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# Harnack inequalities for critical 4-manifolds with a Ricci curvature bound

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Abstract. We study critical Riemannian 4-manifolds with bounded Ricci curvature, but with no apriori analytic constraints such as on Sobolev constants. We derive elliptic-type estimates for the local curvature radius, which itself controls sectional curvature. The method is use degenerating, collapsing metrics to create a noncollapsed blow-up limit, and then use a geometric triviality result for complete Ricci-flat manifolds with a Killing field to rule out such a blow-up. The Cheeger–Tian $\epsilon$-regularity theorem on Einstein manifolds is reproved as a byproduct.

Contents



## 1. Introduction

We develop certain Harnack inequalities for critical metrics on 4-manifolds in terms of local Ricci curvature bounds, and explore some consequences. Our results are of collapsing type, in the sense that volume ratios and Sobolev constants are immaterial. A particular consequence of our main theorem is that we recover the Cheeger–Tian $\epsilon$-regularity result [10] on Einstein 4-manifolds with a simpler argument.

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The main quantities of investigation are the local curvature radius and local Ricci curvature radii:

$$
\begin{aligned}
(1) \qquad & r^s_{\mathcal{R}}(p) = \sup \left\{ \, r \, \in \, (0, \, s) \textit{ and } |\operatorname{Rm}| < r^{-2} \text{ on } B(p, \, r) \, \right\}, \\
& r^s_{\mathcal{RC}}(p) = \sup \left\{ \, r \, \in \, (0, \, s) \textit{ and } |\operatorname{Ric}| < r^{-2} \text{ on } B(p, \, r) \, \right\}.
\end{aligned}
$$

Notice these have a built-in cutoff, so they are often called the $s$-local curvature radii. We use $r_{\mathcal{R}}$, $r_{\mathcal{RC}}$ for $r^{\infty}_{\mathcal{R}}$, $r^{\infty}_{\mathcal{RC}}$, respectively, for the corresponding radii without any cutoff. The geometric meaning of $r_{\mathcal{R}}(p)$ is that if distances are scaled by the quantity $(r_{\mathcal{R}}(p))^{-1}$, then the ball of radius 1 centered on $p$ will have $|\operatorname{Rm}| \leq 1$ throughout, and $|\operatorname{Rm}| = 1$ somewhere; similarly for $r_{\mathcal{RC}}(p)$. The usefulness of the $r_{\mathcal{R}}$ function on critical Riemannian manifolds has been made clear in [10] and elsewhere, where it played a crucial role in collapsing constructions and in proving the Cheeger–Tian structure theorems for Einstein 4-manifolds in [10].

The methods of this paper rely on the interplay between Riemannian geometry, topology, and analysis—the new ingredient in this paper is the "geometric vanishing" theorem of [21], used to show that certain blow-up limits of collapsing manifolds are trivial. This "vanishing" theorem (theorem 1 of [21]) states that a complete Ricci-flat manifold with a Killing field is flat.

We also prove elliptic-type estimates for the behavior of $r_{\mathcal{R}}$, in low energy regions of the manifold, which incidentally recovers the $\epsilon$-regularity result of Cheeger–Tian [10] on Einstein manifolds. Noteworthy is that our method does not require the iterative improvement argument, proposition 8.2, [10].

We work exclusively with "critical" 4-manifolds, which we take to mean simply that the curvature tensor satisfies an elliptic system. These could be any of the following: Einstein, half conformally-flat with constant scalar curvature or otherwise Bach flat with constant scalar curvature, a metric with harmonic curvature, or a scalar-flat or extremal Kähler metric.

**Theorem 1.1** (Main Regularity Estimate)**.** *Assume $(M^4, g)$ is a critical Riemannian manifold. There exist constants $\epsilon_0, \delta_0 > 0$ with the following property. If $\int_{B(p,r)} |\operatorname{Rm}|^2 < \epsilon_0$, then $r_{\mathcal{R}}(q) > \delta_0 \min\{r, \, r_{\mathcal{RC}}(q)\}$ for all $q \in B(p, \frac{1}{2}r)$.*

The conclusion gives an estimate for the Riemannian curvature radius in terms of both the radius of the ball and the Ricci curvature radius. This theorem implies theorem 0.8 of [10]. The constants $\epsilon_0$, $\delta_0$ are universal in the sense that they do not depend on the particular metric (or on Sobolev constants, injectivity radii, etc). An immediate corollary is the following.

**Corollary 1.2.** *There are constants $\epsilon_0$, $\delta_0 < 0$ so that if $\int_{B(p,r)} |\operatorname{Rm}|^2 < \epsilon_0$, then for any $q \in B(q, \frac{1}{2}r)$ we have $|\operatorname{Rm}|_q < \delta_0^{-2} \max\{r^{-2}, r_{\mathcal{RC}}(q)^{-2}\}$.*

Another immediate consequence is the following Harnack-style estimate on the size of the curvature radius; this essentially states that if $r_{\mathcal{R}}$ is not too small at the center of a low-energy ball, it is not too small everywhere.

**Corollary 1.3** (Harnack inequality I)**.** *There exist constants $\epsilon_0, \delta_0 > 0$ with the following property. If $\int_{B(p,r)} |\operatorname{Rm}|^2 < \epsilon_0$ and $r_{\mathcal{R}}(p) < \frac{1}{2}r$, then*

$$r_{\mathcal{R}}(q) > \delta_0 \min\{r_{\mathcal{R}}(p),\, r_{\mathcal{RC}}(q)\}$$

*for any $q \in B(p, \frac{1}{2}r)$.*

**Theorem 1.4** (Harnack inequality II)**.** *Given $k > 0$ and $\mu > 0$, there exist numbers $\epsilon_0 > 0$ and $C = C(\mu, k) < \infty$ so that the following holds. If $r_{\mathcal{RC}}(p) \geq (1+\mu) r_{\mathcal{R}}(p)$ and $\int_{B(p,\,(1+\mu) r_{\mathcal{R}}(p))} |\operatorname{Rm}|^2 \;\leq\; \epsilon_0$, then*

$$r_{\mathcal{R}}(p) \;\geq\; C \left( \fint_{B(p, r_{\mathcal{R}}(p))} |\operatorname{Rm}|^k \right)^{-\frac{1}{2k}}. \tag{2}$$

(The dash-through indicates averaging the integral, by dividing by the Riemannian volume of the domain of integration). To explain the relevance of this theorem, note that by definition we have $|\operatorname{Rm}| \;\leq\; r_{\mathcal{R}}(p)^{-2}$ on $B(p, r_{\mathcal{R}}(p))$. Theorem 1.4 says the reverse inequality holds in an *average* sense, provided energy is small. These "Harnack" inequalities are of analytic interest.

We also prove the following elliptic-type estimate on the curvature scale.

**Corollary 1.5** (Local elliptic estimates for the curvature radius)**.** *Given $K < \infty$ and $l \in \mathbb{N}$, there is an $\epsilon_0 = \epsilon_0(K) > 0$ and $C = C(K, l)$ so that if $r_{\mathcal{R}}(p) < K r_{\mathcal{RC}}(p)$ and $\int_{B(p,\, 2r_{\mathcal{R}}(p))} |\operatorname{Rm}|^2 \;\leq\; \epsilon_0$, then*

$$\sup_{B(p, r_{\mathcal{R}}(p))} \left| \nabla^l r_{\mathcal{R}} \right| \;\leq\; C\, \left(r_{\mathcal{R}}(p)\right)^{1-l}. \tag{3}$$

Setting $l = 1$ we see that $|\nabla r_{\mathcal{R}}|$ has an absolute bound—this is expected, as we already noted $r_{\mathcal{R}}$ has an apriori Lipschitz constant (e.g., (1.9) of [10]). When $l > 1$, $\mu \in (0, 1]$ and $\int_{B(p, 2r_{\mathcal{R}}(p))} |\operatorname{Rm}|^2 < \epsilon_0$, then using (17) we have any of the following

$$\begin{aligned} \sup_{B(p, r_{\mathcal{R}}(p))} \left| \nabla^l r_{\mathcal{R}} \right| \;&\leq\; C \fint_{B(q, \mu r_{\mathcal{R}}(q))} |\operatorname{Rm}|^{\frac{l-1}{2}}, \\ \sup_{B(p, r_{\mathcal{R}}(p))} \left| \nabla^l r_{\mathcal{R}} \right| \;&\leq\; C \left( \fint_{B(q, \mu r_{\mathcal{R}}(q))} |\operatorname{Rm}|^2 \right)^{\frac{l-1}{4}}, \quad \textit{and} \\ \sup_{B(p, r_{\mathcal{R}}(p))} \left| \nabla^l r_{\mathcal{R}} \right| \;&\leq\; C\, \left(r_{\mathcal{R}}(p)\right)^{2-l} \left( \fint_{B(q, \mu r_{\mathcal{R}}(q))} |\operatorname{Rm}|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{4}$$

## 2. Definitions and background results

Here we outline prior definitions and results that will be important for us. The first is "standard" $\epsilon$-regularity; the second is an analytic criterion for collapsing from [10]; the third is a criterion for certain 4-manifolds to be flat (from [21]). Finally we outline the (unfortunately rather involved) theory of N-structures.

### 2.1. $\epsilon$-regularity, collapsing, and Ricci-pinched manifolds.

**Lemma 2.1** (Standard $\epsilon$-regularity)**.** *There exists numbers $\epsilon_0 > 0$, $C < \infty$ so that $r \leq r_{\mathcal{RC}}(p)$ and*

$$\frac{1}{\mathrm{Vol}\, B(p,r)} \int_{B(p,r)} |\,\mathrm{Rm}\,|^2 \;\leq\; \epsilon_0\, r^{-4} \tag{5}$$

*imply*

$$\sup_{B(p,r/2)} |\,\mathrm{Rm}\,| \;\leq\; C \left( \frac{1}{\mathrm{Vol}\, B(p,r)} \int_{B(p,r)} |\,\mathrm{Rm}\,|^2 \right)^{\frac{1}{2}}. \tag{6}$$

Among the numerous references with this type of theorem, see [19] [17] [1] [2] [4] [18] [11] [20].

Lemma 2.1 is normally used in a noncollapsing setting, for obvious reasons: if one assumes bounded volume ratios, say

$$\mathrm{Vol}\, B(p,r) \geq \delta r^n,$$

then one may measure $\int_{B(p,r)} |\,\mathrm{Rm}\,|^2$ against the apriori controlled quantity $r^{-4}\mathrm{Vol}\, B(p,r)$. An argument found in [10], effectively a contrapositive, extends its usefulness to the collapsing setting, providing a way of forcing *collapse with locally bounded curvature*. The significance of collapse with locally bounded curvature is explained in Section 2.2.

**Lemma 2.2** (Collapse criterion [10])**.** *Given $\tau > 0$, there is an $\epsilon = \epsilon(\tau) > 0$ so that $r_{\mathcal{R}}(p) \leq \frac{1}{2} r_{\mathcal{RC}}(p)$ and $\int_{B(p,2r_{\mathcal{R}}(p))} |\,\mathrm{Rm}\,|^2 \;\leq\; \epsilon$ imply*

$$\mathrm{Vol}\, B(p,\, r_{\mathcal{R}}(p)) \;\leq\; \tau \cdot r_{\mathcal{R}}(p)^4.$$

**Proof.** There is a point $q \in B(p, r_{\mathcal{R}}(p))$ with $|\,\mathrm{Rm}(q)| = r_{\mathcal{R}}(p)^{-2}$. Now, assuming that $r_{\mathcal{R}}(p)^{-4}\, \mathrm{Vol}\, B(p, r_{\mathcal{R}}(p)) > \tau$, then choosing $\epsilon_0$ small enough we have (5). But then the conclusion of Lemma 2.1 holds, so

$$r_{\mathcal{R}}(p)^{-2} \;\leq\; C \left( \frac{1}{\mathrm{Vol}\, B(p, 2r_{\mathcal{R}}(p))} \int_{B(p,2r_{\mathcal{R}}(p))} |\,\mathrm{Rm}\,|^2 \right)^{\frac{1}{2}}. \tag{7}$$

Thus $r_{\mathcal{R}}(p)^{-4}\mathrm{Vol}\, B(p, r_{\mathcal{R}}(p)) < C^2\epsilon_0$. Possibly choosing $\epsilon_0$ still smaller, we again have $r_{\mathcal{R}}(p)^{-4}\mathrm{Vol}\, B(p, r_{\mathcal{R}}(p)) \leq \tau$. $\square$

The third result, from [20], is a "geometric vanishing" theorem for certain manifolds with a Killing field. We shall use it in a contradition argument to conclude that certain blow-ups of collapsing manifolds, which are apriori nonflat, are in fact flat.

**Lemma 2.3** (Flatness criterion [21]). *Assume $(N^4, g)$ is a complete 4-manifold with a nowhere-zero Killing field. If $N^4$ is Ricci-flat, then $N^4$ is flat.*

The usefulness of this theorem is that, in the case of collapse with bounded curvature, we have N-structures whose associated locally-defined Killing fields have bounded local variation. By passing to appropriate covers, we obtain complete manifolds where the Killing field(s) obtained from the N-structure automatically have asymptotically bounded local variation, which allows us to use this theorem.

**2.2. Collapsing with bounded curvature: F- and N-structures.** The F-structures of Cheeger–Gromov [7] [8] and N-structures of Cheeger–Gromov-Fukaya [6] will be decisive in what follows, so we define them precisely. A number of variant definitions are available; ours is essentially from [9], with one main difference that is explained below.

**2.2.1. Definitions.** An N-structure $\mathfrak{N}$ is a triple $(\Omega, \mathcal{N}, \iota)$ where $\Omega$ is a domain in a differentiable manifold, $\mathcal{N}$ is a sheaf of nilpotent Lie algebras on $\Omega$, and $\iota : \mathcal{N} \to \mathcal{X}(\Omega)$ (called the action) is a sheaf monomorphism from $\mathcal{N}$ into the Lie algebra sheaf $\mathcal{X}(\Omega)$ of differentiable vector fields on $\Omega$, all so that a collection of sub-structures $\mathcal{A} = \{(\mathcal{N}_i, \Omega_i, \iota_i)\}_i$ exists that satisfies the three conditions below. In what follows, if $p \in \Omega_i$, then its $\mathcal{N}_i$-stalk will be denoted $\mathcal{N}_{i,p}$ and its $\mathcal{N}$-stalk will be denoted $\mathcal{N}_p$.

(i) (Completeness of the cover). The collection of sets $\{\Omega_i\}$ is a locally finite cover of $\Omega$, and given $p \in \Omega$ there is at least one $\Omega_i$ so that $\mathcal{N}_{i,p} = \mathcal{N}_p$.

(ii) (Uniformity of the action). The lifted sheaf $\widetilde{\mathcal{N}_i}$ over the universal cover $\widetilde{\Omega_i} \to \Omega_i$ is a constant sheaf (each stalk is canonically isomorphic to the Lie algebra of global sections $\widetilde{\mathcal{N}_i}(\widetilde{\Omega_i})$).

(As a side note, the lifted action $\widetilde{\iota_i} : \widetilde{\mathcal{N}_i} \to \mathcal{X}(\widetilde{\Omega_i})$ is not uniquely defined but depends on a choice of fundamental domain. This manifests on the $\Omega_i$ as a holonomy phenomenon on the stalks.)

(iii) (Integrability of the action). Given $\Omega_i$, there is a connected, simply-connected nilpotent Lie group $G_i$ so that for any choice of $\widetilde{\iota_i}$ there is an action of $G_i$ on $\widetilde{\Omega_i}$ whose derived action is equal to the image of the Lie algebra of sections $\widetilde{\mathcal{N}_i}(\widetilde{\Omega_i})$ under $\widetilde{\iota_i}$.

An N-structure is an called an F-structure if the associated sheaf $\mathcal{N}$ is abelian. The difference between our definition of F-structures and the common definition is that we do not require that a torus acts on a finite normal cover of $\Omega_i$, but rather some $\mathbb{R}^k$ acts on its universal cover. This is a convenience in what follows for the reason that we will make frequent passages to universal covers, and wish to refer to the structures obtained there as F- or N-strucutres, whether orbits are closed or not.

Via the action of the groups $G_i$ on covers, an N-structure partitions $\Omega$ into orbits; the orbit through $p \in \Omega$ is denoted $\mathcal{O}_p$. An orbit $\mathcal{O}_p$ is called *singular* if its dimension is not equal to the dimension of the stalk $\mathcal{N}_p$; the *singular locus* of $\mathcal{N}$ is the union of all its singular orbits. In addition, orbits may be *exceptional*; these are orbits for which nearby orbits are identified to it in a $k$-to-1 fashion. An example would be $\mathbb{S}^3 \subset \mathbb{C}^2$ with a Killing field given by differentiating the action $t \mapsto (e^{2\pi i t/k} z_1,\, e^{2\pi i l t/k} z_2)$, $k, l \in \mathbb{Z}$ relatively prime; the exceptional orbits in this case are the collection of points $(z_1, 0)$ or $(0, z_2)$ in $\mathbb{S}^3$.

The *rank* of an N-structure $\mathfrak{N}$ at $p \in \Omega$ is the dimension of the orbit of $\mathfrak{N}$ through that point. We say $\mathfrak{N}$ has positive rank if it has positive rank at every point. An N-structure is called *pure* if the dimension of its stalks is locally constant.

An N-structure is called *polarized* if it has positive rank and no singular orbits—this does not mean the orbit dimension (the rank) is locally constant, as the stalk and orbit dimensions may vary together. An N-structure is called *polarizable* if it contains a polarized substructure. An example of Cheeger–Gromov [7] shows the existence of a nonpolarizable F-structure on a 4-dimensional manifold.

Let $\Omega$ be a domain that is saturated for some polarized N-structure $\mathfrak{N}$ of positive rank. An *atlas* for $\mathfrak{N}$, denoted $\mathcal{A} = \{(\Omega_i,\, \mathfrak{N}_i)\}_i$, consists of a collection of countably many open sets $\Omega_i$ with $\Omega = \bigcup_i \Omega_i$, so that each $\Omega_i$ is saturated under $\mathfrak{N}$ (not just $\mathfrak{N}_i$), so that $\mathfrak{N}_i = (\Omega_i, \mathcal{N}_i, \iota_i)$ is a pure substructure of $\mathfrak{N}|_{\Omega_i}$, and so that the $\Omega_i$ themselves have universal covers $\pi_i : \widetilde{\Omega}_i \to \Omega_i$ on which the lifted structure $\widetilde{\mathfrak{N}}_i$ is a constant sheaf whose action integrates to a global action of a connected, simply connected Lie group. Further, that each $p \in \Omega$ lies in finitely many of the $\Omega_i$, that the stalks $\mathcal{N}_{i,p}$ at $p$ can be ordered by strict inclusion: $\mathcal{N}_{i_1,p} \subset \cdots \subset \mathcal{N}_{i_k,p}$, and that there is always some $i$ so that $\mathcal{N}_{i,p} = \mathcal{N}_p$. Lemma 1.2 of Cheeger–Gromov [7] states that an atlas always exists. An atlas is called *polarized* if each pure N-structure $\mathfrak{N}_i$ is polarized.

We present some definitions that describe interactions between N-structures and geometry. A metric is called *invariant* under an N-structure if the action of $\mathcal{N}$ is isometric—more precisely, if the image of the monomorphism $\iota : \mathcal{N} \to \mathcal{X}$ lies in the sub-sheaf of Killing fields. A polarized atlas $\mathfrak{A} = \{(\Omega_i,\, \mathfrak{N}_i)\}$ will be called *$C$-regular* if the norm of the second fundamental form of any orbit of $\mathfrak{N}_i$ is bounded from above by $C$, and the multiplicity of the covering $\{\Omega_i\}$ is also bounded by $C$. A polarized atlas will be called

*$C$-regular with locally bounded curvature* if the norm of the second fundamental form of any orbit of $\mathcal{N}_i$ at a point $p_i \in U_i$ is bounded from above by $Cr_{\mathcal{R}}(p_i)^{-1}$ and the multiplicity of the cover $\{U_i\}$ is bounded by $C$.

**Theorem 2.4** (Cheeger–Gromov [8], Cheeger–Fukaya–Gromov [6]). *There exists $\tau = \tau(n, \delta, \alpha) > 0$ so that if $\Omega \subset N^n$ is a domain in a complete Riemannian manifold $N$ with $|\operatorname{Rm}| < 1$ on $\Omega^{(1)}$, and if $\operatorname{Vol} B(p, 1) \leq \tau$ for all $p \in \Omega$, then a neighborhood of $\Omega$ exists (that is within $\Omega^{(1)}$) that is saturated with respect to an N-structure $\mathfrak{N}$, and so that the metric on $N$ is $\delta$-close in the $C^{1,\alpha}$ sense to a metric for which $\mathcal{N}$ is invariant.*

In the case the metric has an elliptic system, $C^{1,\alpha}$-closeness can be improved to $C^{k,\alpha}$-closeness, but where $\tau$ depends also on $k$.

**Theorem 2.5** (Cheeger–Rong [9]). *If, in addition to the hypotheses of Theorem 2.4, $\tau$ is sufficiently small compared to the diameter of $\Omega \subseteq N^n$, then $\mathfrak{N}$ is pure.*

**Theorem 2.6** (Rong [16]). *If, in addition to the hypotheses of Theorem 2.4, $\tau$ is also sufficiently small compared to the diameter of $\Omega \subseteq N^4$ and $\Omega \subset N^4$ is 4-dimensional, then $\mathfrak{N}$ is polarizable.*

**Theorem 2.7** (Cheeger–Fukaya–Gromov [6]). *Under the hypotheses of Theorem 2.4, the center of the resulting N-structure is itself an F-structure of positive rank.*

In addition, there exists a $C < \infty$ so that $\mathfrak{N}$ has a polarized $C$-regular atlas.

**Theorem 2.8** (Naber–Tian [14]). *If $\pi : \Omega \to \Omega'$ (where $\Omega \subseteq N^4$) is the projection onto the orbit space of a pure N-structure $\mathfrak{N}$, then $\mathfrak{N}$ is an orbifold with $C^\infty$ orbifold points.*

**Theorem 2.9** (Cheeger–Fukaya–Gromov [6], Naber–Tian [14], Cheeger–Rong [9]). *Under the hypotheses of Theorem 2.4 if the metric on $N$ is $\mathfrak{N}$-invariant, the quotient $N \to N'$ along the orbits of $\mathfrak{N}$ is a Riemannian orbifold with $C^\infty$ orbifold points, and injectivity radius bounded from below on compact sub-domains.*

### 2.2.2. Additional properties of N-structures.

**Lemma 2.10** (Global integrability for N-structures). *If $\mathfrak{N}$ is any N-structure on a domain $\Omega$ with an invariant metric, and if $\Omega$ is simply connected, then any element $\mathfrak{b} \in \mathcal{N}_p$ of the stalk at any point $p \in \Omega$ extends uniquely to a Killing field $V$ on $\Omega$.*

**Proof.** This is equivalent to showing that $H^0(\Omega, \mathcal{N})$ is canonically isomorphic to the stalk of $\mathcal{N}_p$ at any point $p$. On any (differentiable) Riemannian manifold, a Killing field is locally determined by its germ at a point.

Suppose $\gamma(t)$, $t \in [0, 1]$ is any loop with $\gamma(0) = \gamma(1) = p$. Assume $\mu_s(t)$ is a homotopy of $\gamma(t)$ to the constant path at $p$; that is, $\mu_0(t) = \gamma(t)$, $\mu_1(t) = p$, and $\mu_s(1) = \mu_s(0) = p$. We assume that the map $\mu : [0,1] \times [0,1] \to N$ is $C^0$, and is analytic when restricted to $(0,1) \times (0,1)$.

Letting $\mathfrak{v} \in \mathcal{N}_p$, in any sufficiently small neighborhood of $p$, the element $\mathfrak{v}$ has a unique extension to a Killing field $V$. Covering $\gamma(t)$ by sufficiently small open sets, we obtain an extension of $\mathfrak{v}$ to a Killing field $V$ in some neighborhood of $\gamma(t)$. This extension is unique along $\gamma$ itself; the issue is that $V(\gamma(0))$ might not equal $V(\gamma(1))$.

Repeating this for any $s \in [0,1]$, we can extend $V$ along the path $t \mapsto \mu_s(t)$. We then obtain a function

$$\begin{aligned} [0,\,1] &\to T_pN \\ s &\mapsto V(\mu_s(1)) \end{aligned} \tag{8}$$

given by $V(\mu_s(1))$. Because Killing fields on analytic Riemannian manifolds analytic (for instance they satisfy the system $\triangle V + \mathrm{Ric}(V) = 0$), this map is also analytic. However, when $s$ is small enough that the path $t \mapsto \mu_s(t)$ lies in a neighborhood of $p$ so small that $V$ is uniquely determined, the map $s \mapsto V(\mu_s(1))$ is constant. Since $V$ solves an elliptic system, it cannot have zeros that accumulate; thus it is constant for all $s$.

This shows that given $\mathfrak{v} \in \mathcal{N}_p$, we can define the field $V$ at any point $q$ by connecting $p$ to $q$ with any path and extending $V$ along this path. The argument above is then used to show the vector $V(q)$ is independent of the path chosen. □

Vital to the argument Section 3 is the ability to trivialize the topology of a space with an N-structure, provided the leaf-space is contractible. Specifically, assuming $\Omega^n$ is some saturated region of an $n$-manifold with a positive-rank N-structure and $\pi : \Omega^n \to \Omega'$ being the projection to the orbit space, we wish to say that the universal cover $\widetilde{\Omega^n}$ of $\Omega^n$ has no closed orbits, provided $\Omega'$ is contractible.

If $\Omega^n$ were a fiber bundle, this would be trivial. But the existence of exceptional fibers and/or singular fibers means $\Omega^n \to \Omega'$ is not even a fibration, so arguments based on lifting homotopies form the base to the total space won't work.

Still, we give arguments that N-structures are trivial when the domain is simply connected domains and the quotient by the orbits of the N-structure is contractible, when the dimension of the total space is 2, 3, or 4. The case $n = 2$ is essentially trivial, and if $n = 3$ this basic question has been studied in the context of Seifert fibrations. In the $n = 4$ case, the author cannot immediately find a reference in the literature, so we present our own argument below.

**Lemma 2.11** (Local triviality for $n = 2$). *Assume $\Omega^2$ is a differentiable simply connected 2-dimensional orbifold, possibly with boundary, that is saturated and metrically invariant with respect to a rank-1 N-structure $\mathfrak{N}$, and* *let $\pi : \Omega^2 \to \Omega'$ be the projection to the orbit space. Then $\Omega^2$ is a differentiable manifold (possibly with boundary), and if $\Omega'$ is contractible, the N-structure's orbits are all unbounded.*

**Proof.** Since $\Omega^2$ is a differentiable orbifold, it is a topological manifold. Since it is simply connected, $\mathfrak{N}$ is represented by a Killing field $\widetilde{V}$ on $\Omega^2$. All orbifold points on a 2-dimensional orbifold are isolated or are geodesic boundaries, but are also invariant with respect to $\widetilde{V}$. Since $\widetilde{V}$ has no fixed points, $\Omega^2$ must be a differentiable manifold, possibly with (totally geodesic) boundary.

Since $\widetilde{V}$ has no fixed points, and since a 1-dimensional foliation on a 2-dimensional manifold cannot have exceptional fibers, the projections

$$\Omega^2 \to \Omega'$$

are indeed fiber bundles. If the orbits happen to be compact (in other words, circles) then they must carry a class in $H^1(\Omega^2, \mathbb{Z})$, by the Gysin sequence. But since $H^1(\Omega^2, \mathbb{Z}) = \varnothing$, the manifold $\Omega^2$ must have $\mathbb{R}^1$ fibers. $\square$

Before moving to the 3- and 4-dimensional cases, we require a technical lemma form the theory of Seifert fibered spaces. We reproduce a proof here.

**Lemma 2.12.** *Assume $\Omega^3$ is a Seifert fibered manifold with boundary with projection $\pi : \Omega^3 \to \Omega'$ onto its orbit space, and assume $\Omega'$ is contractible. Then there is a homeomorphism $\Omega^3 = D^2 \times \mathbb{S}^1$ where $D^2$ is a standard 2-disk.*

**Proof.** Since $\Omega'$ is a 2-disk by assumption, the boundary $\partial\Omega^3$ is a 2-torus. Gluing in a solid torus via a $(1, 1)$ torus gluing along this boundary gives us a closed Seifert fibered 3-manifold $M^3$, with projection $M^3 \to M'$, where $M'$ is a sphere. Also, $M^3$ is oriented.

The basic theory of Seifert fibered spaces (e.g., [15]) says that the fundamental group of $M^3$ is

$$\pi_1 = \left\langle g_1, \ldots, g_r, h \;\middle|\; g_i h g_i^{-1} = h,\ g_i^{\alpha_i} h^{\beta_i} = 1,\ g_1 \ldots g_r = h^b \right\rangle \tag{9}$$

for certain integers $0 < \beta_i < \alpha_i$ and $b \in \mathbb{Z}$. The generator $h$ is represented by a regular fiber, and the generators $g_i$ are represented in a canonical way by certain cycles in appropriate neighborhoods of the exceptional fibers.

One easily verifies that this group is finite unless $b + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} = 0$, in which case $\langle h \rangle$ is an infinite cyclic subgroup. Passing to the universal cover $\widetilde{M^3}$, in the case $\pi_1$ was finite we have $\widetilde{M^3} = \mathbb{S}^3$ (by the solution of the Poincare conjecture) and in the case that $\pi_1$ was infinite, we still have a projection $\widetilde{M^3} \to M' = \mathbb{S}^2$, so $\widetilde{M^3}$ is an $\mathbb{R}$-bundle over $\mathbb{S}^2$, and so $\widetilde{M^3} = \mathbb{S}^2 \times \mathbb{R}$.

From the structure of $\widetilde{M^3}$ we can deduce the structure of $M^3$. In the case $\widetilde{M^3} \approx \mathbb{S}^3$, let $\tilde{\pi} : \widetilde{M^3} \to M^3$ be the covering map, and consider $\widetilde{\Omega^3} = \tilde{\pi}^{-1}\Omega^3$. Then $\widetilde{\Omega^3}$ is still fibered, and clearly we retain a projection $\widetilde{\Omega'} \to \Omega'$ onto the disk $\Omega'$. This means $\widetilde{\Omega^3}$ is a solid torus, and since the only orientable quotients of solid tori are solid tori, we have that $\Omega^3$ is a solid torus, and we are done in this case.

In the case $\widetilde{M^3} \approx \mathbb{S}^2 \times \mathbb{R}$, the manifold $M^3$ has $\mathbb{S}^2 \times \mathbb{R}$ geometry, and the classification of such manifolds gives just 4 possibilities: it can be $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$, the mapping torus of the antipodal map on $\mathbb{S}^2$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$. Of these only $\mathbb{R}P^3 \# \mathbb{R}P^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ are oriented. In the case of $\mathbb{R}P^2 \# \mathbb{R}P^3$, removal of a solid torus gives an $\mathbb{S}^1$-bundle over a Möbius strip; however this is impossible because we assumed that passing to the leaf-space gives a disk, not a Möbius strip. In the case of $\mathbb{S}^2 \times \mathbb{S}^1$, removal of a solid torus clearly leaves behind just a solid torus, so the theorem holds. $\square$

**Lemma 2.13** (Local triviality for $n = 3$)**.** *Assume $\Omega^3$ is a simply-connected 3-dimensional differentiable orbifold with boundary and that is saturated and invariant with respect to a rank-1 N-structure $\mathfrak{N}$, and let $\pi : \Omega^3 \to \Omega'$ be the projection to the orbit space. If $\Omega'$ is contractible, then the N-structure's orbits are all unbounded.*

**Proof.** Since $\Omega^3$ is oriented, it has no point-like orbifold points; this is because any such point must be a cone over $\mathbb{R}P^3$, which is nonorientable. Therefore all orbifold points have a neighborhood of the form $D' \times (0, 1)$ where $D'$ is a 2-dimensional disk with a single orbifold point in its center. Since $D'$ is a topological (though not a differentiable) manifold, $\Omega^3$ is also a manifold.

We are now in the situation of Lemma 2.12, and so $\Omega^3$ is homeomorphic to a solid torus. $\square$

**Lemma 2.14** (Local triviality for $n = 4$)**.** *Assume $\Omega^4$ is a simply connected 4-dimensional differentiable manifold, possibly with boundary, that is saturated and invariant with respect to a rank-1 N-structure $\mathfrak{N}$, and let $\pi : \Omega^4 \to \Omega'$ be the projection to the orbit space. If $\Omega'$ is contractible, then the orbits of $\mathfrak{N}$ on $\Omega^4$ are all unbounded.*

**Proof.** For a contradiction, assume there is some regular circle fiber $\mathcal{O}$ in $\Omega^4$; by Lemma 2.10, $\mathfrak{N}$ is generated by some globally defined vector field $\widetilde{V}$. The plan is to use the contractibility of $\mathcal{O}$ to build a simply connected 4-manifold $M^4$ with a circle action and a spherical boundary, and then compute its Euler number to show that actually no such manifold exists.

**Construction of $M^4$.** Instead of $\Omega^4$, we shall consider $\Omega^8 = \Omega^4 \times \mathbb{R}^4$; this will give us more room to adjust disk and ball embeddings. Because $\Omega^8$ is simply connected, there is a map $i : D^2 \to \Omega^8$ from some standard 2-disk to $\Omega^8$, where $i : \partial D \to \mathcal{O}$ is bijective. Projecting down, we also have a map $i' \triangleq \pi \circ i : D^2 \to \Omega' \times \mathbb{R}^4$.

We may assume $i(D^2)$ and $i'(D^2)$ are in general position, which we take to mean $i(D^2)$ and $i'(D^2)$ have no self-intersections, and the intersection of $D^2$ with the locus of exceptional fibers occurs only at isolated points (this can be done because the locus of exceptional fibers on $\Omega^4$ is 2-dimensional, so is 6-dimensional on $\Omega^8$).

Restricted to the boundary circle, we have that $i' : \partial D^2 \to \Omega' \times \mathbb{R}^4$ maps $\partial D^2$ to a point. Quotienting $D^2$ by $\partial D^2$ therefore, we have a map $i'_s : \mathbb{S}^2 \to \Omega' \times \mathbb{R}^4$. Because $\Omega'$ was assumed to be contractible, $\Omega' \times \mathbb{R}^4$ is also contractible, which means there is a map $I' : D^3 \to \Omega' \times \mathbb{R}^4$ (where $D^3$ is a standard 3-ball) such that $I'|_{\partial D^3} = i_s$. We can assume the image of $I'$ is in general position, and since $\Omega' \times \mathbb{R}^4$ is 7-dimensional, this means the image $I'(D^3)$ has no self-intersections.

Now consider the inverse image $M^4 \triangleq \pi^{-1}(I'(D^3)) \subset \Omega^8$; this is a 4-dimensional manifold-with-boundary that is fibered by circles. The map $\pi : M^4 \to M' \subset \Omega' \times \mathbb{R}^4$ is the projection along fibers, and $\Omega'$ is a differentiable, contractible 3-dimensional orbifold.

**Properties of $M^4$.** The boundary $\partial M^4 = \pi^{-1}(I'(\partial D^3)) = \pi^{-1}(i'_s(\mathbb{S}^2))$ is a Seifert-fibered 3-manifold. The projection map $\pi : \partial M^4 \to \Omega' \times \mathbb{R}^4$ is just the image of $i'_s : \mathbb{S}^2 \to \Omega' \times \mathbb{R}^4$; therefore the base of the Seifert fibered space $\partial M^3$ is $\mathbb{S}^2$.

This boundary contains the original fiber $\mathcal{O}$ as well as the image of the original map $i : D^2 \to \widetilde{\Omega^8}$. Therefore $\mathcal{O}$ is contractible within the boundary, and so we have $\pi_1(\partial M^4) = 0$. By the classification of Seifert fibered 3-manifolds, we have $\partial M^4 = \mathbb{S}^3$.

Finally we prove that $M^4$ is simply connected. First note that regular fibers do not carry homology: because all regular fibers in $M^4$ are homotopic to one another and homotopic to $\mathcal{O}$, we have that all regular fibers are contractible. But it is possible that singular fibers carry torsion, or conceivably that some other cycle might exist.

Let $\gamma \in M^4$ be any simple closed curve in $M^4$. By altering the path by an arbitrarily small amount if necessary, we can assume that each intersection of $\gamma$ with any orbit of $M^4$ occurs in a single point. Then the path $\gamma' = \pi \circ \gamma$ in $M'$ is a simple closed curve and so is the boundary of a 2-disk $D' \subset M'$. Then consider $N^3 = \pi^{-1}(D')$; this is a 3-dimensional Seifert fibered manifold-with-boundary within $M^4$. By Lemma 2.12, $N^3$ is therefore a solid torus, and so the path $\gamma \subset N^3$ is either a multiple of an exceptional fiber at the central of the torus, or else it is contractible. This proves that any path $\gamma$ in $M^4$ is either contractible, or is a multiple of some exceptional circle fiber.

Lastly we prove that exceptional fibers are contractible. Multiples of exceptional fibers are homotopic to multiples of regular fibers, and therefore we know that if $\gamma$ is a path along an exceptional fiber and $[\gamma] \in \pi_1(M^4)$ is its class, then $[\gamma]^k = 1$ for some $k \in \mathbb{N}$. The exceptional locus consists of finitely many totally geodesic 2-dimensional submanifolds, and there is some smallest $k$ that makes $[\gamma]^k = 1$ on each component. Therefore $\pi_1(M^4)$ is finite, and the universal cover $\widetilde{M^4}$ is at most finite-sheeted.

We have that $\widetilde{M^4}$ still has a quotient along fibers to a base $\widetilde{M'}$, but certainly $\widetilde{M'}$ is a connected covering of $M'$, the base of $M^4$. Since $M'$ is a 3-ball, we must have $\widetilde{M}' = M'$. Now we can apply the reasoning about the boundary $\partial M^4$ to the boundary $\partial\widetilde{M^4}$. Specifically, $\partial\widetilde{M^4}$ is a compact Seifert fibered 3-manifold with base being $\mathbb{S}^2$, and with all regular fibers still being contractible, meaning $\pi_1(\partial\widetilde{M^4}) = 0$; therefore $\partial\widetilde{M^4} = \mathbb{S}^3$. Since $\partial\widetilde{M^4} \to \partial M^4$ is a covering space and both manifolds are $\mathbb{S}^3$, it is a trivial cover. Therefore $\widetilde{M^4} \to M^4$ is also a trivial cover, so $M^4$ was already simply connected.

**Proof that $\chi(M^4) > 0$.** Because $\partial M \approx \mathbb{S}^3$, the relative cohomology sequence for the pair $(M, \partial M)$ gives exact sequences

$$0 \longrightarrow H^i(M, \partial M) \longrightarrow H^i(M) \longrightarrow 0 \tag{10}$$

for $i \in \{1, 2, 3\}$. From the Universal Coefficient Theorem, we have

$$H^1(M; \mathbb{Z}) = \mathrm{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z})$$

which is zero because $\pi_1(M^4) = 0$. Therefore $H^1(M) = H^1(M, \partial M) = 0$, and by Poincare duality also $H_3(M, \partial M) = H_3(M) = 0$. Letting

$$b_k = \dim(H_k(M; \mathbb{Z}))$$

be the $k^{th}$ Betti number of $M^4$, we have $b_1 = b_3 = b_4 = 0$, and so

$$\chi(M^4) = 1 + b_2 \geq 1.$$

But $\chi(M^4) = 0$ because the N-structure on $M^4$ is represented by a nowhere zero vector field with compact orbits. This contradiction establishes the proof. □

**Lemma 2.15** (Local triviality of pure N-structures)**.** *Assume $\Omega^4$ is a simply connected 4-manifold, possibly with boundary, that is saturated and invariant with respect to a pure N-structure $\mathfrak{N}$. Let $\Omega'$ be the quotient along fibers given by the map $\pi : \Omega^4 \to \Omega'$, and assume $\Omega'$ is contractible. Then if $\mathfrak{N}_1 \subseteq \mathfrak{N}$ is any rank 1 substructure with the property that all of its orbits are closed and nowhere zero, the orbits of $\mathfrak{N}_1$ are unbounded.*

*Likewise, if $\mathfrak{U} \subseteq \mathfrak{N}$ is any substructure with the property that every closed 1-dimensional substructure has no zero orbits, then the orbits of $\mathfrak{U}$ are copies of $\mathbb{R}^k$ for some $k \in \{1, 2, 3, 4\}$.*

**Proof.** If $\mathfrak{N}$ has a rank 1 substructure $\widetilde{\mathfrak{N}}_1$ with closed orbits, then we are in the situation of Lemma 2.14. But since $\Omega^4$ is simply connected, the conclusion of that Lemma shows it is impossible that any rank 1 substructure has closed orbits; therefore its orbits are diffeomorphic to copies of $\mathbb{R}^1$.

The final statement follows from re-applying the 1-dimensional result as many times as necessary. □

## 3. Proof of Theorem 1.1

For convenience we restate the theorem.

**Theorem 3.1** (Main Regularity Estimate)**.** *There exist constants $\epsilon_0, \delta_0 > 0$ so that if $\int_{B(p,r)} |\operatorname{Rm}|^2 < \epsilon_0$, then $r_{\mathcal{R}}(q) > \delta_0 \min\{r,\, r_{\mathcal{RC}}(q)\}$ for all $q \in B(p, \frac{1}{2}r)$.*

It would be nice if the "$\min\{r,\, r_{\mathcal{RC}}(p)\}$" could be replaced simply with "$r$", but there are two reasons why $r_{\mathcal{R}}$ must be compared to $r_{\mathcal{RC}}$. The first is that we must use a Cheeger–Gromoll style splitting theorem to rule out two-ended blowups; this does not use volume comparison, but essentially an analytic barrier-style argument. The second is the result from [21] that a Ricci-flat instanton with a nonvanishing Killing field is flat; this *does* use volume comparison in one place: it is needed to establish the Hardy–Littlewood style "weak-(1,1)" estimate (see lemma 4.1 of [10]).

**3.1. Outline of the proof.** If $r_{\mathcal{R}}$ degenerates at some nearby point $p'$, we can re-choose the point $p'$ so that $r_{\mathcal{R}}(p')$ is "almost" smallest among all sufficiently nearby $p'$. Rescaling, we have $r_{\mathcal{R}}(p') = 1$ and $r_{\mathcal{R}}$ is bounded uniformly from below on a large region $\Omega$. With $\int |\operatorname{Rm}|^2$ small, Theorem 2.4 forces the existence of an N-structure on $\Omega$.

By passing to the universal cover, we would like the collapsing directions to "unwrap," and become unbounded. But this is not immediately clear: the manifold could resemble a 3-sphere crossed with a line where collapse is along Hopf fibers; this is simply connected so passage to the universal cover changes nothing. But in our situation, with Ricci curvature controlled, a Cheeger–Gromoll style splitting theorem implies that the limit is indeed one-ended, and then with a simple homology argument, we rule out behavior like collapse along Hopf fibers. Specifically, we prove that collapsing directions must carry homology.

Now we may pass to the universal cover $\widetilde{\Omega} \to \Omega$, where we know that all orbits of the N-structure are unbounded, and the injectivity radius is bounded from below. Further, Lemma 2.10 now implies that the N-structure is represented by universally defined Killing fields. The domains $\widetilde{\Omega}$ then converge to a complete Ricci-flat manifold with $r_{\mathcal{R}} = 1$ somewhere, and with at least one nowhere-zero Killing field (coming from the center of the N-structure; Lemma 2.7). Lemma 2.3 implies that these are flat, contradicting that $r_{\mathcal{R}}$ is finite.

**3.2. Point reselection, and properties of the sequence of counterexamples.** Proceeding to the formal proof, assume the theorem is false. Then we can choose a sequence $\delta_j \to 0$ where for any of these $\delta_j$ we have a sequence of pointed critical manifolds $(M_i^n, g_i, p_i)$, radii $r_i$ (of any size), and values $\epsilon_i \to 0$ so that a ball $B_i = B(p_i, r_i)$ exists with $\int_{B_i} |\operatorname{Rm}|^2 < \epsilon_i$, but so that a point $q_i \in B(p_i, \frac{1}{2}r_i)$ exists with $r_{\mathcal{R}}(q) < \delta_j \min\{r, r_{\mathcal{RC}}(q)\}$.

The sequence $\{(\delta_j,\epsilon_i)\}_{i,j=1}^{\infty}$ is a double sequence; passing to a diagonal subsequence allows us to find a single sequence of such counterexamples with $\epsilon_i,\delta_i\to 0$ simultaneously.

Obviously $B(q_i,\delta_i r_{\mathcal{R}}(q_i))\subset B(p_i,r_i)$ and $r_{\mathcal{R}}(q_i)<\delta_i r_{\mathcal{RC}}(q_i)$. Rescaling the metric so $r_{\mathcal{R}}(q_i)=1$, we therefore have:

(a) $\int_{B(q_i,\,\delta_i{}^{-1})}|\operatorname{Rm}|^2<\epsilon_i$.
(b) $|\operatorname{Ric}|\le\delta_i{}^2$ on the ball $B(q_i,\delta_i{}^{-1})$.

We wish for, but don't immediately have, a third property:

(c) $|\operatorname{Rm}|$ is uniformly bounded on $B(q_i,\delta_i{}^{-1})$.

To obtain this, we improve the choice of $q_i$ in order to make $r_{\mathcal{R}}$ "almost" minimal at $q_i$ among all nearby points.

For the moment, drop the $i$ from the notation. Assume there is a point $q_1\in B(q,\frac{1}{2}\delta^{-1})$ with $r_{\mathcal{R}}(q_1)<\frac{1}{2}\delta r_{\mathcal{R}}(q)$. For an inductive procedure, supposing points $q_1,\dots,q_n$ have been selected, next select a point $q_{n+1}\in B(q_n,\,\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_n))$ with $r_{\mathcal{R}}(q_{n+1})<\frac{1}{2}r_{\mathcal{R}}(q_n)$, if such a point exists. After finitely many steps, this process necessarily terminates with a finite sequence of points $\{q_0,\dots,q_N\}$, where the final point $q_N$ has the property that $r_{\mathcal{R}}>\frac{1}{2}r_{\mathcal{R}}(q_N)$ on $B(q_N,\,\delta^{-1}r_{\mathcal{R}}(q_N))$.

We wish to show that

$$\int_{B(q_N,\delta^{-1}r_{\mathcal{R}}(q_N))}|\operatorname{Rm}|^2<\epsilon.$$

If we pick any $x\in B\big(q_N,\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N)\big)$ we have

$$\begin{aligned}
&\operatorname{dist}(x,\,p)\\
&\le\operatorname{dist}(x,\,q_N)+\operatorname{dist}(q_N,q_{N-1})+\cdots+\operatorname{dist}(q_1,q)+\operatorname{dist}(q,p)\\
&\le\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N)+\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_{N-1})+\cdots+\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q)+\frac{1}{2}r_{\mathcal{R}}(p)\\
&<\delta^{-1}\sum_{j=1}^{N}2^{-j}r_{\mathcal{R}}(q)+\frac{1}{2}r<r_{\mathcal{R}}(p)+\frac{1}{2}r<r.
\end{aligned}\tag{11}$$

Therefore indeed $B(q_N,\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N))\subset B(p,r)$, so

$$\int_{B(q_N,\frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N))}|\operatorname{Rm}|^2<\epsilon,$$

and by design we now have $|\operatorname{Rm}| < 4r_{\mathcal{R}}(q_N)$ on the ball $B(q_N, \frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N))$. Also

$$\begin{aligned} \text{(12)} \qquad \operatorname{dist}(x, q) &\le \operatorname{dist}(x, q_N) + \operatorname{dist}(q_N, q_{N-1}) + \cdots + \operatorname{dist}(q_1, q) \\ &\le \frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N) + \frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_{N-1}) + \cdots + \frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q) \\ &< \delta^{-1}\sum_{j=1}^{N} 2^{-j} r_{\mathcal{R}}(q) < \delta^{-1} r_{\mathcal{R}}(q) \end{aligned}$$

which means that $B(q_n, \frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N)) \subset B(q, \delta^{-1})$, so we retain $|\operatorname{Ric}| < \delta^2$ on $B(q_n, \frac{1}{2}\delta^{-1}r_{\mathcal{R}}(q_N))$.

Now replace the old $q$ with this new $q_N$ we have found, and scale the metric so that $r_{\mathcal{R}}(q_N) = 1$. Reintroducing $i$ in to the notation, we have the following expanded list of properties:

(a) $\int_{B(q_i, \frac{1}{2}\delta_i{}^{-1})} |\operatorname{Rm}|^2 < \epsilon_i$.
(b) $|\operatorname{Ric}| \le \delta_i{}^2$ on $B(q_i, \frac{1}{2}\delta_i{}^{-1})$.
(c) $|\operatorname{Rm}| \le 4$ on $B(q_i, \frac{1}{2}\delta_i{}^{-1})$.
(d) $|\operatorname{Rm}| = 1$ somewhere on $B(q_i, 1)$.

By (a)–(d) and Lemma 2.2, the sequence

$$\left\{ B\left(q_i,\ \frac{1}{2}\delta_i{}^{-1}\right)\right\}_i$$

of manifolds with boundary collapses with bounded curvature. By Theorem 2.4 and the comment immediately after, we have a polarized, $C$-regular N-structure $\mathfrak{N}_i$ on a saturation of, say, $B(q_i, \frac{7}{16}\delta_i{}^{-1})$. Let the 4 dimensional manifold-with-boundary $N_i^4 \subset B(q_i, \frac{1}{2}\delta_i{}^{-1})$ be this saturation.

We have the following properties for the pointed manifolds $(N_i^4, g_i, q_i)$:

(i) $(N_i^4, g_i, q_i)$ is a pointed, critical Riemannian manifold with boundary.
(ii) $(N_i^4, g_i)$ has $|\operatorname{Ric}| < \delta_i{}^2$.
(iii) $(N_i^4, g_i)$ has $|\operatorname{Rm}| \le 4$.
(iv) $(N_i^4, g_i, q_i)$ has $|\operatorname{Rm}| = 1$ somewhere on $B(q_i, 1)$.
(v) $(N_i^4, g_i)$ is $\tau_i$-collapsed, where $\tau_i \to 0$.
(vi) $N_i^4$ has a pure C-regular N-structure $\mathfrak{N}_i$ of rank 1, 2, or 3.
(vii) The quotient of $N_i^4$ by the orbits of $\mathfrak{N}_i$ is a Riemannian orbifold $N_i{}'$ of dimension 3, 2, or 1, with injectivity radius bounded from below and $|\operatorname{Rm}|$ bounded from above.

**3.3. Properties of the limiting object.** By the Gromov theory, the pointed manifolds $(N_i^4, g_i, q_i)$ collapse with bounded curvature along their N-structures to a limiting length space $(N_\infty, q_\infty)$. By Theorem 2.9 we know $N_\infty$ is an orbifold that has curvature bounded from above and injectivity radius bounded from below on compact subsets.

In a series of lemmas, we prove that for $i$ sufficiently large, the manifolds-with-boundary $N_i$ have just a single boundary component, and therefore the complete orbifold $N_\infty$ has just one end. We also prove that the $N_i^4$ have *positive* Euler number.

**Lemma 3.2.** *The orbifold $N_\infty$ is one-ended, and for all $i$ sufficiently large, after possibly passing to submanifolds of definite size, the 4-manifolds $N_i^4$ have connected boundary.*

**Proof.** If $N_\infty$ is not one-ended, then there exists a line $\gamma_\infty$ in $N_\infty$. A result of Cheeger–Colding (Theorem 6.64 of [5]) says that the limit $N_\infty$ must have the metric structure of $\mathbb{R} \times X_\infty$ for some length space $X_\infty$.[1] In our situation we have a sectional curvature bound, and we are able to prove the stronger conclusion that $X_\infty$ is a flat manifold, and that sectional curvature on the $N_i^4$ converges pointwise to 0, which provides the contradiction.

Choose an exhaustion $\Omega_i$ of $N_\infty$, so that $\Omega_i$ is a domain that satisfies the following three criteria: $\Omega_i$ is connected and has at least 2 ends, $\Omega_i$ contains $B(q_\infty, 2^i)$, and $\Omega_i$ contains at least two boundary components that are separated by a large distance, say at least $2^i$. Now select manifolds-with-boundary $N_i' \subset N_i$ that are saturated and $2^{-i}$-close to $\Omega_i$ in the Gromov–Hausdorff sense (this is always possible after passing to a subsequence of the $\{N_i^4\}$). We have smooth projections $\pi_i : N_i' \to N_\infty$ that collapse the N-structure orbits in $N_i'$ to points in $N_\infty$. Now $\pi_i$ is a $2^{-i}$ Gromov–Hausdorff approximation, and so by Theorem 2.4 the metric $g_i$ on $N_i$ is $2^{-i}$-close in the $C^{k,\alpha}$ sense to a metric for which $\pi_i$ is a Riemannian submersion. Therefore, by changing the metric on $N_i'$ by a very small amount, we can assume the metric is actually invariant.

Since $N_i'$ has (at least) two boundary components separated by a distance of at least $2^i$, there is a unit-parametrized geodesic path $\gamma_i$ of length at least $2^i$ between them. Further, since we know that $N_\infty$ has a line $\gamma_\infty$, we can choose $\gamma_i$ so that $\pi_i(\gamma_i)$ actually lies on $\gamma_\infty$. In particular, $\pi_i(\gamma_i)$ converges to $\gamma_\infty$ as $i \to \infty$.

Let $b_\infty : N_\infty = \mathbb{R} \times X_\infty \to \mathbb{R}$ be the projection onto the line; this is a Buseman function associated to the line $\gamma_\infty$. By adding a constant, we may assume $b_\infty(q_\infty) = 0$. Abusing notation, we will also use $b_\infty$ to indicate the pullback functions $\pi_i^*(b_\infty)$ on $\widetilde{N}_i$. Now $\pi_i$ is a smooth Riemannian submersion, so on $b_\infty$ has uniform $C^{k,\alpha}$ control on $\widetilde{N}_i$. In particular, the gradient is pinched: $||\nabla b_\infty|_{g_i} - 1| \leq 2^{-i}$. We cannot immediately obtain Hessian pinching, as this is essentially the second fundamental forms of the submersion fibers.

So to obtain the Hessian pinching, we will use Lemma 2.15 locally to create noncollapsed manifolds with pointwise lower bounds on Ricci curvature. Associated to $\gamma_i$, we have the usual almost-Buseman functions:

$$b_i^\pm(x) = t_i^\pm - \mathrm{dist}(x, \gamma_i(t_i^\pm)). \tag{13}$$

[1] Results from Cheeger–Colding theory are not strictly necessary in our argument, but provides the function $b_\infty$ which shortens things somewhat.

We have that that $b_\infty(x)$ is $2^{-i}$-close to $b_i^+$ and to $-b_i^-$ (so also $|b_i^+ + b_i^-| \leq 2^{1-i}$). By the usual Laplacian comparison argument, we have in the barrier sense that

$$\triangle b_i^{\pm} \;\geq\; -3\cdot 2^{-2i}. \tag{14}$$

Choose any contractible region $U_\infty \subset N_\infty$. Consider the regions

$$U_i = \pi^{-1}(U_\infty) \subset N_i',$$

The region $U_i$ has an N-structure with contractible base, and therefore by Lemma 2.15 we know that its the universal cover $\widetilde{U}_i$ has just two types of orbits: those that are copies of $\mathbb{R}^n$ for some $n$, and those orbits which are closed, but are represented by a Killing field that has a zero somewhere on $\underline{U_i}$. It therefore has no collapsed directions, and the Gromov–Hausdorff limit $\overline{U_i} \to \overline{U_\infty}$ is a smooth convergence of 4-manifolds.

The almost-Buseman functions $b_i^{\pm}$ lift to $\widetilde{U}_i$ where we retain $|b_i^+ + b_i^-| < 2^{1-i}$, so in the limit we have $\triangle b_\infty^{\pm} \geq 0$ and $b_\infty = b_\infty^+ = -b_\infty^-$, and therefore $\triangle b_\infty = 0$ in $U_\infty$. From above, we have $|\nabla b_\infty| = 1$, so the fact that $|\operatorname{Ric}| = 0$ on $U_\infty$ and usual Böchner–Weitzenbock formula gives $0 = |\nabla^2 b_\infty|^2$.

Now the limit $U_\infty$ has $\operatorname{Ric} = 0$, $|\nabla^2 b_\infty| = 0$, and $|\nabla b_\infty| = 1$. Thus $b_\infty$ is a metric splitting function on $U_\infty$, so $U_\infty$ splits locally into a line segment crossed with a 3-dimensional Ricci-flat manifold, which is therefore a flat manifold.

Since $U_\infty \subset N_\infty$ was chosen arbitrarily, only under the restriction that is be contractible, we have that on compact subsets of $N_i$ the uniform pointwise convergence of $|\operatorname{Rm}|$ to zero. This contradicts point (iii) above, which says that $r_{\mathcal{R}}(q_i') = 1$ which means that $|\operatorname{Rm}| = 1$ at at least one point. Thus indeed $N_\infty$ is one-ended. $\square$

With this lemma, we have in addition to (i)–(vii), now an eighth property:

(viii) The manifolds $N_i$ are one-ended.

**Lemma 3.3.** *Let $\widetilde{N}_i$ be the universal cover of $N_i$. Then $\chi(\widetilde{N}_i) \geq 1$.*

**Proof.** First note that $\widetilde{N}_i$ is also one-ended, as $\widetilde{N}_i$ retains properties (i)–(vii) except possibly for (v). But if (v) does not hold, meaning $\widetilde{N}_i$ is not collapsed, then the existence of a line in the limit already contradicts $|\operatorname{Rm}| = 1$ as some point in the limit, so the proof of Lemma 3.2 is much easier. Therefore each $\widetilde{N}_i$ is one-ended.

By simple connectedness $H_1(\widetilde{N}_i) = 0$ and the Universal Coefficient Theorem gives $H^1(\widetilde{N}_i) = \{0\}$. Poincare duality gives $H^3(N_i, \partial\widetilde{N}_i) = \{0\}$. Therefore the long exact cohomology sequence for the pair $(\widetilde{N}_i, \partial\widetilde{N}_i)$ gives a short exact sequence

$$0 \to H^3(\widetilde{N}_i) \to H^3(\partial\widetilde{N}_i) \to H^4(N_i, \partial\widetilde{N}_i) \to 0. \tag{15}$$

Of course $H^4(\widetilde{N}_i, \partial\widetilde{N}_i) = \mathbb{Z}$ (generated by the fundamental class). By one-endedness $H^3(\partial\widetilde{N}) = \mathbb{Z}$. Then exactness forces $H^3(\widetilde{N}_i) = \{0\}$.

We have shown that the betti numbers $b_1$ and $b_3$ of $N_i$ are zero, and therefore the Euler number of $\widetilde{N}_i$ is $\chi(\widetilde{N}_i) = 1 + b^2(\widetilde{N}_i) \geq 1$. $\square$

**Proof of Theorem 1.1.** If the Theorem 1.1 is false, we can construct, as we have seen, a sequence of pointed manifolds $(N_i, p_i)$, each of which has connected boundary, each of which is saturated with respect to a pure N-structure $\mathfrak{N}_i$, and each of which has $|\,\mathrm{Rm}\,| = 1$ at some point within $B(p_i, 1)$, but also with $\sup_{N_i} |\,\mathrm{Ric}\,| \to 0$.

Passing to the universal covers $\widetilde{N}_i$, we retain an N-structure, but Lemma 3.3 says $\chi(\widetilde{N}_i) = 0$, which means $\widetilde{N}_i$ cannot have an N-structure of positive rank and compact orbits (by proposition 1.5 of [7]). Therefore the covers $\widetilde{N}_i$ are actually noncollapsed, and so converge, in the $C^\infty$ sense, to a 4-manifold $\widetilde{N}_\infty$ that also has an N-structure of positive rank (but without compact orbits).

Lemma 2.10 says this N-structure is represented by globally-defined Killing fields. Since the rank of the N-structure is positive, there is at least one Killing field without zeros. But now we have a complete, Ricci-flat manifold $\widetilde{N}_\infty$ with a nowhere-zero Killing field. By Lemma 2.3 $\widetilde{N}_\infty$ is flat, contradicting the fact that $|\,\mathrm{Rm}\,| = 1$ somewhere. This contradiction establishes Theorem 1.1. $\square$

## 4. Proof of Theorem 1.4 and Corollary 1.5

We restate Theorem 1.4 for convenience.

**Theorem 4.1.** *Given $k > 0$ and $\mu > 0$, there exist numbers $\epsilon_0 > 0$ and $C = C(\mu, k) < \infty$ so that the following holds. If $r_{\mathcal{RC}}(p) \geq (1+\mu) r_{\mathcal{R}}(p)$ and*

$$\int_{B(p,\,(1+\mu)r_{\mathcal{R}}(p))} |\,\mathrm{Rm}\,|^2 \;\leq\; \epsilon_0, \tag{16}$$

*then*

$$r_{\mathcal{R}}(p) \;\geq\; C \left( \fint_{B(p, r_{\mathcal{R}}(p))} |\,\mathrm{Rm}\,|^k \right)^{-\frac{1}{2k}}. \tag{17}$$

Fix $\mu, k$, and assume there is no such $C$, meaning there is a sequence of counterexamples so that the quantity

$$r_{\mathcal{R}}(q_i)^{2k} \fint_{B(q_i, r_{\mathcal{R}}(q_i))} |\,\mathrm{Rm}\,|^k \tag{18}$$

can degenerate to zero, no matter what $\epsilon_0$ was chosen. By Theorem 1.3, we can choose $\epsilon_0$ small enough that there is a $\delta_0$ so that $r_{\mathcal{R}} \geq \delta_0 r_{\mathcal{R}}(q_i)$ on $B(q_i,\, (1+\mu/2)) r_{\mathcal{R}}(q_i))$.

In particular, the exponential map has no conjugate points on some ball of radius definitely (though very slightly) larger that $r_{\mathcal{R}}(q_i)$. Namely, there is some $\eta > 0$ so that if $o_i \in T_{q_i}M^4$ is the origin in the tangent space over $q_i$, then $\exp_{q_i} : \overline{B}(o_i, (1+\eta)r_{\mathcal{R}}(q_i)) \to B(q_i, (1+\eta)r_{\mathcal{R}}(q_i))$ is a local homeomorphism, where $\eta$ is independent of $i$ and $q_i$, and $\overline{B}$ indicates a ball in the tangent space. Lifting to the tangent space at $q_i$, we have a ball $B((1+\eta)r_{\mathcal{R}}(q_i))$ that is contractible. Finally scale so that $r_{\mathcal{R}}(q_i) = 1$.

Now the exponential map $\exp_{q_i} : B(o_i, 1+\eta) \to B(q_i, 1+\eta)$ does not evenly cover the target, but since $|\operatorname{Rm}| \leq 1$, the deviation is not too large, and can still assume that

$$r_{\mathcal{R}}(o_i)^{2k} \fint_{\overline{B}(o_i, r_{\mathcal{R}}(q_i))} |\operatorname{Rm}|^k \tag{19}$$

degenerates to zero on the ball in the tangent space itself.

Rescaling so $r_{\mathcal{R}}(p_i) = 1$, we have contractible balls $\overline{B}(o_i, 1+\eta)$ with energy integrals $\int_{\overline{B}(o_i,1+\eta)} |\operatorname{Rm}|^k$ converging to 0 as $i \to 0$. We have bounded Ricci curvature on $\overline{B}(o_i, 1+\eta)$ (as long as $\eta$ is chosen smaller then $\mu$), and so the metric automatically converges in the $C^{1,\alpha}$ sense there; a bootstrapping argument shows is converges in the $C^\infty$ sense.

So we get convergence to a manifold-with-boundary $\overline{B}(o_\infty, 1_\eta)$, and since the integrals $\int_{\overline{B}(o_i,1+\eta)} |\operatorname{Rm}|^k$ converge to 0. But $r_{\mathcal{R}}(o_i) = 1$ for each $i$, meaning $|\operatorname{Rm}| = 1$ at some point in the interior of $\overline{B}(o_i, 1+\eta)$. But since the metric is critical and therefore obeys an elliptic system, the function $|\operatorname{Rm}|$ converges in the $C^\infty$ sense to a $C^\infty$ function; therefore in the limit $|\operatorname{Rm}| = 1$ somewhere in $\overline{B}(o_\infty, 1+\eta)$, whereas also $\int |\operatorname{Rm}|^2 = 0$, which is a contradiction. Thus Theorem 1.4 is established.

Corollary 1.5 is proved similarly. Choose $l$, pick counterexamples

$$B(p_i, r_{\mathcal{R}}(p_i)),$$

and scale so $r_{\mathcal{R}}(p_i) = 1$. Again passing to the tangent spaces of the $p_i$, we have convergence of the metrics on the slightly larger, contractible manifolds $B(o_i, 1+\eta)$. The limiting metric on $B(o_\infty, 1+\eta)$ has definite bounds on the quantities $|\nabla^l r_{\mathcal{R}}|$ within $B(o_\infty, 1+\frac{1}{2}\eta)$, so by $C^\infty$ convergence, these bounds must hold on the $B(o_i, 1+\frac{1}{2}\eta)$, and so on the original $B(p_i, r_{\mathcal{R}}(p_i))$.

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