New York Journal of Mathematics

New York J. Math. **23** (2017) 1185–1203.

# On the Medvedev–Scanlon conjecture for minimal threefolds of nonnegative Kodaira dimension

**Jason P. Bell, Dragos Ghioca, Zinovy Reichstein and Matthew Satriano**

Abstract. Motivated by work of Zhang from the early '90s, Medvedev and Scanlon formulated the following conjecture. Let $F$ be an algebraically closed field of characteristic 0 and let $X$ be a quasiprojective variety defined over $F$ endowed with a dominant rational self-map $\phi$. Then there exists a point $x \in X(F)$ with Zariski dense orbit under $\phi$ if and only if $\phi$ preserves no nontrivial rational fibration, i.e., there exists no nonconstant rational functions $f \in F(X)$ such that $\phi^*(f) = f$. The Medvedev–Scanlon conjecture holds when $F$ is uncountable. The case where $F$ is countable (e.g., $F = \overline{\mathbb{Q}}$) is much more difficult; here the Medvedev–Scanlon conjecture has only been proved in a small number of special cases. In this paper we show that the Medvedev–Scanlon conjecture holds for all varieties of positive Kodaira dimension, and explore the case of Kodaira dimension 0. Our results are most definitive in dimension 3.

Contents



Received January 18, 2017; revised August 3, 2017.

2010 *Mathematics Subject Classification.* 14E05, 14C05, 37F10.

*Key words and phrases.* algebraic dynamics, orbit closures, rational invariants, Medvedev–Scanlon conjecture.

The authors have been partially supported by Discovery Grants from the National Science and Engineering Board of Canada.

ISSN 1076-9803/2017

## 1. Introduction

Consider a dominant rational self-map $\phi\colon X \dashrightarrow X$ of an irreducible variety $X$, defined over a field $k$. For an integer $n \geq 0$, we will denote by $\phi^n$ the $n$-th compositional power of $\phi$. Given a point $x \in X$, we define its orbit under $\phi$ (denoted $\mathcal{O}_\phi(x)$) to be the set of all $\phi^n(x)$ (as $n$ ranges over the nonnegative integers) whenever $x$ is not in the indeterminacy locus for $\phi^n$.

The Medvedev–Scanlon Conjecture predicts when there is a point in $X(\overline{\mathbb{Q}})$ with dense $\phi$-orbit. Certainly, no such $\overline{\mathbb{Q}}$-point can exist if $\phi$ *preserves a rational fibration*, i.e., if there is a dominant rational map $\pi : X \dashrightarrow Y$ with $\dim Y > 0$ such that $\pi \circ \phi = \pi$. The Medvedev–Scanlon conjecture asserts that this necessary condition is also sufficient.

**Conjecture 1.1** ([MS14, 7.14])**.** *Let $X$ be an irreducible variety over an algebraically closed field $F$ of characteristic 0 and $\phi\colon X \dashrightarrow X$ be a dominant rational self-map. If $\phi$ does not preserve a rational fibration, then there is a point $x \in X(F)$ with Zariski dense forward orbit under $\phi$.*

In the case, where $F$ is uncountable, Conjecture 1.1 was proved earlier by Amerik and Campana [AC08, Theorem 4.1] (and under the stronger hypothesis that $\phi$ is an automorphism of $X$ independently by Bell, Rogalski and Sierra [BRS10, Theorem 1.2]). Conjecture 1.1 was, in fact, motivated by this theorem and by an older conjecture of Zhang [Zha06, Conjecture 4.1.6] about Zariski dense orbits for polarizable endomorphisms.

For the rest of the introduction we will assume that $F$ is a countable algebraically closed field of characteristic 0 (e.g., $F = \overline{\mathbb{Q}}$). Here the Medvedev–Scanlon conjecture has only been proved in a few special cases, using subtle diophantine techniques:

(1) Medvedev and Scanlon [MS14, Theorem 7.16] established Conjecture 1.1 for endomorphisms $\phi$ of $X = \mathbb{A}^m$ of the form

$$\phi(x_1, \ldots, x_m) = (f_1(x_1), \ldots, f_m(x_m)),$$

where $f_1, \ldots, f_m \in F[x]$. Their proof combines techniques from model theory, number theory and polynomial decomposition theory to obtain a complete description of all periodic subvarieties.

(2) In the case where $X$ is an abelian variety and $\phi\colon X \to X$ is a dominant self-map, Conjecture 1.1 was proved by Ghioca and Scanlon [GS17]. The proof uses an explicit description of endomorphisms of an abelian variety and relies on the Mordell–Lang conjecture, due to Faltings [Fal94].

(3) In the case where $\dim(X) \leq 2$ and $\phi\colon X \dashrightarrow X$ is a birational isomorphism, Conjecture 1.1 was established by Xie [Xie15]. We remark that in [Xie15, Theorem 1.4], this result is stated under the additional assumption that the first dynamical degree of $\phi$ is greater

than 1; however, the same proof goes through without this assumption. We will not use [Xie15, Theorem 1.4] in this paper, but we will appeal to the case of regular automorphisms of surfaces, which was settled earlier in [BGT15, Theorem 1.3]. These results are proved by $p$-adic techniques, in particular, the so-called $p$-adic arc lemma. For details on the $p$-adic arc lemma and its applications we refer the reader to [BGT16, Chapter 4], [A15, Section 1].

(4) Xie [Xie, Theorem 1.1] recently proved Conjecture 1.1 for all polynomial endomorphisms of $\mathbb{A}^2$. The proof relies on valuation-theoretic techniques.

In this paper we will explore Conjecture 1.1 in the case where

$$\phi\colon X \dashrightarrow X$$

is a birational automorphism and $\dim(X) \geq 3$ by using techniques of higher-dimensional algebraic geometry. We begin by observing that if $X$ is an irreducible projective variety of Kodaira dimension $\kappa(X) > 0$, then every dominant rational self-map $\phi\colon X \dashrightarrow X$ preserves a rational fibration; see Proposition 2.3. In particular, the Medvedev–Scanlon Conjecture is vacuously true in this case.

For the remainder of this paper we will consider the case of Kodaira dimension 0. Recall that a smooth projective variety $X$ over $\overline{\mathbb{Q}}$ is called *hyperkähler* if its complex analytification is simply connected and $H^0(\Omega_X^2)$ is spanned by a symplectic form. In dimension 2, hyperkähler varieties are nothing more than K3 surfaces.

We use the convention that a smooth projective variety of dimension $\geq 3$ defined over $\overline{\mathbb{Q}}$ is *Calabi–Yau* if the complex analytification $X_{\mathbb{C}}$ is simply connected, $K_X \simeq \mathcal{O}_X$, and $H^p(\mathcal{O}_X) = 0$ for $0 < p < \dim X$. Since we are working over $\overline{\mathbb{Q}}$, by the symmetry of the Hodge diamond, this latter condition is equivalent to requiring $H^0(\Omega_X^p) = 0$ for $0 < p < \dim X$.

We are now ready to state the main results of this paper.

**Theorem 1.2.** *Fix an integer $n \geq 1$. Then the following conditions are equivalent.*

(a) *The Medvedev–Scanlon Conjecture* 1.1 *holds for all birational self-maps of smooth projective minimal $n$-folds $X$ over $\overline{\mathbb{Q}}$ such that the canonical divisor $K_X$ is numerically trivial,*

(b) *The Medvedev–Scanlon Conjecture* 1.1 *holds for all birational self-maps of smooth projective minimal $n$-folds $X$ of the form*

$$X = A \times \prod_i Y_i \times \prod_j Z_j,$$

*where $A$ is an abelian variety, the $Y_i$ are Calabi–Yau, and the $Z_j$ are hyperkähler.*

**Remark 1.3.** The Abundance Conjecture [KM98, Conjecture 3.12] implies that $K_X$ is numerically trivial for every smooth projective minimal variety $X$ of Kodaira dimension 0.

In the case of threefolds, we obtain the following stronger result

**Theorem 1.4.** *The Medvedev–Scanlon Conjecture* 1.1 *holds for birational self-maps of smooth projective minimal threefolds over* $\overline{\mathbb{Q}}$ *of Kodaira dimension* 0 *if and only if it holds for smooth Calabi–Yau threefolds.*

Finally, we handle the case of Calabi–Yau threefolds, contingent on conjectures in the minimal model program. Via the intersection product, the second Chern class $c_2(X)$ defines a linear form on the nef cone $\mathrm{Nef}(X)$. Miyaoka [Miy87] shows that this linear form always assumes nonnegative values on the nef cone. We separately consider the cases where $c_2(X)$ is strictly positive and where it is not.

**Theorem 1.5.** *Let* $X$ *be a smooth projective Calabi–Yau threefold over* $\overline{\mathbb{Q}}$*. Then the Medvedev–Scanlon Conjecture* 1.1 *holds for all (regular) automorphisms* $\phi\colon X \to X$ *if either:*

(1) $c_2(X)$ *is positive on* $\mathrm{Nef}(X)$*, or*
(2) *there is a semi-ample divisor* $D \neq 0$ *on* $X$ *such that* $c_2(X) \cdot D = 0$.

Here by "divisor" we mean that $D$ is an integral point of $\mathrm{Nef}(X)$, i.e., $D$ is the linear combination of classes of codimension 1 irreducible subvarieties of $X$ with integer coefficients. Note also that here $c_2(X) \neq 0$. Indeed, otherwise there would exist a finite étale cover $A \to X$, where $A$ is an abelian variety. Since we are assuming that $X$ is simply connected, this cannot happen.

**Remark 1.6** (Concerning the hypothesis in Theorem 1.5(2))**.** If the hypothesis in Theorem 1.5(1) fails, then as mentioned above, Miyaoka's theorem implies $Z := c_2(X)^{\perp} \cap \mathrm{Nef}(X)$ is a nonzero face of $\mathrm{Nef}(X)$. A priori, $Z$ could be irrational. If $Z$ contains a nonzero rational class $D$, then the semi-ampleness conjecture [LOP, Conjecture 2.1] implies that some scalar multiple $mD$ is a semi-ample divisor, and so the hypothesis in (2) holds.

Thus, assuming the semi-ampleness conjecture, the only Calabi–Yau varieties $X$ that Theorem 1.5 does not apply to are those for which $Z$ is nonzero and contains no nonzero rational classes. If [Ogu01, Question-Conjecture 2.6] of Oguiso is true over $\overline{\mathbb{Q}}$, then this situation never occurs when the Picard number $\rho(X)$ is sufficiently large.

In light of Remark 1.6, we have the following result.

**Corollary 1.7.** *If the semi-ampleness conjecture* [LOP, Conjecture 2.1] *and* [Ogu01, Question-Conjecture 2.6] *are true over* $\overline{\mathbb{Q}}$*, then the Medvedev–Scanlon Conjecture* 1.1 *is true for all automorphisms of smooth minimal threefolds of nonnegative Kodaira dimension and sufficiently large Picard number.*

**Acknowledgments.** We thank our colleagues Ekaterina Amerik, Donu Arapura, Stéphane Druel, Najmuddin Fakhruddin, Fei Hu, Jesse Kass, Brian Lehman, John Lesieutre, Sándor Kovács, Tom Scanlon, Alan Thompson, Burt Totaro, Tom Tucker, Junyi Xie, and Yi Zhu for stimulating conversations. We are also grateful to the anonymous referee for helpful and constructive comments.

## 2. The case of positive Kodaira dimension

We begin with two useful lemmas.

**Lemma 2.1.** *In order to prove Conjecture* 1.1 *for the dynamical system* $(X,\phi)$*, it is sufficient to prove Conjecture* 1.1 *for an iterate* $(X,\phi^m)$*, for some* $m \in \mathbb{N}$.

**Proof.** It is clear that if $\phi^m$ has a Zariski dense orbit, then so does $\phi$.

It remains to show that if $\phi$ does not preserve a nonconstant fibration, then neither does $\phi^m$. Indeed, suppose there exists a nonconstant $f \in F(X)$ such that $(\phi^m)^*(f) = f$. Then $\phi$ preserves the symmetric function $g_i$ in the rational functions $f, \phi^*(f), \dots, (\phi^{m-1})^*(f)$, for each $i = 1, \dots, m$. Since $f$ is nonconstant, then at least one of $g_1, \dots, g_m$ is nonconstant. In other words, there exists a nonconstant function $g_i$ fixed by $\phi^*$, as desired. □

**Lemma 2.2.** *Let* $\phi : X \dashrightarrow X$ *be a birational automorphism defined over a field* $k$*. Let* $F$ *be an uncountable algebraically closed field containing* $k$*. Then the following conditions are equivalent:*

(1) $k(X)^\phi = k$.
(2) *There exists a* $F$*-point* $x \in X(F)$ *such that the orbit*

$$\{\phi^n(x) \mid n = 0, 1, 2, \dots\}$$

*is dense in* $X$.
(3) $F(X)^\phi = F$.

**Proof.** The implication (1) $\Longrightarrow$ (2) follows from [BGR17, Theorem 1.2]. The remaining implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are obvious. □

**Proposition 2.3.** *If* $X$ *is an irreducible projective variety of Kodaira dimension* $\kappa(X) > 0$ *defined over a field* $k$ *of characteristic* 0 *and* $\phi\colon X \dashrightarrow X$ *is a dominant rational self-map, then* $\phi$ *preserves a rational fibration. In particular, the Medvedev–Scanlon Conjecture* 1.1 *is vacuously true in this case.*

**Proof.** Let $k_0$ be a finitely generated subfield of $k$ such that both $X$ and $\phi$ are defined over $k_0$. After replacing $k$ by $k_0$ we may assume that $k$ is finitely generated over $\mathbb{Q}$ and thus is isomorphic to a subfield of $\mathbb{C}$. We want to show that $\phi$ preserves a rational fibration $X \dashrightarrow Y$ or equivalently, $k(X)^\phi \neq k$. By Lemma 2.2, we may assume without loss of generality that $k = \mathbb{C}$.

Note also that we may replace $X$ by a birationally equivalent variety; this does not change $\mathbb{C}(X)$ or $\mathbb{C}(X)^{\phi}$. After resolving the singularities of $X$, we may also assume that $X$ is smooth.

Next, consider the Iitaka fibration, i.e., the rational map $f : X \dashrightarrow W_m$ defined by the complete linear system $|mK_X|$ for $m$ sufficiently divisible. By a theorem of Nakayama and Zhang [NZ09, Theorem A], there exists an automorphism $\psi\colon W_n \to W_n$ of finite order such that the diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ W_n & \xrightarrow{\psi} & W_n, \end{array}$$

commutes. (In the case, where $\phi$ is an automorphism of $X$, this was proved earlier by Deligne and Ueno [Uen75, Thm 14.10].) By Lemma 2.1, we may replace $\phi$ by $\phi^e$, where $e$ is the order of $\psi$, and thus assume that $\psi = \mathrm{id}_{W_n}$. In other words, $f$ is a rational fibration preserved by $\phi$. □

## 3. The Beauville–Bogomolov decomposition theorem over $\overline{\mathbb{Q}}$

We now recall the Beauville–Bogomolov decomposition theorem. Suppose $X$ is a smooth complex projective variety with numerically trivial canonical divisor $K_X$. Beauville [Bea83, p. 9] defines $\pi : \widetilde{X} \to X$ to be a *minimal split cover* if it is a finite étale Galois cover, $\widetilde{X} \simeq A \times S$, where $A$ is an abelian variety and $S$ is simply connected, and there is no nontrivial element of the Galois group that simultaneously acts as translation on $A$ and the identity on $S$. The main theorem together with Proposition 3 of [Bea83] show that every such $X$ has a minimal split covering and that it is unique up to nonunique isomorphism.

In the sequel we will need a variant of the Beauville–Bogomolov decomposition theorem [Bea83] over $\overline{\mathbb{Q}}$. For lack of a suitable reference, we will prove it below.

**Proposition 3.1.** *Let $X$ be a smooth projective minimal variety over $\overline{\mathbb{Q}}$ with $K_X$ numerically trivial. Then there exists a finite étale Galois cover $\pi\colon \widetilde{X} \to X$ defined over $\overline{\mathbb{Q}}$ such that:*

(1) *$\widetilde{X} = A \times \prod_i Y_i \times \prod_j Z_j$, where $A$ is an abelian variety, the $Y_i$ are Calabi–Yau, and the $Z_j$ are hyperkähler.*
(2) *No element of the Galois group acts simultaneously as translation on $A$ and the identity on all of the $Y_i$ and $Z_j$.*
(3) *If $\pi'\colon \widetilde{X}' \to X$ is a finite étale cover and $\widetilde{X}' = A' \times S'$ with $A'$ an abelian variety and $S'$ a simply connected variety, then there exists a (not necessarily unique) map $\alpha\colon \widetilde{X}' \to \widetilde{X}$ such that $\pi' = \pi \circ \alpha$.*

**Proof.** The Beauville–Bogomolov decomposition theorem tells us that there is a finite group $G$ and a $G$-torsor $B \to X_{\mathbb{C}}$ with $B = A \times \prod_i Y_i \times \prod_j Z_j$, where $A$ is an abelian variety, the $Y_i$ are Calabi–Yau varieties, and the $Z_j$ are hyperkähler varieties. By a standard limit argument, there exists a finitely generated field extension $F/\overline{\mathbb{Q}}$ so that we can descend $B \to X_{\mathbb{C}}$ to a $G$-torsor $B' \to X_F$, the abelian variety $A$ to an abelian variety $A'$ over $F$, and the $Y_i$ (resp. $Z_j$) to smooth proper $F$-schemes $Y_i'$ (resp. $Z_j'$). Moreover, after possibly enlarging $F$, we can descend the isomorphism $B \simeq A \times \prod_i Y_i \times \prod_j Z_j$ to an isomorphism $B' \simeq A' \times \prod_i Y_i' \times \prod_j Z_j'$. Since $H^0(\Omega^p_{Y_i'}) \otimes_F \mathbb{C} = H^0(\Omega^p_{Y_i})$, we have $H^0(\Omega^p_{Y_i'}) = 0$ for $0 < p < \dim Y_i'$. By similar reasoning, we see $H^0(\Omega^2_{Z_j'})$ is 1-dimensional and that $K_{Y_i'} \simeq \mathcal{O}_{Y_i'}$; the latter statement can be proved by using the fact that a line bundle $\mathcal{L}$ on a projective variety is trivial if and only if $H^0(\mathcal{L})$ and $H^0(\mathcal{L}^\vee)$ are both nonzero. Choosing a generator $\omega_j \in H^0(\Omega^2_{Z_j'})$, we have an induced map $T_{Z_j'} \to \Omega^1_{Z_j'}$ and nondegeneracy of $\omega_j$ is equivalent to this map being an isomorphism. Since this is true after a field extension from $F$ to $\mathbb{C}$, it is true over $F$.

Next, let $V$ be a smooth $\overline{\mathbb{Q}}$-variety with function field $F$. After possibly shrinking $V$, we can extend $B' \to X_F$ to a $G$-torsor $B'' \to X_V$, extend $A'$ to an abelian scheme $A'' \to V$, $Y_i'$ and $Z_j'$ to smooth proper $V$-schemes $Y_i''$ and $Z_j''$, and can assume $B'' \simeq A'' \times \prod_i Y_i'' \times \prod_j Z_j''$ over $V$. Let $\pi_i : Y_i'' \to V$ and $\psi_j : Z_j'' \to V$ be the structure maps. After suitably shrinking $V$, we may assume that $(\pi_i)_* \Omega^p_{Y_i''/V} = 0$ for $0 < p < \dim Y_i''$, that $(\psi_j)_* \Omega^2_{Z_j''/V} \simeq \mathcal{O}_V$, and that there is a nonvanishing section $\omega_j$ of $(\psi_j)_* \Omega^2_{Z_j''/V}$ whose induced map $T_{Z_j''/V} \to \Omega^1_{Z_j''/V}$ is an isomorphism.

Finally, we show that for any $\mathbb{C}$-point $t : \operatorname{Spec} \mathbb{C} \to V$, the complex analytifications of the $(Y_i'')_t$ and $(Z_j'')_t$ are simply connected. First note that by the Beauville–Bogomolov decomposition theorem, these varieties have virtually abelian fundamental groups; specifically, if $W$ denotes one of these varieties, then there is a finite Galois cover $A \times S \to W$ with $A$ an abelian variety and $S$ simply connected, so $\pi_1(W)$ contains $\pi_1(A) \simeq \mathbb{Z}^r$ as a finite index subgroup. Next, note that if the étale fundamental group $\pi_1^{et}(W)$ is trivial, then so is $\pi_1(W)$. Indeed, if $\pi_1^{et}(W) = 0$, then $r = 0$, so $\pi_1(W)$ is finite and therefore, $\pi_1(W) = \pi_1^{et}(W) = 0$. Thus, it suffices to prove that for every geometric point $\overline{v}$ of $V$, the étale fundamental groups $\pi_1^{et}((Y_i'')_{\overline{v}})$ and $\pi_1^{et}((Z_j'')_{\overline{v}})$ are trivial. Since the étale fundamental groups of the geometric generic fibers $(Y_i'')_{\overline{\eta}} = Y_i$ and $(Z_j'')_{\overline{\eta}} = Z_j$ are trivial, this follows immediately from specialization results of the étale fundamental group [StPrj, Proposition 0C0Q].

Choosing any $\overline{\mathbb{Q}}$-point $v \in V$ gives our desired $G$-torsor of $B_v'' \to X$. Lastly, condition (3) follows word-for-word from the proof of [Bea83, Proposition 3]: since $\pi$ and $\pi'$ are defined over $\overline{\mathbb{Q}}$, each of the Galois covers in Beauville's proof is also defined over $\overline{\mathbb{Q}}$. □

## 4. Proof of Theorem 1.2

In this section we will prove Theorem 1.2. The key ingredients of the proof are supplied by Lemma 4.1 and Proposition 4.4 below.

**Lemma 4.1.** *Consider the commutative diagram*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \overset{\psi}{\dashrightarrow} & Y, \end{array}$$

*where* $\pi\colon X \to Y$ *is a dominant morphism of irreducible varieties,* $\phi$ *and* $\psi$ *are birational isomorphisms of* $X$ *and* $Y$*, respectively, and the entire diagram is defined over* $\overline{\mathbb{Q}}$*. Further suppose that* $\dim(X) = \dim(Y)$ *and* $\overline{\mathbb{Q}}(X)^{\phi} = \overline{\mathbb{Q}}$*. Then:*

(a) $\overline{\mathbb{Q}}(Y)^{\psi} = \overline{\mathbb{Q}}$.

*In parts* (b) *and* (c)*, assume further that* $\pi\colon X \to Y$ *is a* $G$*-torsor for some finite smooth group scheme* $G$*.*

(b) *If* $\phi$ *is regular at* $x \in X$*, then* $\psi$ *is regular at* $y := \pi(x) \in Y$*.*

(c) *If the Medvedev–Scanlon Conjecture holds for* $(X, \phi)$*, then there exists a point* $y \in Y(\overline{\mathbb{Q}})$ *whose* $\psi$*-orbit is dense in* $Y$*.*

**Proof.** (a) Viewing $\overline{\mathbb{Q}}(Y)$ as a subfield $\overline{\mathbb{Q}}(X)$ via $\pi^*$, we see that

$$\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}(Y)^{\psi} \subset \overline{\mathbb{Q}}(X)^{\phi} = \overline{\mathbb{Q}},$$

and part (a) follows.

(b) The composition $\pi \circ \phi\colon X \dashrightarrow Y$ is a $G$-invariant rational map which is regular at $x$. Hence, it descends to a rational map $Y \dashrightarrow Y$ which is regular at $y$. Clearly, this map coincides with $\psi$. In other words, $\psi$ is regular at $y$, as claimed.

(c) Since the Medvedev–Scanlon Conjecture holds for $\phi$, there exists a point $x \in X(\overline{\mathbb{Q}})$ such that the $\phi$-orbit of $x$ is dense in $X$. Using part (b) for each iterate of $\phi$, we conclude that for each $n \in \mathbb{N}$ such that $\phi^n$ is defined at $x$, we have that $\psi^n$ is defined at $y := \pi(x)$. Furthermore, since the orbit of $x$ under $\phi$ is dense in $X$, we conclude that the orbit of $y$ under $\psi$ is dense in $Y$ as well. □

The next two technical lemmas are used to prove Proposition 4.4.

**Lemma 4.2.** *Let* $X$ *be a smooth projective minimal variety over* $\overline{\mathbb{Q}}$ *with* $K_X$ *numerically trivial. Suppose* $G$ *is a finite group and that*

$$\begin{array}{ccc} \widetilde{X}' & \overset{\varphi}{\dashrightarrow} & \widetilde{X} \\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi} \\ X & \overset{\phi}{\dashrightarrow} & X \end{array}$$

*is a commutative diagram, where $\pi$ and $\pi'$ are $G$-torsors, and $\phi$ and $\varphi$ are birational maps. Then there are finite groups $H$ and $\Gamma$, and a commutative diagram*

$$\begin{array}{ccc} Y' & \overset{\psi}{\dashrightarrow} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \widetilde{X}' & \overset{\varphi}{\dashrightarrow} & \widetilde{X} \end{array}$$

*such that:*

(i) *$\psi$ is a birational $H$-equivariant map.*
(ii) *$p$ and $p'$ are $H$-torsors.*
(iii) *$\pi \circ p$ and $\pi' \circ p'$ are $\Gamma$-torsors.*
(iv) *$Y' \simeq A' \times S'$ with $A'$ an abelian variety and $S'$ simply connected.*

**Proof.** Since $\pi'$ is étale, we see $(\pi')^* K_X = K_{\widetilde{X}'}$ and so $K_{\widetilde{X}'}$ is numerically trivial. Thus, by Proposition 3.1, there is a minimal split cover $q' \colon Z' \to \widetilde{X}'$ defined over $\overline{\mathbb{Q}}$. Taking a further étale cover $Y' \to Z'$, we can assume that the composite map $Y' \to X'$ is Galois with group $\Gamma$. Since $Z'$ is the product of an abelian variety and a simply connected variety, and since $Y' \to Z'$ is étale, we see $Y' = A' \times S'$ with $A'$ an abelian variety and $S'$ simply connected. Let $p'$ denote the composite map $Y' \to \widetilde{X}'$ and let $H$ be its Galois group.

Next, $\phi$ is a birational automorphism of $X$, so as Lazić shows in [Laz13, p. 197] between Remarks 6.1 and 6.2, $\phi$ is a pseudo-automorphism, i.e., neither $\phi$ nor $\phi^{-1}$ contracts a divisor. We can therefore find open subsets $U$ and $V$ of $X$ whose complements have codimension at least 2 such that $\phi|_U \colon U \to V$ is an isomorphism. Since $p'$ is an $H$-torsor, we see

$$Y'|_U := Y' \times_X U \to \widetilde{X}' \times_X U =: \widetilde{X}'|_U$$

is as well. Pulling this torsor back via the isomorphism $\widetilde{X}|_V \to \widetilde{X}'|_U$, we obtain an $H$-torsor $W \to \widetilde{X}|_V$ and thus a Cartesian diagram

$$\begin{array}{ccccccc} Y' & \supseteq & Y'|_U & \xrightarrow{\simeq} & W & & \\ \downarrow{\scriptstyle p'} & & \downarrow & & \downarrow & & \\ \widetilde{X}' & \supseteq & \widetilde{X}'|_U & \xrightarrow{\simeq} & \widetilde{X}|_V & \subseteq & \widetilde{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \supseteq & U & \xrightarrow{\simeq} & V & \subseteq & X. \end{array}$$

Since $\pi$ is étale and the complement of $V$ in $X$ has codimension at least 2, we see the complement of $\widetilde{X}|_V$ in $\widetilde{X}$ has codimension at least 2. Then by [Ols12, Proposition 3.2], the $H$-torsor $W \to \widetilde{X}|_V$ extends uniquely to an $H$-torsor $p \colon Y \to \widetilde{X}$. Since $X \smallsetminus V$ has codimension at least 2, another application of [Ols12, Proposition 3.2] shows that $\pi \circ p$ is a $\Gamma$-torsor. We have therefore obtained a commutative diagram

$$\begin{array}{ccc} Y' & \overset{\psi}{\dashrightarrow} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \widetilde{X}' & \overset{\varphi}{\dashrightarrow} & \widetilde{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \overset{\phi}{\dashrightarrow} & X. \end{array}$$

We have shown properties (ii)–(iv). Since the $H$-torsor $W \to \widetilde{X}|_V$ was obtained as the pullback of the $H$-torsor $Y' \to \widetilde{X}'|_U$ it follows by construction that $Y'|_U \to W$ is $H$-equivariant. Thus, $\psi$ is $H$-equivariant, proving property (i). □

**Lemma 4.3.** *Under the hypotheses of Lemma* 4.2*, if* $\pi\colon \widetilde{X} \to X$ *is the minimal split cover of Proposition* 3.1*, then* $\widetilde{X}'$ *is the product of an abelian variety and a simply connected variety.*

**Proof.** Let $Y$, $Y'$, $p$, $p'$, $\psi$, $H$, and $\Gamma$ be as in the conclusion of Lemma 4.2. In particular, $Y' = A' \times S'$ where $A'$ is an abelian variety and $S'$ is simply connected. By construction, $\widetilde{X}$ is a product of an abelian variety and a simply connected variety, and since $Y$ is a finite étale cover of $\widetilde{X}$, we also see that $Y = A \times S$ with $A$ an abelian variety and $S$ a simply connected variety. Moreover, since $\widetilde{X}$ is the minimal split covering of $X$, the proof of [Bea83, Proposition 3] tells us that the $H$-action on $Y$ realizes $H$ as the normal subgroup of elements in $\Gamma$ acting simultaneously as translation on $A$ and the identity on $S$. As a result, $\widetilde{X} = (A/H) \times S$.

To finish the proof, it suffices to show that $H$ acts on $Y'$ through translation on $A'$ and the identity on $S'$. Indeed, provided we can show this, we then know that $\widetilde{X}' = (A'/H) \times S'$, as desired. To prove that $H$ acts on $Y'$ as stated, we compare it with the $H$-action on $Y = A \times S$. Since $\psi$ is an $H$-equivariant map, it induces an $H$-equivariant birational map $\overline{\psi} : A' \dashrightarrow A$ on Albanese varieties. Every rational map of abelian varieties is regular, so $\overline{\psi}$ is in fact an isomorphism. Moreover, after suitable choice of origin, it respects the group structure. Given $\gamma \in H$, we know it acts on $A$ as translation $t_z$ by some $z$, so $\gamma$ acts on $A'$ as $\overline{\psi}^{-1} t_z \overline{\psi}$ which is translation by $\overline{\psi}^{-1}(z)$. Now, choosing a general point $a \in A'$, $\psi$ induces a birational map on fibers $S' = Y'_a \dashrightarrow Y_{\psi(a)} = S$ that commutes with the $H$-action. Since each $\gamma \in H$ acts as the identity on $S$, the action of $\gamma$ on $S'$ is an automorphism that agrees with the identity map on a dense open. As a result, it is the identity map. □

**Proposition 4.4.** *Let* $X$ *be a smooth projective minimal variety over* $\overline{\mathbb{Q}}$ *with* $K_X$ *numerically trivial, and let* $\pi : \widetilde{X} \to X$ *be a minimal split cover provided by Proposition* 3.1*. Then for every birational automorphism* $\phi$ *of*

*$X$ over $\overline{\mathbb{Q}}$, there exists a birational automorphism $\widetilde{\phi}$ of $\widetilde{X}$ over $\overline{\mathbb{Q}}$ such that $\pi \circ \widetilde{\phi} = \phi \circ \pi$.*

**Proof.** We know that $\pi\colon \widetilde{X} \to X$ is a $G$-torsor for a finite group $G$, and that $\widetilde{X} = A \times S$ with $S$ simply connected and $A$ an abelian variety. Since $X$ is smooth, $\phi$ is regular on an open subset $U \subseteq X$ with $X \smallsetminus U$ having codimension at least 2. Consider the Cartesian diagram

$$\begin{array}{ccc} \widetilde{X} \times_X U & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\phi|_U} & X. \end{array}$$

Since $X \smallsetminus U$ has codimension at least 2, by [Ols12, Proposition 3.2], the $G$-torsor $\widetilde{X} \times_X U \to U$ extends uniquely to a $G$-torsor $\pi' : \widetilde{X}' \to X$. We therefore have a commutative diagram

$$\begin{array}{ccccc} \widetilde{X}' & \supseteq & \widetilde{X} \times_X U & \longrightarrow & \widetilde{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \supseteq & U & \xrightarrow{\phi|_U} & X. \end{array}$$

In other words, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{X}' & \overset{\varphi}{\dashrightarrow} & \widetilde{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \overset{\phi}{\dashrightarrow} & X. \end{array}$$

So, Lemma 4.3 tells us that $\widetilde{X}'$ is the product of an abelian variety and a simply connected variety. Then by Proposition 3.1(3), there exists a map $\alpha\colon \widetilde{X}' \to \widetilde{X}$ such that $\pi' = \pi \circ \alpha$. Since $\pi$ and $\pi'$ are both $G$-torsors, hence finite maps of the same degree, $\alpha$ must be an isomorphism. Therefore, $\widetilde{\phi} = \varphi \circ \alpha^{-1}$ is our desired birational map. □

**Proof of Theorem 1.2.** The implication (a) $\Longrightarrow$ (b) is obvious. To show that (b) $\Longrightarrow$ (a), let $X$ be a smooth projective minimal variety defined over $\overline{\mathbb{Q}}$ with numerically trivial canonical divisor, and let $\phi$ be a birational automorphism of $X$. By Proposition 3.1, there exists a minimal split cover $\pi : \widetilde{X} \to X$ defined over $\overline{\mathbb{Q}}$. By Proposition 4.4, $\phi$ lifts to a birational automorphism $\widetilde{\phi}$ of $\widetilde{X}$. By Lemma 4.1(c), it is then enough to show that Medvedev–Scanlon holds for $\widetilde{\phi}$. □

## 5. Proof of Theorem 1.4

Our proof will rely on the following lemma.

**Lemma 5.1.** *Consider the commutative diagram*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\psi} & Y, \end{array}$$

*where $\pi\colon X \to Y$ is a dominant morphism of irreducible varieties, $\phi$ is birational isomorphisms of $X$, $\psi$ is an automorphism of $Y$, and the entire diagram is defined over $\overline{\mathbb{Q}}$. Suppose $\overline{\mathbb{Q}}(X)^\phi = \overline{\mathbb{Q}}$ (and hence, $\overline{\mathbb{Q}}(Y)^\phi = \overline{\mathbb{Q}}$; see Lemma* 4.1(a)*), and there exists a $y \in Y(\overline{\mathbb{Q}})$ whose $\psi$-orbit is dense in $Y$. Assume further that either*

(a) *$\pi$ is birational, or*
(b) *$\phi$ is a (regular) automorphism and* $\dim(X) = \dim(Y) + 1$.

*Then there exists an $x \in X(\overline{\mathbb{Q}})$ whose $\phi$-orbit is dense in $X$.*

**Proof.** (a) Suppose $\pi$ restricts to an isomorphism between dense open subsets $X_0$ of $X$ and $Y_0$ of $Y$. After replacing $y$ by an iterate, we may assume that $y \in Y_0$. We claim that the preimage $x \in X_0$ of $y$ has a dense $\phi$-orbit in $X$. Indeed, set $y_n := \psi^n(y) \in Y$. Then there is a sequence $i_1 \leqslant i_2 \leqslant \dots$ such that the points $y_{i_1}, y_{i_2}, \dots,$ all lie in $Y_0$ and are dense in $Y$. Then $x_n := \phi^n(x)$ are well defined for $n = i_1, i_2, \dots$ and are dense in $X$. This proves the claim.

(b) By [BRS10, Theorem 1.2], $X$ has only finitely many $\phi$-invariant codimension 1 subvarieties. Denote their union by $H \subset X$. Once again, set $y_n := \psi^n(y) \in Y$. The union of the fibers $\pi^{-1}(y_n)$, as $n$ ranges over the non-negative integers, is dense in $X$. Hence, one of these fibers is not contained in $H$. After replacing $y$ by an iterate, we may assume that $\pi^{-1}(y) \not\subset H$. Choose a $\overline{\mathbb{Q}}$-point $x \in \pi^{-1}(y)$ which does not lie in $H$. We claim that the $\phi$-orbit of $x$ is dense in $X$. Indeed, denote Zariski closure of the orbit of $x$ by $Z$. By our construction $\pi(Z)$ contains the $\psi$-orbit of $y$ and thus is dense in $Y$. Hence, $\dim(Y) \leqslant \dim(Z) \leqslant \dim(X) = \dim(Y) + 1$. On the other hand, since $x \notin H$, $Z$ cannot be a hypersurface in $X$. Thus $\dim(Z) = \dim(X) = \dim(Y) + 1$, i.e., $Z = X$, as desired. □

We now proceed with the proof of Theorem 1.4. Since the abundance conjecture is known for threefolds [Kaw92], Theorem 1.2 tells us that the Medvedev–Scanlon Conjecture 1.1 holds for all smooth projective minimal threefolds of Kodaira dimension 0 if and only if it holds for products of Calabi–Yau varieties, hyperkähler varieties, and abelian varieties over $\overline{\mathbb{Q}}$; see Remark 1.3. We are therefore reduced to three possibilities:

(i) $X$ is an abelian threefold.
(ii) $X$ is a product $E \times S$, where $E$ is an elliptic curve and $S$ is a K3 surface.
(iii) $X$ is a smooth Calabi–Yau 3-fold.

The Medvedev–Scanlon conjecture holds in case (i) by [GS17]. The main result of this section, Proposition 5.3, asserts that Conjecture 1.1 also holds in case (ii). This will leave us with case (iii), thus completing the proof of Theorem 1.4.

**Lemma 5.2.** *Suppose $X = E \times S$, where $E$ an elliptic curve and $S$ is a smooth minimal surface with trivial Albanese and $\kappa(S) \geq 0$. Every birational isomorphism $\phi\colon X \dashrightarrow X$ is of the form $\phi = \phi_E \times \phi_S$ with $\phi_E$ an automorphism of $E$ and $\phi_S$ an automorphism of $S$. In particular, every birational isomorphism of $X$ is regular.*

**Proof.** The projection $\pi : X \to E$ is the Albanese map for $X$. Thus $\phi$ induces a birational automorphism $\phi_E$ of $E$ such that $\pi \circ \phi = \phi_E \circ \pi$. Since $E$ is a smooth curve, $\phi_E$ is an automorphism of $E$. Replacing $\phi$ by $\phi \circ (\phi_E^{-1}, \mathrm{id}_S)$, we see that to prove the lemma, we may assume $\phi_E = \mathrm{id}_E$.

Since $X$ is smooth, the indeterminacy locus $I(\phi)$ of $\phi$ has codimension at least 2, and so $I(\phi) \cap X_t$ has codimension at least 1 for all $t \in E$. We therefore obtain a map $f\colon E \to \mathrm{Bir}(S)$ given by $t \mapsto \phi|_{X_t}$. Since $\kappa(S) \geq 0$, $S$ is not ruled, so $S$ is a unique smooth minimal surface in its birational class, and $\mathrm{Bir}(S) = \mathrm{Aut}(S)$, see for example [Bea96, Theorem V.19]. Our goal is to show that the resulting map $f\colon E \to \mathrm{Aut}(S)$ is constant. Choose a point $t_0 \in E$ and let $\sigma := f(t_0) \in \mathrm{Aut}(S)$. After composing $\phi$ with $(1, \sigma^{-1})\colon E \times S \to E \times S$, we may assume that $f(t_0) = 1 \in \mathrm{Aut}(S)$. Since $E$ is irreducible, this implies that the image of $f$ lies in $\mathrm{Aut}^0(S)$. Since $S$ has trivial Albanese, by [Fuj78, Corollary 5.8], $\mathrm{Aut}^0(S)$ is an affine algebraic group. Thus, $f$ must be a constant map, as claimed. We now define $\phi_S$ to be the image of this map. $\square$

**Proposition 5.3.** *Suppose $X = E \times S$, were $E$ an elliptic curve and $S$ is a surface with trivial Albanese and $\kappa(S) \geq 0$. Let $\phi\colon X \dashrightarrow X$ be a birational isomorphism such that $\overline{\mathbb{Q}}(X)^{\phi} = \overline{\mathbb{Q}}$. Then Conjecture* 1.1 *holds for $(X, \phi)$.*

**Proof.** Let $\pi\colon S \to S_{min}$ be the minimal model of $S$. By Lemma 5.1, $\phi$ descends to an automorphism $E \times S_{min} \to E \times S_{min}$ of the form $(\phi_E, \phi_{min})$, where $\phi_E$ is an automorphism of $E$ and $\phi_{min}$ is an automorphism of $S_{min}$. Now consider the commutative diagram

$$\begin{array}{ccc} E \times S & \overset{\phi}{\dashrightarrow} & E \times S \\ \downarrow{\scriptstyle \mathrm{id}\times\pi} & & \downarrow{\scriptstyle \mathrm{id}\times\pi} \\ E \times S_{min} & \xrightarrow{\phi_E \times \phi_{min}} & E \times S_{min} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ S_{min} & \xrightarrow{\phi_{min}} & S_{min}, \end{array}$$

By [BGT15, Theorem 1.3] the Medvedev–Scanlon conjecture holds for the automorphism $\phi_{min}$ of the surface $S_{min}$. By Lemma 5.1(b), $E \times S_{min}$ has a

$\overline{\mathbb{Q}}$-point with a dense $(\phi_E, \phi_{min})$-orbit. Applying Lemma 5.1(a), we conclude that $E \times S$ has a $\overline{\mathbb{Q}}$-point with a dense $\phi$-orbit, as desired. □

## 6. Pseudo-automorphisms that preserve a line bundle

The following result will be used in the proof of Theorem 1.5 in the next section.

**Proposition 6.1.** *Suppose $\phi\colon X \dashrightarrow X$ is a pseudo-automorphism of a smooth projective variety defined over a field $k$ of characteristic 0, $L$ is a line bundle such that $\phi^*(L) \simeq L$, and $Y$ is the closure of the image of the natural rational map $i\colon X \dashrightarrow \mathbb{P}(H^0(X,L)^*)$. Here, as usual $H^0(X,L)$ denotes the finite-dimensional space of global sections of $L$, and $H^0(X,L)^*$ denotes the dual space. Then:*

(a) *$\phi$ induces a linear automorphism $\bar{\phi}$ of the projective space*

$$\mathbb{P}(H^0(X,L)^*)$$

*preserving $Y$.*

*Moreover, assume $k(X)^\phi = k$. Then:*

(b) *There is a dense $\bar{\phi}$-invariant subset $U$ of $Y$ such that the $\bar{\phi}$-orbit of $y$ is dense in $Y$ for every $y \in U$.*

(c) *$Y$ is a rational variety over the algebraic closure $\overline{k}$.*

Note that since $\phi$ is a pseudo-automorphism, it induces an automorphism $\phi^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X)$.

**Proof.** (a) We begin with the following preliminary observation. Suppose $L$ and $L'$ are isomorphic line bundles on a complete variety $X$ defined over $k$. We claim that there is a canonically defined linear isomorphism between the finite-dimensional projective spaces $\mathbb{P}(H^0(X,L))$ and $\mathbb{P}(H^0(X,L'))$. To define this linear isomorphism, write $L = \mathcal{O}_X(D)$ and $L' = O_X(D')$, where $D$ and $D'$ are divisors on $X$. Since $L$ and $L'$ are isomorphic, these divisors are linearly equivalent. That is,

$$D' = D + (f), \tag{6.2}$$

where $(f)$ denotes the divisor associated to a rational function $f \in k(X)$. Once $f$ is chosen, we can define an isomorphism of vector spaces

$$H^0(X,L) \to H^0(X,L')$$
$$\alpha \mapsto f\alpha.$$

The rational function $f$ in (6.2) is uniquely determined by $L$ and $L'$ up to a nonzero scalar factor. The induced isomorphism of projective spaces $\mathbb{P}(H^0(X,L)) \to \mathbb{P}(H^0(X,L'))$ depends only on $L$ and $L'$ and not on the choice of $f$. This proves the claim.

We now apply this claim in the setting of the proposition, with

$$L' := \phi^*(L).$$

The line bundles $L$ and $L'$ are isomorphic by our assumption. On the other hand, $\phi$ induces an isomorphism

$$\phi^* : H^0(X, L')^* \to H^0(X, L)^*$$

via pull-back. Composing with the dual $\mathbb{P}(H^0(X, L)^*) \to \mathbb{P}(H^0(X, L')^*)$ of the isomorphism $\mathbb{P}(H^0(X, L')) \to \mathbb{P}(H^0(X, L))$ constructed above, we obtain a desired automorphism

$$\bar{\phi} \colon \mathbb{P}(H^0(X, L)^*) \to \mathbb{P}(H^0(X, L)^*)$$

such that the diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{P}(H^0(X, L)^*) & \xrightarrow{\bar{\phi}} & \mathbb{P}(H^0(X, L)^*) \end{array}$$

commutes.

(b) Let $Y$ be the closure of image of $X$ in $\mathbb{P}(V)$ under $i$, where

$$V := H^0(X, L)^*.$$

Since $k(X)^\phi = k$, clearly $k(Y)^\phi = k$ as well.

Set $G$ to be the subgroup of $\mathrm{PGL}(V)$ of automorphisms of $\mathbb{P}(V)$ which preserve $Y$. Then $\bar{\phi} \in G$, and $G$ is a closed subgroup of $\mathrm{PGL}(V)$ and hence, a linear algebraic group. Let $G_0$ be the Zariski closure of the subgroup generated by $\bar{\phi}$ inside $G$. Then $G_0$ is an abelian linear algebraic group. Moreover, for any $y \in Y$, the orbit of $y$ under $\phi$ has the same closure in $Y$ as the orbit of $y$ under $G_0$. So, it suffices to show that there is a dense open subset $U \subset Y$ such that every $y \in U$ has a dense orbit under $G_0$. The last assertion is a consequence of Rosenlicht's theorem; see [Ro56, Theorem 2], *cf.* also [BGR17, Theorem 1.1] and [BRS10, Proposition 7.4(1)]; in fact, we can take $U$ to be a dense $G_0$-orbit in $Y$.

(c) Since $U$ is a $G_0$-orbit, it is isomorphic to the homogeneous space $G_0/H_0$, for some subgroup $H_0 \subset G_0$. Since $G_0$ is abelian, $H_0$ is normal in $G_0$. Hence, as a variety, $U$ is isomorphic to the abelian linear algebraic group $G_0/H_0$. Every abelian linear irreducible algebraic group over $\overline{k}$ is isomorphic to a direct product of copies of $\mathbb{G}_a$ and $\mathbb{G}_m$; we conclude that $U$ is rational over $\overline{k}$ and hence, so is $Y$. □

## 7. Proof of Theorem 1.5

Let $X$ be a minimal threefold with $K_X$ torsion. The automorphism $\phi \colon X \to X$ induces an automorphism $\phi^*$ of the nef cone $\mathrm{Nef}(X)$. Every minimal Gorenstein threefold $Y$ with $c_1(Y) = c_2(Y) = 0$ has an étale cover by an abelian variety [SBW94]. So, if $X$ is a Calabi–Yau variety (hence simply connected) we must have $c_2(X) \neq 0$. As mentioned in the introduction, a theorem of Miyaoka [Miy87] then tells us that $c_2(X)$ is positive on the ample cone $\mathrm{Amp}(X)$ and nonnegative on $\mathrm{Nef}(X)$. We first consider the case where $c_2(X)$ is strictly positive on the nef cone. Our proof of Theorem 1.5(1) was motivated by the arguments given in Chapter 4 of [Ki10].

**Lemma 7.1.** *Suppose $\ell : \mathbb{R}^n \to \mathbb{R}$ is a linear function and $C$ is a closed cone in $\mathbb{R}^n$ such that $\ell(z) > 0$ for any $z \in C$ other than the origin. Then for any real number $M \geq 0$, the region $C_M := \{z \in C \mid \ell(z) \leq M\}$ is compact.*

**Proof.** Let $S$ be the intersection of $C$ with the unit sphere. Clearly $S$ is compact. Define the function $f : S \to \mathbb{R}$ given as follows. For $p \in S$, let $I_p$ be the intersection of the line through $p$ and the origin with the strip $0 \leq \ell(z) \leq M$. Since $\ell$ is positive on $C$, $I_p$ is an interval of finite length. Let $f(p)$ be the length of $I_p$. Since $f$ is continuous and $S$ is compact, $f$ attains its maximal value $r$ on $S$. Consequently, $C_M$ is contained in the ball of radius $r$ centered at the origin. Thus $C_M$ is closed and bounded, hence compact. □

**Proof of Theorem 1.5.** (1) Since $c_2(X)$ is strictly positive on $\mathrm{Nef}(X)$, Lemma 7.1 shows that for all $M \geq 0$, the region

$$\{D \in \mathrm{Nef}(X) \mid c_2(X) \cdot D \leq M\}$$

is compact. As a result, $c_2(X)$ achieves a minimum positive value on $\mathrm{Pic}(X) \cap \mathrm{Amp}(X)$ and this value is achieved by only finitely many $D_i$. Taking the sum of these finitely many $D_i$, we obtain an ample class $A$ which is fixed by $\phi^*$. Let $\mathcal{M}$ be an ample line bundle representing the class of $A$. Since the Albanese of $X$ is trivial, rational equivalence is the same as linear equivalence. Since $\phi^* A = A$ in $\mathrm{NS}(X) \otimes \mathbb{C}$, we have $\phi^* \mathcal{M} \simeq \mathcal{M} \otimes \mathcal{N}$ where $\mathcal{N}$ is a torsion line bundle. Replacing $A$ by a scalar multiple, we may assume that $\phi^*(A)$ is isomorphic to $A$ and that $A$ is very ample. If $\phi$ preserves a rational fibration, we are done. Otherwise, with notation as in Proposition 6.1(b), there is a dense set of $y \in Y$ with dense orbit under $\bar{\phi}$. However, $A$ is very ample, so $Y = X$ which gives the desired conclusion.

(2) We will now consider the case where there is a semi-ample divisor $D \neq 0$ on $X$ such that $c_2(X) \cdot D = 0$. Let $\pi : X \to Y$ be the associated $c_2$-contraction. Oguiso shows ([Ogu01, Theorem 4.3]) that there are only finitely many $c_2$-contractions, and so after replacing $\phi$ by a further iterate, we can assume $\phi^*[D] = [D]$. By Proposition 6.1(a), $\phi$ descends to an automorphism $\overline{\phi}$ of $Y$. Since $D$ is nonzero, $Y$ is not a point. We now consider three cases.

**Case 1.** $\dim(Y) = 3$, i.e., $D$ is big. Since contractions have connected fibers, $\pi$ is birational. If $X$ preserves a rational fibration, we are done. Otherwise, Proposition 6.1(c) tells us that $Y$ is rational over $\overline{\mathbb{Q}}$, which is not possible since $X$ has Kodaira dimension 0. So, the Medvedev–Scanlon Conjecture for $\phi$ holds in this case.

**Case 2.** $\dim(Y) = 2$. By [BGT15, Theorem 1.3], the Medvedev–Scanlon conjecture holds for $Y$. Applying Lemma 5.1(b) to the $c_2$-contraction

$$\pi \colon X \to Y,$$

we see that the Medvedev–Scanlon conjecture holds for $X$ as well.

**Case 3.** $\dim(Y) = 1$. By Proposition 6.1(c), $Y \simeq \mathbb{P}^1$ (over $\overline{\mathbb{Q}}$). Let $Z \subseteq \mathbb{P}^1$ be the locus of points $t$ where the fiber $X_t$ is singular. Then $\overline{\phi}(Z) = Z$. Since $Z$ is a finite set, after replacing $\phi$ by a further iterate, we can assume $\overline{\phi}$ fixes $Z$ point-wise. By [VZ01, Theorem 0.2], we know that $Z$ contains at least 3 points. It follows that $\overline{\phi}$ is the identity since it fixes at least three points of $\mathbb{P}^1$. In other words, there exists a rational function on $X$ which is invariant under some iterate of $\phi$, as desired. □

## References


[A15] Amerik, Ekaterina. Some applications of $p$-adic uniformization to algebraic dynamics. *Rational points, rational curves, and entire holomorphic curves on projective varieties*, 3–21. Contemp. Math., 654, Centre Rech. Math. Proc. *Amer. Math. Soc., Providence, RI.* MR3477538, arXiv:1407.1558, doi: 10.1090/conm/654.

[AC08] Amerik, Ekaterina; Campana, Frédéric. Fibrations méromorphes sur certain variétés à fibrè canonique trivial. *Pure Appl. Math. Q.* **4** (2008), no. 2, 509–545. MR2400885 (2009k:14022), Zbl 1143.14035, arXiv:math/0510299, doi: 10.4310/PAMQ.2008.v4.n2.a9.

[Bea83] Beauville, Arnaud. Some remarks on Kähler manifolds with $c_1 = 0$. *Classification of algebraic and analytic manifolds* (Katata, 1982), 1–26, Progr. Math., 39. *Birkhäuser Boston, Boston, MA*, 1983. MR0728605 (86c:32031), Zbl 0537.53057.

[Bea96] Beauville, Arnaud. Complex algebraic surfaces. Second edition. London Mathematical Society Student Texts, 34. *Cambridge University Press, Cambridge*, 1996. x+132 pp. ISBN: 0-521-49510-5; 0-521-49842-2. MR1406314, Zbl 0849.14014, doi: 10.1017/CBO9780511623936.

[BGR17] Bell, Jason P.; Ghioca, Dragos; Reichstein, Zinovy. On a dynamical version of a theorem of Rosenlicht. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **XVII** (2017), no. 1, 187–204. arXiv:1408.4744.

[BGT15] Bell, Jason Pierre; Ghioca, Dragos; Tucker, Thomas John. Applications of $p$-adic analysis for bounding periods for subvarieties under étale maps. *Int. Math. Res. Not. IMRN*, **2015**, no. 11, 3576–3597. MR3373060, Zbl 1349.14002, arXiv:1310.5775, doi: 10.1093/imrn/rnu046.

[BGT16] Bell, Jason P.; Ghioca, Dragos; Tucker, Thomas J. The dynamical Mordell–Lang conjecture. Mathematical Surveys and Monographs, 210. *American Mathematical Society, Providence, RI*, 2016. xiii+280 pp. ISBN: 978-1-4704-2408-4. MR3468757, Zbl 1362.11001, doi: 10.1090/surv/210.

[BRS10] Bell, Jason P.; Rogalski, Daniel; Sierra, Susan J. The Dixmier–Moeglin equivalence for twisted homogeneous coordinate rings. *Israel J. Math.* **180** (2010), 461–507. MR2735073 (2011m:14005), Zbl 1219.16030, arXiv:0812.3355, doi: 10.1007/s11856-010-0111-0.

[Fal94] Faltings, Gerd. The general case of S. Lang's conjecture. *Barsotti Symposium in Algebraic Geometry* (Abano Terme, 1991), 175–182, Perspect. Math.,

15. *Academic Press, San Diego, CA*, 1994. MR1307396 (95m:11061), Zbl 0823.14009, doi: 10.1016/B978-0-12-197270-7.50012-7.

[Fuj78] Fujiki, Akira. On automorphism groups of compact Kähler manifolds. *Invent. Math.* **44** (1978), no. 3, 225–258. MR0481142 (58 #1285), Zbl 0367.32004, doi: 10.1007/BF01403162.

[GS17] Ghioca, Dragos; Scanlon, Thomas. Density of orbits of endomorphisms of abelian varieties. *Tran. Amer. Math. Soc.* **369** (2017), no. 1, 447–466. MR3557780, Zbl 1354.11045, arXiv:1412.2029, doi: 10.1090/tran6648.

[Kaw92] Kawamata, Yujiro. Abundance theorem for minimal threefolds. *Invent. Math.* **108** (1992), no. 2, 229–246. MR1161091 (93f:14012), Zbl 0777.14011, doi: 10.1007/BF02100604.

[Ki10] Kirson, Antonio. Wild automorphisms of varieties with Kodaira dimension 0. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **56** (2010), no. 2, 327–333. MR2733417 (2011m:14069), Zbl 1205.14053, doi: 10.1007/s11565-010-0104-2.

[KM98] Kollár, János; Mori, Shigefumi. Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. *Cambridge University Press, Cambridge*, 1998. viii+254 pp. ISBN: 0-521-63277-3. MR1658959 (2000b:14018), Zbl 1143.14014.

[Laz13] Lazić, Vladimir. Around and beyond the canonical class. *Birational geometry, rational curves, and arithmetic*, 171–203. Simons Symp., *Springer, Cham*, 2013. MR3114929, Zbl 1281.14010, arXiv:1210.7382, doi: 10.1007/978-1-4614-6482-2_9.

[LOP] Lazić, Vladimir; Oguiso, Keiji; Peternell, Thomas. Nef line bundles on Calabi–Yau threefolds, I. Preprint, 2016. arXiv:1601.01273.

[MS14] Medvedev, Alice; Scanlon, Thomas. Invariant varieties for polynomial dynamical systems. *Ann. Math.* (2) **179** (2014), no. 1, 81–177. MR3126567, Zbl 1347.37145. arXiv:0901.2352, doi: 10.4007/annals.2014.179.1.2.

[Miy87] Miyaoka, Yoichi. The Chern classes and Kodaira dimension of a minimal variety. *Algebraic geometry, Sendai, 1985*, 449–476, Adv. Stud. Pure Math., 10. *North-Holland, Amsterdam*, 1987. MR0946247 (89k:14022), Zbl 0648.14006.

[NZ09] Nakayama, Noboru; Zhang, De-Qi. Building blocks of étale endomorphisms of complex projective manifolds. *Proc. Lond. Math. Soc.* (3) **99** (2009), no. 3, 725–756. MR2551469, Zbl 1185.14012, arXiv:0903.3729, doi: 10.1112/plms/pdp015.

[Ogu93] Oguiso, Keiji. On algebraic fiber space structures on a Calabi–Yau 3-fold. *Internat. J. Math.* **4** (1993), no. 3, 439–465. MR1228584 (94g:14019), Zbl 0793.14030, doi: 10.1142/S0129167X93000248.

[Ogu01] Oguiso, Keiji. On the finiteness of fiber-space structures on a Calabi–Yau 3-fold. Algebraic geometry, 11. *J. Math. Sci. (New York)*, **106** (2001), no. 5, 3320–3335. MR1878052 (2003e:14032), Zbl 1087.14509, doi: 10.1023/A:1017959510524.

[OS01] Oguiso, Keiji; Sakurai, Jun. Calabi–Yau threefolds of quotient type. *Asian J. Math.* **5** (2001), no. 1, 43–77. MR1868164 (2002i:14043), Zbl 1031.14022, arXiv:math/9909175, doi: 10.4310/AJM.2001.v5.n1.a5.

[Ols12] Olsson, Martin. Integral models for moduli spaces of $G$-torsors. *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 4, 1483–1549. MR3025749, Zbl 1293.14004, doi: 10.5802/aif.2728.

[Ro56] Rosenlicht, Maxwell. Some basic theorems on algebraic groups. *Amer. J. Math.* **78** (1956), 401–443. MR0082183 (18,514a), Zbl 0073.37601, doi: 10.2307/2372523.

[SBW94] Shepherd-Barron, N. I.; Wilson, P. M. H. Singular threefolds with numerically trivial first and second Chern classes. *J. Algebraic Geom.* **3** (1994), no. 2, 265–281. MR1257323 (95h:14033), Zbl 0807.14031.

[StPrj] The Stacks Project Authors. Stacks Project. http://math.columbia.edu/algebraic_geometry/stacks-git.

[Uen75] Ueno, Kenji. Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack. Lecture Notes in Mathematics, 439. *Springer-Verlag, Berlin-New York*, 1975. xix+278 pp. MR0506253 (58 #22062), Zbl 0299.14007, doi: 10.1007/BFb0070570.

[VZ01] Viehweg, Eckart; Zuo, Kang. On the isotriviality of families of projective manifolds over curves. *J. Algebraic Geom.* **10** (2001), no. 4, 781–799. MR1838979 (2002g:14012), Zbl 1079.14503, arXiv:math/0002203.

[Xie] Xie, Junyi. The existence of Zariski dense orbits for polynomial endomorphisms on the affine plane. To appear in *Compositio Math.* Preprint, 2015. arXiv:1510.07684, doi: 10.1112/S0010437X17007187.

[Xie15] Xie, Junyi. Periodic points of birational transformations on projective surfaces. *Duke Math. J.* **164** (2015), no. 5, 903–932. MR3332894, Zbl 06436784, arXiv:1106.1825, doi: 10.1215/00127094-2877402.

[Zha06] Zhang, Shou-Wu. Distributions in algebraic dynamics. *Surveys in differential geometry*, X, 381–430, Surv. Differ. Geom., 10. *Int. Press, Somerville, MA*, 2006. pp. 381–430. MR2408228 (2009k:32016), Zbl 1207.37057.

(Jason P. Bell) Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada
jpbell@uwaterloo.ca

(Dragos Ghioca) Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
dghioca@math.ubc.ca

(Zinovy Reichstein) Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
reichst@math.ubc.ca

(Matthew Satriano) Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada
msatriano@uwaterloo.ca

This paper is available via http://nyjm.albany.edu/j/2017/23-51.html.