New York Journal of Mathematics

New York J. Math. **24** (2018) 848–855.

On the index of certain standard congruence subgroups

Thomas Morgan

Abstract. For an epimorphism of the free group on two generators onto a finite group G, one can associate a finite index subgroup of the automorphism group of the free group called the standard congruence subgroup. We calculate the index of this group when G is a non-abelian semi-direct product of cyclic groups of prime order.

Contents

1. Introduction

There is a well-known surjective representation of the automorphism group of a free group $\rho_0 : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/[F_n, F_n]) \cong \mathrm{GL}_n(\mathbb{Z})$. The kernel of this representation is called the *Torelli subgroup*, denoted $IA(F_n)$, and the subgroup of $\mathrm{Aut}(F_n)$ whose elements have determinant 1 under ρ_0 is called the *special automorphism group*, denoted $\mathrm{Aut}^+(F_n)$. While this linear representation of $\mathrm{Aut}^+(F_n)$ is well studied, only a few other representations were studied until 2006, when Grunewald and Lubotzky[GL09] published a paper detailing the construction of a family of virtual linear representations of $\mathrm{Aut}(F_n)$ indexed by finite groups G and surjective homomorphisms $\pi : F_n \to G$. This gave rise to a generalization of the Torelli subgroup different from the Johnson filtration and proved that $\mathrm{Aut}(F_3)$ is large, which implies that it does not have Kazdhan's property (T).

Received April 23, 2018.
2010 *Mathematics Subject Classification.* 20F28.
Key words and phrases. automorphism groups, free groups, congruence subgroups.

ISSN 1076-9803/2018

In constructing these representations, Grunewald and Lubotzky used the subgroup $\Gamma(G,\pi) \leqslant \mathrm{Aut}(F_n)$. This subgroup is called the *standard congruence subgroup* of $\mathrm{Aut}(F_n)$ associated to G and π. The subgroup is defined as follows: let $R := \ker(\pi)$. Then the action of F_n on R by conjugation leads to an action of G on the relation module $\bar{R} := R/[R,R]$. Define $\Gamma(G,\pi) := \{\varphi \in \mathrm{Aut}(F_n) \mid \varphi(R) = R,\ \varphi \text{ induces identity on } F_n/R \cong G\}$. This is exactly the G-equivariant automorphisms under this action. It is also analogous to congruence subgroups, which are extensively studied for arithmetic groups.

Following Grunewald and Lubotzky's paper, Appel and Ribnere [AR09] began a more systematic study of these standard congruence subgroups in the case where $n = 2$. First they restricted themselves to $\Gamma^+(G,\pi) = \Gamma(G,\pi) \cap \mathrm{Aut}^+(F_2)$. Then they computed the index $[\mathrm{Aut}^+(F_2) : \Gamma^+(G,\pi)]$ for G abelian or dihedral. In doing so, and with some further analysis, they gave some partial results to the congruence subgroup problem for $\mathrm{Aut}^+(F_2)$. Appel and Ribnere also posed a conjecture stating the index when G is the non-abelian semidirect product of two cyclic groups of prime order. We prove their conjecture.

Theorem 1.1. *Let G be the non-abelian semidirect product of two cyclic groups, $G = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$, where p and q are primes with $p \equiv_q 1$. Then*

$$[Aut^+(F_2) : \Gamma^+(G,\pi)] = |G| \cdot [SL_2(\mathbb{Z}) : \Gamma_1(q)] = pq(q^2-1).$$

Here $\Gamma_1(q) := \left\{ \begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) \;\middle|\; \delta \equiv_q 0, \epsilon \equiv_q 1 \right\}$. Note that this is true independent of the choice of π. We prove this in Section 2. The second equality follows from the study of congruence subgroups (see for example [DS05]). Then in Section 3, we prove the first equality by using the primitive elements constructed in [OZ81] to construct enough automorphisms in $\Gamma^+(G,\pi)$ to prove that $\rho_0(\Gamma^+(G,\pi)) = \Gamma_1(q)$. The equality then follows from the following proposition from [AR09].

Proposition 1.2 (Appel, Ribnere)**.** *Let $\pi : F_2 \to G$ be an epimorphism of F_2 onto a finite group G. Then*

$$[Aut^+(F_2) : \Gamma^+(G,\pi)] = [SL_2(\mathbb{Z}) : \rho_0(\Gamma^+(G,\pi))] \cdot [G : Z(G)].$$

Here $\rho_0 : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/[F_n,F_n]) \cong \mathrm{GL}_n(\mathbb{Z})$ is the representation introduced earlier.

2. Independence of the choice of π

2.1. A description of $\mathrm{Aut}(G)$. Let $p,q \in \mathbb{N}$ be primes such that $p \equiv_q 1$. Then the non-abelian semi-direct product $G := \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ has the following presentation:

$$G = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^\lambda \rangle$$

for some $1 < \lambda < p$ and $\lambda^q \equiv_p 1$ ($\lambda = 1$ would be the abelian case). Indeed, let $\mathbb{Z}/p\mathbb{Z} = \langle a \rangle$ and $\mathbb{Z}/q\mathbb{Z} = \langle b \rangle$. Then the above presentation comes from the outer semi-direct product associated with the homomorphism $\varphi : \mathbb{Z}/q\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ such that $\varphi(b)(a) = a^\lambda$.

Let $F_2 = \langle x, y \rangle$ be the free group on two generators. To see that the index we wish to calculate is independent of the choice of $\pi : F_2 \to G$, we look at an action of $\mathrm{Aut}^+(F_2)$ on the following set. We define

$$\mathbf{R}_2(G) := \{\ker(\pi) \mid \pi : F_2 \to G \text{ is an epimorphism}\}.$$

We can define an action of $\mathrm{Aut}^+(F_2)$ on $\mathbf{R}_2(G)$ by

$$\varphi.R := \varphi(R) \text{ for } \varphi \in \mathrm{Aut}^+(F_2), R \in \mathbf{R}_2(G).$$

If the action is transitive, then $[\mathrm{Aut}^+(F_2) : \Gamma^+(G, \pi)]$ is independent of the choice of π. Indeed, let $\pi, \pi' : F_2 \to G$ be epimorphisms, and assume that there is some $\varphi \in \mathrm{Aut}^+(F_2)$ such that $\varphi.\ker(\pi) = \ker(\pi')$. Since $\varphi.\ker(\pi) = \ker(\pi \circ \varphi^{-1})$, we have that $\Gamma^+(G, \pi') = \{\varphi \circ \psi \circ \varphi^{-1} \mid \psi \in \Gamma^+(G, \pi)\}$. Since $\Gamma^+(G, \pi)$ and $\Gamma^+(G, \pi')$ are conjugate subgroups of $\mathrm{Aut}^+(F_2)$, we conclude that $[\mathrm{Aut}^+(F_2) : \Gamma^+(G, \pi)] = [\mathrm{Aut}^+(F_2) : \Gamma^+(G, \pi')]$.

In order to show that this action is transitive, it is helpful to first understand the automorphisms of the group G. To that effect, we prove the following lemma.

Lemma 2.1. *For each $0 < i < p, 0 \leqslant j < p$ there is a unique automorphism $\varphi_{i,j} : G \to G$ such that $\varphi(a) = a^i$, $\varphi(b) = a^j b$. Moreover $Aut(G) = \{\varphi_{i,j} \mid 0 < i < p, 0 \leqslant j < p\}$.*

Proof. Let $0 < i < p$, $0 \leqslant j < p$. Since $\{a, b\}$ is a generating set of G, we can define $\varphi_{i,j}$ on $\{a, b\}$ as above and extend to a homomorphism in a unique way. To see this is a well-defined homomorphism, we will check that it satisfies the relations. First, we have that

$$\varphi(a)^p = (a^i)^p = a^{ip} = (a^p)^i = 1^i = 1.$$

Since $p \nmid (1 - \lambda)$ and $\lambda^q \equiv_p 1$, it follows that

$$\varphi(b)^q = (a^j b)^q = a^j a^{j\lambda} \ldots a^{j\lambda^{q-1}} b^q = a^r,$$

where

$$r = \sum_{i=0}^{q-1} j\lambda^i = j \sum_{i=0}^{q-1} \lambda^i = j\Big(\frac{1 - \lambda^q}{1 - \lambda}\Big) \equiv_p 0.$$

Finally, we have that

$$\varphi(b)\varphi(a)\varphi(b)^{-1} = a^j b a^i b^{-1} a^{-j} = a^j a^{\lambda i} a^{-j} = (a^i)^\lambda = \varphi(a)^\lambda.$$

Thus φ is a homomorphism.

To see this map is surjective, note that a^i is a generator of $\langle a \rangle$ for $0 < i < p$. It follows that $a \in \mathrm{Im}(\varphi_{i,j})$. Then $a, a^j b \in \mathrm{Im}(\varphi_{i,j}) \implies b \in \mathrm{Im}(\varphi_{i,j})$. This shows that $\mathrm{Im}(\varphi_{i,j})$ contains a generating set, so $\varphi_{i,j}$ is surjective. Because G is finite, this is enough to show that $\varphi_{i,j}$ is an automorphism.

Now let H denote the set $\{\varphi_{i,j} \mid 0 < i < p, 0 \leqslant j < p\}$. We have shown that $H \subset \mathrm{Aut}(G)$. Thus it remains to show the reverse inclusion.

Let $\varphi \in \mathrm{Aut}(G)$. Since $\langle a \rangle$ contains all of the elements of order p in G,

$$\varphi(a) = a^i \text{ for some } 0 < i < p.$$
$$\varphi(b) = a^j b^k \text{ for some } 0 \leqslant j < p, 0 < k < q.$$

On the one hand,

$$\varphi(bab^{-1}) = \varphi(a^\lambda) = a^{i\lambda}.$$

On the other hand,

$$\varphi(bab^{-1}) = \varphi(b)\varphi(a)\varphi(b)^{-1} = a^j b^k a^i b^{-k} a^{-j} = a^j a^{i\lambda^k} b^k b^{-k} a^{-j} = a^{i\lambda^k}.$$

Thus $a^{i\lambda} = a^{i\lambda^k}$. Since a is of order p and p does not divide i, it follows that $\lambda^{k-1} \equiv_p 1$. We know that λ is of order q, so $q \mid (k-1)$. But $0 < k < q$, so $k = 1$. Thus $\mathrm{Aut}(G) \leqslant H$.

□

2.2. Independence of the choice of $\boldsymbol{\pi}$. We will now show that the action of $\mathrm{Aut}^+(F_2)$ on $\mathbf{R}_2(G)$ defined above is transitive. This will mean that the index $[\mathrm{Aut}^+(F_2) : \Gamma^+(G,\pi)]$ is independent of the choice of π. Thus we will be able to compute the index using the epimorphism

$$\begin{aligned} \pi_0 : \quad & F_2 \to G \\ & x \mapsto a \\ & y \mapsto b \end{aligned}$$

Lemma 2.2. *The action of $Aut^+(F_2)$ on the set $\mathbf{R}_2(G)$ is transitive.*

Proof. Let $\pi : F_2 \to G$ be an arbitrary epimorphism of F_2 onto G. Then for some $i, j, k, \ell \in \mathbb{Z}$, we have $\pi(x) = a^i b^j$, $\pi(y) = a^k b^\ell$. Let $\beta : G \to \mathbb{Z}/q\mathbb{Z}$, $\beta(a^m b^n) = n$. Then $\beta \circ \pi$ is an epimorphism of F_2 onto $\mathbb{Z}/q\mathbb{Z}$. The following diagram commutes:

$$F_2 \xrightarrow{\alpha} (\mathbb{Z}/q\mathbb{Z})^2 \xrightarrow{\delta} (\mathbb{Z}/q\mathbb{Z}), \qquad \beta\circ\pi : F_2 \to (\mathbb{Z}/q\mathbb{Z})$$

where $\alpha(x) = (1,0)$, $\alpha(y) = (0,1)$, and $\delta(m,n) = mj + n\ell$. Since $\delta \circ \alpha$ is surjective, there is an element $(m,n) \in (\mathbb{Z}/q\mathbb{Z})^2$ such that $\delta(m,n) = 1$. Furthermore, since δ is not injective, there is a non-zero element (u,v) such that $\delta(u,v)=0$. It is clear that (m,n) and (u,v) are linearly independent, so $d := \begin{vmatrix} u & m \\ v & n \end{vmatrix} \not\equiv_q 0$. Thus we can choose a $\tilde{d} \in \mathbb{Z}$ such that $d\tilde{d} \equiv_q 1$. The vector $(\tilde{d}u, \tilde{d}v)$ is in the kernel of δ since (u,v) is, and $M := \begin{pmatrix} \tilde{d}u & m \\ \tilde{d}v & n \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}/q\mathbb{Z})$.

Choose a matrix $N \in \mathrm{Sl}_2(\mathbb{Z})$ such that $N \equiv_q M$. Let $\rho_0 : \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/[F_n,F_n]) \cong \mathrm{GL}_n(\mathbb{Z})$ be the homomorphism described in the introduction. Because ρ_0 is surjective, we may choose an automorphism $\phi \in \mathrm{Aut}^+(F_2)$ such that $\rho_0(\phi) = N$. It follows that $\delta \circ \alpha \circ \phi(x) = 0$, $\delta \circ \alpha \circ \phi(y) = 1$. Thus $\pi \circ \phi(x) = a^g$ and $\pi \circ \phi(y) = a^h b$ for some $g, h \in \mathbb{Z}$. By Lemma 2.1, $\varphi_{g,h}^{-1} \circ \pi \circ \phi = \pi_0$. It follows that $\ker(\pi)$, $\ker(\pi_0)$ lie in the same $\mathrm{Aut}^+(F_2)$ orbit. □

3. Proof of Theorem 1.1

3.1. Image of primitive elements. Let p, q be as above. Now that we know the action of $\mathrm{Aut}^+(F_2)$ on $\mathbf{R}_2(G)$ is transitive, we need to show that $\rho_0(\Gamma^+(G,\pi_0)) = \Gamma_1(q)$. Here $\Gamma_1(q) := \left\{ \begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) \;\middle|\; \delta \equiv_q 0, \epsilon \equiv_q 1 \right\}$. To do this, we use the description of primitive elements from [OZ81] to construct elements of $\Gamma^+(G,\pi_0)$. In Section 2 of $[OZ]$, given $\alpha, \delta \in \mathbb{Z}^+$ such that $\gcd(\alpha,\delta)=1$, Osbourne and Zieschang outline a geometric construction of a primitive element $v_{\alpha,\delta}$ containing α copies of x and δ copies of y which we reproduce here. Draw a directed line segment from $(0,0)$ to (α,δ). We use this line segment to generate a word in F_2. Starting at $(0,0)$, every time the segment passes a vertical integer grid line write an x, and every time the segment passes a horizontal integer grid line write a y. Call the resulting word $v'_{\alpha,\delta}$. Define $v_{\alpha,\delta} := xyv'_{\alpha,\delta}$. Lemma 2.3 of [OZ81] shows that $v_{\alpha,\delta}$ is primitive by relating it to a construction earlier in the paper which is algebraically shown to be primitive. We use these primitive elements and their geometric construction in the following lemma:

Lemma 3.1. *Let $\alpha, \delta \in \mathbb{Z}^+$ such that $gcd(\alpha,\delta) = 1$ and $q|\delta$. Then $\pi_0(v_{\alpha,\delta}) = a^z b^\delta$ for some $z \in \mathbb{Z}$ which depends only on α (mod q).*

Proof. Note that the x's in $v'_{\alpha,\delta}$ correspond to points $(i, i\delta/\alpha)$ for $0 < i < \alpha$. Given each x in $v'_{\alpha,\delta}$ we want to consider what it will take to move it to the front of the word. That is, we want to count the number of y's that occur before each x. For the ith x, this is precisely the integer part of $i\delta/\alpha$. This is equal to $(i\delta - r_i)/\alpha$ where $0 \leqslant r_i < \alpha$ is the remainder when $i\delta$ is divided by α. In terms of the image of $v'_{\alpha,\delta}$ under π, we only care about the number of y's mod q. Since $\gcd(\alpha,\delta)=1$ and $q|\delta$, we have $\gcd(q,\alpha)=1$. Thus there exists an $\tilde{\alpha} \in \mathbb{Z}$ such that $\alpha\tilde{\alpha} \equiv_q 1$. It follows that $(i\delta - r_i)/\alpha \equiv_q -\tilde{\alpha} r_i$ since $q|\delta$. Therefore, after commuting the ith a in the image of $v'_{\alpha,\delta}$ to the front of the word it becomes $a^{\lambda^{-\tilde{\alpha} r_i}}$. Thus the image of $v'_{\alpha,\delta}$ under π is $a^s b^{\delta-1}$ where

$$s = \sum_{i=1}^{\alpha-1} \lambda^{-\tilde{\alpha} r_i} = \sum_{i=1}^{\alpha-1} (\lambda^{-\tilde{\alpha}})^{r_i}.$$

Consider the list of remainders $(r_1, r_2, \ldots, r_{\alpha-1})$. By the above considerations, this list completely determines the power of a in $\pi(v'_{\alpha,\delta})$. Since $\gcd(\alpha, \delta) = 1$, $[\delta] \in \mathbb{Z}/\alpha\mathbb{Z}$ is a generator. Furthermore, by definition $[r_i] = i[\delta]$ in $\mathbb{Z}/\alpha\mathbb{Z}$. Thus up to reordering, $(r_1, r_2, \ldots r_{\alpha-1}) = (1, 2, \ldots, \alpha - 1)$. As this is true for any choice of δ meeting our requirements, for fixed α the power of a in $v'_{\alpha,\delta}$ and hence $v_{\alpha,\delta}$ is independent of our choice of δ.

Now fix δ. If $\alpha_2 = \alpha_1 + q$, then we get two different lists of remainders. They are $(1, 2, \ldots, \alpha_1 - 1)$ and $(1, 2, \ldots, \alpha_2 - 1) = (1, 2, \ldots, q, q + 1, q + 2, \ldots, q + \alpha_1 - 1)$. Let $\lambda' := \lambda^{-\tilde{\alpha}_2}$. Note that $\alpha_1 \equiv_q \alpha_2 \implies \tilde{\alpha}_1 \equiv_q \tilde{\alpha}_2$. Thus $\lambda' = \lambda^{-\tilde{\alpha}_1}$. Plugging this into our formula for the power of a in $\pi(v'_{\alpha,\delta})$ and noting that $(\lambda')^q \equiv_p 1$, we get

$$\begin{aligned}
\sum_{i=1}^{\alpha_2-1} (\lambda')^{r_i} &= \sum_{i=1}^{q} (\lambda')^{r_i} + \sum_{i=q+1}^{q+\alpha_1-1} (\lambda')^{r_i} \\
&\equiv_p \lambda' \Big(\frac{1 - (\lambda')^q}{1 - \lambda'} \Big) + \sum_{i=1}^{\alpha_1-1} (\lambda')^{r_i} \\
&\equiv_p \sum_{i=1}^{\alpha_1-1} (\lambda')^{r_i}.
\end{aligned}$$

Thus $\pi_0(v'_{\alpha_1,\delta}) = \pi_0(v'_{\alpha_2,\delta})$. By induction, we see that $\pi_0(v'_{\alpha,\delta})$ and hence $\pi_0(v_{\alpha,\delta})$ only depends upon α mod q. □

3.2. Proof of Theorem 1.1. With Proposition 1.2 and our lemmas, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.2, we may assume that $\pi = \pi_0$. Noting that $Z(G) = 1$ so $[G : Z(G)] = |G|$, by Proposition 1.2 it suffices to show that $\rho_0(\Gamma^+(G, \pi)) = \Gamma_1(q)$. Since $\rho_0(\Gamma^+(G, \pi)) \leqslant \rho_0(\Gamma^+(G^{\text{ab}}, \bar{\pi})) = \Gamma_1(q)$ where $f \circ \pi = \bar{\pi}$ and $f : G \to G^{\text{ab}}$ is the abelianization map, it remains to show $\rho_0(\Gamma^+(G, \pi)) \geqslant \Gamma_1(q)$.

Let $A = \begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \in \Gamma_1(q)$. Then let φ be the automorphism determined by

$$\begin{aligned}
\varphi : \quad & F_2 \to F_2 \\
& x \mapsto v_{\alpha,\delta} \\
& y \mapsto v_{\beta,\epsilon}
\end{aligned}$$

By Theorem 1.2 of [OZ81], we have that $v_{\alpha,\delta}$ and $v_{\beta,\epsilon}$ generate F_2. Thus $\varphi \in \text{Aut}^+(F_2)$. Let $\pi(v_{\beta,\epsilon}) = a^j b$. It must be of this form since $\epsilon \equiv_q 1$ and b

has order q.

First assume $\alpha, \delta > 0$. Since $p \nmid (\lambda - 1)$, there exists an $\ell \in \mathbb{Z}$ such that $(\lambda-1)\ell \equiv_p j$. Then

$$\pi(x^\ell v_{\beta,\epsilon} x^{-\ell}) = a^\ell a^j b a^{-\ell} = a^\ell a^j a^{-\lambda\ell} b = a^{(1-\lambda)\ell + j} b = b$$

By Lemma 3.1, the a exponent of $\pi(v_{\alpha,\delta})$ only depends upon α mod q. But $\alpha \equiv_q 1$ and $v_{1,\delta} = xy^\delta$ by direct computation. Thus

$$\pi(x^\ell v_{\alpha,\delta} x^{-\ell}) = a^\ell a a^{-\ell} = a$$

This shows that $c \circ \varphi \in \Gamma^+(G,\pi)$ where c is conjugation by x^ℓ. Since $\rho_0(c \circ \varphi) = A$ by construction, this shows $A \in \rho_0(\Gamma^+(G,\pi))$.

We now consider the case where α and δ are arbitrary. By the above argument, $\begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix} \in \rho_0(\Gamma^+(G,\pi))$. Furthermore, for the correct choice of ℓ as above, the automorphism

$$\begin{aligned} \psi : \quad & F_2 \to F_2 \\ & x \mapsto x \\ & y \mapsto x^{\ell+1} y x^{-\ell} \end{aligned}$$

is in $\Gamma^+(G,\pi)$, and $\rho_0(\psi) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \rho_0(\Gamma^+(G,\pi))$. For general α, δ, we may write A as a product of powers of these matrices and some $A' = \begin{bmatrix} \alpha' & \beta' \\ \delta' & \epsilon' \end{bmatrix} \in \Gamma_1(q)$ with $\alpha', \delta' > 0$. This shows that $A \in \rho_0(\Gamma^+(G,\pi))$. □

References

[AR09] Appel, Daniel; Ribnere, Evija. On the index of congruence subgroups of Aut(F_n). *J. Algebra* **321** (2009), no. 10, 2875–2889. MR2512632 (2010d:20039), Zbl 1178.20036, arXiv:0804.2578, doi: 10.1016/j.jalgebra.2009.01.022. 849

[DS05] Diamond, Fred; Shurman, Jerry. A first course in modular forms. Graduate Texts in Mathematics, 228. *Springer-Verlag, New York*, 2005. xvi+436 pp. ISBN: 0-387-23229-X. MR2112196 (2006f:11045), Zbl 1062.11022, doi: 10.1007/978-0-387-27226-9. 849

[GL09] Grunewald, Fritz; Lubotzky, Alexander. Linear representations of the automorphism group of a free group. *Geom. Funct. Anal.* **18** (2009), no. 5, 1564–1608. MR2481737 (2010i:20039), Zbl 1175.20028, arXiv:math/0606182, doi: 10.1007/s00039-009-0702-2. 848

[LS01] Lyndon, Roger C.; Schupp, Paul E. Combinatorial group theory. Reprint of the 1977 edition. Classics in Mathematics. *Springer-Verlag, Berlin*, 2001. xiv+339 pp. ISBN: 3-540-41158-5. MR1812024 (2001i:20064), Zbl 0997.20037, doi: 10.1007/978-3-642-61896-3.

[OZ81] Osborne, Richard P.; Zieschang, Heiner. Primitives in the free group on two generators. *Invent. Math.* **63** (1981), no. 1, 17–24. MR0608526 (82i:20042), Zbl 0438.20017, doi: 10.1007/BF01389191. 849, 852, 853

(Thomas Morgan) Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA
tmorgan@ou.edu

This paper is available via http://nyjm.albany.edu/j/2018/24-38.html.