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Sparse bounds for oscillatory and random singular integrals

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Abstract. Let $T_P f(x) = \int e^{iP(y)} K(y) f(x-y)\, dy$, where $K(y)$ is a smooth Calderón–Zygmund kernel on $\mathbb{R}^n$, and P be a polynomial. We show that there is a sparse bound for the bilinear form $\langle T_P f, g\rangle$. This in turn easily implies A_p inequalities. The method of proof is applied in a random discrete setting, yielding the first weighted inequalities for operators defined on sparse sets of integers.

Contents

1. Introduction

Singular integral operators can be pointwise dominated by sparse operators, which are positive localized operators, something that singular integrals are not. This paper extends this theme to the settings of:

(a) oscillatory singular integrals, and
(b) discrete random operators.

In both cases, we easily derive weighted inequalities. In the latter case, these are the first such weighted inequalities known. We state our results before providing a broader context.

Call a collection of cubes $\mathcal{S}$ in $\mathbb{R}^n$ a *sparse* collection if there is a set $E_Q \subset Q$ for each $Q \in \mathcal{S}$ so that:

(a) $|E_Q| > c|Q|$ for each $Q \in \mathcal{S}$, and
(b) the collection of sets $\{E_Q : Q \in \mathcal{S}\}$ are pairwise disjoint.

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Here $0 < c < 1$ will be a dimensional constant that we do not track. Define a *sparse bilinear form* to be

$$\Lambda_{r,s}(f,g) = \sum_{Q\in\mathcal{S}} \langle f\rangle_{Q,r}\langle g\rangle_{Q,s}|Q|, \qquad 1 \le r,s < \infty.$$

Above, $\langle f\rangle_{Q,r}^r := |3Q|^{-1}\int_{3Q}|f|^r\ dx$, and if $r = s$, then $\Lambda_r = \Lambda_{r,r}$. We frequently suppress the collection of sparse cubes $\mathcal{S}$.

We consider Calderón–Zygmund singular integral operators T, defined to be an $L^2(\mathbb{R}^n)$ bounded convolution operator given by

$$\langle Tf, g\rangle = \int\int K(x-y)f(y)g(x)\,dx\,dy.$$

for compactly supported functions f, g with disjoint supports. Moreover, the kernel $K(y)$ satisfies

$$|\nabla^t K(x,y)| \le C_t|x-y|^{-n-t}, \qquad x \neq y \in \mathbb{R}^n,$$

for $t \in \{0,1\}$. Key examples are $K(y) = 1/y$ in dimension one, and the Riesz transform kernels $y/|y|^{n+1}$, in dimension n.

Such operators are of course nonlocal, and involve subtle cancellative effects. It is thus something of a surprise that such operators are dominated by sparse operators, which have none of these features. This is a special case of [6, 18, 22].

Theorem A. *For each Calderón–Zygmund singular integral operator T and bounded compactly supported function f, there is a sparse operator $\Lambda = \Lambda_{T,f}$ so that $|Tf| \lesssim \Lambda_1 f$.*

An immediate corollary are weighted inequalities that are sharp in the A_p characteristic. See [6, 18, 20].

We consider polynomials of a fixed degree d, given by

$$P(x,y) = \sum_{\alpha,\beta:|\alpha|+|\beta|\le d} \lambda_{\alpha,\beta}x^\alpha y^\beta,$$

where we use the usual multi-index notation. The polynomial modulated Calderón–Zygmund operators are

$$T_P f(x) = \int e^{iP(x,y)}K(y)f(x-y)\,dy.$$

The L^p result below is a special case of the results of Ricci and Stein [26, 27], and the weak-type result is due to Chanillo and Christ [5].

Theorem B. *For $1 < p < \infty$, the operator T_P is bounded on L^p, that is*

$$\|T_P : L^p \mapsto L^p\| \lesssim 1,$$

where the implied constant depends on the degree of P, and in particular is independent of λ. Moreover, T_P maps L^1 to weak L^1, with the same bound.

The dependence on the polynomial being felt only through the degree of P is important to the application of these bounds to the setting of nilpotent groups, like the Heisenberg group, see [27]. This dependence continues to hold in the theorems below.

Theorem 1.1. *For each $1 < r < 2$ Calderón–Zygmund operator T, polynomial $P = P(y)$ of degree d and bounded supported functions f, g there is a bilinear form Λ_r so that*

$$|\langle T_P f, g\rangle| \lesssim \Lambda_r(f, g).$$

The implied constant depends only on T, the degree d, and dimension n and choice of $r > 1$.

The bound above continues to hold for polynomials P of two variables, but we suppress the details, as the estimate above can most likely be improved. And, as written is quite easy to prove, yet yields a nontrivial corollary.

Corollary 1.2. *For $1 < p < \infty$, the operator T_P, where $P = P(y)$ is of degree d, is bounded on $L^p(w)$, where w is a Muckenhoupt weight $w \in A_p$.*

Weak-type and weighted estimates for oscillatory singular integrals have been studied in this and more general contexts by various authors, see for instance [9–12, 29]. Y. Ding and H. Liu [9] were interested in $L^p(w)$ inequalities for more general operators T. The approach of these authors entails many complications.

The method of proof of Theorem 1.1 is very simple. And, so we suspect that stronger results are possible. For instance, the following conjecture would imply nearly sharp A_p bounds, for all $1 < p < 2$.

Conjecture 1.3. *For $1 < r < \infty$, the operator T_P, where $P = P(y)$ is of degree d, for each bounded compactly supported function f, there is a sparse operator $\Lambda_{1,r}$ so that*

$$|\langle T_P f, g\rangle| \lesssim \Lambda_{1,r}(f, g).$$

It seems likely that the weak type argument of Chanillo and Christ [5] would establish the conjecture for $r = 2$. Also see [16].

We turn to weighted inequalities for *discrete random Hilbert transforms* acting on functions on $\ell^2(\mathbb{Z})$. Define a sequence of Bernoulli rvs $\{X_n : n \neq 0\}$ with $\mathbb{P}(X_n = 1) = |n|^{-\alpha}$, where $0 \leq \alpha < 1$. Then, the set $\{n : X_n = 1\}$ is a.s. infinite, by the Borel–Cantelli Lemma. Then, we consider the random Hilbert transform, and maximal function below.

$$H_\alpha f(x) = \sum_{n \neq 0} \frac{X_n}{n^{1-\alpha}} f(x - n). \tag{1.4}$$

$$M_\alpha f(x) = \sup_{n>0} \Big| \frac{1}{S_N} \sum_{n=1}^{N} X_n f(x - n) \Big|, \qquad S_N = \sum_{n=1}^{N} X_n.$$

Our sparse bound here is more restrictive, with the value of the sparse index r depending upon random parameter α.

Theorem 1.5. *For any $0 < \alpha < 1$, $1 + \alpha < r < 2$, almost surely, the following holds: For all functions f, g finitely supported on $\mathbb{Z}$, there is a bilinear sparse operator Λ_r so that*

$$|\langle H_\alpha f, g\rangle| \lesssim \Lambda_r(f, g).$$

The same inequality holds for M_α. (The sparse operator can be taken non-random, but the implied constant is random.)

Weighted inequalities are a corollary. They are *the first we know of* holding for operators defined on sets of the integers with zero asymptotic density.

Corollary 1.6. *For any $0 < \alpha < 1$, almost surely, the following holds: For all $1 + \alpha < p < \frac{1+\alpha}{\alpha}$, and weights w so that*

$$w^{1+\alpha} \in A_{(1+\alpha)(p-1)+1}, \qquad w \in A_{1+\frac{1}{(1+\alpha)(p'-1)}}, \tag{1.7}$$

we have $\|H_\alpha : \ell^p(w) \mapsto \ell^p(w)\| < \infty$. The implied constant only depends upon $[w^{1+\alpha}]_{A_{(1+\alpha)(p-1)+1}}$, and $[w]_{A_{1+\frac{1}{\alpha(p'-1)}}}$. The same inequality holds for M_α.

The study of these questions was initiated by Bourgain [3], as an elementary example of a sequence of integers for which one could derive ℓ^p inequalities, with the sequence of integers also having asymptotic density zero. Various aspects of these questions have been studied, both in ℓ^p, in the weak $(1,1)$ endpoints [4, 17, 24, 28, 31]. We are not aware of any result in the literature that proves a weighted estimate in this sort of discrete setting. (If the set of integers has full density, it is easy to transfer weighted estimates.)

There is a subtle difference between the Hilbert transform and the maximal function in this random setting. In particular, more should be true for the maximal function. Prompted by the work of LaVictoire [17], we pose:

Conjecture 1.8. *For $0 < \alpha < 1/2$, almost surely, for all $1 < r < 2$, and finitely supported functions f, g, there is a sparse operator $\Lambda_{1,r}$ so that*

$$\langle M_\alpha f, g\rangle \lesssim \Lambda_{1,r}(f, g).$$

We turn to the context for our paper. The concept of sparse operators arose from Lerner's remarkable median inequality [19]. It's application to weighted inequalities was advanced by several authors, with a high point of this development being Lerner's argument [20] showing that the weighted norm of Calderón–Zygmund operators is comparable to that of the norms of sparse operators. This lead to the question of pointwise control, namely Theorem A. First established by Conde-Alonso and Rey [6], also see Lerner and Nazarov [22], the author [18] established Theorem A with a stopping time argument. The latter argument was extended by Bernicot, Frey and Petermichl [2] to a setting where the operators are generated by semigroups, including examples outside the scope of classical Calderón–Zygmund theory. For closely related developments see [14, 21]. The sparse bounds for commutators [8, 23] are remarkably powerful. Edging beyond the Calderón–Zygmund context, Benau, Bernicot and Frey [1] have supplied sparse bounds for certain Bochner–Riesz multipliers.

Very recently, Culiuc, di Plinio and Ou [7] have established a sparse domination result in a setting far removed from the extensions above: The trilinear form associated to the bilinear Hilbert transform is dominated by a sparse form. This is a surprising result, as the bilinear Hilbert transform has all the difficult features of the Hilbert transform, with additional oscillatory and arithmetic-like aspects. This paper is an initial effort on our part to understand how general a technique 'domination by sparse' could be. There are plenty of additional directions that one could think about.

For instance, the interest in the oscillatory singular integrals is driven in part by their application to singular integrals defined on nilpotent groups. Implications of the sparse bound in this setting are unexplored.

There are two approaches to sparse bounds, the bilinear form method [7], and the use of the maximal truncation inequality [18]. We use neither approach. After applying the known sparse bounds for singular integrals, for the remaining parts of the operator, there is a very simple interpolation argument which you can use in the bilinear setting. The notable point about the proofs is that they are quite easy, and yet deliver striking applications.

2. Proof of Theorem 1.1

Our conclusion is invariant under dilations of the operator. Hence, we can proceed under the assumption that $\|P\| = \sum_\alpha |\lambda_\alpha| = 1$. We can also assume that the polynomial P has no linear term, as it can be absorbed into the function f. Under these assumptions we prove:

Theorem 2.1. *Let P be a polynomial without linear terms, and $\|P\| = 1$. Then, for bounded compactly supported functions f, g and $1 < r < \infty$, there is a sparse form Λ_1 and a $\eta > 0$ so that*

$$|\langle T_P f, g\rangle| \lesssim \Lambda_1(f, g) + \sum_{Q \in \mathcal{D} : |Q| \geq 1} \langle f \rangle_{Q,r} \langle g \rangle_{Q,r} |Q|^{1-\eta}. \tag{2.2}$$

It is easy to see that this implies Theorem 1.1, since the second term on the right is restricted to dyadic cubes of volume at least one, and there is a gain of $|Q|^{-\eta}$. Moreover, we will see that this theorem implies the weighted result.

Let $e(\lambda) = e^{i\lambda}$ for $\lambda \in \mathbb{R}$. If the kernel K of T is supported on

$$2B = \{y : |y| \leq 2\},$$

then we have
$$|e(P(y))K(y) - K(y)| \lesssim \mathbf{1}_{2B}(y)|y|^{-n+1},$$
so that $|T_P f - Tf| \lesssim Mf$. Both T and M admit pointwise domination by sparse forms, hence also by bilinear forms. (This is the main result of [18].)

Thus, we can proceed under the assumption that the kernel K is not supported on B. We can then write
$$K = \sum_{j=1}^{\infty} \varphi_j$$
where φ_j is supported on $2^{j-1}B \setminus 2^{j-2}B$, with $\|\nabla^s \varphi_j\|_\infty \lesssim 2^{-nj-sj}$, for $s = 0, 1$.

We use shifted dyadic grids, $\mathcal{D}_t$, for $1 \le t \le 3^n$. These grids have the property that
$$\{\tfrac{1}{3}Q : Q \in \mathcal{D}_t,\ \ell Q = 2^k, 1 \le t \le 3^n\}$$
form a partition of $\mathbb{R}^n$. Throughout, $\ell Q = |Q|^{1/n}$ is the side length of the cube Q. We fix a dyadic grid $\mathcal{D}_t$ throughout the remainder of the argument, and set $\mathcal{D}_+ = \{Q : \ell Q > 2^{10}\}$. Define
$$I_Q f = \int e(P(y))\varphi_k(y)(\mathbf{1}_{\frac{1}{3}Q} f)(x-y)\, dy, \qquad \ell Q = 2^{k+2}.$$
Note that $I_Q f$ is supported on Q, and that we have suppressed the dependence on P, which we will continue below.

The basic estimate is then the following lemma:

Lemma 2.3. *For each cube Q with $|Q| \ge 1$ and $1 < r < 2$, there holds*
$$|\langle I_Q f, g\rangle| \lesssim 2^{-\eta k}\langle f\rangle_{Q,r}\langle g\rangle_{Q,r}|Q|, \tag{2.4}$$
where $\eta = \eta(d, n, r) > 0$.

Theorem 2.1 follows immediately from this lemma. The oscillatory nature of the problem exhibits itself in the next lemma. Write
$$I_Q^* I_Q \phi(x) = \mathbf{1}_{\frac{1}{3}Q}(x) \cdot \int_{\frac{1}{3}Q} K_Q(x,y)\phi(y)\, dy.$$

Lemma 2.5. *For each cube $Q \in \mathcal{D}_+$, and $x \in \frac{1}{3}Q$, we have*
$$|K_Q(x,y)| \lesssim |Q|^{-1}\mathbf{1}_{Z_Q}(x-y) + |Q|^{-1-\epsilon}\mathbf{1}_Q(x)\mathbf{1}_Q(y),$$
where $Z_Q \subset Q$ has measure at most $(\ell Q)^{-\epsilon}|Q|$, where $\epsilon = \epsilon(n, d) > 0$.

This lemma is well known, see for instance [30, Lemma 4.1]. Here is how we use the lemma. Using Cauchy–Schwartz, we have
$$\begin{aligned}\|I_Q f\|_2^2 &\lesssim |Q|^{-1}\int_Q\int_{Z_Q} |f(x)||f(x-y)|\, dydx + |Q|^{-\epsilon}\langle f\rangle_{Q,1}^2|Q| \\ &\lesssim |Q|^{-\epsilon/n}\|f\mathbf{1}_Q\|_2^2.\end{aligned}$$

We also have the trivial but rarely used $\|I_Q f\|_\infty \lesssim |Q|^{-1}\|f\mathbf{1}_Q\|_1$. By Riesz Thorin interpolation, there holds with $\ell Q = 2^k$,

$$\|I_Q f\|_{r'} \lesssim 2^{-\eta k}|Q|^{-1+2/r'}\|f\mathbf{1}_Q\|_r, \qquad 1 < r \leq 2,\ r' = \tfrac{r}{r-1}.$$

Above, $\eta = \eta(\epsilon, r)$ But, this immediately implies (2.4). Namely,

$$\begin{aligned} |\langle I_Q f, g\rangle| &\lesssim \|I_Q f\|_{r'}\|g\mathbf{1}_Q\|_r \\ &\lesssim 2^{-\eta k}|Q|^{-1+2/r'}\|f\mathbf{1}_Q\|_r\|g\mathbf{1}_Q\|_r \\ &= 2^{-\eta k}\langle f\rangle_{Q,r}\langle g\rangle_{Q,r}|Q|. \end{aligned}$$

(Alternatively, one can just use bilinear interpolation.)

We now give the weighted result.

Proof of Corollary 1.2. The qualitative result that T_P is bounded on $L^p(w)$ for $w \in A_p$, $1 < p < \infty$ is as follows. Given $w \in A_p$, recall that the dual weight is $\sigma = w^{1-p'}$. Then, it is equivalent to show that

$$|\langle T_P(f\sigma), gw\rangle| \lesssim C_{[w]_{A_p}}\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(w)}.$$

Using the sparse domination from (2.2), we see that we need to prove the corresponding bound for the terms on the right in (2.2). Now, it is well known [20] that

$$\Lambda_1(f,g) \lesssim [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

Indeed, this is a key part of the proof of the A_2 Theorem by sparse operators.

So, it remains to consider the second term on the right in (2.2). For each $k \in \mathbb{N}$, we have by Proposition 4.1, $k \in \mathbb{Z}$,

$$\sum_{Q\in\mathcal{D}:|Q|=2^{nk}} \langle f\rangle_{Q,r}\langle g\rangle_{Q,r}|Q| \lesssim [w]_{A_p}^{1/p}[w]_{RH_r}[\sigma]_{RH_r}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

As we recall in § 4, there is a $r = r([w]_{A_p}) > 1$ so that $[w]_{RH_r}[\sigma]_{RH_r} < 4$. And so the proof of the corollary is complete.

Indeed, it is easy enough to make this step quantitative. For $2 < p < \infty$, the choice of r can be taken to satisfy $r - 1 > c[w]_{A_p}^{-1}$, which then means that the choice of $\eta = \eta(r)$ in (2.2) is at least as big is $c[w]_{A_p}^{-1}$. Then, our bound is

$$\langle T_P(\sigma f), gw\rangle \lesssim [w]_{A_p}^{1+\frac{1}{p}}\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(w)}, \qquad 2 < p < \infty.$$

We have no reason to believe that this estimate is sharp. □

3. Random Hilbert transforms

The discrete Hilbert transform

$$Hf(x) = \sum_{n\neq 0}\frac{f(x-n)}{n}$$

satisfies a sparse bound: For all finitely supported functions f and g, there is a sparse operator Λ so that

$$|\langle Hf, g\rangle| \lesssim \Lambda_{1,1}(f,g). \tag{3.1}$$

This is a consequence of the main results of Theorem A. Recalling the definition of H_α in (1.4), we see that $\mathbb{E}H_\alpha f = Hf$, so it remains to consider the difference

$$\begin{aligned} H_\alpha f(x) - Hf(x) &:= \sum_{k=1}^{\infty} \sum_{n: 2^{k-1} \leq |n| < 2^k} \frac{X_n - n^{-\alpha}}{n^{1-\alpha}} f(x-n) \\ &:= \sum_{k=1}^{\infty} T_k f(x). \end{aligned}$$

Above, we have passed directly to the distinct scales of the operator. We will subsequently write $Y_n = X_n - n^{-\alpha}$, which are independent mean zero random variables.

The crux of the matter are these two estimates:

Lemma 3.2. *Almost surely, for all $0 < \epsilon < 1$, and for all integers k, and f, g supported on an interval I of length 2^k, we have*

$$|\langle T_k f, g\rangle| \lesssim \begin{cases} 2^{-k\frac{1-\alpha}{2}+\epsilon} \langle f\rangle_{I,2} \langle g\rangle_{I,2} |I| \\ 2^{k\alpha} \langle f\rangle_{I,1} \langle g\rangle_{I,1} |I|. \end{cases}$$

The implied constant is random, but independent of $k \in \mathbb{N}$ and the choice of functions f, g.

Proof. The second bound follows trivially from

$$|Y_n|/n^{1-\alpha} \mathbf{1}_{2^{k-1} \leq |n| < 2^k} \lesssim 2^{k(\alpha-1)}.$$

For the first bound, we clearly have

$$|\langle T_k f, g\rangle| \leq \|T_k : \ell^2 \to \ell^2\| \cdot \langle f\rangle_{I,2} \langle g\rangle_{I,2} |I|,$$

so it suffices to estimate the operator norm above. The assertion is that with high probability, the operator norm is small:

$$\mathbb{P}\big(\|T_k : \ell^2 \to \ell^2\| > C\sqrt{k} 2^{-k\frac{1-\alpha}{2}}\big) \lesssim 2^{-k},$$

provided C is sufficiently large. Combine this with the Borel–Cantelli Lemma to prove the lemma as stated.

By Plancherel's Theorem, the operator norm is equal to $\|Z(\theta)\|_{L^\infty(d\theta)}$, where

$$Z(\theta) := \sum_{n: 2^k \leq |n| < 2^{k+1}} Y_n \frac{e^{2\pi i \theta}}{n^{1-\alpha}}.$$

The expression above is a random Fourier series, with frequencies at most 2^{k+2}. By Bernstein's Theorem for trigonometric polynomials, the $L^\infty(d\theta)$

norm can be estimated by testing the norm on at most 2^{k+3} equally spaced points in $\mathbb{T}$, that is, we have

$$\mathbb{P}\big(\|Z(\theta)\|_\infty > C\sqrt{k}2^{-k\frac{1-\alpha}{2}}\big) \lesssim 2^k \sup_\theta \mathbb{P}\big(|Z(\theta)| > C\sqrt{k}2^{-k\frac{1-\alpha}{2}}\big),$$

where we have simply used the union bound.

Now, $Z(\theta)$ is the sum of independent, mean zero random variables, which are bounded by one, and have standard deviation bounded by $c2^{-k\frac{1-\alpha}{2}}$. So by, for instance, the Bernstein inequality, it follows that

$$\mathbb{P}\left(|Z(\theta)| > C\sqrt{k}2^{-k\frac{1-\alpha}{2}}\right) \lesssim 2^{-2k},$$

for appropriate C. This completes the proof. $\square$

From the previous lemma, we have the corollary below. It with the sparse bound for the Hilbert transform (3.1) completes the proof of Theorem 1.5, for the random Hilbert transform. The case for maximal averages is entirely similar.

Corollary 3.3. *Almost surely, for $1+\alpha < r < 2$, there is a $\eta > 0$ so that for all integers k, and all functions f, g supported on an interval I of length 2^k, we have*

$$|\langle T_k f, g\rangle| \lesssim 2^{-\eta k}\langle f\rangle_{I,r}\langle g\rangle_{I,r}|I|. \tag{3.4}$$

Proof. This follows from Lemma 3.2 by interpolation. The relevant interpolation parameter θ_0 at which we have only an epsilon loss in the interpolated estimate is given by

$$(1-\theta_0)\alpha = \theta_0\frac{1-\alpha}{2},$$
$$\text{and then}\quad \frac{1}{r_0} = \frac{1-\theta_0}{1} + \frac{\theta_0}{2}.$$

We see that $r_0 = 1+\alpha$. And so we conclude that for $r_0 = 1+\alpha < r < 2$, we have the required gain in the interpolated bound, which proves the corollary. $\square$

We now turn to the weighted inequalities of Corollary 1.6.

Proof of Corollary 1.6. For the deterministic Hilbert transform, we have the sharp bound of Petermichl [25], namely

$$\|H : \ell^p(w) \mapsto \ell^p(w)\| \lesssim [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}.$$

So, it remains to bound the terms in (3.4). By Proposition 4.1, we then need to see that the hypotheses on w, namely (1.7), imply that for some choice of $r > 1+\alpha$, we have

$$w \in A_p, \quad w \in RH_r, \quad \sigma = w^{1-p'} \in RH_r.$$

Recall that $v \in A_q \cap RH_s$ if and only if $v^s \in A_{s(q-1)+1}$. Now, by assumption, $w^{1+\alpha} \in A_{(1+\alpha)(p-1)+1}$. So, there is a $t > 1$ so that $w^{t(1+\alpha)} \in A_{(1+\alpha)(p-1)+1}$, and the A_q classes increase in q, so we conclude that $w \in A_p \cap RH_r$, for a $r > 1+\alpha$.

The second hypothesis is $w \in A_{1+\frac{1}{(1+\alpha)(p'-1)}}$. This is equivalent to

$$(w^{(1-p')})^{1+\alpha} \in A_{(1+\alpha)(p'-1)+1}.$$

Now, $w^{1-p'} = \sigma$ is the dual weight. So by the argument in the previous paragraph, $\sigma \in RH_r$, for some $r > 1+\alpha$. So the proof is complete. □

4. Sparse bounds and weighted inequalities

Let us recall the weighted estimates that we need for our corollaries. A function $w > 0$ is a *Muckenhoupt A_p weight* if

$$[w]_{A_p} = \sup_Q \Big[\frac{w^{\frac{1}{1-p}}(Q)}{|Q|}\Big]^{p-1} \frac{w(Q)}{|Q|} < \infty.$$

Above, we are conflating w as a measure and a density, thus

$$w^{\frac{1}{1-p}}(Q) = \int_Q w(x)^{\frac{1}{1-p}} \, dx.$$

We have these estimates, which are sharp in the A_p characteristic. They are an element of the sparse proof of the A_2 conjecture. (See [20] for a proof.)

Theorem C. *These estimates hold for all $1 < p < \infty$.*

$$\|\Lambda_{1,1} : L^p(w) \mapsto L^p(w)\| \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}.$$

For our applications, we have a second class of operators, a simplified form of those introduced by Benau–Bernicot–Petermichl [1]. For our purposes, we need a much simplified version of their result. Define an additional characteristic of a weight, namely the *reverse Hölder* property.

$$[w]_{RH_r} = \sup_Q \frac{\langle w \rangle_{Q,r}}{\langle w \rangle_Q}.$$

Proposition 4.1. *Fix an integer k, and $1 < r < 2$. We have the bound below for all $w \in A_p$, where $r \le p \le r' = \frac{r}{r-1}$.*

$$\sum_{Q \in \mathcal{D} : |Q| = 2^{nk}} \langle f \rangle_{Q,r} \langle g \rangle_{Q,r} |Q| \lesssim [w]_{A_p}^{1/p} [w]_{RH_r} [\sigma]_{RH_r} \|f\|_{L^p(w)} \|g\|_{L^{p'}(w)}$$

where $\sigma = w^{1-p'}$ is the 'dual' weight to w.

Let us recall these well known facts.

(1) We always have $[w]_{A_p}, [w]_{RH_r} \ge 1$.

(2) For $w \in A_p$ and $\sigma = w^{1-p'}$, the weight σ is locally finite, its 'dual' weight is w, and $[\sigma]_{A_{p'}} = [w]_{A_p}^{p'-1}$.
(3) For every $w \in A_p$ there is a $r = r([w]_{A_p}) > 1$ so that $w \in RH_r$. (In particular, we can take r so that $r - 1 \simeq [w]_{A_p}^{-1}$, by [13, Thm 2.3].)
(4) For every $w \in A_p$, there is a $r = r([w]_{A_p}) > 1$ so that $w^r \in A_p$.
(5) We have $w \in A_p \cap RH_r$ if and only if $w^r \in A_{r(p-1)+1}$, by [15].

Proof of Proposition 4.1. This inequality is rephrased in the self-dual way, namely setting $\sigma = w^{1-p'}$, it is equivalent to show that for $k \in \mathbb{Z}$,

$$\sum_{\substack{Q\in\mathcal{D}\\|Q|=2^{nk}}} \langle f\sigma\rangle_{Q,r}\langle gw\rangle_{Q,r}|Q| \lesssim [w]_{A_p}^{\frac{1}{p}}[\sigma]_{RH_r}[w]_{RH_r}\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(w)}. \tag{4.2}$$

Fix the integer k. We can assume that for $|Q| = 2^{nk}$, if f is not zero on Q, then $f\mathbf{1}_{3Q\setminus Q} \equiv 0$, and we assume the same for g. Then, set

$$f' = \sum_{Q\in\mathcal{D}:|Q|=2^{nk}} \mathbf{1}_Q\Big[\frac{1}{\sigma(Q)}\int_Q |f|^r\, d\sigma\Big]^{1/r}$$

and likewise for g'. It is immediate that $\|f'\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$, thus in (4.2), it suffices to assume that $f = f'$. Then, we can even assume that f and g are supported on a single cube Q, and take the value 1 on that cube.

Then, write

$$\begin{aligned}\langle\sigma\mathbf{1}_Q\rangle_{Q,r}\langle w\mathbf{1}_Q\rangle_{Q,r}|Q| &\leq [\sigma]_{RH_r}[w]_{RH_r}\langle\sigma\mathbf{1}_Q\rangle_{Q,1}\langle w\mathbf{1}_Q\rangle_{Q,1}|Q|\\ &\leq [\sigma]_{RH_r}[w]_{RH_r}\langle\sigma\mathbf{1}_Q\rangle_{Q,1}^{1/p'}\langle w\mathbf{1}_Q\rangle_{Q,1}^{1/p}\cdot\sigma(Q)^{1/p}w(Q)^{1/p'}\\ &\leq [\sigma]_{RH_r}[w]_{RH_r}[w]_{A_p}^{1/p}\sigma(Q)^{1/p}w(Q)^{1/p'}.\end{aligned}$$

This is the inequality claimed. □

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