

Følner’s condition and expansion of Cayley graphs for group actions

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ABSTRACT. Suppose G is a group acting on a set X . If G is finitely generated and A and B are two finite symmetric generating sets, then we show that the Cayley graph $\text{Cay}_A(G, X)$ is amenable if and only if $\text{Cay}_B(G, X)$ is amenable. We prove that (G, X) satisfies the Følner’s condition if and only if for every finitely generated subgroup H of G , $\text{Cay}(H, X)$ is amenable. If G is finitely generated, we show that (G, X) and $\text{Cay}(G, X)$ have the same Følner’s sequences.

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1. Introduction and preliminaries

Følner’s condition for group actions and its relation to the existence of invariant means have been studied by Rosenblatt [17]. Among the extensive literature on Følner’s condition, invariant means and their applications we may mention Følner [6], Namioka [15], Lau [10, 11], Eymard [5], Rosenblatt [18], Lau and Takahashi [12], Stokke [21], Dales and Polyakov [2], and Willson [22]. Rosenblatt’s results have been used by McMullen [14] in the study of covering spaces. A covering $p: Y \rightarrow X$ of connected manifolds is called amenable if there exists an invariant mean for the canonical action of the fundamental group $\pi_1(X)$ on the coset space $\pi_1(X)/\pi_1(Y)$. This amenability is then characterized (among other things) in terms of the expansion of the coset graphs of $K/(K \cap \pi_1(Y))$, for finitely generated subgroups K of $\pi_1(X)$ ([14, Proposition 3.1]).

In this paper we show that a similar characterization of Følner’s condition exists in the more general setting where a group G acts on a set X . Since we

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do not require the transitivity of the group action, our results are applicable to spaces more general than coset spaces. Our results will also extend some known results in the literature for particular case that $X = G$, and G acts on itself by group multiplication.

Before summarizing the results of this paper, let us recall some terminology. If $\mathcal{G} = (V, E)$ is a graph (finite or infinite) and F is a subset of V , then the *border* of F is the set of all vertices in $V - F$ that are connected to F by an edge. We denote the border of F by $b(F)$. There exist several definitions for the expansion of a graph in the literature. In this paper we follow McMullen [14, p. 98] to define the *expansion* of \mathcal{G} by

$$(1) \quad \gamma(\mathcal{G}) = \inf \left\{ \frac{|b(F)|}{|F|} : F \subset V, 0 < |F| < \infty \right\},$$

where $|\cdot|$ denotes the number of elements. (For infinite graphs, this definition is consistent with the one given by Bekka et al. [1, p. 254]. For infinite k -regular graphs, it is easily seen that $\gamma(\mathcal{G}) \leq h(\mathcal{G}) \leq k\gamma(\mathcal{G})$, where $h(\mathcal{G})$ is the Cheeger constant. For more on Cheeger constant, see Lubotzky [13], Davidoff et al. [3].) A graph \mathcal{G} is called *amenable* if $\gamma(\mathcal{G}) = 0$ (McMullen [14]). Thus a graph is amenable if and only if there exists a sequence $(F_n)_n$ of finite subsets of V such that $\lim_{n \rightarrow \infty} |b(F_n)|/|F_n| = 0$. Such a sequence is called a *Følner's sequence* of \mathcal{G} . Note that all finite graphs are amenable.

Let G be a group and X be a nonempty set. We say that G acts on X if there exists a mapping $G \times X \rightarrow X$, $(s, x) \mapsto s \cdot x$, such that (i) $e \cdot x = x$, and (ii) $s \cdot (t \cdot x) = (st) \cdot x$, for all $x \in X$ and $s, t \in G$. Let $\ell_{\mathbb{R}}^{\infty}(X)$ denote the Banach space of all bounded real functions on X . A *mean* for (G, X) is a positive linear functional $m \in \ell_{\mathbb{R}}^{\infty}(X)^*$ with norm 1. If m satisfies the condition $m(L_s f) = m(f)$ for all $s \in G$, $f \in \ell_{\mathbb{R}}^{\infty}(X)$ (where $(L_s f)(x) = f(s \cdot x)$, $x \in X$), then m is called an *invariant mean* for (G, X) . In this paper we shall need the following characterizations of the existence of invariant means, due to Rosenblatt [17, Theorem 4.10]. (We shall state only a special case of Rosenblatt's result, which is nonetheless sufficient for our purposes in this paper.) In the following, Δ denotes the symmetric difference of sets.

Theorem 1.1 (Rosenblatt). *Let G be a group acting on a set X . The following statements are equivalent.*

- (i) (G, X) has an invariant mean.
- (ii) For every $\epsilon > 0$ and every finite subset A of G , there exists a finite set $F \subset X$ such that for all $a \in A$,

$$(2) \quad \frac{|a \cdot F \Delta F|}{|F|} \leq \epsilon \quad (\text{Følner's condition}).$$

- (iii) There exists a net $(F_{\alpha})_{\alpha}$ of finite subsets of X such that for all $s \in G$,

$$(3) \quad \lim_{\alpha} \frac{|s \cdot F_{\alpha} \Delta F_{\alpha}|}{|F_{\alpha}|} = 0 \quad (\text{Følner's net}).$$

We call (G, X) *amenable* if any one of the above equivalent conditions holds. In the special case that $X = G$ and the action of G is the group multiplication, amenability of (G, G) is the same as the amenability of the group G (Day [4], Runde [19]). We remark that if G is finitely generated, then Følner's net can be replaced by a Følner's sequence in the above theorem.

Suppose G acts on a set X and G is finitely generated with $A = A^{-1}$ a finite symmetric set of generators of G . The *Cayley graph* $\text{Cay}_A(G, X)$ (also known as the *Schreier graph*) is defined as follows: the vertices of the graph are the points in X and two vertices $x, y \in X$ are connected by an edge if $a \cdot x = y$ for some $a \in A$. (The assumption that A is symmetric ensures that the graph is undirected.) Although Cayley graphs depend on the generating sets A , the results in this paper are in effect independent of a particular choice of A (see Theorem 2.5). For this reason we shall usually drop the subscript A from the notation and denote a Cayley graph by $\text{Cay}(G, X)$. In the special case that $X = G$ and the action of G on itself is the group multiplication, $\text{Cay}(G, G)$ will be called the *Cayley graph of G* and denoted by $\text{Cay}(G)$. (We should remark that some authors use a different convention to define $\text{Cay}(G)$: two vertices $x, y \in G$ are connected by an edge if $xa = y$ for some $a \in A$. The convention used in this paper is consistent with the one used for group actions.)

The main results of this paper are as follows. In Theorem 2.2 we show that the expansion of a graph is determined by the expansions of its components. More precisely, if $(\mathcal{G}_i)_{i \in I}$ are the components of a graph \mathcal{G} , then

$$\gamma(\mathcal{G}) = \inf_{i \in I} \gamma(\mathcal{G}_i).$$

Suppose G is a finitely generated group acting on a set X and A and B are two finite symmetric generating sets of G . In Theorem 2.5 we show that $\text{Cay}_A(G, X)$ is amenable if and only if $\text{Cay}_B(G, X)$ is amenable (in other words, the amenability of a Cayley graph is independent of its generating sets).

In Section 3 we show that if G acts on a set X , then the amenability of (G, X) can be characterized in terms of the amenability of $\text{Cay}(H, X)$, where H is a finitely generated subgroup of G (Theorem 3.2). As an interesting consequence, it follows that if G is an amenable group acting on a set X , then $\text{Cay}(H, X)$ is amenable for every finitely generated subgroup H of G (Corollary 3.5). For finitely generated groups G , the result in Theorem 3.2 is further strengthened in Theorem 3.7, where we show that (G, X) and $\text{Cay}(G, X)$ have the same Følner's sequences. As an example, we give an explicit construction of a Følner's sequence for both \mathbb{Z}^n and $\text{Cay}(\mathbb{Z}^n)$ (Example 3.8).

2. The expansion $\gamma(\mathcal{G})$

Recall that a graph $\mathcal{G} = (V, E)$ is *connected* if every two vertices in V can be connected by a path. A *component* of \mathcal{G} is a connected subgraph in

which no vertex is connected to a vertex outside the subgraph. To compute $\gamma(\mathcal{G})$ we only need to consider the components of \mathcal{G} . To prove this fact we need the following:

Lemma 2.1. *If $x_i > 0$, $x'_i \geq 0$, for $i = 1, \dots, n$, then*

$$\frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_n} \geq \min \left\{ \frac{x'_1}{x_1}, \dots, \frac{x'_n}{x_n} \right\}.$$

Proof. This follows by a simple induction. The case of $n = 2$ is easy to check. For the general case we write

$$\frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_n} \geq \min \left\{ \frac{x'_1 + \dots + x'_{n-1}}{x_1 + \dots + x_{n-1}}, \frac{x'_n}{x_n} \right\} \geq \min \left\{ \frac{x'_1}{x_1}, \dots, \frac{x'_n}{x_n} \right\}. \quad \square$$

Theorem 2.2. *If \mathcal{G} is a graph and $\{\mathcal{G}_i\}_{i \in I}$ are its components, then*

$$\gamma(\mathcal{G}) = \inf_{i \in I} \gamma(\mathcal{G}_i).$$

Proof. Let $i \in I$ and V_i be the set of vertices of \mathcal{G}_i . Since \mathcal{G}_i is a component of \mathcal{G} , if F is a finite subset of V_i , then the border points of F in \mathcal{G} (if any) are all inside V_i , and from this it follows that $\gamma(\mathcal{G}) \leq \gamma(\mathcal{G}_i)$. Thus we have $\gamma(\mathcal{G}) \leq \inf_{i \in I} \gamma(\mathcal{G}_i)$.

To prove the reverse inequality, let F be a finite set of vertices of \mathcal{G} . Then there are indices i_1, \dots, i_n , such that F has nonempty intersection with each V_{i_j} and $F \subset V_{i_1} \cup \dots \cup V_{i_n}$. Let x_j be the number of elements of $F \cap V_{i_j}$, and x'_j be the number of its bordering points in V_{i_j} . If $b(F)$ is the border of F in \mathcal{G} , then $|b(F)| = x'_1 + \dots + x'_n$, and we have

$$\frac{|b(F)|}{|F|} = \frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_n} \geq \min \left\{ \frac{x'_1}{x_1}, \dots, \frac{x'_n}{x_n} \right\} \geq \inf_{i \in I} \gamma(\mathcal{G}_i).$$

The inequality $\gamma(\mathcal{G}) \geq \inf_{i \in I} \gamma(\mathcal{G}_i)$ follows. \square

The following corollary will be used in the next section (cf. Corollary 3.3).

Corollary 2.3. *If H is a finitely generated subgroup of a group G , then the graphs $\text{Cay}(H, G)$ and $\text{Cay}(H)$ have the same expansion.*

Proof. The components of the graph $\text{Cay}(H, G)$ are of the form $\text{Cay}(H, Hz)$, where Hz ($z \in G$) are the right cosets of H in G . Let F be a subset of H , $b(Fz)$ be the border of Fz in $\text{Cay}(H, Hz)$, and $b(F)$ be the border of F in $\text{Cay}(H)$. Then it is easy to check that $b(Fz) = b(F)z$. It follows that the graphs $\text{Cay}(H, Hz)$ and $\text{Cay}(H)$ have the same expansion for every $z \in G$. Now the corollary follows from Theorem 2.2. \square

To prove our next result, let us state the following:

Lemma 2.4. *Let G be a finitely generated group and A be a finite symmetric generating subset of G . Let G act on a set X and F be a nonempty subset*

of X . Suppose that $x \in F$ and $a_1, \dots, a_n \in A$ and $y = a_1 a_2 \cdots a_n \cdot x$. Then

$$(4) \quad y \in \bigcup_{j=0}^n b^{(j)}(F),$$

where $b^{(0)}(F) = F$ and $b^{(j)}(F) = b(b(\cdots(b(F))\cdots))$, j -times.

Proof. We use induction on n . The case $n = 1$ follows immediately from the definition of $b(F)$. Suppose, as induction hypothesis, that

$$(5) \quad a_2 a_3 \cdots a_n \cdot x \in \bigcup_{j=0}^{n-1} b^{(j)}(F).$$

To prove (4), assume that

$$(6) \quad y \notin \bigcup_{j=0}^{n-1} b^{(j)}(F),$$

we must then show that $y \in b^{(n)}(F)$. For every $0 \leq j \leq n - 2$, (6) implies that $y \notin b^{(j)}(F) \cup b^{(j+1)}(F)$, and hence $a_2 a_3 \cdots a_n \cdot x \notin b^{(j)}(F)$. Now it follows from (5) that $a_2 a_3 \cdots a_n \cdot x \in b^{(n-1)}(F)$. We know from (6) that $y \notin b^{(n-1)}(F)$, thus we must have $y \in b(b^{(n-1)}(F)) = b^{(n)}(F)$, as we wanted to show. \square

The following theorem states that the amenability of a Cayley graph (associated to a group action) is independent of its generating set (for related results see Soardi [20, Theorem 7.34], Bekka et al. [1, Example 3.6.2(ii)], and Grigorchuk [7, p. 5]).

Theorem 2.5. *Let G be a finitely generated group and A and B be two symmetric sets of generators of G . Suppose G acts on a set X . Then $\text{Cay}_A(G, X)$ is amenable if and only if $\text{Cay}_B(G, X)$ is amenable.*

Proof. Expressing each $a \in A$ as a reduced word in B , let M be the length of the longest such words. Similarly, by expressing each $b \in B$ as a reduced word in A , let us define N to be the length of the longest such words.

Let F be a finite subset of X and $b_A(F)$ and $b_B(F)$ denote the border of F in $\text{Cay}_A(G, X)$ and $\text{Cay}_B(G, X)$, respectively. Let $y \in b_B(F)$, so that $y \notin F$ but $y = b \cdot x$ for some $b \in B$, $x \in F$. If $b = a_1 a_2 \cdots a_n$ is a representation of b as a reduced word in A , then $n \leq N$, $y = a_1 a_2 \cdots a_n \cdot x$, and by Lemma 2.4, we have $y \in \bigcup_{j=1}^n b_A^{(j)}(F)$ (note that $y \notin F = b_A^{(0)}(F)$). Since the above holds for every $y \in b_B(F)$, we obtain $b_B(F) \subset \bigcup_{j=1}^N b_A^{(j)}(F)$, and consequently

$$|b_B(F)| \leq \sum_{j=1}^N |b_A^{(j)}(F)|.$$

It follows from the definition of the border that $|b_A(b_A(F))| \leq |A||b_A(F)|$, and by repeated applications of this inequality we get

$$|b_A^{(j)}(F)| \leq |A|^{j-1}|b_A(F)|, \quad 1 \leq j \leq N.$$

Hence

$$|b_B(F)| \leq \left(\sum_{j=0}^{N-1} |A|^j \right) |b_A(F)|.$$

By a similar argument, changing the roles of A and B , we can obtain

$$|b_A(F)| \leq \left(\sum_{j=0}^{M-1} |B|^j \right) |b_B(F)|.$$

If we write $C_1 = (\sum_{j=0}^{M-1} |B|^j)^{-1}$ and $C_2 = \sum_{j=0}^{N-1} |A|^j$, then clearly both C_1 and C_2 are nonzero, and the last two inequalities can be written as

$$(7) \quad C_1|b_A(F)| \leq |b_B(F)| \leq C_2|b_A(F)|.$$

If we denote the expansion of $\text{Cay}_A(G, X)$ by γ_A and the expansion of $\text{Cay}_B(G, X)$ by γ_B , then it follows from (7) that $C_1\gamma_A \leq \gamma_B \leq C_2\gamma_A$. The claim of the theorem follows immediately. \square

3. Amenable group actions

We begin this section with a result estimating $|b(F)|$. For related results in the case that G acts on itself see Følner [6, Theorem, p. 245], and Bekka et al. [1, Corollary G.5.6].

Lemma 3.1. *Let G be a finitely generated group and A be a finite symmetric set of generators of G . Suppose G acts on a set X and $\text{Cay}(G, X)$ is the corresponding Cayley graph. Then for every nonempty finite subset F of X ,*

$$(8) \quad \frac{1}{2|A|} \sum_{a \in A} |a \cdot F \Delta F| \leq |b(F)| \leq \frac{1}{2} \sum_{a \in A} |a \cdot F \Delta F|.$$

Proof. For each $a \in A$,

$$(9) \quad \begin{aligned} |a \cdot F \Delta F| &= |a \cdot F - F| + |F - a \cdot F| \\ &= |a \cdot F - F| + |a \cdot (a^{-1} \cdot F - F)| \\ &= |a \cdot F - F| + |a^{-1} \cdot F - F|. \end{aligned}$$

Since $\bigcup_{a \in A} a \cdot F$ consists of all vertices that are adjacent to vertices in F , we have $\bigcup_{a \in A} (a \cdot F - F) = b(F)$. Using (9) and the fact that $A^{-1} = A$, we obtain

$$|b(F)| \leq \sum_{a \in A} |a \cdot F - F| = \frac{1}{2} \sum_{a \in A} |a \cdot F \Delta F|.$$

To prove the first inequality in (8), we write

$$\sum_{a \in A} |a \cdot F \Delta F| = 2 \sum_{a \in A} |a \cdot F - F| \leq 2|A| \max_{a \in A} |a \cdot F - F| \leq 2|A| |b(F)|. \quad \square$$

Theorem 3.2. *Let G be a group acting on a set X . Then (G, X) is amenable if and only if for every finitely generated subgroup H of G , $\text{Cay}(H, X)$ is amenable.*

Proof. If (G, X) is amenable, then (H, X) is amenable since Følner's condition (2) for (G, X) clearly implies the Følner's condition for (H, X) . Now let A be a finite symmetric generating set for H . Følner's condition for (H, X) implies that for a given $\epsilon > 0$ there exists a finite set $F \subset X$ such that

$$\frac{|a \cdot F \Delta F|}{|F|} \leq \frac{\epsilon}{|A|} \quad (a \in A).$$

It follows from (8) that

$$\frac{|b(F)|}{|F|} \leq \frac{1}{2} \sum_{a \in A} \frac{|a \cdot F \Delta F|}{|F|} \leq \frac{1}{2} \sum_{a \in A} \frac{\epsilon}{|A|} = \frac{\epsilon}{2},$$

thus $\text{Cay}(H, X)$ is amenable.

To prove the converse, suppose $\text{Cay}(H, X)$ is amenable for every finitely generated subgroup H of G . We will show that the Følner's condition holds for (G, X) . Let $\epsilon > 0$ and A be a finite subset of G . By enlarging A if necessary, we may assume that A is symmetric. Let H be the subgroup of G generated by A . By assumption $\text{Cay}(H, X)$ is amenable, and hence there exists a finite set $F \subset X$ with the property that

$$\frac{|b(F)|}{|F|} \leq \frac{\epsilon}{2|A|}.$$

Then using (8), for each $a \in A$,

$$\frac{|a \cdot F \Delta F|}{|F|} \leq 2|A| \frac{|b(F)|}{|F|} \leq 2|A| \frac{\epsilon}{2|A|} = \epsilon.$$

Thus Følner's condition holds and (G, X) is amenable. □

By applying Theorem 3.2 to the special case that $X = G$ and using Corollary 2.3, we obtain the following interesting result:

Corollary 3.3. *A group G is amenable if and only if $\text{Cay}(H)$ is amenable for every finitely generated subgroup H of G .*

Example 3.4. Let F_n ($n \geq 2$) be the free nonabelian group on n generators. This group is nonamenable (Paterson [16]) and hence by Corollary 3.3, $\text{Cay}(F_n)$ has nonzero expansion. It is not difficult to verify that $\text{Cay}(F_n)$ is a $2n$ -regular infinite tree, i.e., a connected, infinite, acyclic graph in which each vertex has degree $2n$. As a result, $\text{Cay}(F_n)$ has expansion $\gamma = 2n - 2$ (cf. McMullen [14, p. 98]). □

In view of Theorem 3.2, it is interesting to note that if G is amenable then so is (G, X) whenever G acts on X . To see this, let $x_0 \in X$ be fixed and for each $f \in \ell_{\mathbb{R}}^{\infty}(X)$ define $\tilde{f} \in \ell_{\mathbb{R}}^{\infty}(G)$ by $\tilde{f}(s) = f(s \cdot x_0)$ ($s \in G$). If m is a left invariant mean on G , then $m' \in \ell_{\mathbb{R}}^{\infty}(X)^*$ defined by $m'(f) = m(\tilde{f})$ is an invariant mean for (G, X) . It is possible, however, that G is a nonamenable group acting on a set X such that (G, X) is amenable. For example, we may take $G = F_2$ the free nonabelian group on two generators, H a subgroup of finite index, and $X = G/H$ the space of left cosets which is equipped with the canonical action of G on G/H ; in that case $1_X/|X|$ is an invariant mean for (G, X) .

We may now state the following:

Corollary 3.5. *If G is an amenable group acting on X , then $\text{Cay}(H, X)$ is amenable for every finitely generated subgroup H of G .*

In preparation for our next theorem, we have:

Lemma 3.6. *Let G be a finitely generated group and A be a finite symmetric set of generators of G . Suppose G acts on a set X and F is a finite subset of X .*

(i) *For every $a \in A$ and $s \in G$,*

$$(10) \quad as \cdot F \Delta F \subset (as \cdot F \Delta s \cdot F) \cup (s \cdot F \Delta F).$$

(ii) *For every $a \in A$,*

$$(11) \quad |b(a \cdot F)| \leq (3 + |A|)|b(F)|.$$

Proof. (i) The inclusions

$$as \cdot F - F \subset (as \cdot F - s \cdot F) \cup (s \cdot F - F),$$

and

$$F - as \cdot F \subset (F - s \cdot F) \cup (s \cdot F - as \cdot F),$$

imply directly that

$$(as \cdot F - F) \cup (F - as \cdot F) \subset (as \cdot F \Delta s \cdot F) \cup (s \cdot F \Delta F),$$

which proves (10).

(ii) It follows from the equality $a \cdot F = (a \cdot F \cap F) \cup (a \cdot F \cap F^c)$ that

$$(12) \quad b(a \cdot F) \subset b(a \cdot F \cap F) \cup b(a \cdot F \cap F^c).$$

Next we estimate the sizes of the two sets on the right side of (12). It is easy to check that

$$b(a \cdot F \cap F) \subset b(F) \cup (F - a \cdot F) = b(F) \cup a \cdot (a^{-1} \cdot F - F).$$

From this it follows that

$$(13) \quad |b(a \cdot F \cap F)| \leq |b(F)| + |a^{-1} \cdot F - F| \leq 2|b(F)|.$$

Since $a \cdot F \cap F^c \subset b(F)$ it follows that

$$b(a \cdot F \cap F^c) \subset b(b(F)) \cup b(F),$$

and hence

$$(14) \quad |b(a \cdot F \cap F^c)| \leq |b(b(F))| + |b(F)| \leq |A||b(F)| + |b(F)| = (|A| + 1)|b(F)|.$$

Combining (12), (13) and (14), we get $|b(a \cdot F)| \leq (3 + |A|)|b(F)|$. □

For the case of finitely generated groups, the following theorem improves the result in Theorem 3.2 by showing that (G, X) and $\text{Cay}(G, X)$ have the same Følner's sequences.

Theorem 3.7. *Let G be a finitely generated group acting on a set X and A be a finite symmetric generating set of G . A sequence $(F_n)_n$ of finite subsets of X is a Følner's sequence of (G, X) if and only if it is a Følner's sequence of $\text{Cay}(G, X)$. In particular, (G, X) is amenable if and only if $\text{Cay}(G, X)$ is amenable.*

Proof. First we prove the 'only if' part. Let $(F_n)_n$ be a Følner's sequence for (G, X) . Using (8) and (3), we have

$$\lim_{n \rightarrow \infty} \frac{|b(F_n)|}{|F_n|} \leq \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{a \in A} \frac{|a \cdot F_n \Delta F_n|}{|F_n|} = \frac{1}{2} \sum_{a \in A} \lim_{n \rightarrow \infty} \frac{|a \cdot F_n \Delta F_n|}{|F_n|} = 0.$$

Thus $(F_n)_n$ is a Følner's sequence for $\text{Cay}(G, X)$.

To prove the 'if' part, suppose $(F_n)_n$ is a Følner's sequence for $\text{Cay}(G, X)$. Note that for each $a \in A$, (8) implies that $|a \cdot F_n \Delta F_n| \leq 2|A||b(F_n)|$, from which it follows that

$$\lim_{n \rightarrow \infty} \frac{|a \cdot F_n \Delta F_n|}{|F_n|} \leq 2|A| \lim_{n \rightarrow \infty} \frac{|b(F_n)|}{|F_n|} = 0.$$

It remains to show that the above limit holds if $a \in A$ is replaced by an arbitrary $s \in G$. Let $s = a_1 a_2 \cdots a_k$ ($a_i \in A$) be an arbitrary but fixed element of G . Let also $s_i = a_i a_{i+1} \cdots a_k$, for $i = 1, \dots, k$, so that $s_1 = s$ and $s_k = a_k$. Put $s_{k+1} = e$. Then for each $n \in \mathbb{N}$, using (10) repeatedly, we can write

$$(15) \quad s \cdot F_n \Delta F_n \subset \sum_{i=1}^k (s_i \cdot F_n \Delta s_{i+1} \cdot F_n).$$

Furthermore, by letting $C = 3 + |A|$, and using (8) and (11), we get

$$(16) \quad |s_i \cdot F_n \Delta s_{i+1} \cdot F_n| \leq \sum_{a \in A} |as_{i+1} \cdot F_n \Delta s_{i+1} F_n| \leq 2|A||b(s_{i+1} \cdot F_n)| \leq 2|A|C^{k-i}|b(F_n)|.$$

It follows from (15) and (16) that

$$|s \cdot F_n \Delta F_n| \leq \sum_{i=1}^k |s_i \cdot F_n \Delta s_{i+1} \cdot F_n| \leq 2|A| \sum_{i=1}^k C^{k-i} |b(F_n)|.$$

If we let

$$R = 2|A| \sum_{i=1}^k C^{k-i} = \frac{2|A|(C^k - 1)}{C - 1},$$

then for each $n \in \mathbb{N}$, we obtain

$$|s \cdot F_n \Delta F_n| \leq R|b(F_n)|,$$

where R is independent of n . Thus

$$\lim_{n \rightarrow \infty} \frac{|s \cdot F_n \Delta F_n|}{|F_n|} \leq R \cdot \lim_{n \rightarrow \infty} \frac{|b(F_n)|}{|F_n|} = 0,$$

which completes the proof that $(F_n)_n$ is a Følner's sequence for (G, X) . \square

An interesting application of the above theorem is that the 'geometry' of the graph $\text{Cay}(G, X)$ can be used in finding a Følner's sequence for (G, X) . This is illustrated in the following example.

Example 3.8. Consider the abelian group \mathbb{Z}^n , generated by

$$A = \{\pm e_i : 1 \leq i \leq n\},$$

where e_i is the n -tuple with 1 in the i th coordinate and 0 elsewhere. The Cayley graph $\text{Cay}(\mathbb{Z}^n)$ is the infinite lattice in \mathbb{R}^n whose vertices are the points in \mathbb{Z}^n and whose edges are line segments of unit length parallel to the axes, joining the vertices. By Corollary 3.3, $\text{Cay}(\mathbb{Z}^n)$ is amenable. To construct a Følner's sequence, let F_r be the set of vertices of this graph that are on or inside the closed ball B_r in \mathbb{R}^n of radius $r > 0$ and center 0. Let $N(r) = |F_r|$ be the number of lattice points that belong to B_r . We recall that if $|B_r|$ is the volume (i.e., the n -dimensional Lebesgue measure) of B_r , then

$$|B_r| = |B_1|r^n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}r^n.$$

We can find estimates of $N(r)$ with the help of a classical argument due to Gauss (cf. Hardy and Wright [8, pp. 270–271], de la Harp [9, pp. 5–6]). Each $v = (m_1, \dots, m_n) \in \mathbb{Z}^n$ uniquely identifies a unit cell

$$S_v = [m_1 - 1, m_1] \times \cdots \times [m_n - 1, m_n]$$

in \mathbb{R}^n which has v as its upper-right corner. If $v \in B_r$, then $S_v \subset B_{r+\sqrt{n}}$, and hence

$$N(r) \leq |B_{r+\sqrt{n}}| = |B_1|(r + \sqrt{n})^n.$$

Similarly, if $S_v \cap B_{r-\sqrt{n}} \neq \emptyset$, ($r > \sqrt{n}$), then $S_v \subset B_r$, and hence

$$N(r) \geq |B_{r-\sqrt{n}}| = |B_1|(r - \sqrt{n})^n.$$

Thus

$$(17) \quad |B_1|(r - \sqrt{n})^n \leq N(r) \leq |B_1|(r + \sqrt{n})^n,$$

from which it follows that

$$N(r) = |B_1|r^n + O(r^{n-1}).$$

Next we estimate $|b(F_r)|$. If $w \in b(F_r)$, then $w \notin B_r$, but w is connected by an edge to some vertex $v \in B_r$. Thus $w \in B_{r+\sqrt{n}}$. Then using (17),

$$|b(F_r)| \leq N(r + \sqrt{n}) - N(r) \leq |B_1|(r + 2\sqrt{n})^n - |B_1|(r - \sqrt{n})^n,$$

from which it follows that $|b(F_r)| = O(r^{n-1})$. Therefore

$$\lim_{r \rightarrow \infty} \frac{|b(F_r)|}{|F_r|} = \lim_{r \rightarrow \infty} \frac{O(r^{n-1})}{|B_1|r^n + O(r^{n-1})} = 0.$$

It follows that $(F_r)_{r \in \mathbb{N}}$ is a Følner's sequence for both \mathbb{Z}^n and $\text{Cay}(\mathbb{Z}^n)$. \square

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