi(g^{-1}\, \gamma_{j,\Lambda,\Delta,\xi}\, g) = \varphi(\eta_{j,\Lambda,\Delta,\xi})$ as elements of $\varphi(H_{\Lambda,\Delta,\xi})$. Thus, $\varphi(\gamma_{j,\Lambda,\Delta,\xi}) \sim \varphi(\eta_{j,\Lambda,\Delta,\xi})$. Since we have exhausted all possibilities, it follows that

$$\mathrm{CD}_\Gamma(\gamma_{j,\Lambda,\Delta,\xi}, \eta_{j,\Lambda,\Delta,\xi}) = (p_{j,\Lambda,\Delta,\xi})^{\psi_{\mathrm{RF}}(\Gamma)}.$$

Hence,

$$n^{\psi_{\mathrm{RF}}(\Gamma)(c(\Gamma)-1)} \preceq \mathrm{Conj}_\Gamma(n).$$

Now suppose that Γ is an infinite, finitely generated nilpotent group where $|T(\Gamma)| > 1$. There exists a finite index, torsion-free, subgroup of Γ which we denote as Δ. Note that all torsion-free, finite index subgroups of Γ have the same step length. Let Δ/Λ be a maximal admissible quotient of Δ. Using above reasoning, there exists an element $x_\Lambda \in \Delta$ such that Δ/Λ is an admissible quotient with respect to x_Λ where $x_\Lambda^t = [y, z]$ for some $y \in \Delta_{c(\Delta)-1}$ and $z \in \Delta$. In particular, there exists a cyclic series $\{K_i\}_{i=1}^{h(\Delta)}$ and a compatible generating subset $\{\xi_i\}_{i=1}^{h(\Delta)}$ that together satisfy Proposition 3.6 for Λ where $\xi_1 = x_\Lambda$.

Let $\{p_{j,\Lambda,K,\xi}\}$ be an enumeration of primes greater than $\mathrm{B}(\Delta/\Lambda, \bar{K}, \bar{\xi})$. There exist an $a_{\Lambda,K,\xi} \in \Delta_{c(\Delta)-1}$ and $b_{\Lambda,K,\xi} \in \Delta$ such that

$$[a_{\Lambda,K,\xi}, b_{\Lambda,K,\xi}] = \xi_1^{t\ \mathrm{B}(\Delta/\Lambda,\bar{K},\bar{\xi})}.$$

Let

$$\gamma_{j,\Lambda,K,\xi} = (a_{\Lambda,K,\xi})^{p_{j,\Delta,K,\xi}}\ \xi_1^{t\ \mathrm{B}(\Delta/\Lambda,\bar{K},\bar{\xi})}$$

and

$$\eta_{j,\Lambda,\Delta,\xi} = (a_{\Lambda,K,\xi})^{p_{j,\Lambda,\bar{K},\bar{\xi}}}\ \xi_1^{2\,t\ \mathrm{B}(\Delta/\Lambda,\bar{K},\bar{\xi})}$$

be the elements from the above construction for Δ. Let

$$\rho : \Gamma \to \Gamma/T(\Gamma) \cdot \Gamma^{p_{j,\Lambda,K,\xi}}$$

be the natural projection. We have that

$$\rho(\gamma_{j,\Lambda,K,\xi}) \neq \rho(\eta_{j,\Lambda,K,\xi}) \quad \text{and} \quad \rho(\gamma_{j,\Lambda,K,\xi}), \rho(\eta_{j,\Lambda,K,\xi}) \neq 1$$

by construction. Additionally, $\pi_{T(\Gamma)}(\Delta)$ is a finite index subgroup of $\Gamma/T(\Gamma)$. Thus, [17, Lem 4.8(c)] implies that

$$Z(\pi_{T(\Gamma)}(\Delta)) = \pi_{T(\Gamma)}(\Delta) \cap Z(\Gamma/Z(\Gamma)).$$

Hence, $\pi_{T(\Gamma)}(\xi_1) \in Z(\Gamma/T(\Gamma))$. Since $\rho(\gamma_{j,\Lambda,K,\xi})$ and $\rho(\eta_{j,\Lambda,K,\xi})$ are unequal, central elements of $\Gamma/T(\Gamma) \cdot \Gamma^{p_{j,\Lambda,K,\xi}}$, we have that $\gamma_{j,\Lambda,K,\xi} \nsim \eta_{j,\Lambda,K,\xi}$.

Proposition 13.1 implies that

$$\mathrm{CD}_\Delta(\gamma_{j,\Lambda,K,\xi}, \eta_{j,\Lambda,K,\xi}) \leq \mathrm{CD}_\Gamma(\gamma_{j,\Lambda,K,\xi}, \eta_{j,\Lambda,K,\xi}).$$

By the above construction, we have that

$$(p_{j,\Lambda,K,\xi})^{\psi_{\mathrm{RF}}(\Delta)(c(\Delta)-1)} \leq \mathrm{CD}_\Gamma(\gamma_{j,\Lambda,K,\xi}, \eta_{j,\Lambda,K,\xi})$$

where

$$\|\gamma_{j,\Lambda,K,\xi}\|_S, \|\eta_{j,\Lambda,K,\xi}\|_S \approx (p_{j,\Lambda,K,\xi})^{1/(c(\Delta)-1)}.$$

If S' is a finite generating subset of Γ, then

$$\|\gamma_{j,\Lambda,K,\xi}\|_S \approx \|\gamma_{j,\Lambda,K,\xi}\|_{S'} \quad \text{and} \quad \|\eta_{j,\Lambda,K,\xi}\|_S \approx \|\eta_{j,\Lambda,K,\xi}\|_{S'}.$$

Hence,

$$\|\gamma_{j,\Lambda,K,\xi}\|_{S'}, \|\eta_{j,\Lambda,K,\xi}\|_{S'} \approx (p_{j,\Lambda,K,\xi})^{1/(c(\Delta)-1)},$$

and

$$(n_{j,\Lambda,K,\xi})^{\psi_{\mathrm{RF}}(\Delta)(c(\Delta)-1)} \preceq \mathrm{CD}_\Gamma(\gamma_{j,\Lambda,K,\xi}, \eta_{j,\Lambda,K,\xi}).$$

Since the projection to the torsion-free quotient $\pi_{T(\Gamma)} : \Gamma \to \Gamma/T(\Gamma)$ is injective when restricted to Δ, Δ is isomorphic to a finite index subgroup of $\Gamma/T(\Gamma)$, and thus, Theorem 1.3 implies that $\psi_{\mathrm{RF}}(\Delta) = \psi_{\mathrm{RF}}(\Gamma)$. Since $c(\Gamma/T(\Gamma)) = c(\Delta)$, we have that

$$n^{\psi_{\mathrm{Lower}}(\Gamma)} \preceq \mathrm{Conj}_\Gamma(n). \qquad \square$$

Theorem 1.9. *Let Γ and Δ be infinite, finitely generated nilpotents of step size greater than or equal to 2, and suppose that $\Gamma/T(\Gamma)$ and $\Delta/T(\Delta)$ have isomorphic Mal'tsev completions. Then*

$$n^{\psi_{\mathrm{Lower}}(\Gamma)} \preceq \mathrm{Conj}_\Delta(n) \quad \textit{and} \quad n^{\psi_{\mathrm{Lower}}(\Delta)} \preceq \mathrm{Conj}_\Gamma(n).$$

Proof. Suppose that Γ and Δ are two infinite, finitely generated nilpotent groups of step size 2 or greater such that $\Gamma/T(\Gamma)$ and $\Delta/T(\Delta)$ has isomorphic Mal'tsev completions. Proposition 7.8 implies that

$$\psi_{\mathrm{RF}}(\Gamma/T(\Gamma)) = \psi_{\mathrm{RF}}(\Delta/T(\Delta)).$$

By definition of $\psi_{\mathrm{RF}}(\Gamma)$ and $\psi_{\mathrm{RF}}(\Delta)$, we have that $\psi_{\mathrm{RF}}(\Gamma) = \psi_{\mathrm{RF}}(\Delta)$. Since $c(\Gamma/T(\Gamma)) = c(\Delta/T(\Delta))$, our theorem is now evident. $\square$

14. Proof of Theorem 1.10

Theorem 1.10. *For natural numbers $c > 1$ and $k \geq 1$, there exists an irreducible, torsion-free, finitely generated nilpotent group Γ of step length c such that*

$$n^k \preceq \mathrm{Conj}_\Gamma(n).$$

Proof. For each $s \in \mathbb{N}$, let Λ_s be the group given in Definition 8.7, and let $\theta : Z(\Lambda_s) \to Z(\Lambda_s)$ be the identity morphism. Let $c > 1$ and $m \geq 1$, and consider the group $\Gamma_{cm} = (\Lambda_{c+1} \circ_\theta)_{i=1}^m$ with a finite generating subset S_{cm}. Proposition 8.6 implies that $h(\Gamma_{cm}) = c\, m^2 + c\, m - 1$, and since $c^2\, m^2 + c^2\, m - 1 \geq m$, Theorem 1.8(ii) implies that $n^m \preceq \mathrm{Conj}_{\Gamma_{cm}}(n)$ as desired. □

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