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Compactness of Hankel operators with conjugate holomorphic symbols on complete Reinhardt domains in $\mathbb{C}^2$

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Abstract. In this paper we characterize compact Hankel operators with conjugate holomorphic symbols on the Bergman space of bounded convex Reinhardt domains in $\mathbb{C}^2$. We also characterize compactness of Hankel operators with conjugate holomorphic symbols on smooth bounded pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$.

Contents

1. Introduction 1265
2. Preliminary lemmas 1267
3. Proof of Theorem 1 1270
4. Proof of Theorem 2 1272
References 1272

1. Introduction

We assume $\Omega \subset \mathbb{C}^2$ is a bounded convex Reinhardt domain. We denote the Bergman space with the standard Lebesgue measure on Ω as $A^2(\Omega)$. Recall that the Bergman space $A^2(\Omega)$ is the space of holomorphic functions on Ω that are square integrable on Ω under the standard Lebesgue measure. The Bergman space is a closed subspace of $L^2(\Omega)$. Therefore there exists an orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$ called the Bergman projection. The Hankel operator with symbol ϕ is defined as $H_\phi g = (I - P)(\phi g)$ for all $g \in A^2(\Omega)$. If $\phi \in L^\infty(\Omega)$, then H_ϕ is a bounded operator, however, the converse is not necessarily true. In one complex variable on the unit disk, Axler in [1] showed that the Hankel operator with conjugate holomorphic symbol ϕ is bounded if and only if $\overline{\phi}$ is in the Bloch space. There are unbounded, holomorphic functions in the Bloch space, as it only specifies a

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growth rate of the derivative of the function near the boundary of the disk. Namely, an analytic function ϕ is in the Bloch space if

$$\sup\left\{(1-|z|^2)|\phi'(z)| : z \in \mathbb{D}\right\} < \infty.$$

Let $h \in A^2(\Omega)$ so that the Hankel operator $H_{\overline{h}}$ is compact on $A^2(\Omega)$. The Hankel operator with an $L^2(\Omega)$ symbol may only be densely defined, since the product of L^2 functions may not be in L^2. However, if compactness of the Hankel operator is also assumed, then the Hankel operator with an L^2 symbol is defined on all of $A^2(\Omega)$.

We wish to use the geometry of the boundary of Ω to give conditions on h. For example, if Ω is the bidisk, Le in [5, Corollary 1] shows that if $h \in A^2(\mathbb{D}^2)$ such that $H_{\overline{h}}$ is compact on $A^2(\mathbb{D}^2)$ then $h \equiv c$ for some $c \in \mathbb{C}$. In one variable, Axler in [1] showed that $H_{\overline{g}}$ is compact on $A^2(\mathbb{D})$ if and only if g is in the little Bloch space. That is, $\lim_{|z|\to 1^-}(1-|z|^2)|g'(z)| = 0$. If the symbol h is smooth up to the boundary of a smooth bounded convex domain in $\mathbb{C}^2$, Čučković and Şahutoğlu in [3] showed that Hankel operator H_h is compact if and only if h is holomorphic along analytic disks in the boundary of the domain.

In this paper we will use the following notation.

$$\begin{aligned} S_t &= \{z \in \mathbb{C} : |z| = t\}, \\ \mathbb{T}^2 = S_1 \times S_1 &= \{z \in \mathbb{C} : |z| = 1\} \times \{w \in \mathbb{C} : |w| = 1\}, \\ \mathbb{D}_r &= \{z \in \mathbb{C} : |z| < r\} \end{aligned}$$

for any $r, t > 0$. If $r = 1$ we write

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

We say $\Delta \subset b\Omega$ is an analytic disk if there exists a function

$$h = (h_1, h_2) : \mathbb{D} \to b\Omega$$

so that each component function is holomorphic on $\mathbb{D}$ and the image

$$h(\mathbb{D}) = \Delta.$$

An analytic disk is said to be trivial if it is degenerate (that is, $\Delta = (c_1, c_2)$ for some constants c_1 and c_2).

In [2] we considered bounded convex Reinhardt domains in $\mathbb{C}^2$. We characterized nontrivial analytic disks in the boundary of such domains.

We defined

$$\Gamma_\Omega = \overline{\bigcup\{\phi(\mathbb{D}) : \phi : \mathbb{D} \to b\Omega \text{ are holomorphic, nontrivial}\}}$$

and showed that

$$\Gamma_\Omega = \Gamma_1 \cup \Gamma_2$$

where either $\Gamma_1 = \emptyset$ or

$$\Gamma_1 = \overline{\mathbb{D}}_{r_1} \times S_{s_1}$$

and likewise either $\Gamma_2 = \emptyset$ or

$$\Gamma_2 = S_{s_2} \times \overline{\mathbb{D}}_{r_2}$$

for some $r_1, r_2, s_1, s_2 > 0$.

Remark 1. We only consider domains in $\mathbb{C}^2$ as opposed to domains in $\mathbb{C}^n$ for $n \geq 3$ because a full geometric characterization of analytic structure in higher dimensions is unknown.

The main results are the following theorems.

Theorem 1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Let $f \in A^2(\Omega)$ so that $H_{\overline{f}}$ is compact on $A^2(\Omega)$. If $\Gamma_1 \neq \emptyset$, then f is a function of z_2 alone. If $\Gamma_2 \neq \emptyset$, then f is a function of z_1 alone.*

Corollary 1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Suppose $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. Let $f \in A^2(\Omega)$ so that $H_{\overline{f}}$ is compact on $A^2(\Omega)$. Then there exists $c \in \mathbb{C}$ so that $f \equiv c$.*

Theorem 2. *Let $\Omega \subset \mathbb{C}^2$ be a C^∞-smooth bounded pseudoconvex complete Reinhardt domain. Let $f \in A^2(\Omega)$ such that $H_{\overline{f}}$ is compact on $A^2(\Omega)$. Suppose either of the following conditions hold:*

1. *There exists a holomorphic function $F = (F_1, F_2) : \mathbb{D} \to b\Omega$ so that both F_1 and F_2 are not identically constant.*
2. *$\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.*

Then $f \equiv c$ for some $c \in \mathbb{C}$.

2. Preliminary lemmas

As a bit of notation to simplify the reading, we will use the multi-index notation. That is, we will write

$$z = (z_1, z_2)$$

and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$$

and $|\alpha| = \alpha_1 + \alpha_2$. We say $\alpha = \beta$ if $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. If either $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$ we say $\alpha \neq \beta$.

It is well known that for bounded complete Reinhardt domains in $\mathbb{C}^2$, the monomials

$$\left\{ \frac{z^\alpha}{\|z^\alpha\|_{L^2(\Omega)}} : \alpha \in \mathbb{Z}_+^2 \right\}$$

form an orthonormal basis for $A^2(\Omega)$.

We denote

$$\frac{z^\alpha}{\|z^\alpha\|_{L^2(\Omega)}} = e_\alpha(z)$$

Definition 1. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$, we define

$$G_\beta := \left\{ \psi \in L^2(\Omega) : \psi(\zeta z) = \zeta^\beta \psi(z) \,\text{a.e.}\, z \in \Omega \,\text{a.e}\, \zeta \in \mathbb{T}^2 \right\}.$$

Note this definition makes sense in the case Ω is a Reinhardt domain, and is the same as the definition of quasi-homogeneous functions in [5].

Lemma 1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded complete Reinhardt domain. G_α as defined above are closed subspaces of $L^2(\Omega)$ and for $\alpha \neq \beta$,*

$$G_\alpha \perp G_\beta.$$

Proof. The proof that G_β is a closed subspace of $L^2(\Omega)$ is similar to [5]. Without loss of generality, suppose $\alpha_1 \neq \beta_1$. Since Ω is a complete Reinhardt domain, one can 'slice' the domain similarly to [4]. That is,

$$\Omega = \bigcup_{z_2 \in H_\Omega} (\Delta_{|z_2|} \times \{z_2\})$$

where $H_\Omega \subset \mathbb{C}$ is a disk centered at 0 and

$$\Delta_{|z_2|} = \{z \in \mathbb{C} : |z| < r_{|z_2|}\}$$

is a disk with radius depending on $|z_2|$. As we shall see, the proof relies on the radial symmetry of both H_Ω and $\Delta_{|z_2|}$.

Let $f \in G_\alpha$, $g \in G_\beta$, $z_1 = r_1\zeta_1$, $z_2 = r_2\zeta_2$ for $(\zeta_1, \zeta_2) \in \mathbb{T}^2$, and $r_1, r_2 \geq 0$. Then we have

$$\begin{aligned}
&\langle f, g \rangle \\
&= \int_\Omega f(z)\overline{g(z)} dV(z) \\
&= \int_{H_\Omega} \int_{0 \leq r_1 \leq r_{|z_2|}} \int_{\mathbb{T}} \zeta_1^{\alpha_1} \overline{\zeta_1}^{\beta_1} f(r_1, z_2)\overline{g(r_1, z_2)} r_1 d\sigma(\zeta_1) dr_1 dV(z_2).
\end{aligned}$$

Since $\alpha_1 \neq \beta_1$,

$$\int_{\mathbb{T}} \zeta_1^{\alpha_1} \overline{\zeta_1}^{\beta_1} d\sigma(\zeta_1) = 0.$$

This completes the proof. $\square$

In the case of a bounded convex Reinhardt domain in $\mathbb{C}^2$, one can use the 'slicing' approach in [4] to expilictly compute $P(\overline{z}^j e_n)$.

Lemma 2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded complete Reinhardt domain. Then the Hankel operator with symbol $\overline{z}^j \overline{w}^k$ applied to the orthonormal basis vector e_n has the following form:*

$$H_{\overline{z}^j} e_n(z) = \frac{\overline{z}^j z^n}{\|z^n\|}$$

if either $n_1 - j_1 < 0$ or $n_2 - j_2 < 0$. If $n_1 - j_1 \geq 0$ and $n_2 - j_2 \geq 0$ then we can express the Hankel operator applied to the standard orthonormal basis as

$$H_{\overline{z}^j} e_n(z) = \frac{\overline{z}^j z^n}{\|z^n\|} - \frac{z^{n-j}\|z^n\|}{\|z^{n-j}\|^2}.$$

Furthermore, for any monomial

$$\overline{w}^j w^n \in G_{n-j}$$

the projection

$$(I-P)(\overline{w}^j w^n) \in G_{n-j}.$$

Proof. We have

$$\begin{aligned}
&P(\overline{z}^j e_n)(z) \\
&= \int_\Omega \overline{w}^j \frac{w^n}{\|w^n\|} \sum_{l \in \mathbb{Z}_+^2} \overline{e_l(w)} e_l(z) dV(z,w) \\
&= \int_{H_\Omega} \int_{w_1 \in \Delta_{|w_2|}} \overline{w_1}^{j_1} \overline{w_2}^{j_2} \frac{w_1^{n_1} w_2^{n_2}}{\|z^n\|} \\
&\quad \cdot \sum_{l_1, l_2 = 0}^{\infty} \overline{e_{l_1,l_2}(w_1,w_2)} e_{l_1,l_2}(z_1,z_2) dA_1(w_1) dA_2(w_2) \\
&= \sum_{l_1,l_2=0}^{\infty} \frac{z_1^{l_1} z_2^{l_2}}{\|z^n\| \| \|z^l\|^2} \int_{H_\Omega} \overline{w_2}^{j_2+l_2} w_2^{n_2} \int_{w_1 \in \Delta_{|w_2|}} \overline{w_1}^{j_1+l_1} w_1^{n_1} dA_1(w_1) dA_2(w_2).
\end{aligned}$$

Converting to polar coordinates and using the orthogonality of $\{e^{in\theta} : n \in \mathbb{Z}\}$ and the fact that

$$\int_{w_1 \in \Delta_{|w_2|}} \overline{w_1}^{j_1+l_1} w_1^{n_1} dA_1(w_1)$$

is a radial function of w_2 and H_Ω is radially symmetric, we have the only nonzero term in the previous sum is when $n_2 - j_2 = l_2$ and $n_1 - j_1 = l_1$. Therefore, we have $P(\overline{w}^j e_n)(z) = 0$ if $n_2 - j_2 < 0$ or $n_1 - j_1 < 0$. Otherwise, if $n_2 - j_2 \geq 0$ and $n_1 - j_1 \geq 0$, we have

$$P(\overline{w}^j e_n)(z) = \frac{z^{n-j}\|z^n\|}{\|z^{n-j}\|^2}.$$

Therefore, we have

$$H_{\overline{w}^j} e_n(z) = \frac{\overline{z}^j z^n}{\|z^n\|} - \frac{z^{n-j}\|z^n\|}{\|z^{n-j}\|^2}$$

if $n_2 - k \geq 0$ and $n_1 - j \geq 0$ otherwise

$$H_{\overline{w}^j} e_n(z) = \frac{\overline{z}^j z^n}{\|z^n\|}$$

if either $n_2 - k < 0$ or $n_1 - j < 0$. This also shows that the subspaces G_α remain invariant under the projection $(I-P)$, at least for monomial symbols. □

Lemma 3. *For every $\alpha \geq 0$, the product Hankel operator*

$$H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha} : A^2(\Omega) \to A^2(\Omega)$$

is a diagonal operator with respect to the standard orthonormal basis

$$\{e_j : j \in \mathbb{Z}^2_+\}.$$

Proof. Assume without loss of generality, $j \neq l$. We have

$$\begin{aligned}\langle H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha} e_j, e_l\rangle &= \langle H_{\overline{z}^\alpha} e_j, H_{\overline{z}^\alpha} e_l\rangle \\ &= \langle (I-P)(\overline{z}^\alpha e_j), \overline{z}^\alpha e_l\rangle.\end{aligned}$$

We have $\overline{z}^\alpha e_j \in G_{j-\alpha}$, $\overline{z}^\alpha e_l \in G_{l-\alpha}$. By Lemma 2,

$$(I-P)\overline{z}^\alpha e_j \in G_{j-\alpha}.$$

By Lemma 1, G_α are mutually orthogonal. Therefore,

$$\langle (I-P)(\overline{z}^\alpha e_j), \overline{z}^\alpha e_l\rangle = 0$$

unless $j = l$. □

Using Lemma 2 and Lemma 3, let us compute the eigenvalues of

$$H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha}.$$

Let us first assume $n - \alpha \geq 0$. We have

$$\begin{aligned}\langle H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha} e_n, e_n\rangle &= \left\langle \frac{\overline{z}^\alpha z^n}{\|z^n\|} - \frac{z^{n-\alpha}\|z^n\|}{\|z^{n-\alpha}\|^2}, \frac{\overline{z}^\alpha z^n}{\|z^n\|}\right\rangle \\ &= \frac{\|\overline{z}^\alpha z^n\|^2}{\|z^n\|^2} - \frac{\|z^n\|^2}{\|z^{n-\alpha}\|^2}.\end{aligned}$$

If $n - \alpha < 0$, we have

$$\langle H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha} e_n, e_n\rangle = \frac{\|\overline{z}^\alpha z^n\|^2}{\|z^n\|^2}.$$

3. Proof of Theorem 1

Proof. Assume $f \in A^2(\Omega)$ and $H_{\overline{f}}$ is compact on $A^2(\Omega)$. Then, we can represent

$$f = \sum_{j,k=0}^{\infty} c_{j,k,f}\, z_1^j z_2^k$$

almost everywhere (with respect to the Lebesgue volume measure on Ω). Let

$$\{e_m : m \in \mathbb{Z}^2_+\}$$

be the standard orthonormal basis for $A^2(\Omega)$. Then

$$\left\|H_{\overline{f}} e_m\right\|^2 \to 0$$

as $|m| \to \infty$. Using the mutual orthogonality of the subspaces G_α, we get

$$\begin{aligned}
\|H_{\overline{f}} e_m\|^2 &= \langle (I-P)(\overline{f} e_m), \overline{f} e_m \rangle \\
&= \left\langle \sum_{j,k=0}^{\infty} (I-P)(\overline{c_{j,k,f} z_1}^{j} \overline{z_2}^{k} e_m), \sum_{s,p=0}^{\infty} \overline{c_{s,p,f} z_1}^{s} \overline{z_2}^{p} e_m \right\rangle \\
&= \sum_{j,k=0}^{\infty} \left\| H_{\overline{c_{j,k,f} z_1^j z_2^k}} e_m \right\|^2 \\
&\geq \left\| H_{\overline{c_{j,k,f} z_1^j z_2^k}} e_m \right\|^2
\end{aligned}$$

for every $(j,k) \in \mathbb{Z}_+^2$. Taking limits as $|m| \to \infty$, we have

$$\lim_{|m| \to \infty} \left\| H_{\overline{c_{j,k,f} z_1^j z_2^k}} e_m \right\|^2 = 0$$

for all $(j,k) \in \mathbb{Z}_+^2$. The Hankel operators

$$H^*_{\overline{c_{j,k,f} z_1^j z_2^k}} H_{\overline{c_{j,k,f} z_1^j z_2^k}}$$

are diagonal by Lemma 3, with eigenvalues

$$\lambda_{j,k,m} = \left\| H_{\overline{c_{j,k,f} z_1^j z_2^k}} e_m \right\|^2 .$$

This shows that

$$H^*_{\overline{c_{j,k,f} z_1^j z_2^k}} H_{\overline{c_{j,k,f} z_1^j z_2^k}}$$

are compact for every $(j,k) \in \mathbb{Z}_+^2$. Then

$$H_{\overline{c_{j,k,f} z_1^j z_2^k}}$$

are compact on $A^2(\Omega)$.

Without loss of generality, assume $\Gamma_1 \neq \emptyset$. Then there exists a holomorphic function $F = (F_1, F_2) : \mathbb{D} \to b\Omega$ so that F_2 is identically constant and F_1 is nonconstant. Therefore, by [2], the composition

$$\overline{c_{j,k,f} F_1(z)^j F_2(z)^k}$$

must be holomorphic in z. This cannot occur unless $\overline{c_{j,k,f}} = 0$ for $j > 0$. Therefore, using the representation

$$f = \sum_{j,k=0}^{\infty} c_{j,k,f} z_1^j z_2^k$$

we have $f = \sum_{k=0}^{\infty} c_{0,k,f} z_2^k$ almost everywhere. By holomorphicity of f and the identity principle, this implies

$$f \equiv \sum_{k=0}^{\infty} c_{0,k,f} z_2^k.$$

Hence f is a function of only z_2. The proof is similar if $\Gamma_2 \neq \emptyset$. $\square$

4. Proof of Theorem 2

Using the same argument in the proof of Theorem 1, one can show compactness of $H_{\overline{f}}$ implies compactness of

$$H_{\overline{c_{j,k,f} z_1^j z_2^k}}$$

for every $j, k \in \mathbb{Z}_+$. Hence by [3, Corollary 1], for any holomorphic function $\phi = (\phi_1, \phi_2) : \mathbb{D} \to b\Omega$, we have

$$\overline{c_{j,k,f}}\,\overline{\phi_1}^j\,\overline{\phi_2}^k$$

must be holomorphic. If we assume condition two in Theorem 2, then it follows that $f \equiv c_{0,0,f}$. Assuming condition one in Theorem 2, we may assume ϕ_1 and ϕ_2 are not identically constant. Thus $c_{j,k,f} = 0$ for $j > 0$ or $k > 0$ and so $f \equiv c_{0,0,f}$.

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