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Simplicial complexity: piecewise linear motion planning in robotics

Jesús González

Abstract. Using the notion of contiguity of simplicial maps, and its relation (via iterated subdivisions) to the notion of homotopy between continuous maps, we adapt Farber's topological complexity to the realm of simplicial complexes. We show that, for a finite simplicial complex K, our discretized concept recovers the topological complexity of the realization $\|K\|$. Our approach lays the theoretical grounds for designing and implementing algorithms that search for optimal motion planners for autonomous systems in real-life applications.

Contents

1. Introduction

For a topological space X, let $P(X)$ stand for the free path space on X endowed with the compact-open topology. Farber's topological complexity $\mathrm{TC}(X)$ is defined as the sectional category of the evaluation map $e\colon P(X) \to X \times X$, $e(\gamma) = (\gamma(0), \gamma(1))$. Here we use the reduced form of the resulting homotopy invariant, namely a contractible space has zero topological complexity. In other words, $\mathrm{TC}(X) + 1$ is the smallest cardinality of open covers $\{U_i\}_i$ of $X \times X$ so that e admits a continuous section σ_i on each U_i. The open sets U_i in such an open cover are called *local domains*, the corresponding sections σ_i are called *local rules*, and the family of pairs $\{(U_i, \sigma_i)\}$ is called a *motion planner* for X. A motion planner is said to be optimal if it has $\mathrm{TC}(X) + 1$ local domains. In view of the continuity requirement on local rules, an optimal motion planner for the configuration

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space of a given robot gives us a way to minimize the possibility of accidents in the programming of the robot's performance in noisy environments. In short, topological complexity provides us with a topological framework for studying the motion planning problem in robotics.

Due to its homotopy nature, Farber's idea quickly attracted the attention of topologists, who began to develop the theoretical aspects of topological complexity. In particular, a number of methods have emerged to estimate $\mathrm{TC}(X)$ for families of spaces X. For instance, (co)homological methods have proven to be useful (and accesible) for bounding from below $\mathrm{TC}(X)$, while sophisticated (but hard-to-deal-with) obstruction-theoretic methods have been used to get upper bounds. In some cases, the power of the algebraic topology toolbox leads to the actual computation of $\mathrm{TC}(X)$ —usually, however, without giving a clue about how to construct explicit optimal motion planners. Such successful cases hold most notably when X is a symplectic simply-connected closed manifold. In those cases the cohomology lower bound agrees with the simplest possible homotopy-obstruction upper bound (that is, the scenario in which all possible obstructions lie in groups which vanish just by simple dimensional reasons). But in other less fortunate cases the cohomology lower bound falls far from the simplest obstruction-theory upper bound. In such cases, as is well known by experts, trying to improve the upper bound by direct analysis of homotopy obstructions can be a major (and potentially inaccessible) task, especially when several "layers" of obstructions are involved. Such characteristics of the current TC development have been a main obstacle for the actual applicability of the TC ideas to problems arising from real-life needs.

The present paper aims at mending the above situation. Our goal is to lay the theoretical grounds for an eventual construction of (potentially optimal) motion planners through computer-implementable algorithms. The idea is to combine computational topology methods with heuristic processes in order to replace the hard (non-algorithmic, and often prohibitive) analysis of homotopy obstructions for estimating $\mathrm{TC}(X)$ from above. Indeed, the ultimate goal would be that powerful computer resources become a real option to inaccessible theoretic calculations.

A previous attempt to discretize (rather to approach combinatorially) Farber's TC appeared in [6, Example 4.5], where topological complexity is developed in the context of finite spaces. However, the resulting concept appears to be too rigid, and in fact it fails to detect the well known equality $\mathrm{TC}(S^1) = 1$. Indeed, the best estimate coming from Tanaka's model is $\mathrm{TC}(S^1) \leq 3$.

Another viewpoint for discretizing Farber's topological complexity has recently been proposed in [3]. Although Fernández-Ternero, Macías-Virgós, Minuz and Vilches employ techniques based on the notion of contiguity (as we do), there is a substantial difference between our approach and theirs. Namely, the authors of [3] use Barmak-Minian's concept of strong homotopy type to *import* homotopy-like notions of continuous maps into the combinatorics realm. On the other hand, in our case, the homotopy notion is *translated* into combinatorial terms by using a well known fact in combinatorial topology: Homotopy classes of (continuous) maps between the geometric realizations of abstract simplicial complexes can be recovered (via simplicial approximation) as contiguity classes of simplicial maps between the (suitably subdivided) original complexes. In our model, the use of the barycentric subdivision functor yields a much more flexible invariant. For instance, while the model in [3] improves Tanaka's estimate to $\mathrm{TC}(S^1) \leq 2$, we are able to recast the equality $\mathrm{TC}(S^1) = 1$ in purely combinatorial terms. In fact, we prove that, for any abstract complex K, our discretized topological complexity of K agrees with Farber's topological complexity of the geometric realization $\|K\|$.

Our approach is fully algorithmizable and can be implemented in a computer in order to explore the topological complexity of compact polyhedra. In this regard, the reader should be aware that the resulting search space grows exponentially with the number of iterated barycentric subdivisions used (cf. Remark 4.3). Such a computational characteristic leads to the need of designing and implementing heuristic algorithms for the search and optimization of discretized motion planners. The final section in this paper provides a benchmark for testing and comparing eventual implementations.

Our idea rests on the observation that the sectional category of a fibration $p\colon E \to B$ over a CW complex B can be defined in terms of the existence of "local" sections of p on the elements of a covering of B by Euclidean neighborhood retracts (e.g. subcomplexes) —instead of by open sets. In particular, in the case of the fibration defining $\mathrm{TC}(X)$, the following result, whose proof is elementary (compare to [2, Lemma 4.21]), allows us to reduce the resulting sectioning problem to a standard homotopy problem, which will be translated in the next section into purely simplicial terms.

Lemma 1.1. *The evaluation map $e\colon P(X) \to X \times X$ admits a section on a subset A of $X \times X$ if and only if the two compositions $A \hookrightarrow X \times X \xrightarrow{\pi_1} X$ and $A \hookrightarrow X \times X \xrightarrow{\pi_2} X$ are homotopic.*

2. Preliminaries

This section is devoted to fixing notation and reviewing homotopy-type properties of the categories of simplicial complexes and their realizations. For details, the reader should consult standard references, such as [5, Chapter 3].

We work with abstract simplicial complexes K, referred here as "complexes", with simplicial maps φ between complexes, and with their corresponding topological realizations $\|K\|$ and $\|\varphi\|$. With an eye on applications, we will only care about finite complexes. The finiteness hypothesis will allow us to prove that our discrete model for topological complexity recasts the original concept.

For a point $x \in \|K\|$, the barycentric coordinate of x with respect to a vertex[1] v of K is denoted by $x(v) \in [0,1]$. The simplex $\sigma_x \in K$ carrying x consists of those vertices $v \in K$ having $x(v) > 0$. For instance, $\sigma_v = v$ when v is a vertex of K. For a simplex $\sigma \in K$, we think of $\|\sigma\|$ as the obvious subspace of $\|K\|$; the corresponding open simplex $\langle\sigma\rangle \subseteq \|\sigma\|$ consists of the points $x \in \|K\|$ having $\sigma_x = \sigma$. In other words,

$$\langle\sigma\rangle = \{x \in K\colon x(v) > 0 \text{ if and only if } v \in \sigma\}.$$

Note that $\|\sigma\|$ is the closure of $\langle\sigma\rangle$, that $\langle\varnothing\rangle = \varnothing$, and that the set underlying $\|K\|$ is the disjoint union $\coprod_{\sigma\in K}\langle\sigma\rangle$.

Definition 2.1. Let K and L be complexes. A (simplicial) approximation of a continuous map $f\colon \|K\| \to \|L\|$ is a simplicial map $\varphi\colon K \to L$ such that $\|\varphi\|(x) \in \|\sigma\|$ whenever $x \in \|K\|$ and $f(x) \in \langle\sigma\rangle$.

Example 2.2. A simplicial map $\varphi\colon K \to L$ is the only approximation of the geometric realization $\|\varphi\|$.

Example 2.3. For any subdivision K' of a complex K, the standard piecewise linear homeomorphism $\|K'\| \xrightarrow{=} \|K\|$ admits an approximation $K' \to K$. Indeed any map ι from the vertices of K' to the vertices of K with the property that $v'(\iota(v')) > 0$, for any vertex v' of K', is in fact an approximation of $\|K'\| \xrightarrow{=} \|K\|$. Actually, such vertex-maps ι are the only approximations of $\|K'\| \xrightarrow{=} \|K\|$.

Example 2.3 will be most important for $K' = \mathrm{Sd}(K)$, the barycentric subdivision of K and, more generally, for $K' = \mathrm{Sd}^b(K)$, the b-fold iterated barycentric subdivision of K ($\mathrm{Sd}^{b+1}(K) = \mathrm{Sd}(\mathrm{Sd}^b(K))$).

Remark 2.4. Approximations behave well with respect to compositions: If

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} M$$

are respective approximations of

$$\|K\| \xrightarrow{f} \|L\| \xrightarrow{g} \|M\|,$$

then $\psi\varphi$ is an approximation of gf.

Next we recast the notion of contiguity of simplicial maps in a form suitable for the computational applications we have in mind.

Definition 2.5. Let c be a positive integer. Two simplicial maps φ, φ' : $K \to L$ are called:

(1) contiguous (or 1-contiguous) provided $\varphi(\sigma) \cup \varphi'(\sigma)$ is a simplex of L for any simplex σ of K.

[1] We do not make a distinction between a vertex v of K, the corresponding 0-dimensional simplex $\{v\}$, and the corresponding point $v \in \|K\|$.

(2) c-contiguous if there is a sequence of maps $\varphi_0, \varphi_1, \cdots, \varphi_c : K \to L$, with $\varphi_0 = \varphi$ and $\varphi_c = \varphi'$, such that φ_{i-1} and φ_i are contiguous for each $i \in \{1, 2, \ldots, c\}$.

Note that it is enough to verify the condition in part (1) of Definition 2.5 on maximal simplexes σ of K. We will say that the sequence of maps φ_j in part (2) of Definition 2.5 is a contiguity chain of length c between φ and φ', and we will then write $\varphi \sim_c \varphi'$. We will also write $\varphi \sim \varphi'$ to mean $\varphi \sim_c \varphi'$ for some c. This defines an equivalence relation in the set of simplicial maps $K \to L$. The corresponding equivalence class of φ is denoted by $[\varphi]$, and is called the contiguity class of φ.

Remark 2.6. Composition of contiguity classes is well defined at the level of representatives. Indeed, for simplicial maps

$$J \xrightarrow{\psi} K \xrightarrow{\varphi,\varphi'} L \xrightarrow{\omega} M,$$

$\omega\varphi\psi$ and $\omega\varphi'\psi$ are c-contiguous provided φ and φ' are so.

The importance of the notion of contiguity of simplicial maps stems from its close relationship to the notion of homotopy of continuous maps. The relationship becomes a full translation when iterated barycentric subdivisions are allowed. Informally, the topological realization construction yields a one-to-one correspondence between the contiguity classes of simplicial maps $K \to L$ (with a "highly enough" subdivided K) and the homotopy classes of continuous maps $\|K\| \to \|L\|$. The explicit property is stated in the following omnibus result. Recall we only consider finite complexes.

Theorem 2.7. *Existence and uniqueness of approximations:*

(1) *Two approximations of the same continuous map are* 1*-contiguous. Consequently, if it exists, the contiguity class of approximations of a given continuous map is unique.*
(2) *For any continuous map $f\colon \|K\| \to \|L\|$ there is some non-negative integer b_0 such that, for each $b \geq b_0$, f admits an approximation $\varphi_b\colon \mathrm{Sd}^b(K) \to L$. (Recall that $\|\mathrm{Sd}^b(K)\| = \|K\|$.) Consequently, if $\iota\colon \mathrm{Sd}^{b+1}(K) \to \mathrm{Sd}^b(K)$ is any approximation of the identity on $\|K\|$, then $\varphi_b\iota$ and φ_{b+1} are 1-contiguous.*

Relationship between contiguity and homotopy:

(3) *Simplicial maps in the same contiguity class have homotopic topological realizations.*
(4) *Given homotopic maps $f_0, f_1\colon \|K\| \to \|L\|$, there is a non-negative integer b_0 such that, for each $b \geq b_0$, any pair of approximations $\varphi_0, \varphi_1\colon \mathrm{Sd}^b(K) \to L$ of f_0, f_1, respectively, satisfy $\varphi_0 \sim_c \varphi_1$ for some $c \geq 0$ (c depends on φ_0 and φ_1).*

The facts reviewed in this section will be freely used throughout the paper.

3. Simplicial complexity

As suggested by Lemma 1.1, we will need to consider a simplicial structure on a product of complexes in such a way that the topological realization of the product recovers the product of the topological realizations of the factors. Consequently, all complexes we deal with will be assumed to be ordered, and their product will be taken in the category of ordered complexes (see for instance [1] for the classical details on the construction). However, we stress that maps of complexes will *not* be required to preserve the given orderings.

A collection $\mathcal{C}$ of subcomplexes of K is a cover if $K = \bigcup_{L\in\mathcal{C}} L$ (of course, in such a case, $\bigcup_{L\in\mathcal{C}} \|L\| = \|K\|$). For a non-negative integer b, fix an approximation

$$\iota\colon \mathrm{Sd}^{b+1}(K\times K) \to \mathrm{Sd}^{b}(K\times K) \tag{3.1}$$

of the identity on $\|K\|\times\|K\|$. By abuse of notation, iterated compositions of these maps will also be denoted by $\iota\colon \mathrm{Sd}^{b}(K\times K) \to \mathrm{Sd}^{b'}(K\times K)$. Lastly, for $i\in\{1,2\}$, let

$$\pi_i\colon \mathrm{Sd}^{b}(K\times K) \to K \tag{3.2}$$

denote the composition of $\iota\colon \mathrm{Sd}^{b}(K\times K) \to K\times K$ with the i-th projection $K\times K\to K$. As reviewed in the previous section, the contiguity class of each of these maps is well-defined.

Definition 3.1. For non-negative integers b and c, the (b,c)-simplicial complexity $\mathrm{SC}^b_c(K)$ of an (ordered) complex K is one less than the smallest cardinality of finite covers of $\mathrm{Sd}^b(K\times K)$ by subcomplexes J for each of which the compositions

$$J\hookrightarrow \mathrm{Sd}^{b}(K\times K) \xrightarrow{\pi_1} K \quad\text{and}\quad J\hookrightarrow \mathrm{Sd}^{b}(K\times K) \xrightarrow{\pi_2} K \tag{3.3}$$

are c-contiguous. When no such finite coverings exist, we set $\mathrm{SC}^b_c(K)=\infty$.

In analogy with the topological situation, the subcomplexes J appearing in the covers of Definition 3.1 are called piecewise linear local domains, the contiguity chains connecting the two maps in (3.3) are called piecewise linear local rules, and a system of covering piecewise linear local domains with corresponding piecewise linear local rules is called a piecewise linear motion planner. The term “piecewise linear” comes from the obvious fact that 1-contiguous simplicial maps are homotopic through a piecewise linear homotopy.

Remark 3.2. The ordering in K is used only for the construction of the simplicial structure on $K\times K$; the value of $\mathrm{SC}^b_c(K)$ is clearly independent of the chosen ordering.

Lemma 1.1 and [2, Proposition 4.12 and Remark 4.13] yield

$$\mathrm{SC}^b_c(K) \geq \mathrm{TC}(\|K\|). \tag{3.4}$$

The obvious monotonic sequence $\mathrm{SC}^b_0(K) \geq \mathrm{SC}^b_1(K) \geq \mathrm{SC}^b_2(K) \geq \cdots \geq 0$ is eventually constant, and we let $\mathrm{SC}^b(K)$ stand for the corresponding stable value. Note that the sequence of numbers $\mathrm{SC}^b_c(K)$ depends, in principle, on the chosen approximations (3.1). However, we prove:

Lemma 3.3. *The stabilized value* $\mathrm{SC}^b(K)$ *is independent of the chosen approximations (*3.1*).*

Proof. This is an easy consequence of the main result in the previous section (Theorem 2.7). For the benefit of the non-specialist reader, we spell out details.

Let $\overline{\mathrm{SC}}{}^b_c$ stand for the invariant defined in terms of a second set of approximations $\bar\iota\colon \mathrm{Sd}^b(K \times K) \to \mathrm{Sd}^{b-1}(K \times K)$ of the identity. Remark 2.6 and part (1) of Theorem 2.7 imply that the corresponding compositions

$$\pi_i, \overline{\pi_i}\colon \mathrm{Sd}^b(K \times K) \to K,$$

as well as their restrictions $\pi_i \circ j$ and $\overline{\pi_i} \circ j$, are 1-contiguous, where $i \in \{1, 2\}$ and $j\colon J \hookrightarrow \mathrm{Sd}^b(K \times K)$ is the inclusion of some subcomplex J. If $\varphi_0, \varphi_1, \ldots, \varphi_c\colon J \to K$ is a contiguity chain of length c between $\pi_1 \circ j$ and $\pi_2 \circ j$, then $\overline{\pi_1} \circ j, \varphi_0, \varphi_1, \ldots, \varphi_c, \overline{\pi_2} \circ j$ is a contiguity chain of length $c + 2$ between $\overline{\pi_1} \circ j$ and $\overline{\pi_2} \circ j$. Consequently $\mathrm{SC}^b_c(K) \geq \overline{\mathrm{SC}}{}^b_{c+2}(K)$. Likewise $\overline{\mathrm{SC}}{}^b_c(K) \geq \mathrm{SC}^b_{c+2}(K)$, and the result follows. □

Following the idea in the previous proof, note that in the setting of Definition 3.1, if $\lambda_J\colon \mathrm{Sd}(J) \to J$ is an approximation of the identity, then the two compositions in the diagram

$$\begin{tikzcd} J \arrow[r, hook] & \mathrm{Sd}^b(K \times K) \\ \mathrm{Sd}(J) \arrow[u, "\lambda"] \arrow[r, hook] & \mathrm{Sd}^{b+1}(K \times K) \arrow[u, "\iota"'] \end{tikzcd}$$

are 1-contiguous, as they are approximations of the inclusion $\|J\| \subseteq \|K\| \times \|K\|$. Consequently $\mathrm{SC}^b_c(K) \geq \mathrm{SC}^{b+1}_{c+2}(K)$ and

$$\mathrm{SC}^0(K) \geq \mathrm{SC}^1(K) \geq \mathrm{SC}^2(K) \geq \cdots . \tag{3.5}$$

Definition 3.4. The simplicial complexity $\mathrm{SC}(K)$ of a complex K is the stabilized value of the monotonic sequence (3.5).

We stress the fact that the equality $\mathrm{SC}(K) = \mathrm{SC}^b_c(K)$ holds for large enough indices b and c (depending on K), so that (3.4) becomes

$$\mathrm{SC}(K) \geq \mathrm{TC}(\|K\|). \tag{3.6}$$

The parameters b and c in the definition of $\mathrm{SC}(K)$ can be thought of as playing a measurement role in the simplicial motion planning problem. The parameter b gives a notion of "complexity": the more twisted some (topological) local rule R is, the larger b would have to be in order to approach R by a piecewise linear local rule. On the other hand, c gives a notion of "distance", as it counts the number of piecewise linear segments needed to combinatorially realize motion planners. These considerations are illustrated by the computational results in Section 4.

Next we prove that the equality in (3.6) is sharp.

Theorem 3.5. *Equality holds in (*3.6*) for any finite K.*

Proof. Let $\mathrm{TC}(\|K\|) = k$ and choose a motion planner

$$\{(U_0, \sigma_0), (U_1, \sigma_1), \ldots, (U_k, \sigma_k)\}$$

for $\|K\|$ with $k{+}1$ local domains. Choose a large positive integer b so that the realization of each simplex of $\mathrm{Sd}^b(K \times K)$ is contained in some U_j $(0 \leq j \leq k)$ —this uses the finiteness assumption on K. For each $j \in \{0, 1, \ldots, k\}$, let L_j be the subcomplex of $\mathrm{Sd}^b(K \times K)$ consisting of those simplices whose realization is contained in U_j. Then $L_0, L_1, \ldots, L_k$ cover $K \times K$.

By Lemma 1.1 the two projections $\pi_1, \pi_2 \colon \|K\| \times \|K\| \to \|K\|$ are homotopic over each U_i and, in particular, over the (realization of the) corresponding subcomplex L_i. Therefore there are positive integers b' and c such that, for each $j \in \{0, 1, \ldots, k\}$, the two simplicial composites

$$\mathrm{Sd}^{b+b'}(L_i) \hookrightarrow \mathrm{Sd}^{b+b'}(K \times K) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} K$$

are c-contiguous. It follows that $\mathrm{SC}_c^{b+b'}(K) \leq k$, which implies equality in (3.6). $\square$

Theorem 3.5 asserts that, when the configuration space of a robot has a simplicial structure, it is always possible to produce optimal motion planners whose local rules are piecewise linear. The key point, then, is that the search of such motion planners can be done with the aid of a computer. Exhaustive search, however, is most likely doomed to fail, as the size of the search space increases exponentially with every subdivision (cf. Remark 4.3). We believe that heuristic-based algorithms should play an important role in approaching this problem.

Remark 3.6. Concerning the complexity issue discussed in the previous paragraph, it is convenient to keep in mind that the performance of a computer-assisted estimation of the simplicial complexity of a simplicial complex K can be substantially improved by considering non-necessarily barycentric subdivisions. For instance, assume that S is a subdivision of $K \times K$ such that $\mathrm{Sd}^b(K \times K)$ is in turn a subdivision of S. Fix approximations

$$\mathrm{Sd}^b(K \times K) \xrightarrow{\iota''} S \quad \text{and} \quad S \xrightarrow{\iota'} K \times K$$

of the identity on $\|K \times K\|$, and take $\iota \colon \mathrm{Sd}^b(K \times K) \to K \times K$ to be the composite approximation $\iota'\iota''$. If S admits a covering by $k + 1$ subcomplexes

J for each of which the two compositions on the top row of the diagram

$$\begin{tikzcd} J \arrow[r, hook] & S \arrow[r, "\iota'"] & K \times K \arrow[r, "\pi_j"] & K \\ (\iota'')^{-1}(J) \arrow[u, "\iota''"] \arrow[r, hook] & \mathrm{Sd}^b(K \times K) \arrow[u, "\iota''"] \arrow[ur, "\iota"'] & & \end{tikzcd} \qquad (j = 1, 2) \tag{3.7}$$

are c-contiguous, then we get a similar situation at the level of $\mathrm{Sd}^b(K \times K)$ — following the bottom composition in (3.7). In particular $\mathrm{SC}^b_c(K) \leq k$. This observation is used in the next section in order to simplify an estimation of $\mathrm{SC}^b_c(\partial\Delta^2)$ for small values of b and c.

Theorem 3.5 allows us to extrapolate all the nice properties of Farber's topological complexity to the simplicial realm. For instance:

(1) Two complexes whose topological realizations are homotopy equivalent necessarily have the same simplicial complexity.
(2) A complex K has $\mathrm{SC}(K) = 0$ ($\mathrm{SC}(K) = 1$) if and only if the realization $\|K\|$ is contractible (has the homotopy type of an odd sphere).
(3) The simplicial complexity of an (ordered) product of (ordered) complexes K_i is bounded from above by $\sum_i \mathrm{SC}(K_i)$.

Remark 3.7. In Section 1 we have commented on the similarities (and differences) of our approach with the one developed in [3]. A similar situation holds regarding [4], where the invariant $\mathrm{scat}(K)$ —a discretized model for the Lusternik-Schnirelmann category of topological spaces— is proposed in terms of the notion of contiguity of simplicial maps. In fact, the methods in the present paper make it clear that the Lusternik-Schnirelmann category of the topological realization of a given complex K agrees with $\mathrm{scat}(\mathrm{Sd}^b(K))$, as long as b is sufficiently large. However, it does not seem to be the case that the topological complexity of $\|K\|$ would have to be recovered as the discrete TC model in [3] of a highly subdivided K. The main problem comes from the fact that the product of barycentrically subdivided complexes is not as fine as the barycentric subdivision of the product of the complexes.

4. An example: the circle

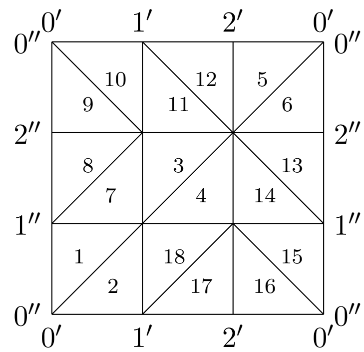
Throughout this section we let K stand for the 1-dimensional skeleton of the 2-dimensional simplex Δ^2, so that $\|K\|$ is homeomorphic to the circle S^1. Our aim is to show that

$$\mathrm{SC}^b_c(K) \leq 1 \tag{4.1}$$

for some set of approximations (3.1) as long as $b \geq 1$ and $c \geq 5$. In particular, since the equality $\mathrm{TC}(S^1) = 1$ is well known, it follows that, in the present case, the sequence (3.5) stabilizes from its second term.

The inequality in (4.1) was first noted through semi-automatized computer experimentation guided by the author's geometric insight. This led to the streamlined combinatorial proof presented in this section which, should be emphasized, can be checked independently of any pre-existing geometric aid. Our intention reporting the inequality in (4.1) —and other related result(s)— is two fold. On the one hand, we hope to motivate the development of fully automatized implementations that would search for optimal piecewise linear motion planners for general polyhedra. Additionally, the analysis presented here shows that the case of the circle provides a benchmark for testing and comparing such eventual implementations (cf. Remark 4.3).

Let the vertices of K be 0, 1, and 2, ordered in the obvious way. The (realization of the ordered) product structure on $K \times K$ can be depicted as



(4.2)

where opposite sides of the external square are identified as indicated. The enumeration shown of the 18 2-simplexes is used to define subcomplexes J_i of $K \times K$ ($i = 1, 2, 3$). Namely, J_i is generated by the 2-simplexes corresponding to the triangles in (4.2) with numbering from $6i - 5$ to $6i$. Note that $\|J_i\|$ is contractible for $i = 2, 3$, whereas $\|J_1\|$ strongly deformation retracts to the diagonal in $\|K \times K\| = \|K\| \times \|K\|$. Since the fibration $e \colon P(\|K\|) \to \|K\| \times \|K\|$ has an obvious section on any singleton as well as on the diagonal, and since $\{J_1, J_2, J_3\}$ cover $K \times K$, Lemma 1.1 and the considerations in Section 3 immediately yield $\mathrm{SC}^b_c(K) \leq 2$ for b and c large enough. This inequality actually holds for small values of b and c, a fact that we prove (in Proposition 4.1) before addressing (4.1) (in Proposition 4.2).

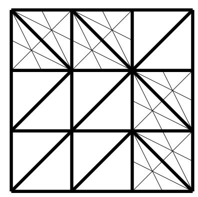
Proposition 4.1. $\mathrm{SC}^0_3(K) \leq 2$.

Proof. Recall the projections $\pi_j \colon K \times K \to K$ ($j = 1, 2$) in (3.2). Direct inspection shows that the restriction of π_1 and π_2 to J_1 are 1-contiguous. Likewise, a contiguity chain $\pi_1|J_2 = \varphi_0, \varphi_1, \varphi_2, \varphi_3 = \pi_2|J_2$ of maps $J_2 \to K$ is obtained with φ_i ($i = 1, 2$) the map that sends every vertex of J_2 into the vertex i of K. The situation for the restriction of π_1 and π_2 to J_3 is completely similar (actually symmetric) to the one just described for J_2. $\square$

Proposition 4.2. *There is an approximation* (3.1) *for which* $\mathrm{SC}^1_5(K) \leq 1$.

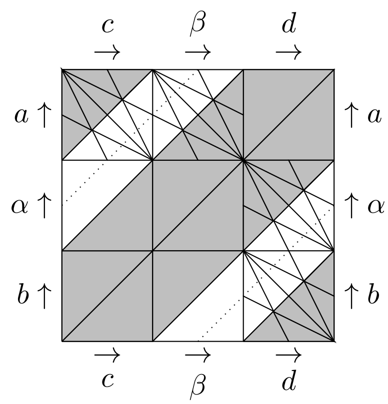
As with Proposition 4.1, before proving Proposition 4.2, we give a theoretical explanation of (a weaker form of) this phenomenon; in addition, this will allow us to introduce some auxiliary notation.

The first barycentric subdivision $\mathrm{Sd}(K \times K)$ starts as



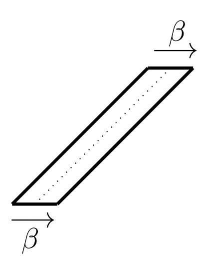
(with the identifications indicated in (4.2)) where we have only shown the barycentric subdivision of the four "squares" in (4.2) whose diagonal has a negative slope —but all other squares are to be subdivided in a similar way.

Consider the subcomplex J_+ (respectively J_-) of $\mathrm{Sd}(K \times K)$ whose topological realization is given by the shaded (respectively unshaded) region in (4.3).

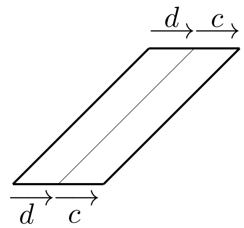


(4.3)

Note that $\|J_+\|$ (respectively $\|J_-\|$) is homeomorphic to a cylinder which strongly deformation retracts to the diagonal $\Delta_+ = \{(x, x)\}$ (respectively anti-diagonal $\Delta_- = \{(x, -x)\}$) in $S^1 \times S^1 = \|\mathrm{Sd}(K \times K)\|$. The assertion is easiest for J_-: topologically, $\|J_-\|$ is obtained from the two unshaded strips in (4.3) by identifying the two edges α and the two edges β. Identification of the edges α yields the longer strip

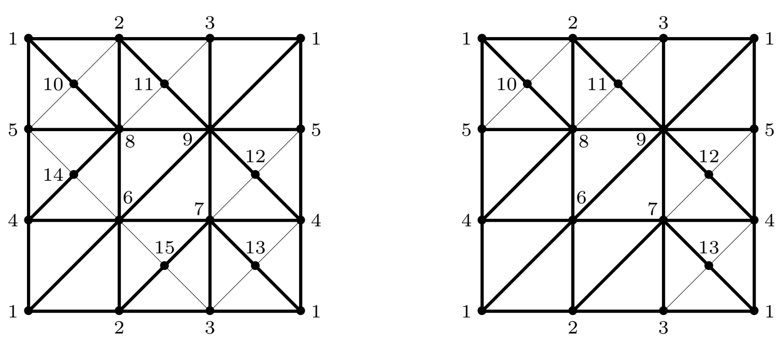


and the identification of the edges β then yields a cylinder. In turn, this cylinder strongly deformation retracts to its middle dotted line, which corresponds to the anti-diagonal Δ_- (i.e., the two dotted lines in (4.3)). The case of $\|J_+\|$ is similar: gluing the lower right shaded triangle to the main body of the shaded region in (4.3) along the common edge b, and gluing the upper left shaded triangle to the main body of (4.3) along the common edge a, yields the cylinder:



This cylinder strongly deformation retracts to its middle thin slanted line, which corresponds to the diagonal Δ_+. Now, we have already noted that S^1 has an obvious motion planner on Δ_+, whereas a motion planner on Δ_- is given by rotating (in any direction) half a twist the circle. So, Lemma 1.1 and the considerations in Section 3 yield the inequality $\mathrm{SC}^b_c(K) \leq 1$ for b and c large enough. Our actual proof (below) of the more specific Proposition 4.2 is independent of knowing about motion planning rules on the diagonal and antidiagonal; the argument boils down to exhibiting the explicit contiguity chains (4.5) and (4.6).

Proof of Proposition 4.2. As noted in Remark 3.6, it suffices to prove the corresponding assertion replacing $\mathrm{Sd}(K \times K)$ by a "coarser" subdivision of $K \times K$. Actually, we can simplify calculations by working with the two (nested) subdivisions S' and S'' of $K \times K$ whose topological realizations have the combinatorial structure shown in (4.4), and whose vertices have been numbered for notational convenience in what follows.



(4.4) subdivision S'' subdivision S'

Explicitly, S' is obtained from $K \times K$ by (non-barycentrically) subdividing the four "squares" in $K \times K$ whose diagonal has a negative slope. In turn, S'' is obtained from S' by doing the corresponding subdivision with the other two "squares" which are not crossed by the diagonal. Of course, $\mathrm{Sd}(K \times K)$ is then obtained as a subdivision of S''.

Next we use Example 2.3 to choose approximations $\iota'' \colon S'' \to S'$ and $\iota' \colon S' \to K \times K$ of the identity on $\|K \times K\|$. Namely, ι'' and ι' (are forced to) behave as the identity map on vertices that are common to their domains and ranges, while

$$\iota'(10) = \iota'(13) = 1 \quad \text{and} \quad \iota'(11) = \iota'(12) = 9,$$

and

$$\iota''(14) = 8 \quad \text{and} \quad \iota''(15) = 7.$$

As a last preparatory ingredient, consider the obvious subcomplexes J'_+ and J'_- of S' whose topological realizations correspond to those indicated in (4.3). For instance, J'_+ has 16 2-simplexes and uses all of the 13 vertices of S', while J'_- has only 10 2-simplexes and does not use the "diagonal" vertices 1, 6 and 9. Further, the additional subdivisions in S'' (not present in S') yield corresponding subcomplexes J''_+ and J''_- of S''. In the situation of Remark 3.6, $J''_\pm = (\iota'')^{-1}(J'_\pm)$.

The two compositions in (3.7) for $S = S'$, $J = J'_+$ and $b = 1$ are described in the second and fourth rows of (4.5) where, as the reader can easily check, a contiguity chain of length 2 between these two compositions is indicated[2]. In particular, we get a corresponding contiguity chain of length 2 defined on J''_+ (at the level of S'').

$$\begin{array}{rccccccccccccc}
\text{vertex number} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
(\pi_1\,\iota')|J'_+ = \varphi_0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 2 & 0 \\
\varphi_1 & 0 & 1 & 2 & \mathbf{1} & 0 & 1 & \mathbf{1} & 1 & 2 & 0 & 2 & 2 & 0 \\
(\pi_2\,\iota')|J'_+ = \varphi_2 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 0 & 2 & 2 & 0
\end{array} \tag{4.5}$$

Lastly, in order to describe a piecewise linear motion planner on $\|J_-\|$, we work directly at the level of the finer subdivision S'': The two compositions in (3.7) for $S = S''$, $J = J''_-$ and $b = 1$ are described in the second and last rows of (4.6), where a contiguity chain of length 5 between these two compositions is indicated.

(4.6)

$$\begin{array}{rccccccccccccccc}
\text{vertex number} & - & 2 & 3 & 4 & 5 & - & 7 & 8 & - & 10 & 11 & 12 & 13 & 14 & 15 \\
(\pi_1\,\iota'\iota'')|J''_- = \varphi_0 & - & 1 & 2 & 0 & 0 & - & 2 & 1 & - & 0 & 2 & 2 & 0 & 1 & 2 \\
\varphi_1 & - & 1 & 2 & 0 & 0 & - & 2 & 1 & - & 1 & 1 & 0 & 2 & 0 & 2 \\
\varphi_2 & - & 1 & 1 & 2 & 0 & - & 2 & 0 & - & 0 & 1 & 2 & 2 & 0 & 1 \\
\varphi_3 & - & 0 & 1 & 2 & 2 & - & 1 & 0 & - & 0 & 0 & 2 & 1 & 2 & 1 \\
\varphi_4 & - & 0 & 0 & 1 & 2 & - & 1 & 2 & - & 2 & 0 & 1 & 1 & 2 & 0 \\
(\pi_2\,\iota'\iota'')|J''_- = \varphi_5 & - & 0 & 0 & 1 & 2 & - & 1 & 2 & - & 0 & 2 & 2 & 0 & 2 & 1
\end{array}$$

□

[2]The two boldface 1's in the third row of (4.5) are the only difference between φ_0 and φ_1; this represents a small initial piecewise linear homotopy in preparation for the main one between the indicated maps φ_1 and φ_2.

Remark 4.3. The piecewise linear homotopies in this section have been found through the combined efforts of a computer and human intuition, in part because a brute force search by computer (without the human component) is just out of the question. For instance, the exhaustive search of the homotopy described in (4.6) would have to consider a search space of 3^{72} instances —too large for the current computer technology. For any practical usage, an eventual fully automatized search would have to replace human geometric insight by an algorithm with some type of heuristic component (stochastic methods, machine learning, etc.) that would allow to perform a "smart" non-exhaustive search.

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(Jesús González) Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Av. IPN 2508, Zacatenco, México City 07000, México

jesus@math.cinvestav.mx

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