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Curvature decompositions on Einstein four-manifolds

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Abstract. For Einstein four-manifolds with positive scalar curvature, we investigate relations among various positivity conditions on the curvature tensor, some of which are of great importance in the study of the Ricci flow. These relations suggest possible new ideas to study the well-known rigidity conjecture for positively curved Einstein four-manifolds.

Contents

1. Introduction

A Riemannian metric is called an Einstein metric if $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$. A central problem in differential geometry is to study the existence, rigidity, and moduli space of Einstein metrics. In dimension four, a well-known conjecture states that Einstein four-manifolds with positive sectional curvature are isometric to (S^4, g_0) or $(\mathbb{C}P^2, g_{FS})$. Many authors have made important progress on this conjecture, cf. Berger [Berg61], Derdzinski [Der83], Hitchin [Bes87], Gursky and LeBrun [GL99], Yang [Yang00], and Costa [Cos04]. Curvature decompositions are basic tools to understand the structure of the curvature tensor. The three curvature decompositions on Einstein four-manifolds: the standard curvature decomposition [Bes87], the duality curvature decomposition [Bes87], and the Berger curvature decomposition [Berg61], are essential in these works.

The positivity of the curvature operator is of great importance in the study of the Ricci flow. Recall that a curvature operator $\mathfrak{R}$ is k-positive (k-nonnegative), if the sum of its k smallest eigenvalues is positive (nonnegative). In a pioneering work, Hamilton [Ham86] proved that the space

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of positive curvature operator is preserved along the Ricci flow, and compact four-manifolds with positive curvature operator are diffeomorphic to spherical space forms. Chen [Chen91] later relaxed Hamilton's condition to 2-positive curvature operator. In a recent breakthrough, Böhm and Wilking [BW08] proved that compact n-dimensional manifolds with 2-positive curvature operator are diffeomorphic to spherical space forms.

Unfortunately, as Böhm and Wilking [BW08] pointed out, the space of 3-positive curvature operator is not preserved along the Ricci flow. However as the curvature operator of $(\mathbb{C}P^2, g_{FS})$ is 3-positive and the curvature operator of $(S^2 \times S^2, g_0 \oplus g_0)$ is 5-positive, it is natural to study the rigidity of Einstein four-manifolds with 3-positive or 4-positive curvature operator. As the first step, we investigate the relationship among k-positive curvature operator, positive sectional curvature, and positive isotropic curvature (see Section 2 for the definition).

Theorem 1.1. *Let (M^4, g) be an Einstein four-manifold with $\mathrm{Ric} = \lambda g$, $\lambda > 0$. Then we have:*

(1) *$\mathfrak{R}$ is 2-positive if and only if the isotropic curvature is positive.*
(2) *If $K > \frac{\lambda}{12}$, then $\mathfrak{R}$ is 3-positive; if $\mathfrak{R}$ is 3-positive, then $K > \frac{\lambda}{30}$.*
(3) *$\mathfrak{R}$ is 4-positive if and only if $K < \lambda$, and it implies $K > (4 - \sqrt{17})\lambda$.*

Remark 1.2. Costa [Cos04] proved that Einstein four-manifolds with $K \geq \frac{\lambda}{3(2+\sqrt{2})}$ are isometric to (S^4, g_0) or $(\mathbb{C}P^2, g_{FS})$.

Remark 1.3. If moreover the metric is Hermitian, then 4-positive curvature operator is equivalent to positive orthogonal bisectional curvature.

The rigidity of Einstein manifolds with positive curvature operator and positive isotropic curvature have been studied by Tachibana [Tach74] and Brendle [Bre10]. Tachibana [Tach74] proved that Einstein manifolds with positive curvature operator are isometric to spherical space forms. Brendle [Bre10] proved that Einstein manifolds with positive isotropic curvature are isometric to spherical space forms.

The basic idea of the proof, motivated by the work of Brendle [Bre10], is to apply the maximum principle to an equation of the curvature tensor, and reduce the problem to constrained optimizations. The new ingredient in the proof is to combine an analog of Brendle's argument [Bre10] and the Berger curvature decomposition [Berg61].

Notice that $K > \frac{\lambda}{12}$ implies $K < \frac{5\lambda}{6}$. Using the same argument as in Theorem 1.1, we can show that a slightly smaller upper bound also implies 3-positive curvature operator:

Proposition 1.4. *Let (M, g) be an Einstein four-manifold with $\mathrm{Ric} = \lambda g$, $\lambda > 0$. If $K < \frac{14-\sqrt{19}}{12}\lambda \approx (\frac{5}{6} - \frac{3}{100})\lambda$, then $\mathfrak{R}$ is 3-positive.*

The proof of Theorem 1.1 shows that on Einstein four-manifolds, the upper bound and lower bound of the sectional curvature are asymmetric. For simplicity, we assume $\lambda = 1$. On one hand, $K \geq \delta$ implies $K \leq 1 - 2\delta$. For example $\delta = \frac{1}{6}$ for $(\mathbb{C}P^2, g_{FS})$. On the other hand, $K \leq \delta$ (naively) implies $K \geq 1 - 2\delta$. However by our argument, the lower bound can be made much larger than $1 - 2\delta$. For example, 4-nonnegative curvature operator (equivalently $K \leq 1$) implies $K \geq -1$, but from Theorem 1.1 we can make $K \geq 4 - \sqrt{17}$. This suggests that $K < 1$ may be equivalent to $K > 0$. Half Weyl curvature and half curvature operator have a similar asymmetric property. We denote eigenvalues of W^+ by $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Notice that $-2\lambda_3 \leq \lambda_1 \leq -\frac{1}{2}\lambda_3$ since W^+ is traceless.

Proposition 1.5. *Let (M, g) be an Einstein four-manifold with* $\mathrm{Ric} = g$. *Suppose the minimum of λ_1 is achieved at p. Then*

$$\lambda_1(p) \geq \frac{1}{2}\left(2\lambda_3 + 1 - \sqrt{12\lambda_3^2 + 4\lambda_3 + 1}\right)(p) > (1 - \sqrt{3}\,)\lambda_3(p).$$

The proof of Theorem 1.1 also provides an alternative proof of the Weitzenböck formula for Einstein metrics on four-manifolds by Derdzinski [Der83]. Moreover the alternative proof directly extends from Einstein metrics on four-manifolds to "Einstein metrics" on four-dimensional smooth metric measure spaces, including gradient Ricci solitons, quasi-Einstein metrics, etc (see [Wu13, Wu17] for details).

For readers' convenience, we now provide the following table of curvature conditions for Einstein metrics on four-manifolds:

$$\boxed{\begin{array}{ccc}
\mathfrak{R}\text{ positive} \Rightarrow\ \text{2-positive} \Rightarrow K > \frac{1}{12} \Rightarrow \text{3-positive} \Rightarrow K > \frac{1}{30} \Rightarrow & & K > 0 \\
\quad\Updownarrow & & \Downarrow \\
\quad\text{PIC} & & \text{4-positive} \\
\quad\Downarrow & & \Updownarrow \\
\text{half 2-positive} \Leftrightarrow \text{half PIC} & & K < 1 \\
\quad\Downarrow & & \Downarrow \\
\text{conf. half PIC} & & R > 0 \Leftrightarrow \text{6-positive}
\end{array}}$$

TABLE 1. Curvature table for Einstein four-manifolds.

Here R is the scalar curvature; PIC denotes positive isotropic curvature; half PIC means PIC for orthonormal four-frame of a fixed orientation; and conformally half PIC means that there is a metric with half PIC in the conformal class of the Einstein metric; half 2-positive curvature operator means $\mathfrak{R}^{\pm} = \frac{R}{12}g + W^{\pm}$ is 2-positive.

From above relations, it is natural to ask the following questions for Einstein four-manifolds.

(1) If the curvature operator is 3-positive, is (M, g) isometric to (S^4, g_0) or $(\mathbb{C}P^2, g_{FS})$?
(2) If the sectional curvature is positive, is the curvature operator 3-positive?

(3) If the curvature operator is 4-positive, is the sectional curvature positive?

Question (1) is answered in a sequel [Wu13] to the author's thesis, yet the other two remain open.

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2. Proof of results

We first summarize the three curvature decompositions on Einstein four-manifolds: the standard curvature decomposition, the duality curvature decomposition, and the Berger curvature decomposition.

On a Riemannian manifold (M^n, g), the irreducible decomposition of the representations of the orthogonal group induces the standard curvature decomposition of the curvature tensor [Bes87]

$$\begin{aligned}\mathrm{Rm} =& W + \frac{1}{n-2}\mathrm{Ric} \odot g - \frac{R}{2(n-1)(n-2)} g \odot g\\ =& W + \frac{1}{n-2}\overset{\circ}{\mathrm{Ric}} \odot g + \frac{R}{2n(n-1)} g \odot g.\end{aligned}$$

On an oriented four-manifold (M^4, g), the Hodge star operator

$$\star : \wedge^2 TM \to \wedge^2 TM$$

induces a natural decomposition of the vector bundle of 2-forms $\wedge^2 TM$,

$$\wedge^2 TM = \wedge^+ M \oplus \wedge^- M,$$

where $\wedge^{\pm} M$ are eigenspaces of ± 1 respectively, sections of which are called self-dual, anti-self-dual 2-forms. It further induces a decomposition for the curvature operator $\mathfrak{R} : \wedge^2 TM \to \wedge^2 TM$ [Bes87]

$$\mathfrak{R} = \begin{pmatrix} \frac{R}{12} g + W^+ & \overset{\circ}{\mathrm{Ric}} \\ \overset{\circ}{\mathrm{Ric}} & \frac{R}{12} g + W^- \end{pmatrix},$$

where $\overset{\circ}{\mathrm{Ric}}$ is the traceless Ricci curvature, R is the scalar curvature. In particular if (M^4, g) is an Einstein manifold, then

$$\mathfrak{R} = \begin{pmatrix} \frac{R}{12} g + W^+ & 0 \\ 0 & \frac{R}{12} g + W^- \end{pmatrix} \triangleq \begin{pmatrix} \mathfrak{R}^+ & 0 \\ 0 & \mathfrak{R}^- \end{pmatrix}.$$

In [Berg61], Berger discovered another curvature decomposition for Einstein four-manifolds (see also Singer and Thorpe [ST69]):

Proposition 2.1. *Let (M,g) be an Einstein four-manifold with* $\mathrm{Ric} = \lambda g$. *For any $p \in M$, there exists an orthonormal basis $\{e_i\}_{1\le i\le 4}$ of T_pM, such that relative to the corresponding basis $\{e_i \wedge e_j\}_{1\le i<j\le 4}$ of $\wedge^2 T_pM$, $\mathfrak{R}$ takes the form*

$$\mathfrak{R} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A = \mathrm{diag}\{a_1, a_2, a_3\}$, $B = \mathrm{diag}\{b_1, b_2, b_3\}$, and they satisfy the following properties,

(1) $a_1 = K(e_1, e_2) = K(e_3, e_4) = \min\{K(\sigma)\}$,
$a_3 = K(e_1, e_4) = K(e_2, e_3) = \max\{K(\sigma)\}$,
$a_2 = K(e_1, e_3) = K(e_2, e_4)$, *and* $a_1 + a_2 + a_3 = \lambda$.
(2) $b_1 = R_{1234}$, $b_2 = R_{1342}$, $b_3 = R_{1423}$.
(3) $|b_i - b_j| \le |a_i - a_j|$, $1 \le i, j \le 3$.

The Berger curvature decomposition corresponds to a special duality curvature decomposition, because eigenvectors of $a_i \pm b_i$ are self-dual and anti-self-dual 2-forms, respectively.

By diagonalizing the matrix in the Berger curvature decomposition, we get eigenvalues of the curvature operator $\mathfrak{R}$ and half curvature operators $\mathfrak{R}^{\pm}$ in the following order,

$$\begin{cases} a_1 + b_1 \le a_2 + b_2 \le a_3 + b_3, \\ a_1 - b_1 \le a_2 - b_2 \le a_3 - b_3. \end{cases} \tag{2.1}$$

Therefore on an Einstein four-manifold, we have:

(1) Positive sectional curvature is equivalent to $(a_1+b_1)+(a_1-b_1) > 0$, that is, the sum of the smallest eigenvalues of $\mathfrak{R}^+$ and $\mathfrak{R}^-$ is positive.
(2) 2-positive curvature operator is equivalent to $(a_1+a_2)\pm(b_1+b_2) > 0$ and $a_1 > 0$.
(3) Positive isotropic curvature implies $(a_1 + a_2) \pm (b_1 + b_2) > 0$.
(4) 3-positive curvature operator is equivalent to $2a_1 + a_2 \pm b_2 > 0$.
(5) 4-positive curvature operator is equivalent to

$$a_1 + a_2 > 0 \quad \text{and} \quad \lambda + (a_1 \pm b_1) > 0.$$

Recall that (M,g) is said to have positive isotropic curvature [MM88], if for any orthonormal four-frame $\{e_i, e_j, e_k, e_l\}$, the curvature tensor satisfies

$$R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} > 2R_{ijkl}.$$

Proof of Theorem 1.1. Without loss of generality we assume $\lambda = 1$. We start with some simple observations. It is well known that 2-positive curvature operator implies positive isotropic curvature. By property (3) in the Berger curvature decomposition, we have

$$a_1 - a_2 \le b_2 - b_1 \le a_2 - a_1,$$
$$a_2 - a_3 \le b_2 - b_3 \le a_3 - a_2.$$

Taking the sum we get $|b_2| \le \frac{1}{3}(a_3 - a_1)$. If $a_1 > \frac{1}{12}$, then

$$2a_1 + a_2 - |b_2| \ge 2a_1 + a_2 - \frac{1}{3}(a_3 - a_1) \ge 4a_1 - \frac{1}{3} > 0,$$

therefore $\mathfrak{R}$ is 3-positive. If $\mathfrak{R}$ is 4-positive, it is obvious that $a_1 + a_2 > 0$, therefore $K < 1$.

Recall that for Einstein manifolds (see Hamilton [Ham82]),

$$\Delta R(e_i, e_j, e_k, e_l) + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) = 2R_{ijkl}, \tag{2.2}$$

where $B_{ijkl} = g^{mn}g^{pq}R_{imjp}R_{knlq}$. Applying the Berger curvature decomposition, we get

$$\begin{cases} \Delta R(e_1, e_2, e_1, e_2) + 2(a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3) = 2a_1, \\ \Delta R(e_1, e_3, e_1, e_3) + 2(a_2^2 + b_2^2 + 2a_1a_3 + 2b_1b_3) = 2a_2, \\ \Delta R(e_1, e_4, e_1, e_4) + 2(a_3^2 + b_3^2 + 2a_1a_2 + 2b_1b_2) = 2a_3. \end{cases}$$

Suppose that the minimum of the sectional curvature is attained at p by the tangent plane spanned by $\{e_1, e_2\}$. Recall that

$$\begin{aligned} 2\min K = 2a_1(p) &= (a_1 + b_1)(p) + (a_1 - b_1)(p) \\ &= \min_{\|\omega\|=1} (\mathfrak{R}^+(\omega, \omega) + \mathfrak{R}^-(\omega, \omega)), \end{aligned}$$

so the minimum of the sum of eigenvalues of $\mathfrak{R}^+$ and $\mathfrak{R}^-$ is attained at p. For any $v \in T_pM$ and the geodesic $\gamma(t)$ with $\gamma(0) = p$, $\gamma'(0) = v$, let $\{e_1, e_2, e_3, e_4\}$ be a parallel orthornormal frame along $\gamma(t)$, then we have

$$(D^2_{v,v}R)(e_1, e_2, e_1, e_2)(p) = D^2_{v,v}(R(e_1, e_2, e_1, e_2))(p) \ge 0.$$

Taking the trace we have $(\Delta R)(e_1, e_2, e_1, e_2)(p) \ge 0$, therefore at p we get

$$\boxed{a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \le a_1.} \tag{2.3}$$

First we prove that 2*-positive curvature operator is equivalent to positive isotropic curvature.* It suffices to show that $(a_1 + a_2) \pm (b_1 + b_2) > 0$ implies $a_1 > 0$. In fact if $(a_1 + a_2) \pm (b_1 + b_2) > 0$, then

$$a_2 \pm b_2 > 0, \qquad a_3 \pm b_3 > 0.$$

Therefore by (2.3), we have

$$\begin{aligned} a_1(p) \ge& a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \\ >& a_1^2 + b_1^2 \ge 0. \end{aligned}$$

Next we prove that 3*-positive curvature operator implies positive sectional curvature.* If $\mathfrak{R}$ is 3-positive, then

$$a_2 \pm b_2 > -2a_1, \qquad a_3 \pm b_3 > -2a_1.$$

Assuming that $a_1(p) \le 0$, then $a_2 \pm b_2 > 0$ and $a_3 \pm b_3 > 0$, then by (2.3), we have

$$\begin{aligned} a_1(p) \ge & a_1^2 + b_1^2 + 2(a_2 a_3 + b_2 b_3) \\ > & a_1^2 + b_1^2 \ge 0, \end{aligned}$$

which leads to a contradiction. Therefore $a_1(p) > 0$, i.e., (M, g) has positive sectional curvature.

Next we derive a lower bound for the sectional curvature when $\mathfrak{R}$ *is* 3*-positive.* Let $a_2(p) = k a_1(p),\ k \ge 1$. If $b_2 b_3 \ge 0$, then from (2.3) we get,

$$a_1 \ge a_1^2 + 2a_2 a_3 \ge a_1^2 + 2a_1(1 - 2a_1) = 2a_1 - 3a_1^2,$$

which implies that $a_1 = \frac{1}{3}$.

If $b_2 b_3 < 0$, without loss of generality, we assume $b_2 < 0,\ b_3 > 0$. On one hand, by 3-positivity of the curvature operator, $|b_2| < a_2 + 2a_1 = (k+2)a_1$, so we get

$$\begin{aligned} b_1^2 + 2b_2 b_3 = & b_2^2 + b_3^2 + 4b_2 b_3 \\ = & (b_3 + 2b_2)^2 - 3b_2^2 \\ > & -3(k+2)^2 a_1^2. \end{aligned}$$

Plugging into (2.3), we have

$$\begin{aligned} a_1 & \ge a_1^2 + b_1^2 + 2(a_2 a_3 + b_2 b_3) \\ & > a_1^2 + 2k a_1[1 - (k+1)a_1] - 3(k+2)^2 a_1^2 \\ & = 2k a_1 - (5k^2 + 14k + 11)a_1^2. \end{aligned}$$

Therefore we get

$$a_1 > \frac{2k - 1}{5k^2 + 14k + 11}. \tag{2.4}$$

On the other hand, by the Berger curvature decomposition,

$$|b_3 - b_2| \le a_3 - a_2 = 1 - (2k+1)a_1,$$

so we have

$$\begin{aligned} b_1^2 + 2b_2 b_3 = & \frac{3}{2} b_1^2 - \frac{1}{2}(b_3 - b_2)^2 \\ \ge & -\frac{1}{2}(a_3 - a_2)^2 = -\frac{1}{2}[1 - (2k+1)a_1]^2. \end{aligned} \tag{2.5}$$

Therefore,

$$\begin{aligned} a_1 & \ge a_1^2 + b_1^2 + 2(a_2 a_3 + b_2 b_3) \\ & \ge a_1^2 + 2k a_1[1 - (k+1)a_1] - \frac{1}{2}[1 - (2k+1)a_1]^2 \\ & = -\left(4k^2 + 4k - \frac{1}{2}\right) a_1^2 + (4k+1)a_1 - \frac{1}{2}, \end{aligned}$$

which implies

$$a_1 \le \frac{4k - \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1}, \quad \text{or} \quad a_1 \ge \frac{4k + \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1}. \tag{2.6}$$

If $a_1 \ge \frac{4k+\sqrt{8k^2-8k+1}}{8k^2+8k-1}$, then $a_1 = \frac{1}{3}$ if $k = 1$; and if $k > 1$ direct computation shows that,

$$a_2 - a_3 = (2k+1)a_1 - 1 \ge (2k+1)\frac{4k + \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1} - 1 > 0,$$

which contradicts to $a_2 \le a_3$. Therefore from (2.4) and (2.6), we have either $a_1 = \frac{1}{3}$, or

$$\frac{2k - 1}{5k^2 + 14k + 11} < a_1 \le \frac{4k - \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1},$$

which holds only if $1 \le k \le 4$, so we get

$$a_1 > \min_{1 \le k \le 4} \frac{2k - 1}{5k^2 + 14k + 11} = \frac{1}{30}.$$

At last we prove that $a_1 + a_2 > 0$ implies $\mathfrak{R}$ is 4-positive. It suffices to prove that $a_1 + a_2 > 0$ implies $1 + (a_1 \pm b_1) > 0$. From the Berger decomposition we have $|b_1| \le \frac{1}{3} - a_1$, so $a_1 > -\frac{1}{3}$ implies $1 + (a_1 \pm b_1) > 0$. We will show that in fact $a_1 + a_2 > 0$ implies $a_1 > 4 - \sqrt{17}$.

Assuming $a_1(p) = \min a_1$. Plugging (2.5) into (2.3), we have

$$\begin{aligned} a_1(p) \ge & a_1^2 + b_1^2 + 2(a_2 a_3 + b_2 b_3) \\ \ge & a_1^2 + 2a_2 a_3 - \frac{1}{2}(a_3 - a_2)^2. \end{aligned} \tag{2.7}$$

Since $a_3 + a_2 = 1 - a_1$, and $a_2 > -a_1, a_3 < 1$, we have (the minimum is achieved on the boundary)

$$\begin{aligned} 2a_2 a_3 - \frac{1}{2}(a_3 - a_2)^2 = & -\frac{1}{2}a_2^2 - \frac{1}{2}a_3^2 + 3a_2 a_3 \\ > & -\frac{1}{2}a_1^2 - \frac{1}{2} - 3a_1. \end{aligned} \tag{2.8}$$

Plugging (2.8) into (2.7), we get that $a_1 > 4 - \sqrt{17}$. □

Remark 2.2. In the author's thesis [Wu12], there was a naive mistake that "by Berger curvature decomposition $a_1 + a_2 > 0$ automatically implies $1 + (a_1 \pm b_1) > 0$". The author caught and corrected this (see the last step in the proof of Theorem 1.1) in August 2012 when he arrived at Cornell University as a postdoctoral fellow and prepared for seminar talks on his thesis and the work of Gursky and LeBrun [GL99] and Yang [Yang00].

Proof of Proposition 1.4. The proof of Proposition 1.4, similar to the proof of 3-positive curvature operator implying $K > \frac{\lambda}{30}$, contains a two-step constrained optimization. We omit the details since the argument is basically the same as the proof of Theorem 1.1. We assume $\lambda = 1$.

Step 1. We show that $K < \frac{14-\sqrt{19}}{12}$ implies $K > \frac{5-\sqrt{19}}{12}$. Recall that at the minimum point of the sectional curvature, we have

$$a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \le a_1.$$

Therefore the constrained optimization is

$$\begin{aligned} &\textit{Minimize} && a_1 \\ &\textit{subject to} && a_3 < \frac{14-\sqrt{19}}{12}, \\ &&& a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \le a_1, \\ &&& a_1 + b_1 \le a_2 + b_2 \le a_3 + b_3, \\ &&& a_1 - b_1 \le a_2 - b_2 \le a_3 - b_3, \\ &&& a_1 + a_2 + a_3 = 1,\ b_1 + b_2 + b_3 = 0. \end{aligned}$$

Step 2. We show that $K < \frac{14-\sqrt{19}}{12}$ *and* $K > \frac{5-\sqrt{19}}{12}$ imply 3-positive curvature operator. To do this, we evaluate Equation (2.2) at eigenvectors of the curvature operator and plug in the Berger decomposition. We denote eigenvalues of $\mathfrak{R}^+$ and $\mathfrak{R}^-$ by $\bar\lambda_i = a_i + b_i$, $\bar\mu_i = a_i - b_i$, and corresponding orthonormal eigenvectors by ω_i^+, ω_i^-, respectively. We get

$$\begin{cases} \Delta R(\omega_1^+, \omega_1^+) + \bar\lambda_1^2 + 2\bar\lambda_2\bar\lambda_3 = \bar\lambda_1, \\ \Delta R(\omega_2^+, \omega_2^+) + \bar\lambda_2^2 + 2\bar\lambda_1\bar\lambda_3 = \bar\lambda_2, \\ \Delta R(\omega_3^+, \omega_3^+) + \bar\lambda_3^2 + 2\bar\lambda_1\bar\lambda_2 = \bar\lambda_3, \\ \Delta R(\omega_1^-, \omega_1^-) + \bar\mu_1^2 + 2\bar\mu_2\bar\mu_3 = \bar\mu_1. \\ \Delta R(\omega_2^-, \omega_2^-) + \bar\mu_2^2 + 2\bar\mu_1\bar\mu_3 = \bar\mu_2. \\ \Delta R(\omega_3^-, \omega_3^-) + \bar\mu_3^2 + 2\bar\mu_1\bar\mu_2 = \bar\mu_3. \end{cases} \tag{2.9}$$

Suppose the minimum of the sum of any three eigenvalues is attained at a point q by $\bar\lambda_1 + \bar\lambda_2 + \bar\mu_1 = 1 - \bar\lambda_3 + \bar\mu_1 = \min_{\|\omega\|=1}(I - \mathfrak{R}^+ + \mathfrak{R}^-)(\omega, \omega)$. Then at q, taking the sum in Equation (2.9) we get

$$\boxed{\bar\mu_1^2 + 2\bar\mu_2\bar\mu_3 - \bar\lambda_3^2 - 2\bar\lambda_1\bar\lambda_2 \le \bar\mu_1 - \bar\lambda_3.} \tag{2.10}$$

Therefore the constrained optimization is

$$\begin{aligned} &\textit{Minimize} && 1 + \bar\mu_1 - \bar\lambda_3 \\ &\textit{subject to} && \bar\lambda_3 + \bar\mu_3 < \frac{14-\sqrt{19}}{6}, \\ &&& \bar\lambda_1 + \bar\mu_1 > \frac{5-\sqrt{19}}{6}, \\ &&& \bar\mu_1^2 + 2\bar\mu_2\bar\mu_3 - \bar\lambda_3^2 - 2\bar\lambda_1\bar\lambda_2 \le \bar\mu_1 - \bar\lambda_3. \\ &&& \bar\lambda_1 \le \bar\lambda_2 \le \bar\lambda_3,\ \bar\mu_1 \le \bar\mu_2 \le \bar\mu_3, \\ &&& \bar\lambda_1 + \bar\lambda_2 + \bar\lambda_3 = 1,\ \bar\mu_1 + \bar\mu_2 + \bar\mu_3 = 1. \end{aligned}$$

We get $(\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\mu}_1)(q) > 0$. If the minimum is attained by $\bar{\lambda}_1 + \bar{\mu}_1 + \bar{\mu}_2$ at some point, then we get the same conclusion. $\square$

The proof of Proposition 1.5 follows from an observation from Equation (2.9) that at the minimum point of $\lambda_1 = \bar{\lambda}_1 - \frac{1}{3}$, we have $\bar{\lambda}_1^2 + 2\bar{\lambda}_2\bar{\lambda}_3 \leq \bar{\lambda}_1$, therefore $\lambda_1^2 + 2\lambda_2\lambda_3 \leq \lambda_1$. $\square$

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