Endpoint results for spherical multipliers on noncompact symmetric spaces

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ABSTRACT. In this paper we prove boundedness results on atomic Hardy type spaces for multipliers of the spherical transform on noncompact symmetric spaces of arbitrary rank. The multipliers we consider satisfy either inhomogeneous or homogeneous Mihlin–Hörmander type conditions. In particular, we are able to treat the case of *strongly singular multipliers* whose convolution kernels are not integrable at infinity. Thus our results apply also to negative and imaginary powers of the Laplacian.

CONTENTS

1. Introduction

Suppose that $\mathcal{M}$ is a bounded translation invariant operator on $L^2(\mathbb{R}^n)$ and denote by m the Fourier transform of its convolution kernel: $\mathcal{M}$ is usually referred to as the *Fourier multiplier operator* associated to the *multiplier*

m. A celebrated result of L. Hörmander [20] states that if m satisfies the following Mihlin type conditions

$$(1.1) \qquad |D^I m(\xi)| \leq C\, |\xi|^{-|I|} \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

for all multiindices I of length $|I| \leq \lfloor n/2 \rfloor + 1$, where $\lfloor n/2 \rfloor$ denotes the largest integer $\leq n/2$, then $\mathcal{M}$ extends to an operator bounded on $L^p(\mathbb{R}^n)$ for all p in $(1, \infty)$, and of weak type 1. This result was complemented by C. Fefferman and E.M. Stein [12], who showed that $\mathcal{M}$ extends to a bounded operator on the classical Hardy space $H^1(\mathbb{R}^n)$.

In the pioneering works of Stein [35] and R.S. Strichartz [36], the authors proposed to extend the aforementioned results to Riemannian manifolds. The purpose of this paper is to do so for *spherical multipliers* on Riemannian symmetric spaces of the noncompact type, which constitute an important generalization of the hyperbolic disc, and are paradigmatic examples of Riemannian manifolds with bounded geometry and exponential volume growth.

Notable contributions to this problem on general symmetric spaces of the noncompact type are [1, 8, 17, 22, 31]. Some of them follow up previous work on rank one symmetric spaces [21, 34] or complex symmetric spaces [3].

A related problem is to obtain results similar to those of Hörmander and Fefferman–Stein for *spectral multipliers* of the Laplace–Beltrami operator on Riemannian manifolds. Since the literature on this subject is huge, without aiming at exhaustiveness we mention that contributions on this problem on classes of manifolds that include noncompact symmetric spaces can be found in [6, 7, 13, 16, 23, 25, 27, 29, 28, 33, 37, 39] and the references therein. Notice that on rank one symmetric spaces spherical multipliers coincide with spectral multipliers of the Laplacian.

In this paper we establish an analogue of the aforementioned result of Fefferman and Stein on symmetric spaces of the noncompact type. This requires identifying a space that plays in this context the same role as the classical Hardy space $H^1(\mathbb{R}^n)$ plays on Euclidean spaces. Observe that noncompact symmetric spaces are *not* spaces of homogeneous type in the sense of Coifman and Weiss [9], i.e., the doubling condition fails (for large balls). Furthermore it can be seen (see Section 4) that the generalizations of $H^1(\mathbb{R}^n)$ to Riemannian manifolds with Ricci curvature bounded from below and spectral gap [5, 32, 38], do not produce an endpoint estimate for the class of strongly singular spherical multipliers we consider.

In order to analyze the state of the art, we need some notation, which is standard (see also Section 2). We denote by G a noncompact connected real semisimple Lie group with finite centre, by K a maximal compact subgroup of G and by $\mathbb{X} = G/K$ the associated noncompact Riemannian symmetric space. It is well known that (G, K) is a Gelfand pair, i.e., the convolution

algebra $L^1(K\backslash G/K)$ of all K–bi-invariant functions in $L^1(G)$ is commutative. The spectrum of $L^1(K\backslash G/K)$ is the tube $T_{\overline{\mathbf{W}}} = \mathfrak{a}^* + i\overline{\mathbf{W}}$, where $\mathbf{W}$ is the open convex polyhedron in $\mathfrak{a}^*$ that is the interior of the convex hull of the Weyl orbit of the half-sum of positive roots ρ. Denote by $\widetilde{\kappa}$ the Gelfand transform (also referred to as the spherical Fourier transform, or the Harish-Chandra transform in this setting) of the function κ in $L^1(K\backslash G/K)$. It is known that $\widetilde{\kappa}$ is a bounded continuous function on $T_{\overline{\mathbf{W}}}$, holomorphic in $T_{\mathbf{W}}$ (i.e., in $\mathfrak{a}^* + i\mathbf{W}$), and invariant under the Weyl group W. The Gelfand transform extends to K–bi-invariant tempered distributions on G (see, for instance, [15, Ch. 6.1]).

It is well known that the Banach algebra $L^\infty(\mathfrak{a}^*)^W$ of all Weyl invariant essentially bounded measurable functions on $\mathfrak{a}^*$ is isomorphic to the space of all G–invariant bounded linear operators on $L^2(\mathbb{X})$. The isomorphism is given by the map $m \mapsto \mathcal{M}$ where

$$\widetilde{\mathcal{M}f}(\lambda) = m(\lambda)\,\widetilde{f}(\lambda) \qquad \forall f \in L^2(\mathbb{X}) \quad \forall \lambda \in \mathfrak{a}^*.$$

Thus $\mathcal{M}f = f * \kappa$, where κ is the K–bi-invariant tempered distribution on G such that $\widetilde{\kappa} = m$. We call κ the *kernel* of $\mathcal{M}$ and m the *spherical multiplier* associated to $\mathcal{M}$. Notice that the space of all G–invariant bounded linear operators on $L^2(\mathbb{X})$ and the spherical Fourier transform are the counterpart on $\mathbb{X}$ of the class of bounded translation invariant operators on $L^2(\mathbb{R}^n)$ and of the Euclidean Fourier transform, respectively.

A well known result of Clerc and Stein [8] states that if $\mathcal{M}$ extends to a bounded operator on $L^p(\mathbb{X})$ for all p in $(1,\infty)$, then m extends to a holomorphic function on $T_{\mathbf{W}}$, bounded on closed subtubes thereof. It is natural to ask for sufficient conditions on m which ensure that $\mathcal{M}$ is bounded on $L^p(\mathbb{X})$ for all p in $(1,\infty)$, and satisfies some endpoint result for $p = 1$.

The most popular kind of requirement is that m admits an extension to the boundary of $T_{\mathbf{W}}$ which satisfies certain differential inequalities on $T_{\overline{\mathbf{W}}}$. We briefly examine the differential inequalities that appear in the literature. J.-Ph. Anker [1], following up earlier results of M. Taylor [37] and J. Cheeger, M. Gromov and Taylor [7] for manifolds with bounded geometry, proved that if m satisfies Mihlin type conditions of the form

$$(1.2) \qquad |D^I m(\zeta)| \leq C \left(1 + |\zeta|\right)^{-|I|} \qquad \forall \zeta \in T_{\overline{\mathbf{W}}}$$

for every multiindex I of length $|I| \leq \lfloor n/2 \rfloor + 1$, then the operator $\mathcal{M}$ is of weak type 1. Here n denotes the dimension of $\mathbb{X}$. This extends previous results on special classes of symmetric spaces [3, 8, 34]. The analysis performed by Anker reveals that the inverse spherical Fourier transform κ of m is integrable at infinity and satisfies a local Hörmander type integral condition. Anker's result was complemented by A. Carbonaro, Mauceri and Meda [5], who showed that if m satisfies (1.2), then $\mathcal{M}$ is bounded from the Hardy space $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$. Here $H^1(\mathbb{X})$ is the space introduced in [5]. The space $H^1(\mathbb{X})$ is defined via atoms supported on balls of radius at

most 1 and satisfying a standard cancellation condition. It is not hard to see that $\mathcal{M}$ is bounded also from the Hardy space $\mathfrak{h}^1(\mathbb{X})$ of Goldberg type, introduced by Taylor in [38], to $L^1(\mathbb{X})$. As a first result in this paper, we improve the endpoint result in [5] by showing that if a multiplier m satisfies condition (1.2), then $\mathcal{M}$ is bounded from the Hardy space $H^1(\mathbb{X})$ to $H^1(\mathbb{X})$ (see Theorem 3.4(i)).

Notice that condition (1.2) is a nonhomogeneous Mihlin-Hörmander condition; in particular, neither the multiplier nor its derivatives can have local singularities on $T_{\overline{\mathbf{W}}}$. However, it is known that certain classes of multipliers which have local singularities on $T_{\overline{\mathbf{W}}}$ are bounded on $L^p(\mathbb{X})$, $p \in (1, \infty)$, and of weak type 1. For instance, if $\mathcal{L}$ is the Laplace–Beltrami operator on $\mathbb{X}$, Anker and L. Ji [2] showed that the operator $\mathcal{L}^{-\tau}$ is of weak type 1 as long as τ is in $(0, 1]$, and it is not of weak type 1 when $\tau > 1$. The multiplier corresponding to $\mathcal{L}^{-\tau}$ is $Q(\zeta)^{-\tau}$, where $Q(\zeta) := \langle \zeta, \zeta \rangle + \langle \rho, \rho \rangle$ is the Gelfand transform of the Laplacian. Notice that $Q(\zeta)^{-\tau}$ is unbounded, together with its derivatives, near $i\rho$ (and the points in its Weyl orbit). This proves that there exist operators bounded on $L^2(\mathbb{X})$ and of weak type 1 such that the associated spherical Fourier multiplier is unbounded on $T_{\overline{\mathbf{W}}}$, i.e., on the Gelfand spectrum of $L^1(K \backslash G / K)$. This, of course, cannot happen for Euclidean Fourier multipliers. A slightly less singular example is given by $\mathcal{L}^{iu}$ when u is real and nonzero. The corresponding multiplier is bounded on $T_{\overline{\mathbf{W}}}$, but its derivatives are unbounded near $i\rho$ (and the points in its Weyl orbit). It may be worth observing that the kernels of these operators are not integrable at infinity.

A breakthrough in the problem was established by A.D. Ionescu [21] in the case where the rank of $\mathbb{X}$ is one. He was interested in sharp L^p bounds and did not consider weak type 1 estimates. However, by slightly modifying Ionescu's argument, it is not hard to see that if m satisfies the following estimates on $T_{\overline{\mathbf{W}}}$

$$(1.3) \qquad \left| D^I m(\zeta) \right| \leq \begin{cases} C\,|\zeta - i\rho|^{-I} & \text{if } |\zeta - i\rho| \leq 10^{-1} \\ C\,|\zeta + i\rho|^{-I} & \text{if } |\zeta + i\rho| \leq 10^{-1} \\ C\,|\zeta|^{-I} & \text{otherwise} \end{cases}$$

for $0 \leq I \leq N$, with N large enough, then $\mathcal{M}$ is of weak type 1. Remember that $T_{\overline{\mathbf{W}}}$ reduces to the strip $\{\zeta \in \mathbb{C} : \left|\mathrm{Im}(\zeta)\right| \leq \rho\}$ in this case. Notice that if m satisfies (1.3), then, in particular, its restriction to the boundary of $T_{\mathbf{W}}$ satisfies the homogeneous Mihlin–Hörmander conditions (1.1) on $\mathbb{R}$. Note, however, that the spherical multiplier associated to $\mathcal{L}^{-\tau}$ does not satisfy (1.3) when $\tau > 0$, and the aforementioned result of Anker and Ji cannot be deduced from Ionescu's result even when the rank of $\mathbb{X}$ is one.

An analogue of (1.3) on higher rank symmetric spaces was introduced by Ionescu [22]. However, these new conditions do not apply, for instance, to $\mathcal{L}^{iu}$, $u \neq 0$ (see [31] for details). Inspired by Ionescu's results, two of us introduced the so-called *strongly singular multipliers* [31]. These are Weyl

invariant functions m on $T_{\overline{\mathbf{W}}}$ satisfying the following conditions

$$(1.4) \qquad |D^I m(\zeta)| \leq \begin{cases} C\,|Q(\zeta)|^{-\tau - d(I)/2} & \text{if } |Q(\zeta)| \leq 1 \\ C\,|Q(\zeta)|^{-|I|/2} & \text{if } |Q(\zeta)| > 1, \end{cases}$$

where τ is a nonnegative real number, and $|I|$ and $d(I)$ are the isotropic and an anisotropic length of the multiindex I, respectively (see the beginning of Section 3 for the definition of $d(I)$). Notice that if $\tau > 0$, then the multiplier m itself may be unbounded near $i\rho$. We remark that condition (1.4), although technical, is natural: it is satisfied by a class of functions of the Laplacian that includes the potentials $\mathcal{L}^{-\tau}$ with $\operatorname{Re}\tau \geq 0$ (see Remark 3.3). It is straightforward to check that if $\mathbb{X}$ has rank one and $\tau = 0$, then conditions (1.4) reduce to (1.3). It is known [31] that if m satisfies condition (1.4) and τ is in $[0,1)$, or if m is a function of $\mathcal{L}$ and $\tau \leq 1$, then $\mathcal{M}$ is of weak type 1. This generalizes the result of Anker and Ji. As the aforementioned counterexample of Anker and Ji shows, the weak type 1 estimate fails when $\tau > 1$. It is interesting to notice that if the rank of $\mathbb{X}$ is at least two, and m satisfies (1.4), then its restriction to $\mathfrak{a}^* + i\rho$ does not satisfy Mihlin–Hörmander conditions (1.1); rather, it exhibits the following anisotropic behaviour: there exists a constant C such that

$$(1.5) \quad |D^I m(\xi + i\rho)| \leq \begin{cases} C\,\mathcal{N}(\xi)^{-2\tau - d(I)} & \forall \xi \in \mathfrak{a}^* \text{ such that } |\xi| \leq 1 \\ C\,|\xi|^{-|I|} & \forall \xi \in \mathfrak{a}^* \text{ such that } |\xi| > 1. \end{cases}$$

Here $\mathcal{N}$ denotes a suitable anisotropic "norm" (see Section 3 for its definition). This phenomenon was already noticed in [31].

In this paper we shall prove that all strongly singular multipliers admit endpoint results for $p = 1$, no matter how large the parameter τ is. To obtain these endpoint results we shall use the atomic Hardy type spaces $H^1(\mathbb{X})$ introduced in [5] and $X^k(\mathbb{X})$ introduced in [26], and further investigated in the series of papers [27, 29, 28]. It is important to keep in mind that the following strict continuous containments hold

$$H^1(\mathbb{X}) \supset X^1(\mathbb{X}) \supset X^2(\mathbb{X}) \supset \cdots \supset X^k(\mathbb{X}) \supset \cdots .$$

Moreover, for each positive integer k, $L^p(\mathbb{X})$ is the complex interpolation space between $X^k(\mathbb{X})$ and $L^2(\mathbb{X})$, for every $p \in (1,2)$.

Now we summarize the main results of the paper. We denote by n the dimension of $\mathbb{X}$ and by $\lceil n/2 \rceil$ the smallest integer $\geq n/2$. We prove the following (see Theorem 3.4):

- (i) If m satisfies (1.2) for all I such that $|I| \leq \lceil n/2 \rceil + 2$, then $\mathcal{M}$ is bounded from $H^1(\mathbb{X})$ to $H^1(\mathbb{X})$.
- (ii) If m satisfies (1.4) for all I such that

$$|I| \leq \lceil n/2 \rceil + 2 \quad \text{and} \quad k > \tau + \lceil n/2 \rceil + 2,$$

  then $\mathcal{M}$ is bounded from $X^k(\mathbb{X})$ to $H^1(\mathbb{X})$.

(iii) If m satisfies (1.4) for all I such that $|I| \leq \lceil n/2 \rceil + 2$ and $k > \tau$, then $\mathcal{M}$ is bounded from $X^k(\mathbb{X})$ to $L^1(\mathbb{X})$.

We complement our analysis by showing that if $\mathbb{X}$ is complex, then $\mathcal{L}^{iu}$, whose multiplier satisfies (1.4) for all I, is unbounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$ for all nonzero real u (see Theorem 4.1). *A fortiori*, the same result holds for the larger Goldberg type space $\mathfrak{h}^1(\mathbb{X})$ introduced in [32, 37]. A similar result for $\mathcal{L}^{-\tau}$, $\tau > 0$, was proved in [30, Theorem 4.1].

We emphasize that the results in this paper, as well as those concerning the Riesz transforms in [30], corroborate the fact that $X^k(\mathbb{X})$ does serve as an effective counterpart on $\mathbb{X}$ of the classical Hardy space $H^1(\mathbb{R}^n)$, whereas the effectiveness of the spaces $H^1(\mathbb{X})$ and $\mathfrak{h}^1(\mathbb{X})$ is somewhat limited to operators whose kernels are integrable at infinity.

The paper is organized as follows. Section 2 contains the basic notions of analysis on $\mathbb{X}$ and the definitions of the Hardy spaces $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$. Section 3 contains the positive results for spherical multipliers (Theorem 3.4). Finally, in Section 4 we prove that if $\mathbb{X}$ is complex, then $\mathcal{L}^{iu}$, $u \in \mathbb{R} \setminus \{0\}$, is unbounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$ (see Theorem 4.1).

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

2. Preliminaries

2.1. Preliminaries on symmetric spaces. In this subsection we recall the basic notions of analysis on noncompact symmetric spaces that we shall need in the sequel. Our main references are the books [18, 19] and the papers [1, 2]. For the sake of the reader we recall also the notation, which is quite standard.

We denote by G a noncompact connected real semisimple Lie group with finite centre, by K a maximal compact subgroup and by $\mathbb{X} = G/K$ the associated noncompact Riemannian symmetric space. The point $o = eK$, where e is the identity of G, is called the *origin* in $\mathbb{X}$. Let θ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan involution and Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote by Σ the restricted root system of $(\mathfrak{g}, \mathfrak{a})$ and by W the associated Weyl group. Once a positive Weyl chamber $\mathfrak{a}^+$ has been selected, Σ^+ denotes the corresponding set of positive roots, Σ_s the set of simple roots in Σ^+ and Σ_0^+ the set of positive indivisible roots. As usual, $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ denotes the sum of the positive root spaces. Denote by m_α the dimension of $\mathfrak{g}_\alpha$ and set $\rho := (1/2) \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. We denote by $\mathbf{W}$ the interior of the convex hull of the points $\{w \cdot \rho : w \in W\}$. Clearly $\mathbf{W}$ is an open convex polyhedron in $\mathfrak{a}^*$. By $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$ we denote the analytic subgroups of G corresponding to $\mathfrak{n}$ and $\mathfrak{a}$. The Killing form B induces the K–invariant inner product $\langle X, Y \rangle = -B\big(X, \theta(Y)\big)$ on $\mathfrak{p}$ and hence a G–invariant metric

d on $\mathbb{X}$. The closed ball with centre $x \cdot o$ and radius r will be denoted by $B_r(x \cdot o)$. The map $X \mapsto \exp X \cdot o$ is a diffeomorphism of $\mathfrak{p}$ onto $\mathbb{X}$. The distance of $\exp X \cdot o$ from the origin in $\mathbb{X}$ is equal to $|X|$, and will be denoted by $|\exp X \cdot o|$. We denote by n the dimension of $\mathbb{X}$ and by ℓ its rank, i.e., the dimension of $\mathfrak{a}$.

We identify functions on the symmetric space $\mathbb{X}$ with K–right-invariant functions on G, in the usual way. If $E(G)$ denotes a space of functions on G, we define $E(\mathbb{X})$ and $E(K\backslash\mathbb{X})$ to be the closed subspaces of $E(G)$ of the K–right-invariant and the K–bi-invariant functions, respectively. We write $\mathrm{d}x$ for a Haar measure on G, and let $\mathrm{d}k$ be the Haar measure on K of total mass one. The Haar measure of G induces a G–invariant measure μ on $\mathbb{X}$ for which

$$\int_{\mathbb{X}} f(x \cdot o) \, \mathrm{d}\mu(x \cdot o) = \int_G f(x) \, \mathrm{d}x \qquad \forall f \in C_c(\mathbb{X}).$$

We shall write $|E|$ instead of $\mu(E)$ for a measurable subset E of $\mathbb{X}$. We recall that

$$(2.1) \qquad \int_G f(x) \, \mathrm{d}x = \int_K \int_{\mathfrak{a}^+} \int_K f\big(k_1 \exp H \, k_2\big) \, \delta(H) \, \mathrm{d}k_1 \, \mathrm{d}H \, \mathrm{d}k_2 \,,$$

where $\mathrm{d}H$ denotes a suitable nonzero multiple of the Lebesgue measure on $\mathfrak{a}$, and

$$(2.2) \qquad \delta(H) = \prod_{\alpha \in \Sigma^+} \big(\sinh \alpha(H) \big)^{m_\alpha} \leq C \, \mathrm{e}^{2\rho(H)} \qquad \forall H \in \mathfrak{a}^+.$$

We recall the Iwasawa decomposition of G, which is $G = K\,A\,N$. For every x in G we denote by $H(x)$ the unique element of $\mathfrak{a}$ such that $x \in K \exp H(x)N$. For any linear form $\lambda : \mathfrak{a} \to \mathbb{C}$, the elementary spherical function φ_λ is defined by the rule

$$\varphi_\lambda(x) = \int_K \mathrm{e}^{-(i\lambda+\rho)H(x^{-1}k)} \, \mathrm{d}k \qquad \forall x \in G.$$

In the sequel we shall use the following estimate of the spherical function φ_0 [2, Proposition 2.2.12]:

$$(2.3) \qquad \varphi_0(\exp H \cdot o) \leq (1 + |H|)^{|\Sigma_0^+|} \, \mathrm{e}^{-\rho(H)} \qquad \forall H \in \mathfrak{a}^+.$$

The spherical transform $\mathcal{H}f$ of an $L^1(G)$ function f, also denoted by $\widetilde{f}$, is defined by the formula

$$\mathcal{H}f(\lambda) = \int_G f(x) \, \phi_{-\lambda}(x) \, \mathrm{d}x \qquad \forall \lambda \in \mathfrak{a}^*.$$

Harish-Chandra's inversion formula and Plancherel formula state that for "nice" K–bi-invariant functions f on G

$$(2.4) \qquad f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda) \, \phi_\lambda(x) \, \mathrm{d}\nu(\lambda) \qquad \forall x \in G$$

and

$$\|f\|_2 = \left[\int_{\mathfrak{a}^*} |\widetilde{f}(\lambda)|^2 \, \mathrm{d}\nu(\lambda) \right]^{1/2} \qquad \forall f \in L^2(K\backslash G/K),$$

where $\mathrm{d}\nu(\lambda) = c_G |\mathbf{c}(\lambda)|^{-2} \, \mathrm{d}\lambda$, and $\mathbf{c}$ denotes the Harish-Chandra $\mathbf{c}$-function. We do not need the exact form of $\mathbf{c}$. It will be enough to know that there exists a constant C such that

(2.5) $$|\mathbf{c}(\lambda)|^{-2} \leq C \left(1 + |\lambda| \right)^{n-\ell},$$

[18, IV.7]. The spherical transform can be factored as follows $\mathcal{H} = \mathcal{F}\mathcal{A}$, where $\mathcal{A}$ is the Abel transform, defined by

$$\mathcal{A}f(H) = \mathrm{e}^{\rho(H)} \int_N f\big((\exp H)n\big) \, \mathrm{d}n \qquad \forall H \in \mathfrak{a},$$

and $\mathcal{F}$ denotes the Euclidean Fourier transform on $\mathfrak{a}$.

Next, we recall the Cartan decomposition of G, which is $G = K \exp \overline{\mathfrak{a}^+} K$. In fact, for almost every x in G, there exists a unique element $A^+(x)$ in $\mathfrak{a}^+$ such that x belongs to $K \exp A^+(x)K$.

Lemma 2.1. *The map $A^+ : G \to \mathfrak{a}$ is Lipschitz with respect to both left and right translations of G. More precisely*

$$|A^+(yx) - A^+(y)| \leq \; d(x \cdot o, o) \qquad and \qquad |A^+(xy) - A^+(y)| \leq \; d(x \cdot o, o),$$

for all x and y in G.

Proof. The first inequality follows from $|A^+(yx) - A^+(y)| \leq \; d(yx \cdot o, y \cdot o)$, see [2, Lemma 2.1.2], and the G–invariance of the metric d on $\mathbb{X}$.

The second inequality follows from the first, for $A^+(x^{-1}) = -\sigma \, A^+(x)$, where σ is the element of the Weyl group that maps the negative Weyl chamber $-\mathfrak{a}^+$ to the positive Weyl chamber $\mathfrak{a}^+$. $\qquad\square$

For every positive r we define

(2.6) $$\begin{cases} \mathfrak{b}_r = \{H \in \mathfrak{a} : |H| \leq r\} \\ B_r = K(\exp \mathfrak{b}_r)K \\ \mathfrak{b}'_r = \{H \in \mathfrak{a} : (w \cdot \rho)(H) \leq |\rho|r \text{ for all } w \in W\} \\ B'_r = K(\exp \mathfrak{b}'_r)K. \end{cases}$$

The set B_r is the inverse image under the canonical projection $\pi : G \to \mathbb{X}$ of the ball $B_r(o)$ in the symmetric space $\mathbb{X}$. Thus, a function f on $\mathbb{X}$ is supported in $B_r(o)$ if and only if, as a K–right-invariant function on G, is supported in B_r. We shall use the following properties of the sets defined above [1, Proposition 4].

Proposition 2.2. *The Abel transform is an isomorphism between*

$$C_c^\infty(K \setminus \mathbb{X}) \quad and \quad C_c^\infty(\mathfrak{a})^W.$$

Moreover the following hold:

(i) *supp $f \subset B_r$ if and only if supp $(\mathcal{A}f) \subset \mathfrak{b}_r$.*

(ii) supp $f \subset B'_r$ *if and only if* supp $(\mathcal{A}f) \subset \mathfrak{b}'_r$.

We shall also need the following lemma.

Lemma 2.3. *The following hold:*

(i) *There exists a constant C such that $|B'_r| \leq C\, r^{\ell-1}\, e^{2|\rho|r}$ for all $r \geq 1$.*

(ii) *There exists an integer M such that $B_1 \cdot B'_r \subseteq B'_{r+M}$ for every $r > 0$.*

Proof. To prove (i) we apply the integration formula in Cartan coordinates

$$|B'_r| = |W| \int_{\mathfrak{a}^+ \cap \mathfrak{b}'_r} \delta(H)\, \mathrm{d}H,$$

and use the estimate (2.2) for the density function δ. The conclusion follows by choosing orthogonal coordinates $(H_1, \ldots, H_\ell)$ on $\mathfrak{a}$ such that $H_\ell = \rho(H)/|\rho|$, and observing that there exists a constant C such that $|(H_1, \ldots, H_{\ell-1})| \leq C\, H_\ell$ on $\mathfrak{a}^+ \cap \mathfrak{b}'_r$.

Next we prove (ii). Denote by ς the Minkowski functional of the set $\mathfrak{b}'_1$. Since $\mathfrak{b}'_1$ is convex and absorbing, ς is a norm on $\mathfrak{a}$. Define $\tilde{\varsigma}(x) = \varsigma\big(A^+(x)\big)$ for all x in G. By Lemma 2.1, $\tilde{\varsigma}$ is left (and right) uniformly continuous on G. Thus there exists $\varepsilon > 0$ such that $|\tilde{\varsigma}(xy) - \tilde{\varsigma}(y)| \leq 1$ for all x in B_ε and all y in G. Therefore

(2.7) $$\tilde{\varsigma}(xy) \leq \tilde{\varsigma}(y) + 1 \qquad \forall x \in B_\varepsilon,\ \forall y \in G.$$

Now, if $x \in B_1$ there exist M elements $x_1, \ldots, x_M$ in B_ε such that $x = x_1 x_2 \cdots x_M$. Thus, iterating (2.7), we get

$$\tilde{\varsigma}(xy) \leq \tilde{\varsigma}(y) + M \qquad \forall x \in B_1,\ \forall y \in G.$$

Since $B'_r = \{x \in G : \tilde{\varsigma}(x) \leq r\}$ for all $r > 0$, this proves that $B_1 \cdot B'_r \subseteq B'_{r+M}$, as required. $\square$

2.2. Hardy spaces on $\mathbb{X}$. In this subsection we briefly recall the definitions and properties of $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$. For more about $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$ we refer the reader to [5] and [26, 27, 29], respectively.

Definition 2.4. An H^1-*atom* is a function a in $L^2(\mathbb{X})$, with support contained in a ball B of *radius at most* 1, and such that:

(i) $\int_B a\, \mathrm{d}\mu = 0$.

(ii) $\|a\|_2 \leq |B|^{-1/2}$.

Definition 2.5. The *Hardy space* $H^1(\mathbb{X})$ is the space of all functions g in $L^1(\mathbb{X})$ that admit a decomposition of the form $g = \sum_{j=1}^{\infty} c_j\, a_j$, where a_j is an H^1-atom, and $\sum_{j=1}^{\infty} |c_j| < \infty$. Then $\|g\|_{H^1}$ is defined as the infimum of $\sum_{j=1}^{\infty} |c_j|$ over all decompositions above of g.

Remark 2.6. A straightforward consequence of [26, Lemma 5.7] that we shall use repeatedly in the sequel is the following. If f is in $L^2(\mathbb{X})$, its support

is contained in $B_R(o)$ for some $R > 1$, and its integral vanishes, then f is in $H^1(\mathbb{X})$, and

$$\left\|f\right\|_{H^1} \leq C\, R \left|B_R(o)\right|^{1/2} \left\|f\right\|_2 .$$

An easy adaptation of the proof of [26, Lemma 5.7] shows that if f is in $L^2(\mathbb{X})$, its support is contained in B'_R for some $R > 1$, and its integral vanishes, then f is in $H^1(\mathbb{X})$, and

$$\left\|f\right\|_{H^1} \leq C\, R \left|B'_R\right|^{1/2} \left\|f\right\|_2 .$$

The Hardy type spaces $X^k(\mathbb{X})$ were introduced in [26] as certain Banach spaces isometrically isomorphic to $H^1(\mathbb{X})$. An atomic characterization of $X^k(\mathbb{X})$ was then established in [27], and refined in [29]. In this paper we adopt the latter as the definition of $X^k(\mathbb{X})$. We say that a (smooth) function q on $\mathbb{X}$ is k–quasi-harmonic if $\mathcal{L}^k q$ is constant on $\mathbb{X}$.

Definition 2.7. Suppose that k is a positive integer. An X^k-*atom* is a function A, with support contained in a ball B of *radius at most* 1, such that:

  (i) $\int_{\mathbb{X}} A\, q\, d\mu = 0$ for every k–quasi-harmonic function q.
  (ii) $\|A\|_2 \leq \left|B\right|^{-1/2}$.

Note that condition (i) implies that $\int_{\mathbb{X}} A\, d\mu = 0$, because the constant function 1 is k–quasi-harmonic on $\mathbb{X}$.

Definition 2.8. The space $X^k(\mathbb{X})$ is the space of all functions F of the form $\sum_j c_j\, A_j$, where A_j are X^k-atoms and $\sum_j |c_j| < \infty$, endowed with the norm

$$\|F\|_{X^k} = \inf \left\{ \sum_j |c_j| : F = \sum_j c_j\, A_j, \quad \text{where } A_j \text{ is an } X^k\text{-atom} \right\} .$$

2.3. Estimate of operators. We shall encounter various occurrences of the problem of estimating the $H^1(\mathbb{X})$ norm of functions of the form $a * \gamma$, where a is an $H^1(\mathbb{X})$-atom with support in $B_R(o)$ for some $R \leq 1$, and γ is a K–bi-invariant function with support contained in the ball $B_\beta(o)$. The following lemma contains a version of such an estimate that we shall use frequently in the sequel.

Lemma 2.9. *Suppose that a and γ are as above. The following hold:*

  (i) *There exists a constant C such that*

$$\|a * \gamma\|_{H^1}$$
$$\leq \begin{cases} \left|B_{R+\beta}(o)\right|^{1/2} \min\left(\left\|\gamma\right\|_2, C\, R\, \left\|\nabla\gamma\right\|_2\right) & \text{if } R + \beta \leq 1 \\ C\,(R+\beta) \left|B_{R+\beta}(o)\right|^{1/2} \left\|\gamma\right\|_2 & \text{if } R + \beta > 1. \end{cases}$$

(ii) *Suppose further that γ is of the form $\mathcal{A}^{-1}(\Phi\mathcal{A}\kappa)$, where Φ is a smooth function with compact support, and define $s := (n - \ell)/2$. Then there exists a constant C such that*

$$\left\|\mathcal{A}^{-1}(\Phi\mathcal{A}\kappa)\right\|_2 \leq C \left\|\Phi\mathcal{A}\kappa\right\|_{H^s(\mathfrak{a})}$$

and

$$\left\|\nabla\left[\mathcal{A}^{-1}(\Phi\mathcal{A}\kappa)\right]\right\|_2 \leq C \left\|\Phi\mathcal{A}\kappa\right\|_{H^{s+1}(\mathfrak{a})},$$

where $H^s(\mathfrak{a})$ denotes the standard Sobolev space of order s on $\mathfrak{a}$.

Proof. First we prove (i). Notice that the support of $a * \gamma$ is contained in $B_{R+\beta}(o)$ and that its integral vanishes. Furthermore $\left\|a * \gamma\right\|_2 \leq \left\|\gamma\right\|_2$.

In the case where $R + \beta > 1$ the required estimate follows from Remark 2.6 above. Thus, we may assume that $R + \beta \leq 1$. Observe that

$$a * \gamma = \frac{a * \gamma}{\left\|a * \gamma\right\|_2 \left|B_{R+\beta}(o)\right|^{1/2}} \left\|a * \gamma\right\|_2 \left|B_{R+\beta}(o)\right|^{1/2},$$

unless $a * \gamma = 0$, in which case there is nothing to prove. Then

$$\frac{a * \gamma}{\left\|a * \gamma\right\|_2 \left|B_{R+\beta}(o)\right|^{1/2}}$$

is an H^1-atom, whence its norm in $H^1(\mathbb{X})$ is at most 1. Therefore

$$\left\|a * \gamma\right\|_{H^1} \leq \left\|a * \gamma\right\|_2 \left|B_{R+\beta}(o)\right|^{1/2}.$$

To conclude the proof of the estimate when $R + \beta \leq 1$, it remains to prove that

$$\left\|a * \gamma\right\|_2 \leq \min\left(\left\|\gamma\right\|_2, C\, R \left\|\nabla\gamma\right\|_2\right).$$

Clearly $\left\|a * \gamma\right\|_2 \leq \left\|a\right\|_1 \left\|\gamma\right\|_2 \leq \left\|\gamma\right\|_2$. To prove that $\left\|a * \gamma\right\|_2 \leq C\, R \left\|\nabla\gamma\right\|_2$, we argue as follows. The function a has vanishing integral. Hence

$$a * \gamma(x \cdot o) = \int_{B_R(o)} a(y \cdot o) \left[\gamma(y^{-1}x \cdot o) - \gamma(x \cdot o)\right] \mathrm{d}\mu(y \cdot o).$$

Then, by the generalized Minkowski inequality,

(2.8)

$$\left\|a * \gamma\right\|_2$$

$$\leq \int_{B_R(o)} \mathrm{d}\mu(y \cdot o) \left|a(y \cdot o)\right| \left[\int_{B_{R+\beta}(o)} \left|\gamma(y^{-1}x \cdot o) - \gamma(x \cdot o)\right|^2 \mathrm{d}\mu(x \cdot o)\right]^{1/2}.$$

Take a vector field Y in $\mathfrak{p}$ such that $y^{-1} = \exp Y$. Then, for all x in $B_{R+\beta}(o)$

$$(2.9) \quad \left| \gamma(y^{-1}x \cdot o) - \gamma(x \cdot o) \right| \leq \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} \gamma \left(x \exp \mathrm{Ad}(x^{-1})tY \cdot o \right) \right| \mathrm{d}t$$

$$= \int_0^1 \left| \mathrm{Ad}(x^{-1})Y \gamma \left(x \exp \mathrm{Ad}(x^{-1})tY \cdot o \right) \right| \mathrm{d}t$$

$$\leq \int_0^1 \left| \mathrm{Ad}(x^{-1})Y \right| \left| \nabla \gamma \left(x \exp \mathrm{Ad}(x^{-1})tY \cdot o \right) \right| \mathrm{d}t$$

$$\leq C \left| Y \right| \int_0^1 \left| \nabla \gamma \left(\exp(tY)x \cdot o \right) \right| \mathrm{d}t;$$

in the last inequality we have used the fact that $\sup_{x \in \overline{B}_1(o)} \left| \mathrm{Ad}(x^{-1}) \right| < \infty$. Thus, by (2.9), Minkowski's integral inequality, and the fact that $|Y| \leq R$,

$$\left[\int_{B_{R+\beta}(o)} \left| \gamma(y^{-1}x \cdot o) - \gamma(x \cdot o) \right|^2 \mathrm{d}\mu(x \cdot o) \right]^{1/2}$$

$$\leq C \left| Y \right| \int_0^1 \mathrm{d}t \left[\int_{B_{R+\beta}(o)} \left| \nabla \gamma \left(\exp(tY)x \cdot o \right) \right|^2 \mathrm{d}\mu(x \cdot o) \right]^{1/2}$$

$$\leq C R \left[\int_{B_\beta(o)} \left| \nabla \gamma(x \cdot o) \right|^2 \mathrm{d}\mu(x \cdot o) \right]^{1/2},$$

where we have used the fact that $|Y| < R$, because $y \cdot o$ is in $B_R(o)$. This concludes the proof of the required estimate in the case where $R + \beta \leq 1$, and of (i).

Next we prove the second inequality in (ii): the proof of the first inequality is similar, even simpler, and is omitted. Observe that

$$\left\| \nabla \left[\mathcal{A}^{-1}(\Phi \mathcal{A} \kappa) \right] \right\|_2^2 = \left(\mathcal{L} \left[\mathcal{A}^{-1}(\Phi \mathcal{A} \kappa) \right], \mathcal{A}^{-1}(\Phi \mathcal{A} \kappa) \right).$$

Then Plancherel's formula and estimate (2.5) for Plancherel's measure imply that

(2.10)

$$\left\| \nabla \left[\mathcal{A}^{-1}(\Phi \mathcal{A} \kappa) \right] \right\|_2 = \left[\int_{\mathfrak{a}^*} (|\rho|^2 + |\lambda|^2) \left| \mathcal{F}(\Phi \mathcal{A} \kappa)(\lambda) \right|^2 \mathrm{d}\nu(\lambda) \right]^{1/2}$$

$$\leq C \left[\int_{\mathfrak{a}^*} (1 + |\lambda|^2)^{1+(n-\ell)/2} \left| \mathcal{F}(\Phi \mathcal{A} \kappa)(\lambda) \right|^2 \mathrm{d}\lambda \right]^{1/2}$$

$$\leq C \left[\int_{\mathfrak{a}} \left| (\mathcal{I} - \Delta)^{(s+1)/2} (\Phi \mathcal{A} \kappa)(H) \right|^2 \mathrm{d}H \right]^{1/2}$$

$$= C \left\| \Phi \mathcal{A} \kappa \right\|_{H^{s+1}(\mathfrak{a})},$$

as required. Notice that have used the Euclidean Plancherel's formula in the second inequality above. □

3. Spherical multipliers

In this section we consider two classes $H^\infty(T_{\mathbf{W}}; J)$ and $H(T_{\mathbf{W}}; J, \tau)$ of spherical multipliers on $\mathbb{X}$ and the associated convolution operators. We shall investigate endpoint results for these operators that involve either $H^1(\mathbb{X})$ or $X^k(\mathbb{X})$. We find that convolution operators associated to multipliers in $H^\infty(T_{\mathbf{W}}; J)$ and $H(T_{\mathbf{W}}; J, \tau)$ have quite different boundedness properties. The main reason for this is that the convolution kernels associated to multipliers in $H^\infty(T_{\mathbf{W}}; J)$ are integrable at infinity, whereas those associated to multipliers in $H(T_{\mathbf{W}}; J, \tau)$ may be not.

Definition 3.1. Suppose that J is a positive integer. Denote by $H^\infty(T_{\mathbf{W}}; J)$ the space of all Weyl invariant bounded holomorphic functions in the tube $T_{\mathbf{W}}$ such that

$$(3.1) \qquad |D^I m(\zeta)| \leq C \, (1 + |\zeta|)^{-|I|} \qquad \forall \zeta \in T_{\mathbf{W}}$$

for all multiindices I such that $|I| \leq J$. The norm of m in $H^\infty(T_{\mathbf{W}}; J)$ is the infimum of all constants C such that (3.1) holds.

To introduce the second class of multipliers, we need more notation. For every multiindex $I = (i_1, \ldots, i_\ell)$ we write $I = (I', i_\ell)$, where $I' = (i_1, \ldots, i_{\ell-1})$. Denote by $|I|$ the length of I, and by $d(I)$ its *anisotropic length*, which agrees with $|I|$ if $\ell = 1$, and is defined by

$$d(I) = i_1 + \cdots + i_{\ell-1} + 2i_\ell$$

if $\ell \geq 2$. We write any point ξ in $\mathfrak{a}^*$ as $\xi' + \xi_\ell \, \rho/|\rho|$, where ξ' is orthogonal to ρ. We denote by $|\cdot|$ the Euclidean norm $|\xi| = (|\xi'|^2 + \xi_\ell^2)^{1/2}$ and by $\mathcal{N}$ the anisotropic norm $\mathcal{N}(\xi) = (|\xi'|^4 + \xi_\ell^2)^{1/4}$. Since we may identify $\mathfrak{a}^*$ with $\mathfrak{a}$, we can define $\mathcal{N}$ also on $\mathfrak{a}$. There exists a constant c_ℓ such that if a function f on $\mathfrak{a}$ is given by $f(H) = f_0(\mathcal{N}(H))$ for some $f_0 : [0, \infty) \to \mathbb{C}$, then

$$(3.2) \qquad \int_{\mathfrak{a}} f(H) \, \mathrm{d}H = c_\ell \int_0^\infty f_0(s) \, s^\ell \, \mathrm{d}s.$$

Recall that $Q(\lambda) = \langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle$ is the Gelfand transform of $\mathcal{L}$. We recall the definition of *singular spherical multipliers* that was introduced in [31, Definition 3.7].

Definition 3.2. Suppose that J is a positive integer and that τ is in $[0, \infty)$. Denote by $H(T_{\mathbf{W}}; J, \tau)$ the space of all Weyl invariant holomorphic functions m in $T_{\mathbf{W}}$ such that there exists a positive constant C such that

$$(3.3) \qquad |D^I m(\zeta)| \leq \begin{cases} C \, |Q(\zeta)|^{-\tau - d(I)/2} & \text{if } |Q(\zeta)| \leq 1 \\ C \, |Q(\zeta)|^{-|I|/2} & \text{if } |Q(\zeta)| \geq 1, \end{cases}$$

for every $|I| \leq J$ and for all $\zeta \in T_{\mathbf{W}^+}$, where $\mathbf{W}^+ = \overline{(\mathfrak{a}^*)^+} \cap \mathbf{W}$ and $(\mathfrak{a}^*)^+$ is the interior of the fundamental domain of the action of the Weyl group W that contains ρ. The norm $\|m\|_{H(T_{\mathbf{W}};J,\tau)}$ is the infimum of all C such that (3.3) holds.

Hereafter we shall often write $\|m\|_{(J)}$ for $\|m\|_{H^\infty(T_{\mathbf{W}};J)}$, and $\|m\|_{(J;\tau)}$ for $\|m\|_{H(T_{\mathbf{W}};J,\tau)}$.

As observed in [31, Remark 3.8] and in the Introduction, if m is in $H(T_{\mathbf{W}};J,\tau)$, then its restriction to $\mathfrak{a}^* + i\rho$ exhibits the following anisotropic behaviour when the rank of $\mathbb{X}$ is at least two. There exists a constant C such that for every I with $|I| \leq J$ and all $\xi \in \mathfrak{a}^*$

$$(3.4) \qquad |D^I m(\xi + i\rho)| \leq \begin{cases} C\,\|m\|_{(J,\tau)}\,\mathcal{N}(\xi)^{-2\tau - d(I)} & \text{if } |\xi| \leq 1 \\ C\,\|m\|_{(J,\tau)}\,|\xi|^{-|I|} & \text{if } |\xi| > 1. \end{cases}$$

Remark 3.3. The class $H(T_{\mathbf{W}};J,\tau)$ strictly contains an interesting class of functions of the Laplacian. Indeed, suppose that J is a nonnegative integer and that τ is in $[0,\infty)$. and that M is a holomorphic function in the parabolic region in the plane defined by

$$\mathbf{P} = \{(x,y) \in \mathbb{R}^2 : y^2 < 4|\rho|^2\,x\}$$

and there exists a positive constant C such that

$$(3.5) \quad |M^{(j)}(z)| \leq \begin{cases} C\,|z|^{-\tau-j} & \text{if } |z| \leq 1 \\ C\,|z|^{-j} & \text{if } |z| > 1 \end{cases} \qquad \forall z \in \mathbf{P} \quad \forall j \in \{0,1,\ldots,J\}.$$

Then $M \circ Q$ belongs to $H(T_{\mathbf{W}};J,\tau)$ and the associated convolution operator is $M(\mathcal{L})$. This was proved in [31, Proposition 3.9]. Notice that $\mathbf{P}$ is the image of $T_{\mathbf{W}}$ under the function Q.

We shall denote by $\|\!|\mathcal{M}|\!\|_{\mathfrak{A};\mathfrak{B}}$ the operator norm of $\mathcal{M}$ *qua* linear operator between the Banach spaces $\mathfrak{A}$ and $\mathfrak{B}$. In the case where $\mathfrak{A} = \mathfrak{B}$, we shall simply write $\|\!|\mathcal{M}|\!\|_{\mathfrak{A}}$ instead of $\|\!|\mathcal{M}|\!\|_{\mathfrak{A};\mathfrak{A}}$. The main result of this section is the following.

Theorem 3.4. *Suppose that J is an integer $\geq \lceil n/2 \rceil + 2$ and that $\tau \geq 0$. The following hold:*

(i) *There exists a constant C such that*

$$\|\!|\mathcal{M}|\!\|_{H^1} \leq C\,\|m\|_{H^\infty(T_{\mathbf{W}};J)} \qquad \forall m \in H^\infty(T_{\mathbf{W}};J).$$

(ii) *If $k > \tau + J$, then there exists a constant C such that*

$$\|\!|\mathcal{M}|\!\|_{X^k;H^1} \leq C\,\|m\|_{H(T_{\mathbf{W}};J,\tau)} \qquad \forall m \in H(T_{\mathbf{W}};J,\tau).$$

(iii) *If $k > \tau$, then there exists a constant C such that*

$$\|\!|\mathcal{M}|\!\|_{X^k;L^1} \leq C\,\|m\|_{H(T_{\mathbf{W}};J,\tau)} \qquad \forall m \in H(T_{\mathbf{W}};J,\tau).$$

The proof of Theorem 3.4(iii) requires estimating the Fourier transform of multipliers on $\mathfrak{a}^*$, which are compactly supported and satisfy a local anisotropic Mihlin condition. Related estimates were proved by E. Fabes and N. Rivière [11] long ago.

Lemma 3.5. *Suppose that β is a positive real number and that J is an integer $\geq \ell + 1 + \beta$. Assume that m is a smooth function on $\mathfrak{a}^* \setminus \{0\}$ that vanishes outside a compact set, and that there exists a positive constant C_0 such that*

$$\left| D^I m(\lambda) \right| \leq C_0 \, \mathcal{N}(\lambda)^{\beta - d(I)} \qquad \forall \lambda \in \mathfrak{a}^* \setminus \{0\}$$

for every multiindex I with $|I| \leq J$. Then there exists a constant C such that

$$\left| (\mathcal{F}^{-1} m)(H) \right| \leq C \left[1 + \mathcal{N}(H) \right]^{-\ell - 1 - \beta} \qquad \forall H \in \mathfrak{a}.$$

Proof. Observe that m is a distribution with compact support, whence $\mathcal{F}^{-1} m$ is smooth by the Paley–Wiener theorem. Thus, it suffices to prove the required estimate for $\mathcal{N}(H)$ large. For simplicity we assume that the support of m is contained in $\{\lambda : \mathcal{N}(\lambda) \leq 1\}$. For every positive integer j we define the anisotropic annulus

$$F_j := \{\lambda \in \mathfrak{a} : 2^{-j-1} \leq \mathcal{N}(\lambda) \leq 2^{-j+1}\};$$

its Lebesgue measure is approximately $2^{-j(\ell+1)}$. Denote by ψ a smooth function with support contained in $[1/2, 2]$, and such that

$$1 = \sum_{j=1}^{\infty} \psi\left(2^j \mathcal{N}(\lambda)\right) \qquad \forall \lambda \in \mathfrak{a}^* \setminus \{0\} : \mathcal{N}(\lambda) \leq 1.$$

For simplicity, set $\psi_j(\lambda) = \psi\left(2^j \mathcal{N}(\lambda)\right)$, and write $m = \sum_{j=1}^{\infty} \psi_j \, m$. It is straightforward, albeit tedious, to check that there exists a constant C such that

$$\left| D^I (\psi_j \, m)(\lambda) \right| \leq C \mathcal{N}(\lambda)^{\beta - d(I)} \qquad \forall \lambda \in \mathfrak{a}^*$$

for every multiindex I such that $|I| \leq J$ and for every positive integer j. Fix H such that $\mathcal{N}(H)$ is large, and write

$$\mathcal{F}^{-1} m(H) = \sum_{j : \mathcal{N}(H) \leq 2^j} \mathcal{F}^{-1}(\psi_j \, m)(H) + \sum_{j : \mathcal{N}(H) > 2^j} \mathcal{F}^{-1}(\psi_j \, m)(H).$$

By the Euclidean Fourier inversion formula, the assumptions on m, and (3.2), we may estimate each summand in the first sum as follows

$$\left| \mathcal{F}^{-1}(\psi_j \, m)(H) \right| \leq \int_{F_j} |m(\lambda)| \, d\lambda \leq C_0 \int_{F_j} \mathcal{N}(\lambda)^{\beta} \, d\lambda \leq C \, 2^{-j(\beta + \ell + 1)}.$$

In order to estimate the summands in the second sum, we introduce the differential operator $\widetilde{\Delta}$ on $\mathfrak{a}^*$ defined by $\sum_{i=1}^{\ell-1} \partial_{\lambda_i}^2 + \partial_{\lambda_\ell}$, use Fourier's inversion formula, and integrate by parts, using the identity

$$\widetilde{\Delta}^h e^{i\lambda(H)} = \left(i H_\ell - |H'|^2 \right)^{2h} e^{i\lambda(H)}.$$

We obtain

$$\left|\mathcal{F}^{-1}(\psi_j\,m)(H)\right| \le C\left|iH_\ell - |H'|^2\right|^{-2h} \int_{F_j} \left|\widetilde{\Delta}^h(\psi_j m)(\lambda)\right| d\lambda$$

$$\le C\,\mathcal{N}(H)^{-2h} \int_{F_j} \mathcal{N}(\lambda)^{\beta-2h}\, d\lambda$$

$$\le C\,\mathcal{N}(H)^{-2h}\, 2^{-j(\beta-2h+\ell+1)},$$

for every $h \le J/2$. By combining the last two estimates, we see that

$$\left|\mathcal{F}^{-1}m(H)\right| \le C \sum_{j:\mathcal{N}(H)\le 2^j} 2^{-j(\beta+\ell+1)} + C\,\mathcal{N}(H)^{-2h} \sum_{j:\mathcal{N}(H)>2^j} 2^{-j(\beta-2h+\ell+1)}$$

$$\le C\,\mathcal{N}(H)^{-\ell-1-\beta},$$

as required. $\qquad\qquad\square$

We are now ready to prove Theorem 3.4. Following up an idea of Anker [1], in the proof of Theorem 3.4(i) we decompose the Abel transform $\mathcal{A}\kappa$ of the kernel of the operator $\mathcal{M}$, rather than the kernel itself, via a partition of unity that we now describe. For every function ϕ on $\mathfrak{a}$ and every positive number t we denote by ϕ^t and ϕ_t the functions defined by

$$\phi^t(H) = \phi(tH) \qquad \text{and} \qquad \phi_t(H) = t^{-\ell}\phi(H/t) \qquad \forall H \in \mathfrak{a}.$$

Denote by Ψ a smooth radial function on $\mathfrak{a}$ with support contained in the annulus $\{H \in \mathfrak{a} : 2^{-1} \le |H| \le 2\}$ and such that $\sum_{k\in\mathbf{Z}} \Psi^{2^{-k}} = 1$ on $\mathfrak{a} \setminus \{0\}$. Also, set $\Phi := 1 - \sum_{k=1}^{\infty} \Psi^{2^{-k}}$. Notice that the support of Φ is contained in $\mathfrak{b}_2$ (see (2.6) for the definition of $\mathfrak{b}_2$).

Proof of Theorem 3.4. First, we prove (i). By [24, Theorem 4.1] and the translation invariance of $\mathcal{M}$, it suffices to show that there exists a constant C such that

$$(3.6) \qquad\qquad \left\|\mathcal{M}a\right\|_{H^1} \le C \left\|m\right\|_{(J)},$$

for each H^1-atom a supported in $B_R(o)$, with $R \le 1$. We consider the cases where $R \ge 10^{-1}$ and $R < 10^{-1}$ separately.

In the first case we consider a partition of unity on $\mathfrak{a}$ of the form

$$1 = \Phi + \sum_{j=1}^{\infty} \psi_j',$$

where ψ_j' is a smooth Weyl invariant function on $\mathfrak{a}$ with support contained in the "polyhedral annulus" $\mathfrak{b}_{j+2}' \setminus \mathfrak{b}_j'$ (see (2.6) for the definition of $\mathfrak{b}_j'$). Note that H is in $\mathfrak{b}_{j+2}' \setminus \mathfrak{b}_j'$ if and only if $j\,|\rho| \le (w \cdot \rho)(H) \le (j+2)\,|\rho|$, where w is the element in the Weyl group such that $w \cdot H$ belongs to the (closure of the) positive Weyl chamber.

In the second case we consider a partition of unity on $\mathfrak{a}$ of the form

$$1 = \Phi^{1/R} + \sum_{h=1}^{N} \Psi^{1/(2^h R)} + \sum_{j=0}^{\infty} \psi'_j,$$

where N is the least integer for which $2^N R > 1/2$. Note that $\Phi^{1/R}$, $\Psi^{1/(2^h R)}$ and ψ'_0 are smooth Weyl invariant functions on $\mathfrak{a}$ with support contained in $\mathfrak{b}_{2R}$, $\mathfrak{b}_{2^{h+1}R} \setminus \mathfrak{b}_{2^{h-1}R}$ and $\mathfrak{b}_2 \setminus \mathfrak{b}_{10^{-1}}$, respectively. Furthermore ψ'_j, $j \geq 1$, is as in the decomposition above in the case where $R \geq 10^{-1}$.

The proof of estimate (3.6) in the case where $R \geq 10^{-1}$ is simpler than that in the case where $R < 10^{-1}$. We give full details in the second case, and leave the first case to the interested reader.

Thus, suppose that a is an $H^1(\mathbb{X})$ atom with support contained in $B_R(o)$ and that $R < 10^{-1}$. We decompose κ as follows

$$(3.7) \qquad \kappa = \mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big) + \sum_{h=1}^{N} \mathcal{A}^{-1}\big(\Psi^{1/(2^h R)}\mathcal{A}\kappa\big) + \sum_{j=0}^{\infty} \mathcal{A}^{-1}\big(\psi'_j\mathcal{A}\kappa\big),$$

and estimate the $H^1(\mathbb{X})$ norm of the convolution of a with each summand separately. This will be done in Step I–Step III below. It is important to keep in mind that $\mathcal{A}\kappa = \mathcal{F}^{-1}m$, i.e., $\mathcal{A}\kappa$ is related to the multiplier m via the Euclidean Fourier transform.

Step I. We denote by $\big\|\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)\big\|_{Cv_2}$ the norm of the convolution operator $f \mapsto f * \mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)$ on $L^2(\mathbb{X})$. Observe that

$$\big\|\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)\big\|_{Cv_2} = \big\|\mathcal{H}\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)\big\|_{L^{\infty}(\mathfrak{a}^*)}.$$

Since $\mathcal{H} = \mathcal{F}\mathcal{A}$, $\mathcal{H}\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big) = \big(\mathcal{F}\Phi^{1/R}\big) *_{\mathfrak{a}^*} m$. Therefore

$$\big\|\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)\big\|_{Cv_2} \leq \big\|\mathcal{F}\Phi^{1/R}\big\|_{L^1(\mathfrak{a}^*)} \big\|m\big\|_{\infty} \leq \big\|\mathcal{F}\Phi^{1/R}\big\|_{L^1(\mathfrak{a}^*)} \big\|m\big\|_{(J)}.$$

Observe further that $\big\|\mathcal{F}\Phi^{1/R}\big\|_{L^1(\mathfrak{a}^*)} = \big\|(\mathcal{F}\Phi)_{1/R}\big\|_{L^1(\mathfrak{a}^*)} = \big\|\mathcal{F}\Phi\big\|_{L^1(\mathfrak{a}^*)}$, which is finite, and independent of R. By Proposition 2.2(i), the support of $\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)$ is contained in $B_{2R}(o)$, whence

$$(3.8) \quad \big\|a * \mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)\big\|_{H^1} \leq \sqrt{|B_{3R}(o)|} \, \big\|a\big\|_2 \, \big\|\mathcal{A}^{-1}\big(\Phi^{1/R}\mathcal{A}\kappa\big)\big\|_{Cv_2}$$

$$\leq C \sqrt{\frac{|B_{3R}(o)|}{|B_R(o)|}} \, \big\|m\big\|_{(J)}$$

$$\leq C \, \big\|m\big\|_{(J)};$$

we have used the local doubling condition in the last inequality.

Step II. Next we estimate the $H^1(\mathbb{X})$ norm of $\sum_{h=1}^{N} a * \mathcal{A}^{-1}\big(\Psi^{1/(2^h R)}\mathcal{A}\kappa\big)$. By Proposition 2.2(i), the support of $\mathcal{A}^{-1}\big(\Psi^{1/(2^h R)}\mathcal{A}\kappa\big)$ is contained in the ball centred at o with radius $2^{h+1} R$, whence the support of $a * \mathcal{A}^{-1}\big(\Psi^{1/(2^h R)}\mathcal{A}\kappa\big)$

is contained in the ball centred at o with radius $\left(1 + 2^{h+1}\right) R$, which is less than 1. Thus, we may apply the first estimate in Lemma 2.9(i), and conclude that there exists a constant C, independent of R and h, such that

$$
\begin{aligned}
\left\| a * \mathcal{A}^{-1}\left(\Psi^{1/(2^h R)}\mathcal{A}\kappa\right) \right\|_{H^1} &\leq (2^h R)^{n/2}\, C\, R\, \left\| \nabla\{\mathcal{A}^{-1}\left(\Psi^{1/(2^h R)}\mathcal{A}\kappa\right)\} \right\|_2 \\
&\leq C\,(2^h R)^{n/2}\, R\, \left\| \Psi^{1/(2^h R)}\mathcal{A}\kappa \right\|_{H^{s+1}(\mathfrak{a})},
\end{aligned}
$$

where $s = (n - \ell)/2$. We have used the second estimate in Lemma 2.9(ii), with $\Psi^{1/(2^h R)}$ in place of Φ, in the second inequality above.

We shall prove that there exists a constant C, independent of R and h, such that

$$(3.9) \qquad \left\| \Psi^{1/(2^h R)}\mathcal{A}\kappa \right\|_{H^{s+1}(\mathfrak{a})} \leq C\, \left\| m \right\|_{(J)}\,(2^h R)^{-1-n/2}.$$

The last two bounds clearly imply the required estimate, for

$$
\sup_{R < 10^{-1}} \left\| \sum_{h=1}^{N} a * \mathcal{A}^{-1}\left(\Psi^{1/(2^h R)}\mathcal{A}\kappa\right) \right\|_{H^1} \leq C\, \left\| m \right\|_{(J)} \sum_{h=1}^{N} 2^{-h} \leq C\, \left\| m \right\|_{(J)}.
$$

Thus, it remains to prove (3.9). Write t instead of $(2^h R)^{-1}$. Then (3.9) may be rewritten as

$$(3.10) \qquad \left\| \Psi^t \mathcal{A}\kappa \right\|_{H^\nu(\mathfrak{a})} \leq C\, \left\| m \right\|_{(J)}\, t^{\nu + \ell/2},$$

with $\nu = s + 1$. We shall prove (3.10) in the case where ν is a positive integer. The estimate in the case where ν is not an integer will follow from this by interpolation.

Thus, suppose that ν is a positive integer. We set $D := \left(i^{-1}\partial_1, \ldots, i^{-1}\partial_\ell\right)$. By the definition of the Sobolev norm and Leibnitz's rule, we see that

$$
\begin{aligned}
& \left\| \Psi^t \mathcal{A}\kappa \right\|^2_{H^\nu(\mathfrak{a})} \\
&= \int_{\mathfrak{a}} \left| \Psi^t(H)\, \mathcal{A}\kappa(H) \right|^2 \mathrm{d}H + \sum_{|\beta|=\nu} \int_{\mathfrak{a}} \left| D^\beta(\Psi^t \mathcal{A}\kappa)(H) \right|^2 \mathrm{d}H \\
&\leq \int_{A(t)} \left| \mathcal{A}\kappa(H) \right|^2 \mathrm{d}H + C \sum_{|\beta|=\nu} \sum_{\beta' \leq \beta} t^{2|\beta|-2|\beta'|} \int_{A(t)} \left| D^{\beta'}(\mathcal{A}\kappa)(H) \right|^2 \mathrm{d}H,
\end{aligned}
$$

where $A(t)$ denotes the annulus $\{H \in \mathfrak{a} : (2t)^{-1} \leq |H| \leq 2/t\}$. For each β' of length at most $J - \ell/2$ the estimate

$$
\int_{A(t)} \left| D^{\beta'}(\mathcal{A}\kappa)(H) \right|^2 \mathrm{d}H \leq C\, \left\| m \right\|^2_{(J)}\, t^{\ell+2|\beta'|}
$$

is a straightforward consequence of [40, Lemma 4.1, p. 359], which is a well known statement concerning Euclidean Fourier multipliers satisfying Hörmander conditions. Observe that $s + 1 = (n - \ell)/2 + 1 \leq J - \ell/2$, for we are assuming that $J \geq \lceil n/2 \rceil + 2$. Therefore,

$$
\left\| \Psi^t \mathcal{A}\kappa \right\|^2_{H^\nu(\mathfrak{a})} \leq C\, \left\| m \right\|^2_{(J)}\, t^{\ell+2\nu},
$$

thereby proving (3.10), and concluding the proof of Step II.

Step III. It remains to estimate the $H^1(\mathbb{X})$ norm of the terms $a * \mathcal{A}^{-1}(\psi'_j \mathcal{A}\kappa)$, $j = 0, 1, 2, \ldots$ in the decomposition (3.7). These estimates are reminiscent of those obtained by Anker in [1, Proposition 5]. We shall give details in the case $j \geq 1$. The case where $j = 0$ is even simpler, and may be treated similarly, with slight modifications.

By Proposition 2.2(ii), the support of $\mathcal{A}^{-1}(\psi'_j \mathcal{A}\kappa)$, as a K–bi-invariant function on G, is contained in B'_{j+2}, hence that of $a * \mathcal{A}^{-1}(\psi'_j \mathcal{A}\kappa)$ is contained in $B_R \cdot B'_{j+2}$, which, by Lemma 2.3, is contained in B'_{j+3+M} for a suitably large integer M. Thus, the support of $a * \mathcal{A}^{-1}(\psi'_j \mathcal{A}\kappa)$ is in B'_{j+3+M}. Furthermore, its integral vanishes. Remark 2.6 combined with Lemma 2.9(ii) (with ψ'_j in place of Φ) and Lemma 2.3(i), implies that there exists a constant C, independent of a and j, such that

(3.11)
$$\left\| a * \mathcal{A}^{-1}(\psi'_j \mathcal{A}\kappa) \right\|_{H^1} \leq C \, (j + 3 + M) \left| B'_{j+3+M} \right|^{1/2} \, \left\| \mathcal{A}^{-1}(\psi'_j \mathcal{A}\kappa) \right\|_2$$
$$\leq C \, j^{(\ell+1)/2} \, \mathrm{e}^{|\rho| \, j} \, \left\| \psi'_j \mathcal{A}\kappa \right\|_{H^s(\mathfrak{a})} ,$$

where s is equal to $(n - \ell)/2$. We shall estimate the Sobolev norm of order s above, when s is a nonnegative integer. The required estimate for nonintegral s will follow from this by interpolation.

Thus, suppose that s is a nonnegative integer. We need to estimate $\left\| \psi'_j \mathcal{A}\kappa \right\|_2$ and $\left\| D^\beta(\psi'_j \mathcal{A}\kappa) \right\|_2$ for all multiindices β of length s. By Leibnitz's rule, $D^\beta(\psi'_j \mathcal{A}\kappa)$ may be written as a linear combination of terms of the form $D^{\beta_1}\psi'_j \, D^{\beta_2}(\mathcal{A}\kappa)$, where $\beta_1 + \beta_2 = \beta$. By the Euclidean Paley–Wiener theorem, $\mathcal{F}\psi'_j$ is an entire function of exponential type, and recall that m is holomorphic in $T_{\mathbf{W}}$ and bounded on $T_{\overline{\mathbf{W}}}$ together with its derivatives up to the order J. Thus, by Euclidean Fourier analysis,

$$\mathcal{F}\left[D^{\beta_1}\psi'_j \, D^{\beta_2}(\mathcal{A}\kappa) \right](\lambda)$$
$$= \int_{\mathfrak{a}^*} (\lambda - \xi)^{\beta_1} \, \mathcal{F}\psi'_j(\lambda - \xi) \, \xi^{\beta_2} \, m(\xi) \, \mathrm{d}\xi$$
$$= \int_{\mathfrak{a}^*} (\lambda - \xi - i\rho)^{\beta_1} \, \mathcal{F}\psi'_j(\lambda - \xi - i\rho) \, (\xi + i\rho)^{\beta_2} \, m(\xi + i\rho) \, \mathrm{d}\xi,$$

which equals the Fourier transform of $D^{\beta_1}\psi'_j \cdot (D + i\rho)^{\beta_2}\mathcal{F}^{-1}m_\rho$ evaluated at the point $\lambda - i\rho$. Here $m_\rho(\lambda) = m(\lambda + i\rho)$. Thus,

$$\left[D^{\beta_1}\psi'_j \, D^{\beta_2}(\mathcal{A}\kappa) \right](H) = \mathrm{e}^{-\rho(H)} \, D^{\beta_1}\psi'_j(H) \, (D + i\rho)^{\beta_2}\mathcal{F}^{-1}m_\rho(H).$$

Observe that $\rho(H) \geq |\rho|j$ and $\sum_{L=1}^{\ell} |H_L|^{2J} \geq c \, j^{2J} > 0$ for all H in the support of ψ'_j. The first of the two inequalities above follows directly from the definition of the support of ψ'_j, and the second is a consequence of the trivial estimate

$$|\rho|^{2J} \, j^{2J} \leq \rho(H)^{2J} \leq |\rho|^{2J} \, |H|^{2J} \qquad \forall H \in \mathfrak{a}^+ \cap \mathrm{supp}(\psi'_j),$$

and the fact that $|H|^{2J} \leq C \sum_{L=1}^{\ell} |H_L|^{2J}$ (the left and the right hand side are both elliptic polynomials of the same degree). Hence

$$\left\| D^{\beta_1} \psi_j' \, D^{\beta_2}(\mathcal{A}\kappa) \right\|_2^2$$

$$\leq \mathrm{e}^{-2|\rho| j} \int_{\mathfrak{a}} \left| D^{\beta_1} \psi_j'(H) \, (D + i\rho)^{\beta_2} \mathcal{F}^{-1} m_\rho(H) \right|^2 \mathrm{d}H$$

$$\leq C \, j^{-2J} \, \mathrm{e}^{-2|\rho| j} \left\| D^{\beta_1} \psi_j' \right\|_\infty^2 \sum_{L=1}^{\ell} \int_{\mathfrak{a}} |H_L|^{2J} \left| (D + i\rho)^{\beta_2} \mathcal{F}^{-1} m_\rho(H) \right|^2 \mathrm{d}H.$$

It is straightforward to prove that the L^∞ norms of $D^{\beta_1} \psi_j'$ are uniformly bounded with respect to all positive integers j and all multiindices β_1 of length at most J. By the Euclidean Plancherel formula, the last integral is equal to

$$\int_{\mathfrak{a}^*} \left| \partial_L^J \big[(\cdot + i\rho)^{\beta_2} m_\rho \big](\lambda) \right|^2 \mathrm{d}\lambda,$$

which, in turn, may be estimated by

$$C \, \|m\|_{(J)} \int_{\mathfrak{a}^*} \big(1 + |\lambda| \big)^{2(|\beta_2| - J)} \, \mathrm{d}\lambda \leq C \, \|m\|_{(J)} \int_{\mathfrak{a}^*} \big(1 + |\lambda| \big)^{2(s - J)} \, \mathrm{d}\lambda.$$

Since $J - s > \ell/2$, the last integral is convergent. By combining the estimates above, we see that there exists a constant C, independent of j, such that

$$\left\| D^\beta \big(\psi_j' \mathcal{A}\kappa \big) \right\|_2 \leq C \, \|m\|_{(J)} \, j^{-J} \, \mathrm{e}^{-|\rho| j}.$$

Slight modifications in the argument above prove that $\left\| \psi_j' \mathcal{A}\kappa \right\|_2$ satisfies a similar estimate. Then, by (3.11), there exists a constant C, independent of j, such that

$$\left\| a * \mathcal{A}^{-1} \big(\psi_j' \mathcal{A}\kappa \big) \right\|_{H^1} \leq C \, \|m\|_{(J)} \, j^{-J + (\ell+1)/2}.$$

Thus, the $H^1(\mathbb{X})$ norm of $\sum_{j=1}^{\infty} \mathcal{A}^{-1} \big(\psi_j' \mathcal{A}\kappa \big)$ may be estimated by

$$C \, \|m\|_{(J)} \sum_{j=1}^{\infty} j^{-J + (\ell+1)/2}.$$

This series is convergent, for $J > (\ell + 3)/2$ by assumption, and the proof of (i) is complete.

Next we prove (ii). Recall that the operator $\mathcal{L}\,(\mathcal{I} + \mathcal{L})^{-1}$, which we shall denote by $\mathcal{U}$ in the sequel, establishes an isometric isomorphism between $H^1(\mathbb{X})$ and $X^1(\mathbb{X})$. Similarly, $\mathcal{U}^k$ establishes an isometric isomorphism between $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$ (see [26, 27] for details). Thus, to prove that $\mathcal{M}$ is bounded from $X^k(\mathbb{X})$ to $H^1(\mathbb{X})$ is equivalent to showing that $\mathcal{M}\mathcal{U}^k$ is bounded on $H^1(\mathbb{X})$. The multiplier associated to $\mathcal{M}\mathcal{U}^k$ is $m \, Q^k \, (1 + Q)^{-k}$. It is straightforward, though tedious, to check that if $k > \tau + J$, then there exists a constant C, independent of m, such that

$$\left\| m \, Q^k \, (1 + Q)^{-k} \right\|_{(J)} \leq C \, \|m\|_{(J;\tau)} .$$

The required conclusion then follows from (i).

Finally, we prove (iii). Denote by L an integer $> \tau + J$. Define $\mathcal{M}_1$ and $\mathcal{M}_2$ by

$$\mathcal{M}_1 = \mathcal{M}\,(\mathcal{I} - \mathrm{e}^{-\mathcal{L}})^L \quad \text{and} \quad \mathcal{M}_2 = \mathcal{M}\,[\mathcal{I} - (\mathcal{I} - \mathrm{e}^{-\mathcal{L}})^L].$$

A straightforward, albeit tedious, calculation shows that there exists a constant C, independent of m, such that the multiplier m_1 associated to $\mathcal{M}_1$ satisfies

$$\left\|m_1\right\|_{(J)} \leq C\,\left\|m\right\|_{(J;\tau)}.$$

By (i) the operator $\mathcal{M}_1$ is bounded on $H^1(\mathbb{X})$ with norm bounded by $C\,\|m\|_{(J;\tau)}$. Then $\mathcal{M}_1$ is *a fortiori* bounded from $X^k(\mathbb{X})$ to $L^1(\mathbb{X})$, with the required norm bound.

It remains to show that $\mathcal{M}_2$ is bounded from $X^k(\mathbb{X})$ to $L^1(\mathbb{X})$, and that the appropriate norm estimate holds. By [29, Corollary 6.2 and Prop. 6.3], and the translation invariance of $\mathcal{M}_2$, it suffices to show

$$\sup_A \|\mathcal{M}_2 A\|_1 < \infty,$$

where the supremum is taken over all X^k-atoms A with support contained in a ball B centred at o. Recall that the radius of B is at most 1.

Suppose that A is such an atom. By Schwarz's inequality, the L^2-boundedness of $\mathcal{L}^{-k}$ and the size condition of A,

$$\begin{aligned}
\|\mathcal{M}_2 A\|_1 &\leq \left\|\mathcal{M}_2 \mathcal{L}^k\right\|_{L^1}\,\left\|\mathcal{L}^{-k} A\right\|_1 \\
&\leq \left\|\mathcal{M}_2 \mathcal{L}^k\right\|_{L^1}\,\left\|\mathcal{L}^{-k} A\right\|_2\,|B|^{1/2} \\
&\leq \left\|\mathcal{M}_2 \mathcal{L}^k\right\|_{L^1}\,\left\|\mathcal{L}^{-k}\right\|_2\,|B|^{-1/2}\,|B|^{1/2} \\
&\leq C\,\left\|\mathcal{M}_2 \mathcal{L}^k\right\|_{L^1}.
\end{aligned}$$

To conclude the proof of (iii), it then suffices to show that there exists a constant C, independent of m, such that $\left\|\mathcal{M}_2 \mathcal{L}^k\right\|_{L^1} \leq C\,\left\|m\right\|_{(J;\tau)}$. We give details in the case where the rank of $\mathbb{X}$ is ≥ 2. The proof in the rank one case is much easier, and is omitted.

For the rest of the proof of (iii), we shall denote the spherical multiplier and the convolution kernel of $\mathcal{M}_2 \mathcal{L}^k$ by $m_{2,k}$ and $\kappa_{2,k}$, respectively. We shall prove that there exists a constant C, independent of m, such that

$$\left\|\kappa_{2,k}\right\|_1 \leq C\,\left\|m\right\|_{(J;\tau)}.$$

We need the following notation. Define the function $\omega : \mathfrak{a} \to \mathbb{R}$

$$(3.12) \qquad\qquad \omega(H) = \min_{\alpha \in \Sigma_s} \alpha(H) \qquad \forall H \in \mathfrak{a},$$

and, for each $c > 0$, the subset $\mathfrak{s}_c$ of $\overline{\mathfrak{a}^+}$ by

$$(3.13) \qquad\qquad \mathfrak{s}_c = \{H \in \mathfrak{a} : 0 \leq \omega(H) \leq c\};$$

$\mathfrak{s}_c$ is the set of all points in $\overline{\mathfrak{a}^+}$ at "distance" at most c from the walls of $\mathfrak{a}^+$. We shall estimate $\kappa_{2,k}$ in

$$B_2(o), \quad B_1(o)^c \cap K \exp(\mathfrak{s}_1) \cdot o \quad \text{and} \quad [B_1(o) \cup K \exp(\mathfrak{s}_1) \cdot o]^c$$

separately.

We first estimate $\kappa_{2,k}$ in $B_2(o)$. Recall that $m_{2,k} = Q^k \left[1 - (1-\mathrm{e}^{-Q})^L\right] m$. Since $k > \tau$, by condition (3.3) there exists a constant C, independent of m, such that

$$(3.14) \qquad |m_{2,k}(\lambda)| \leq C \left\|m\right\|_{(J;\tau)} (1 + |\lambda|)^{2k} \mathrm{e}^{-\operatorname{Re} Q(\lambda)} \qquad \forall \lambda \in T_{\overline{\mathbf{W}}}.$$

This, the spherical inversion formula and estimate (2.5) for the Plancherel measure entail the pointwise bound

$$|\kappa_{2,k}(\exp H \cdot o)| \leq C \left\|m\right\|_{(J;\tau)} \int_{\mathfrak{a}^*} (1 + |\lambda|)^{2k+n-\ell} \mathrm{e}^{-Q(\lambda)} \, \mathrm{d}\lambda \leq C \left\|m\right\|_{(J;\tau)}$$

for every H in $\mathfrak{b}_2$. Thus, $\kappa_{2,k}$ is integrable in $B_2(o)$, and there exists a constant C, independent of m, such that

$$\int_{B_2(o)} |\kappa_{2,k}| \, \mathrm{d}\mu \leq C \left\|m\right\|_{(J;\tau)}.$$

Next, we indicate how to estimate $\kappa_{2,k}(\exp H \cdot o)$ when H is close to the walls of the Weyl chamber, but off the ball $\mathfrak{b}_1$. We shall argue as in the proof of [31, Theorem 3.2-Step III]. A straightforward computation shows that for each nonnegative integer s and each positive σ there exists a constant C, which does not depend on m, such that

$$\left|D^I m_{2,k}(\lambda + i\eta)\right| (1 + |\lambda|)^s \leq C \left\|m\right\|_{(J;\tau)} \mathrm{e}^{-\operatorname{Re} Q(\lambda)/2}$$

for all $|I| \leq \ell + 1$ and for all $\eta \in \mathbf{W}(\sigma)$. Here $\mathbf{W}(\sigma)$ is the set of all η in $\mathbf{W}$ such that $|\eta - w \cdot \rho| \geq \sigma$ for all $w \in W$. By [31, Lemma 5.6 (ii)], there exist an integer s and positive constants σ and C such that

$$\int_{B_1(o)^c \cap K \exp(\mathfrak{s}_1) \cdot o} |\kappa_{2,k}| \, \mathrm{d}\mu$$
$$\leq C \max_{|I| \leq \ell+1} \sup_{\eta \in \mathbf{W}(\sigma)} \int_{\mathfrak{a}^*} \left|D^I m_{2,k}(\lambda + i\eta)\right| (1 + |\lambda|)^s \, \mathrm{d}\lambda.$$

By combining the last two estimates, we see that

$$\int_{B_1(o)^c \cap K \exp(\mathfrak{s}_1) \cdot o} |\kappa_{2,k}| \, \mathrm{d}\mu \leq C \left\|m\right\|_{(J;\tau)}.$$

Finally, we estimate $\kappa_{2,k}$ away from the walls of the Weyl chamber and off the ball $B_1(o)$. We shall use the Harish-Chandra's expansion of spherical functions away from the walls of the Weyl chamber [19, Theorem 5.5, p. 430].

Denote by Λ the positive lattice generated by the simple roots in Σ^+. For all H in $\mathfrak{a}^+$ and λ in $\mathfrak{a}^*$

(3.15)
$$|\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(\exp H)$$
$$= e^{-\rho(H)} \sum_{q \in \Lambda} e^{-q(H)} \sum_{w \in W} \mathbf{c}(-w \cdot \lambda)^{-1}\, \Gamma_q(w \cdot \lambda)\, e^{i(w \cdot \lambda)(H)}.$$

The coefficient Γ_0 is equal to 1; the other coefficients Γ_q are rational functions, holomorphic, for some t in $\mathbb{R}^-$, in a certain region $T_{\mathbf{W}^t}$ that we now define. For each t in $\mathbb{R}$ we denote by $\mathbf{W}^t$ the set

(3.16)
$$\mathbf{W}^t = \{\lambda \in \mathbf{W} : \ \omega^*(\lambda) > t\},$$

where $\omega^* : \mathfrak{a}^* \to \mathbb{R}$ is defined by

$$\omega^*(\lambda) = \min_{\alpha \in \Sigma_s} \langle \alpha, \lambda \rangle \qquad \forall \lambda \in \mathfrak{a}^*.$$

For each t in $\mathbb{R}^-$ the set $\mathbf{W}^t$ is an open neighbourhood of $\mathbf{W}^+$ that contains the origin. Thus, the tube $T_{\mathbf{W}^t} = \mathfrak{a}^* + i\mathbf{W}^t$ is a neighbourhood of the tube $T_{\mathbf{W}^+} = \mathfrak{a}^* + i\mathbf{W}^+$ in $\mathfrak{a}_{\mathbb{C}}^*$ that contains $\mathfrak{a}^* + i0$.

We denote by $\check{\mathbf{c}}$ the function $\check{\mathbf{c}}(\lambda) = \mathbf{c}(-\lambda)$ which is holomorphic in $T_{\mathbf{W}^t}$ for some negative t and satisfies the following estimate

$$|(\check{\mathbf{c}})^{-1}(\zeta)| \le C \prod_{\alpha \in \Sigma_0^+} \left(1 + |\zeta|\right)^{\sum_{\alpha \in \Sigma_0^+} m_\alpha/2} = C \left(1 + |\zeta|\right)^{(n-\ell)/2} \qquad \forall \zeta \in T_{\mathbf{W}^t}.$$

This, the analyticity of $(\check{\mathbf{c}})^{-1}$ on $T_{\mathbf{W}^t}$, and Cauchy's integral formula imply that for every multiindex I

(3.17)
$$\left|D^I(\check{\mathbf{c}})^{-1}(\zeta)\right| \le C \left(1 + |\zeta|\right)^{(n-\ell)/2} \qquad \forall \zeta \in T_{\mathbf{W}^t}.$$

Observe that there exists a constant d, and, for each positive integer N, another constant C such that

(3.18)
$$|D^I\Gamma_q(\zeta)| \le C \left(1 + |q|\right)^d \qquad \forall \zeta \in T_{\mathbf{W}^t} \quad \forall I : |I| \le N.$$

Indeed, the estimate of the derivatives is a consequence of Gangolli's estimate for Γ_q [14] and Cauchy's integral formula. The Harish-Chandra expansion is pointwise convergent in $\mathfrak{a}^+$ and uniformly convergent in $\mathfrak{a}^+ \setminus \mathfrak{s}_c$ for every $c > 0$.

By proceeding as in the proof of Step II in [31, Theorem 3.2], and using the Harish-Chandra expansion (3.15) of the spherical function φ_λ, we may write $c_G^{-1}\, |W|^{-1}\, \kappa_{2,k} = \kappa_{2,k}^{(0)} + \kappa_{2,k}^{(1)}$, where c_G is as in the inversion formula (2.4), $|W|$ denotes the cardinality of the Weyl group, and

$$\kappa_{2,k}^{(0)}(\exp H \cdot o) = e^{-\rho(H)} \int_{\mathfrak{a}^*} m_{2,k}(\lambda)\, \mathbf{c}(-\lambda)^{-1}\, e^{i\lambda(H)}\, d\lambda$$

$$\kappa_{2,k}^{(1)}(\exp H \cdot o) = \sum_{q \in \Lambda \setminus \{0\}} e^{-\rho(H)-q(H)} \int_{\mathfrak{a}^*} m_{2,k}(\lambda)\, \mathbf{c}(-\lambda)^{-1}\, \Gamma_q(\lambda)\, e^{i\lambda(H)}\, d\lambda.$$

To estimate $\kappa_{2,k}^{(1)}$ on $\left[B_1(o) \cup K \exp(\mathfrak{s}_1) \cdot o\right]^c$, first we move the contour of integration to the space $\mathfrak{a}^* + i\rho$ and obtain

(3.19)
$$\kappa_{2,k}^{(1)}(\exp H \cdot o)$$
$$= \sum_{q \in \Lambda \setminus \{0\}} e^{-2\rho(H) - q(H)}$$
$$\times \int_{\mathfrak{a}^*} m_{2,k}(\lambda + i\rho) \, \mathbf{c}(-\lambda + i\rho)^{-1} \, \Gamma_q(\lambda + i\rho) \, e^{i\lambda(H)} \, d\lambda.$$

We now estimate the absolute value of the integrand. We use the point-wise estimate (3.14) as an upper bound for $|m_{2,k}|$, and estimates (3.17) and (3.18) for $|\mathbf{c}^{-1}|$ and the coefficients $|\Gamma_q|$ of the Harish-Chandra expansion, respectively, and obtain

$$\left| \kappa_{2,k}^{(1)}(\exp H \cdot o) \right|$$
$$\leq C \, \|m\|_{(J;\tau)} \sum_{q \in \Lambda \setminus \{0\}} e^{-2\rho(H) - q(H)} \, (1 + |q|)^d \int_{\mathfrak{a}^*} e^{-\operatorname{Re} Q(\lambda)/2} \, d\lambda$$
$$\leq C \, \|m\|_{(J;\tau)} \sum_{q \in \Lambda \setminus \{0\}} e^{-2\rho(H) - q(H)} \, (1 + |q|)^d.$$

Notice that (see (3.12) for the definition of ω)

$$q(H) = \sum_{\alpha \in \Sigma_s} n_\alpha \, \alpha(H) \geq \omega(H) \sum_{\alpha \in \Sigma_s} n_\alpha = \omega(H) \, |q|.$$

This, and the fact that $\omega(H) \geq 1$ for every H in $\mathfrak{a}^+ \setminus \mathfrak{s}_1$, imply that

$$e^{-q(H)} \leq e^{-|q|\omega(H)} \leq e^{1 - |q| - \omega(H)}.$$

Therefore

$$\left| \kappa_{2,k}^{(1)}(\exp H \cdot o) \right| \leq C \, \|m\|_{(J;\tau)} \, e^{-2\rho(H) - \omega(H)} \sum_{q \in \Lambda \setminus \{0\}} (1 + |q|)^d \, e^{-|q|}$$
$$\leq C \, \|m\|_{(J;\tau)} \, e^{-2\rho(H) - \omega(H)} \qquad \forall H \in \mathfrak{a}^+ \setminus (\mathfrak{b}_1 \cup \mathfrak{s}_1).$$

Then we integrate in polar co-ordinates, use (2.2) to estimate the density function $\delta(H)$, and obtain

$$\int_{[B_1(o) \cup K \exp(\mathfrak{s}_1) \cdot o]^c} \left| \kappa_{2,k}^{(1)} \right| \, d\mu \leq C \, \|m\|_{(J;\tau)} \int_{\mathfrak{a}^+ \setminus (\mathfrak{b}_1 \cup \mathfrak{s}_1)} e^{-\omega(H)} \, dH$$
$$\leq C \, \|m\|_{(J;\tau)},$$

for the last integral is easily seen to be convergent.

It remains to estimate $\kappa_{2,k}^{(0)}$ on $\left[B_1(o) \cup K \exp(\mathfrak{s}_1) \cdot o\right]^c$. Much as before, we move the contour of integration from $\mathfrak{a}^*$ to $\mathfrak{a}^* + i\rho$, and obtain

$$\kappa_{2,k}^{(0)}(\exp H \cdot o) = e^{-2\rho(H)} \int_{\mathfrak{a}^*} m_{2,k}(\lambda + i\rho) \, \mathbf{c}(-\lambda + i\rho)^{-1} \, e^{i\lambda(H)} \, d\lambda.$$

Set $f(\lambda) = m_{2,k}(\lambda + i\rho)\, \mathbf{c}(-\lambda + i\rho)^{-1}$. We may decompose f as the sum of Θf and $(1 - \Theta)\, f$, where Θ is a smooth function of compact support in $\mathfrak{a}^*$, which is equal to 1 near the origin. Observe that $(1 - \Theta)\, f$ is in the Euclidean Sobolev space $H^J(\mathfrak{a}^*)$, with norm dominated by $C \left\| m \right\|_{(J;\tau)}$. Since $J \geq \lfloor \ell/2 \rfloor + 1$, its inverse Fourier transform is in $L^1(\mathfrak{a})$, by a celebrated result of Bernstein, and $\left\| \mathcal{F}^{-1}[(1 - \Theta)f] \right\|_{L^1(\mathfrak{a})} \leq C \left\| m \right\|_{(J;\tau)}$. As to Θf, a straightforward computation together with estimate (3.17) for the derivatives of $\mathbf{c}^{-1}$, and the assumption on m (in particular (3.4)), shows that there exists a constant C, independent of m, such that

$$\left| D^I(\Theta f)(\lambda) \right| \leq C \left\| m \right\|_{(J;\tau)} \mathcal{N}(\lambda)^{2k - 2\tau - d(I)}$$

for all I such that $|I| \leq J$. Set $\beta = \min\{1, 2k - 2\tau\}$. Observe that

$$J \geq \ell + 1 + \beta \quad \text{and} \quad \mathcal{N}(\lambda)^{2k - 2\tau} \leq C \mathcal{N}(\lambda)^\beta$$

on the support of Θ. Thus, Θf satisfies the assumptions of Lemma 3.5, whence

$$\left| \mathcal{F}^{-1}(\Theta f)(H) \right| \leq C \left\| m \right\|_{(J;\tau)} \left[1 + \mathcal{N}(H) \right]^{-\ell - 1 - \beta}.$$

By combining these estimates, we see that

$$\int_{\left[B_1(o) \cup K \exp(\mathfrak{s}_1) \cdot o \right]^c} \left| \kappa_{2,k}^{(0)} \right| \mathrm{d}\mu \leq \int_{[\mathfrak{b}_1 \cup \mathfrak{s}_1]^c} \left| \mathcal{F}^{-1}[(1 - \Theta)f](H) + \mathcal{F}^{-1}(\Theta f) \right| \mathrm{d}H$$

$$\leq C \left\| m \right\|_{(J;\tau)} \left[1 + \int_{\mathfrak{a}} (1 + \mathcal{N}(H))^{-\ell - 1 - \beta} \, \mathrm{d}H \right].$$

By integrating in anisotropic polar co-ordinates (see (3.2)), we obtain

$$\int_{\mathfrak{a}} (1 + \mathcal{N}(H))^{-\ell - 1 - \beta} \, \mathrm{d}H = c_\ell \int_0^\infty (1 + r)^{-\ell - 1 - \beta} \, r^\ell \, \mathrm{d}r < \infty,$$

because $\beta > 0$.

This concludes the proof of (iii), and of Theorem 3.4. $\qquad\square$

4. Unboundedness on $H^1(\mathbb{X})$ of imaginary powers

In this section we prove that if G is complex, then the imaginary powers $\mathcal{L}^{iu}$, $u \in \mathbb{R} \setminus \{0\}$, are unbounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$. Thus the endpoint result in Theorem 3.4(iii) is sharp. We restrict to the case where G is complex, for in this case we are able to obtain asymptotic estimates of the kernel of $\mathcal{L}^{-iu}$. However, we believe that Theorem 4.1 is true for any noncompact symmetric space. A key role in these estimate is played by the fact that the heat kernel is given by the following explicit formula [4]

$$h_t(x \cdot o) = \varphi_0(x \cdot o)\, (4\pi t)^{-n/2}\, \mathrm{e}^{-|\rho|^2 t - |x \cdot o|^2 / 4t} \qquad \forall x \cdot o \in \mathbb{X}.$$

By standard subordination to the heat kernel, the kernel κ_0^{2iu} of $\mathcal{L}^{-iu}$ is given by
(4.1)
$$\kappa_0^{2iu}(x \cdot o) = \frac{1}{\Gamma(iu)} \int_0^\infty t^{iu-1} \, h_t(x \cdot o) \, \mathrm{d}t$$
$$= \frac{\varphi_0(x \cdot o)}{(4\pi)^{n/2}\Gamma(iu)} \int_0^\infty t^{iu-n/2-1} \, \mathrm{e}^{-|\rho|^2 t - |x \cdot o|^2/4t} \, \mathrm{d}t$$
$$= C(n,u) \, \varphi_0(x \cdot o) \, |x \cdot o|^{iu-n/2} \int_0^\infty s^{iu-n/2-1} \, \mathrm{e}^{-(s+s^{-1})|\rho||x \cdot o|/2} \, \mathrm{d}s.$$

Theorem 4.1. *If G is complex and $u \in \mathbb{R} \setminus \{0\}$, then $\mathcal{L}^{-iu}$ does not map $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$.*

We need a couple of technical lemmata. It may be convenient to define the function Ψ on $\mathbb{X} \setminus \{o\}$ by
$$\Psi(x \cdot o) = |x \cdot o|^{-(n+1)/2} \, \varphi_0(x \cdot o) \, \mathrm{e}^{-|\rho||x \cdot o|}.$$

Lemma 4.2. *There exist constants C_0 and C, depending on u, such that*
$$\kappa_0^{2iu}(x \cdot o) \sim C_0 \, |x \cdot o|^{iu} \, \Psi(x \cdot o) \qquad as \, |x \cdot o| \, tends \, to \, \infty,$$
i.e., the ratio between the left and the right hand side tends to 1, and
$$|\nabla \kappa_0^{2iu}(x \cdot o)| \leq C \, \Psi(x \cdot o) \qquad \forall x \cdot o \in B_1(o)^c.$$

Proof. The estimate of κ_0^{2iu} follows from (4.1) and the estimate
$$\int_0^\infty s^{iu-n/2-1} \, \mathrm{e}^{-(s+s^{-1})|\rho||x \cdot o|/2} \, \mathrm{d}s \sim C \, |x \cdot o|^{-1/2} \, \mathrm{e}^{-|\rho||x \cdot o|}$$

as $|x \cdot o|$ tends to ∞, obtained by the Laplace method [10].
   To estimate $\nabla \kappa_0^{2iu}(x \cdot o)$, we differentiate (4.1) and observe that $\nabla \kappa_0^{2iu}(x \cdot o)$ may be written as the sum of three terms containing as factors $\nabla \varphi_0(x \cdot o)$, $\nabla |x \cdot o|$ and
$$\nabla |x \cdot o| \int_0^\infty s^{iu-n/2-1}(s + s^{-1}) \, \mathrm{e}^{-(s+s^{-1})|\rho||x \cdot o|/2} \, \mathrm{d}s.$$

The desired conclusion follows, since $\big| \nabla |x \cdot o| \big| = 1$ for $x \notin K$,
$$\big| \nabla \varphi_0(x \cdot o) \big| \leq C \, \varphi_0(x \cdot o) \qquad \forall x \cdot o \in B_1(o)^c,$$

and
$$\int_0^\infty s^{iu-n/2-1}(s + s^{-1}) \, \mathrm{e}^{-(s+s^{-1})|\rho||x \cdot o|/2} \, \mathrm{d}s \asymp |x \cdot o|^{-1/2} \, \mathrm{e}^{-|\rho||x \cdot o|},$$

i.e., the ratio between the absolute value of the left hand side and the right hand side is bounded and bounded away from 0, again by the Laplace method. $\qquad \square$

Lemma 4.3. *Furthermore, for each $\varepsilon > 0$ the following hold:*

(i) *There exists a positive number η_0 such that*

$$\sup_{B_\eta(y \cdot o)} \Psi \leq (1 + \varepsilon) \inf_{B_\eta(y \cdot o)} \Psi \qquad \forall \eta \leq \eta_0 \quad \forall y \cdot o \in B_2(o)^c.$$

(ii) *For each $R > 0$ there exists a neighbourhood U of the identity in K such that*

$$\Psi(a_R u a \cdot o) \leq (1 + \varepsilon) \Psi(a_R a \cdot o) \qquad \forall u \in U \quad \forall a \in \exp \mathfrak{b}_2^c.$$

Proof. It is straightforward to check that the proof of [30, Lemma 4.2] extends almost *verbatim* with Ψ in place of κ_0^σ. We omit the details. $\square$

Finally we prove Theorem 4.1.

Proof of Theorem 4.1. Fix $\varepsilon > 0$. We shall prove that $\mathcal{L}^{-iu} f$ is not in $L^1(\mathbb{X})$, where $f = b_\eta \mathbf{1}_{B_\eta(o)} - b_\eta \mathbf{1}_{B_\eta(a_R^{-1} \cdot o)}$, $b_\eta = |B_\eta(o)|^{-1}$, $\eta \leq \eta_0$, and η_0 is as in Lemma 4.3(i). By arguing much as in the proof of [30, Lemma 4.2], we see that

$$\left| f * \kappa_0^{2iu}(x \cdot o) \right|$$

$$\geq b_\eta \left| \int_{B_\eta(o)} \kappa_0^{2iu}(x^{-1} y \cdot o) \, d\mu(y \cdot o) \right| - b_\eta \left| \int_{B_\eta(o)} \kappa_0^{2iu}(x^{-1} a_R^{-1} y \cdot o) \, d\mu(y \cdot o) \right|.$$

Observe that the mean value theorem, and Lemmata 4.2 and 4.3 imply that

$$\left| b_\eta \int_{B_\eta(o)} \kappa_0^{2iu}(x^{-1} y \cdot o) \, d\mu(y \cdot o) - \kappa_0^{2iu}(x^{-1} \cdot o) \right| \leq \eta \sup_{B_\eta(x^{-1} \cdot o)} |\nabla \kappa_0^{2iu}|$$

$$\leq C \eta \sup_{B_\eta(x^{-1} \cdot o)} \Psi$$

$$\leq C \eta (1 + \varepsilon) \Psi(x^{-1} \cdot o),$$

so that for $|x \cdot o|$ large

$$\left| b_\eta \int_{B_\eta(o)} \kappa_0^{2iu}(x^{-1} y \cdot o) \, d\mu(y \cdot o) \right| \geq \left| \kappa_0^{2iu}(x^{-1} \cdot o) \right| - C \eta (1 + \varepsilon) \Psi(x^{-1} \cdot o)$$

$$\geq \left[\frac{C_0}{2} - C \eta (1 + \varepsilon) \right] \Psi(x \cdot o).$$

We choose η so small that $C \eta (1 + \varepsilon) \leq C_0/4$. Similarly, we may prove that

$$\left| b_\eta \int_{B_\eta(o)} \kappa_0^{2iu}(x^{-1} a_R^{-1} y \cdot o) \, d\mu(y \cdot o) \right| \leq \left[2C_0 + C \eta (1 + \varepsilon) \right] \Psi(a_R x \cdot o).$$

Altogether, these estimates and the choice of η imply that for $|x \cdot o|$ large enough

$$\left| f * \kappa_0^{2iu}(x \cdot o) \right| \geq \frac{C_0}{4} \Psi(x \cdot o) \left[1 - 9 \frac{\Psi(a_R x \cdot o)}{\Psi(x \cdot o)} \right].$$

The conclusion follows as in the proof of [30, Lemma 4.2]. $\square$

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