Unitary extensions of pairs of commuting isometric operators and their generalized resolvents

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ABSTRACT. In this paper we study a pair V_1, V_2 of commuting isometric operators in a Hilbert space H, which are not necessarily defined on the whole space. An old question: is there a possibility for an extension of V_1, V_2 to a pair of commuting unitary operators U_1, U_2 in a Hilbert space $\widetilde{H} \supseteq H$? In the case of a unitary V_2 we present a transparent criterion in terms of the original space H. The general case is discussed, as well. We introduce a notion of a generalized resolvent for V_1, V_2. Characteristic properties of the generalized resolvent in terms of H are obtained. In the case of a unitary V_2 all generalized resolvents are parametrized.

CONTENTS

1. Introduction

Let V_1, V_2 be closed isometric operators in a Hilbert space H. We emphasize that V_1, V_2 *are not necessarily defined on the whole space H*. Suppose that

$$(1) \qquad V_1 V_2 h = V_2 V_1 h, \quad h \in D(V_1 V_2) \cap D(V_2 V_1).$$

In other words, V_1 and V_2 commute whenever possible.

Question. Do there exist a Hilbert space $\widetilde{H} \supseteq H$ and commuting unitary operators U_1, U_2 in $\widetilde{H}$, such that $U_1 \supseteq V_1$, $U_2 \supseteq V_2$? This question was studied in a series of papers [5], [6], [2], [8], [9], see also references therein. Some applications were considered in the above papers, as well.

The extension theory of single isometric (symmetric) operators to unitary (respectively self-adjoint) operators is classical. For pairs of commuting isometric (symmetric) operators the corresponding extension theory is complicated and it is still developing. For the case of commuting symmetric operators see, for example, historical notes in a recent paper [12].

When dealing with isometric operators on subspaces, it is often useful to introduce the corresponding partial isometries. For the above Question this would lead to confusions. In fact, the corresponding partial isometries can be noncommutative (see Example 4.3 below). Thus, we can not restrict ourselves only investigating extensions or dilations of partial isometries.

We shall investigate this problem by introducing and using the corresponding generalized resolvents. Observe that generalized resolvents are themselves very valuable and useful objects for interpolation problems and differential equations, see a survey in [13].

If the answer on the above question is affirmative, then we may define the following operator-valued function of two complex variables:

$$(2) \quad \mathbf{R}_{z_1,z_2}$$
$$= \mathbf{R}_{z_1,z_2}(V_1, V_2)$$
$$= P_H^{\widetilde{H}}(E_{\widetilde{H}} + z_1 U_1)(E_{\widetilde{H}} - z_1 U_1)^{-1}(E_{\widetilde{H}} + z_2 U_2)(E_{\widetilde{H}} - z_2 U_2)^{-1}\Big|_H ,$$

$z_1, z_2 \in \mathbb{T}_e$.

The function $\mathbf{R}_{z_1,z_2}(V_1, V_2)$ is called *a generalized resolvent of a pair of isometric operators V_1, V_2 (corresponding to extensions U_1, U_2).*

Let $\widetilde{E}_{k,t}$, $t \in [0, 2\pi]$, be the (right-continuous) spectral family of U_k, $\widetilde{E}_{k,0} = 0$, $k = 1, 2$ (we shall use the terminology from [10]). The following operator-valued function of two real variables:

$$(3) \quad\quad\quad\quad \mathbf{E}_{t_1,t_2} = P_H^{\widetilde{H}}\widetilde{E}_{1,t_1}\widetilde{E}_{2,t_2}\Big|_H , \quad\quad t_1, t_2 \in [0, 2\pi],$$

is said to be *a (strongly right-continuous) spectral function of a pair of isometric operators V_1, V_2 (corresponding to extensions U_1, U_2).*

As it follows from their definitions, a generalized resolvent and a spectral function, which correspond to the same extensions U_1, U_2, are related by the following equality:

$$(4) \quad (\mathbf{R}_{z_1,z_2}h, h)_H = \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}} \right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}} \right) d(\mathbf{E}_{t_1,t_2}h, h)_H,$$

$h \in H$, $z_1, z_2 \in \mathbb{T}_e$. Here the "distribution" function $(\mathbf{E}_{t_1,t_2}h, h)_H$ defines a (nonnegative) finite measure σ on $\mathfrak{B}(\mathbb{R}^2)$. Moreover, we have

$$\sigma((0, 2\pi] \times (0, 2\pi]) = \sigma(\mathbb{R}^2) = \|h\|_H^2.$$

(One may define σ on a semi-ring of rectangles of the form

$$\delta = \{a < t_1 \leq b, \ c < t_2 \leq d\}$$

and then extend by the standard procedure).

For the convenience of the reader we shall recall some basic definitions and properties of a generalized resolvent of an isometric operator. Let V be a closed isometric operator in a Hilbert space H. It is well known that there always exists a unitary extension $U \supseteq V$ in a Hilbert space $\widetilde{H} \supseteq H$. The following operator-valued function:

$$(5) \qquad \mathbf{R}_\zeta = \mathbf{R}_\zeta(V) = P_H^{\widetilde{H}} \left(E_{\widetilde{H}} - \zeta U\right)^{-1}\Big|_H, \qquad \zeta \in \mathbb{T}_e,$$

is said to be *a generalized resolvent of an isometric operator V (corresponding to the extension U)*. An arbitrary generalized resolvent $\mathbf{R}_\zeta$ has the following form ([3]):

$$(6) \qquad \mathbf{R}_\zeta = [E_H - \zeta(V \oplus F_\zeta)]^{-1}, \qquad \zeta \in \mathbb{D},$$

where F_ζ is a function from $\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$ (see Notations). Conversely, an arbitrary function $F_\zeta \in \mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$ defines by relation (6) a generalized resolvent $\mathbf{R}_\zeta$ of the operator V. Moreover, to different functions from $\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$ there correspond different generalized resolvents of the operator V. Formula (6) is known as Chumakin's formula for the generalized resolvents of an isometric operator. Observe that the first such type formula was obtained by Shtraus for a densely defined symmetric operator (the history of this subject and an exposition of the corresponding results was given in [13]).

Chumakin established the following characteristic properties of a generalized resolvent of a closed isometric operator ([3]):

Theorem 1.1. *In order that a family of linear operators R_ζ, acting in a Hilbert space H ($D_{R_\zeta} = H$) and depending on complex parameter ζ ($|\zeta| \neq 1$), be a generalized resolvent of a closed isometric operator, it is necessary and sufficient that the following conditions hold:*

(1) *There exists a number $\zeta_0 \in \mathbb{D} \backslash \{0\}$ and a subspace $L \subseteq H$ such that*

$$(\zeta R_\zeta - \zeta_0 R_{\zeta_0})f = (\zeta - \zeta_0) R_\zeta R_{\zeta_0} f,$$

for arbitrary $\zeta \in \mathbb{T}_e$ and $f \in L$.
(2) *The operator R_0 is bounded and $R_0 h = h$, for all $h \in H \ominus \overline{R_{\zeta_0} L}$.*
(3) *For an arbitrary $h \in H$ the following inequality holds:*

$$\mathrm{Re}(R_\zeta h, h)_H \geq \frac{1}{2}\|h\|_H^2, \qquad \zeta \in \mathbb{D}.$$

(4) *For an arbitrary $h \in H$ $R_\zeta h$ is an analytic vector-valued function of a parameter ζ in $\mathbb{D}$.*
(5) *For an arbitrary $\zeta \in \mathbb{D} \backslash \{0\}$ we have:*

$$R_\zeta^* = E_H - R_{\frac{1}{\bar{\zeta}}}.$$

Theorem 1.2. *In order that a family of linear operators R_ζ ($D_{R_\zeta} = H$, $|\zeta| \neq 1$) in a Hilbert space H be a generalized resolvent of a given closed isometric operator V in H, it is necessary and sufficient that the following conditions hold:*

(1) *For all $\zeta \in \mathbb{T}_e$ and for all $g \in D(V)$ the following equality holds:*

$$R_\zeta(E_H - \zeta V)g = g.$$

(2) *The operator R_0 is bounded and $R_0 h = h$, for all $h \in H \ominus D(V)$.*

(3) *For an arbitrary $h \in H$ the following inequality holds:*

$$\mathrm{Re}(R_\zeta h, h)_H \geq \frac{1}{2}\|h\|_H^2, \qquad \zeta \in \mathbb{D}.$$

(4) *For an arbitrary $h \in H$ $R_\zeta h$ is an analytic vector-valued function of a parameter ζ in $\mathbb{D}$.*

(5) *For an arbitrary $\zeta \in \mathbb{D}\backslash\{0\}$ the following equality is true:*

$$R_\zeta^* = E_H - R_{\frac{1}{\bar\zeta}}.$$

Theorems 1.1, 1.2 played a central role in Chumakin's proof of formula (6). One of our purposes is to obtain some analogs of Theorems 1.1, 1.2 for a generalized resolvent of a pair of commuting isometric operators. An important role will be played by the following class H_2 of analytic functions of two complex variables, which was introduced by Korányi in [5] (We use the original notation of Korányi for this class. Since the Hardy space will not appear in this paper, it will cause no confusion).

Definition 1.3. The class H_2 is the class of functions f of two complex variables z_1, z_2 defined and holomorphic for all $|z_1|, |z_2| \neq 1$ (including ∞) and satisfying the conditions:

(a) $f(\overline{z_1}^{-1}, \overline{z_2}^{-1}) = \overline{f}(z_1, z_2)$ for all $|z_1|, |z_2| \neq 1$.
(b) $f(z_1, z_2) - f(\overline{z_1}^{-1}, z_2) - f(z_1, \overline{z_2}^{-1}) + f(\overline{z_1}^{-1}, \overline{z_2}^{-1}) \geq 0$, for $|z_1|, |z_2| < 1$.
(c) $f(z_1, 0) + f(z_1, \infty) = 0$, $f(0, z_2) + f(\infty, z_2) = 0$ for all $|z_1| \neq 1$ and $|z_2| \neq 1$.

Every function $g \in H_2$ admits the following representation (see [5, formula (26)] and considerations on page 532 in [5]):

$$(7) \quad g(z_1, z_2)$$
$$= \frac{1}{4}\left((E + z_1\widehat{U})(E - z_1\widehat{U})^{-1}(E + z_2\widehat{V})(E - z_2\widehat{V})^{-1}\varepsilon_{0,0}, \varepsilon_{0,0}\right)_{\widehat{\mathfrak{B}}},$$

$z_1, z_2 \in \mathbb{T}_e$, where $\widehat{U}, \widehat{V}$ are some commutative unitary operators in a Hilbert space $\widehat{\mathfrak{B}}$; $\varepsilon_{0,0} \in \widehat{\mathfrak{B}}$. Let $\widehat{E}_{1,t}$, $t \in [0, 2\pi]$, be the (right-continuous) spectral family of $\widehat{U}$, $\widehat{E}_{1,0} = 0$. Let $\widehat{E}_{2,t}$, $t \in [0, 2\pi]$, be the (right-continuous) spectral

family of $\widehat{V}$, $\widehat{E}_{2,0} = 0$. As in relation (4) we may write:

$$(8) \quad g(z_1, z_2) = \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}} \right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}} \right) d \left(\widehat{E}_{1,t_1} \widehat{E}_{2,t_2} \frac{1}{2} \varepsilon_{0,0}, \frac{1}{2} \varepsilon_{0,0} \right)_{\widehat{\mathfrak{B}}}$$

$$= \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}} \right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}} \right) d\mu,$$

$z_1, z_2 \in \mathbb{T}_e$, where μ is a (nonnegative) finite measure on $\mathfrak{B}(\mathbb{R}^2)$ generated by the distribution function $\left(\widehat{E}_{1,t_1} \widehat{E}_{2,t_2} \frac{1}{2} \varepsilon_{0,0}, \frac{1}{2} \varepsilon_{0,0} \right)_{\widehat{\mathfrak{B}}}$. Moreover, we have $\mu((0, 2\pi] \times (0, 2\pi]) = \mu(\mathbb{R}^2)$.

Our main results in the present paper are the following:

(i) Characteristic properties of a generalized resolvent of a given pair of isometric operators in terms of the original Hilbert space (Theorem 3.3).

(ii) A criterion for the existence of a commuting unitary extension for a given pair of isometric operators in terms of the original Hilbert space (Corollary 3.4).

(iii) An analytic parameterization of all generalized resolvents for a given pair: isometric+unitary (Theorem 4.1).

(iv) A simple criterion for the existence of a commuting unitary extension for a given pair: isometric+unitary (Corollary 4.2).

The method we use goes back to Shtraus's ideas of 1954 presented in his remarkable paper [11]. Shtraus characterized generalized resolvents of symmetric operators in terms of the original Hilbert space, see Theorems 3–6 in [11]. In particular, he used Naimark's dilation ideas. The characterization was then used to obtain an analytic parameterization of all generalized resolvents of a closed, densely-defined symmetric operator, see Theorem 7 in [11].

Chumakin applied a similar approach for generalized resolvents of a closed isometric operator. He obtained the above Theorems 1.1 and 1.2. These theorems were used to derive an analytic parameterization (6) of all generalized resolvents of a given closed isometric operator.

We modify the above method for the two-dimensional case. Some of conditions (namely conditions (1), (3) of Theorem 1.1 and condition (3) of Theorem 1.2) we could not transmit to the two-dimensional case. They were replaced by other conditions. An important role, as before, is played by Naimark's dilation idea. Its detailed exposition is given in Theorem 3.1. Notice that Theorem 3.1, probably, can be also derived from Koranyi's Corollary in [5, p. 548]. We present a more direct proof, without using of some interpolation problems (as Theorem 5 in [5]). (Koranyi used a general Naimark's dilation theorem, as well. However, some details were hidden in his proof.)

Besides the application of the (general) Naimark's dilation theorem, there exists another problem: provide some conditions which ensure that the constructed unitary operators will extend the prescribed isometric operators V_1, V_2. This problem will be solved by comparing and analyzing the construction of spectral families and operators in the proofs of Chumakin and in the proof of Theorem 3.1.

In Section 4 we apply our characterization for the case Isometric+unitary. In this case it can be verified directly that a generalized resolvent should have form (74). It was natural to assume that an arbitrary generalized resolvent could be produced by this formula. In order to prove this, we check the conditions of Theorem 3.3.

As corollaries, we obtain conditions for the existence of a commuting unitary extension for a given pair of isometric operators in terms of the original Hilbert space. Observe that Corollary 3.4 has no additional assumptions on isometric operators.

Finally, in Section 5 we present an application of our results to some moment problems.

We now compare our results with existing investigations on the subject. Despite there is a huge amount of papers on generalized resolvents for single operators, we do not know papers on generalized resolvents for commuting pairs of operators.

On the other hand, there exists a parameterization by Moran in [8] of minimal unitary extensions of a pair of commuting isometries under some additional assumptions. It was assumed that for a couple of isometries U, V, defined on closed subspaces, the following conditions hold:

$$(9) \qquad U^n D(V) \subset D(U), \quad U^n R(V) \subset D(U), \qquad n = 0, 1, ...;$$

and

$$(10) \qquad (U^n V f, V f') = (U^n f, f'), \qquad \forall f, f' \in D(V), \ n = 1, 2, ...$$

Observe that condition (9) includes the case of a unitary U. The additional condition (10) ensures the existence of unitary extensions. Notice that we did not assume the existence of unitary extensions in Theorem 4.1. This allowed to obtain a new criterion for the existence of unitary extensions (Corollary 4.2). As for Moran's parameterization, we think it can be used to obtain a parameterization of the corresponding generalized resolvents. However, we prefer to use our previous results. Notice that the methods of Moran are quite different.

Let us mention briefly known criteria for the existence of a commuting unitary extension for a given pair of isometric operators. As far as we know, all the existing criteria use some additional assumptions. Thus, Corollary 3.4 provides a new tool for this problem.

In Koranyi's paper [5] the related results are contained in Lemma 2 (p. 253), Corollary, Remark and Lemma 3 (p. 525). The Remark can be compared with our Corollary 4.2: for the case of a unitary operator

U and an isometric operator V it characterize the existence by an algebraic condition involving all powers of U ($\|U^n V x + V U^n y\| = \|x + y\|$, $x \in D(V), y \in U^{-n} D(V)$).

In Markelov's paper [6] different conditions for the existence of unitary extensions are given with various assumptions, see Theorems 1-4, Corollary on page 208 and Lemmas 1-3. All of them are different from our Corollary 4.2.

Arocena presented a criterion for the existence of unitary extensions of given isometries satisfying (9). His criterion contains condition (10), see [2, Theorem A] (see also Theorem on page 329 of this paper).

Notations 1.4. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and nonnegative integers, respectively; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$; $\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\}$; $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$; $\mathbb{T}_e = \{z \in \mathbb{C} : |z| \neq 1\}$. By $k \in \overline{m,n}$ (or $k = \overline{m,n}$) we mean that $k \in \mathbb{Z}_+ : m \leq k \leq n$; for $m, n \in \mathbb{Z}_+$. By $\mathbb{R}^2$ we denote the two-dimensional real Euclidean space. By $\mathfrak{B}(\mathbb{R}^2)$ we mean the set of all Borel subsets of $\mathbb{R}^2$.

In this paper Hilbert spaces are not necessarily separable, operators in them are supposed to be linear.

If H is a Hilbert space then $(\cdot, \cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If A is bounded then $\|A\|$ denotes its norm. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. By $\operatorname{Lin} M$ we mean the set of all linear combinations of elements from M, and $\operatorname{span} M := \overline{\operatorname{Lin} M}$. By E_H we denote the identity operator in H, i.e., $E_H x = x$, $x \in H$. In obvious cases we may omit the index H. If H_1 is a subspace of H, then $P_{H_1} = P_{H_1}^H$ is an operator of the orthogonal projection on H_1 in H. By $[H]$ we denote a set of all bounded operators on H. For a closed isometric operator V in H we denote: $M_\zeta(V) = (E_H - \zeta V)D(V)$, $N_\zeta(V) = H \ominus M_\zeta(V)$, $\zeta \in \mathbb{C}$; $M_\infty(V) = R(V)$, $N_\infty(V) = H \ominus R(V)$. For a unitary operator U in H we denote: $\mathcal{R}_z(U) := (E_H - zU)^{-1}$, $z \in \mathbb{T}_e$.

By $\mathcal{S}(D; N, N')$ we denote a class of all analytic in a domain $D \subseteq \mathbb{C}$ operator-valued functions $F(z)$, which values are linear nonexpanding operators mapping the whole N into N', where N and N' are some Hilbert spaces.

For a unitary operator U in a Hilbert space H we shall use the following notation:

$$U(z) := (E_H + zU)(E_H - zU)^{-1} = -E_H + 2\mathcal{R}_z(U), \qquad z \in \mathbb{T}_e.$$

It is straightforward to check that ([5, p. 531])

$$(11) \qquad (U(z))^* = -U\left(\frac{1}{\bar{z}}\right), \qquad z \in \mathbb{T}_e \backslash \{0\};$$

$$(12) \qquad U(z) - U\left(\frac{1}{\overline{z}}\right) = 2(1 - |z|^2)\mathcal{R}_z^*(U)\mathcal{R}_z(U) \geq 0, \qquad z \in \mathbb{D}\backslash\{0\}.$$

If we set $U(\infty) := -E_H$, then relation (11) will be valid for all $z \in \mathbb{T}_e \cup \{\infty\}$.

2. Preliminary results

We shall need the following elementary lemma.

Lemma 2.1. *Let μ be a (nonnegative) finite measure on $\mathfrak{B}(\mathbb{R}^2)$. Let $\varphi_j(z;t)$ be an analytic of z in a domain $D \subseteq \mathbb{C}$ complex-valued function depending on a parameter $t \in \mathbb{R}$ with all derivatives $(\varphi_j(z;t))_z^{(k)}$, $k \in \mathbb{Z}_+$ being continuous and bounded as a function of t (with an arbitrary fixed $z \in D$); $j = 1, 2$. Suppose that for each $z_0 \in D$ there exists a closed ball*

$$U(z_0) = \{z \in \mathbb{C}: \ |z - z_0| \leq R_{z_0}\} \subseteq D$$

($R_{z_0} > 0$), such that

$$(13) \qquad \left|(\varphi_j(z;t))_z^{(k)}\right| \leq M_{k,j}(z_0), \qquad z \in U(z_0), \ t \in \mathbb{R}, \ k \in \mathbb{Z}_+,$$

where $M_{k,j}(z_0)$ does not depend on t. Here $j = 1, 2$ is a fixed number. Then

$$(14) \qquad \left((g(z_1, z_2))_{z_1}^{(k)}\right)_{z_2}^{(l)} = \int_{\mathbb{R}^2} (\varphi_1(z_1;t_1))_{z_1}^{(k)} (\varphi_2(z_2;t_2))_{z_2}^{(l)} \, d\mu(t_1, t_2),$$

$k, l \in \mathbb{Z}_+$, where

$$(15) \qquad g(z_1, z_2) = \int_{\mathbb{R}^2} \varphi_1(z_1;t_1)\varphi_2(z_2;t_2) d\mu(t_1, t_2), \quad z_1, z_2 \in D,$$

and all derivatives in (14) exist.

Proof. Firstly, we shall check relation (14) with $l = 0$ by the induction (for $k \in \mathbb{Z}_+$). We may use the definition of the derivative, Lagrange's theorem on a finite increment of a function (the mean value theorem), inequality (13) and the Lebesgue dominated convergence theorem to verify the induction step. Secondly, fix an arbitrary $k \in \mathbb{Z}_+$ and check relation (14) by the induction (for $l \in \mathbb{Z}_+$) in a similar manner. $\qquad\square$

By the induction argument we may write:

$$(16) \qquad \left(\frac{1 + ze^{it}}{1 - ze^{it}}\right)_z^{(k)} = 2k!\frac{e^{ikt}}{(1 - ze^{it})^{k+1}} - \delta_{k,0}, \qquad z \in \mathbb{T}_e, \ t \in \mathbb{R}, \ k \in \mathbb{Z}_+;$$

$$(17) \qquad \left(\frac{u + e^{it}}{u - e^{it}}\right)_u^{(l)} = (-1)^l 2l!\frac{e^{it}}{(u - e^{it})^{l+1}} + \delta_{l,0}, \quad u \in \mathbb{T}_e, \ t \in \mathbb{R}, \ l \in \mathbb{Z}_+.$$

Let $g(z_1, z_2)$ be an arbitrary function which admits representation (8) where μ is a (nonnegative) finite measure on $\mathfrak{B}(\mathbb{R}^2)$ with

$$\mu((0, 2\pi] \times (0, 2\pi]) = \mu(\mathbb{R}^2).$$

By Lemma 2.1 and relations (16), (17) we obtain that

$$(18) \qquad \left((g(z_1, z_2))_{z_1}^{(k)} \right)_{z_2}^{(l)} \bigg|_{(z_1, z_2)=(0,0)} = \begin{cases} s_{0,0}, & \text{if } k = l = 0 \\ 2l! s_{0,l}, & \text{if } k = 0,\ l \in \mathbb{N} \\ 2k! s_{k,0}, & \text{if } k \in \mathbb{N},\ l = 0 \\ 4k! l! s_{k,l}, & \text{if } k, l \in \mathbb{N}, \end{cases}$$

$$(19) \qquad \left((g(u_1^{-1}, z_2))_{u_1}^{(k)} \right)_{z_2}^{(l)} \bigg|_{(u_1, z_2)=(0,0)} = \begin{cases} -2k! s_{-k,0}, & \text{if } k \in \mathbb{N},\ l = 0 \\ -4k! l! s_{-k,l}, & \text{if } k, l \in \mathbb{N}, \end{cases}$$

where $g(u_1^{-1}, z_2)|_{u_1=0} := \lim_{u_1 \to 0} g(u_1^{-1}, z_2)$, $z_2 \in \mathbb{D}$; and therefore $g(u_1^{-1}, z_2)$ is defined on $\mathbb{D} \times \mathbb{D}$;

$$(20) \qquad \left((g(z_1, u_2^{-1}))_{z_1}^{(k)} \right)_{u_2}^{(l)} \bigg|_{(z_1, u_2)=(0,0)} = \begin{cases} -2l! s_{0,-l}, & \text{if } k = 0,\ l \in \mathbb{N} \\ -4k! l! s_{k,-l}, & \text{if } k, l \in \mathbb{N}, \end{cases}$$

where $g(z_1, u_2^{-1})|_{u_2=0} := \lim_{u_2 \to 0} g(z_1, u_2^{-1})$, $z_1 \in \mathbb{D}$; and therefore $g(z_1, u_2^{-1})$ is defined on $\mathbb{D} \times \mathbb{D}$;

$$(21) \qquad \left((g(u_1^{-1}, u_2^{-1}))_{u_1}^{(k)} \right)_{u_2}^{(l)} \bigg|_{(u_1, u_2)=(0,0)} = 4k! l! s_{-k,-l}, \quad k, l \in \mathbb{N},$$

where

$$g(u_1^{-1}, u_2^{-1})|_{u_1=0} = \lim_{u_1 \to 0} g(u_1^{-1}, u_2^{-1}), \quad u_2 \in \mathbb{D} \backslash \{0\};$$

$$g(u_1^{-1}, u_2^{-1})|_{u_2=0} = \lim_{u_2 \to 0} g(u_1^{-1}, u_2^{-1}), \quad u_1 \in \mathbb{D} \backslash \{0\};$$

$$g(u_1^{-1}, u_2^{-1})|_{u_1=u_2=0} = \lim_{u_2 \to 0} g(u_1^{-1}, u_2^{-1})|_{u_1=0};$$

and therefore $g(u_1^{-1}, u_2^{-1})$ is defined on $\mathbb{D} \times \mathbb{D}$. Here

$$(22) \qquad s_{k,l} := \int_{\mathbb{R}^2} e^{ikt_1} e^{ilt_2} d\mu, \quad k, l \in \mathbb{Z},$$

are the trigonometric moments of μ. Thus, *all trigonometric moments of μ are uniquely determined by the function $g(z_1, z_2)$.*

Consider the following function:

$$(23) \qquad f_{m,k}(t) = \begin{cases} \left(\left(\frac{1}{k}\right)^m - (2\pi)^m \right) kt + (2\pi)^m, & 0 \le t \le \frac{1}{k} \\ t^m, & \frac{1}{k} < t \le 2\pi, \end{cases}$$

where $m \in \mathbb{Z}_+$, $k \in \mathbb{N}$. Extend $f_{m,k}(t)$ to a continuous function on the real line with the period 2π. By Weierstrass's approximation theorem there exists a trigonometric polynomial $T_{m,k}(t)$ such that

$$(24) \qquad |f_{m,k}(t) - T_{m,k}(t)| < \frac{1}{k}, \qquad t \in \mathbb{R}.$$

Observe that

$$(25) \qquad |f_{m,k}(t)| \le (2\pi)^m, \qquad t \in \mathbb{R}.$$

By (24) it follows that

$$\text{(26)} \qquad |T_{m,k}(t)| \le (2\pi)^m + 1, \qquad t \in \mathbb{R}.$$

For arbitrary $m, n \in \mathbb{Z}_+$ we may write

$$\text{(27)} \quad \left| \int_{\mathbb{R}^2} t_1^m t_2^n d\mu - \int_{\mathbb{R}^2} T_{m,k}(t_1) T_{n,k}(t_2) d\mu \right|$$

$$\le \left| \int_{\mathbb{R}^2} (t_1^m - T_{m,k}(t_1)) t_2^n d\mu \right| + \left| \int_{\mathbb{R}^2} T_{m,k}(t_1) (t_2^n - T_{n,k}(t_2)) d\mu \right|$$

$$\le \left| \int_{\mathbb{R}^2} (t_1^m - f_{m,k}(t_1)) t_2^n d\mu \right| + \left| \int_{\mathbb{R}^2} (f_{m,k}(t_1) - T_{m,k}(t_1)) t_2^n d\mu \right|$$

$$+ \left| \int_{\mathbb{R}^2} T_{m,k}(t_1) (t_2^n - f_{n,k}(t_2)) d\mu \right|$$

$$+ \left| \int_{\mathbb{R}^2} T_{m,k}(t_1) (f_{n,k}(t_2) - T_{n,k}(t_2)) d\mu \right| \to 0,$$

as $k \to \infty$. Therefore all power moments:

$$\text{(28)} \qquad r_{m,n} := \int_{\mathbb{R}^2} t_1^m t_2^n d\mu, \quad m, n \in \mathbb{Z}_+,$$

are uniquely determined by the function $g(z_1, z_2)$. Since the two-dimensional power moment problem which has a solution with a compact support is determinate (e.g., [7, Theorem B, p. 323]), then we conclude that *the measure μ in representation* (8) *is uniquely determined by the function g.*

Proposition 2.2. *Let* σ_j *(*$j = \overline{1,4}$*) be (nonnegative) finite measures on* $\mathfrak{B}(\mathbb{R}^2)$ *with* $\sigma_j((0, 2\pi]^2) = \sigma_j(\mathbb{R}^2)$*. If*

$$\text{(29)} \qquad s_{k,l}(\sigma_1) - s_{k,l}(\sigma_2) + i s_{k,l}(\sigma_3) - i s_{k,l}(\sigma_4) = 0, \qquad k, l \in \mathbb{Z},$$

then

$$\text{(30)} \qquad \sigma_1 - \sigma_2 + i\sigma_3 - i\sigma_4 = 0.$$

Proof. Observe that the measures σ_j ($j = \overline{1,4}$) satisfy the assumptions on the measure μ introduced after (17). Therefore we may apply the above constructions to these measures. Notice that the function $f_{m,k}(t)$ in (23) depends on m, k, t but do not depend on the measure μ. By (27) for arbitrary $m, n \in \mathbb{Z}_+$ we may write

$$\text{(31)} \quad \left| r_{m,n}(\sigma_1) - r_{m,n}(\sigma_2) + i r_{m,n}(\sigma_3) - i r_{m,n}(\sigma_4) \right.$$

$$- \left(\int_{\mathbb{R}^2} T_{m,k}(t_1) T_{n,k}(t_2) d\sigma_1 - \int_{\mathbb{R}^2} T_{m,k}(t_1) T_{n,k}(t_2) d\sigma_2 \right.$$

$$\left. \left. + i \int_{\mathbb{R}^2} T_{m,k}(t_1) T_{n,k}(t_2) d\sigma_3 - i \int_{\mathbb{R}^2} T_{m,k}(t_1) T_{n,k}(t_2) d\sigma_4 \right) \right| \to 0,$$

as $k \to \infty$. By (29) we conclude that the expression in the round brackets in (31) is equal to zero. Therefore

$$(32) \qquad r_{m,n}(\sigma_1) - r_{m,n}(\sigma_2) + i r_{m,n}(\sigma_3) - i r_{m,n}(\sigma_4) = 0, \qquad m, n \in \mathbb{Z}_+.$$

Extracting the real and the imaginary parts we get

$$(33) \qquad\qquad r_{m,n}(\sigma_1) = r_{m,n}(\sigma_2), \qquad m, n \in \mathbb{Z}_+;$$

$$(34) \qquad\qquad r_{m,n}(\sigma_3) = r_{m,n}(\sigma_4), \qquad m, n \in \mathbb{Z}_+.$$

Since the corresponding two-dimensional power moment problem is determinate, we conclude that $\sigma_1 = \sigma_2$ and $\sigma_3 = \sigma_4$. $\qquad \square$

Proposition 2.3. *Let σ_j $(j = \overline{1,4})$ be (nonnegative) finite measures on $\mathfrak{B}(\mathbb{R}^2)$ with $\sigma_j((0, 2\pi]^2) = \sigma_j(\mathbb{R}^2)$. Let $g_j(z_1, z_2)$ be a function which admits representation* (8) *with σ_j instead of μ; $j = \overline{1,4}$. If*

$$(35) \quad g_1(z_1, z_2) - g_2(z_1, z_2) + i g_3(z_1, z_2) - i g_4(z_1, z_2) = 0, \qquad z_1, z_2 \in \mathbb{T}_e,$$

then

$$(36) \qquad\qquad \sigma_1 - \sigma_2 + i\sigma_3 - i\sigma_4 = 0.$$

Proof. The measures σ_j $(j = \overline{1,4})$ satisfy the assumptions on the measure μ introduced after (17). Moreover, the functions $g_j(z_1, z_2)$ for σ_j are introduced in the same way as $g(z_1, z_2)$ for μ. Calculating derivatives of $g_1(z_1, z_2) - g_2(z_1, z_2) + i g_3(z_1, z_2) - i g_4(z_1, z_2)$ at various points and using relations (18)–(21) we obtain that

$$s_{k,l}(\sigma_1) - s_{k,l}(\sigma_2) + i s_{k,l}(\sigma_3) - i s_{k,l}(\sigma_4) = 0, \qquad k, l \in \mathbb{Z}.$$

By Proposition 2.2 we conclude that relation (36) holds. $\qquad \square$

3. Properties of generalized resolvents

The following theorem is an analog of Theorem 1.1.

Theorem 3.1. *Let an operator-valued function R_{z_1, z_2} be given, which depends on complex parameters $z_1, z_2 \in \mathbb{T}_e$ and which values are linear bounded operators defined on a (whole) Hilbert space H. This function is a generalized resolvent of a pair of closed isometric operators in H (satisfying the commutativity relation (1)) if an only if the following conditions are satisfied:*

(1) $R_{0,0} = E_H$;

(2) $R_{z_1, z_2}^* = R_{\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}}$, $z_1, z_2 \in \mathbb{T}_e \backslash \{0\}$;

(3) *For all $h \in H$, for the function $f(z_1, z_2) := (R_{z_1, z_2} h, h)_H$, $z_1, z_2 \in \mathbb{T}_e$, there exist limits:*

$$f(\infty, z_2) := \lim_{z_1 \to \infty} f(z_1, z_2), \quad f(z_1, \infty) := \lim_{z_2 \to \infty} f(z_1, z_2), \quad z_1, z_2 \in \mathbb{T}_e;$$

$$f(\infty, \infty) = \lim_{z_2 \to \infty} \lim_{z_1 \to \infty} f(z_1, z_2),$$

> *and the extended by these relations function $f(z_1, z_2)$, $z_1, z_2 \in \mathbb{T}_e \cup \{\infty\}$ belongs to H_2.*

Proof. *Necessity.* Let V_1, V_2 be closed isometric operators in a Hilbert space H satisfying relation (1). Suppose that there exist commuting unitary extensions $U_k \supseteq V_k$, $k = 1, 2$, in a Hilbert space $\widetilde{H} \supseteq H$, and $R_{z_1, z_2} = \mathbf{R}_{z_1, z_2}$ be the corresponding generalized resolvent. By the definition of the generalized resolvent we see that condition (1) is satisfied. By (11) for arbitrary $z_1, z_2 \in \mathbb{T}_e \backslash \{0\}$ and $h, g \in H$ we may write

$$
\begin{aligned}
(\mathbf{R}_{z_1, z_2} h, g)_H &= \left(P_H^{\widetilde{H}} U_1(z_1) U_2(z_2)|_H h, g \right)_H = (U_1(z_1) U_2(z_2) h, g)_{\widetilde{H}} \\
&= \left(h, U_1(\overline{z_1}^{-1}) U_2(\overline{z_2}^{-1}) g \right)_{\widetilde{H}} = (h, \mathbf{R}_{\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}} g)_H.
\end{aligned}
$$

Therefore condition (2) holds.

Choose an arbitrary $h \in H$ and set

$$
(37) \qquad f(z_1, z_2) = (U_1(z_1) U_2(z_2) h, h)_{\widetilde{H}}, \qquad z_1, z_2 \in \mathbb{T}_e \cup \{\infty\}.
$$

Here $U_1(\infty) = U_2(\infty) := -E_{\widetilde{H}}$. It is easy to check that this definition is consistent with the definition of $f(z_1, z_2)$ from the statement of the theorem. Observe that the set $\mathbb{T}_e \times \mathbb{T}_e$ is a union of four polycircular domains $\mathbb{D} \times \mathbb{D}$, $\mathbb{D} \times \mathbb{D}_e$, $\mathbb{D}_e \times \mathbb{D}$ and $\mathbb{D}_e \times \mathbb{D}_e$. In each of these domains the function $f(z_1, z_2)$ is holomorphic with respect to each variable. By Hartogs's theorem we conclude that $f(z_1, z_2)$ is holomorphic at each point of $\mathbb{T}_e \times \mathbb{T}_e$. For the infinite points we may use the change of variable $u = \frac{1}{z}$ and proceed in the same manner. Conditions (a)–(c) in the definition of the class H_2 can be checked by relations (11), (12), as it was done in [5, p. 531]. Thus, $f(z_1, z_2) \in H_2$ and condition (3) holds.

Sufficiency. Suppose that an operator-valued function R_{z_1, z_2} satisfies the assumptions of the theorem and conditions (1), (2), (3). By condition (3) and relation (8) we may write:

$$
(38) \qquad (R_{z_1, z_2} h, h)_H = \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}} \right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}} \right) d\mu(\delta; h, h),
$$

$z_1, z_2 \in \mathbb{T}_e$, $h \in H$, where $\mu(\delta; h, h)$ is a (nonnegative) finite measure on $\mathfrak{B}(\mathbb{R}^2)$ such that $\mu((0, 2\pi] \times (0, 2\pi]) = \mu(\mathbb{R}^2)$. Set

$$
(39) \quad \mu(\delta; h, g)
$$
$$
= \frac{1}{4} (\mu(\delta; h + g, h + g) - \mu(\delta; h - g, h - g) + i\mu(\delta; h + ig, h + ig)
$$
$$
- i\mu(\delta; h - ig, h - ig)),
$$

$\delta \in \mathfrak{B}(\mathbb{R}^2)$, $h, g \in H$. Then

$$
(40) \qquad (R_{z_1, z_2} h, g)_H = \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}} \right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}} \right) d\mu(\delta; h, g),
$$

$z_1, z_2 \in \mathbb{T}_e$, $h, g \in H$. The integral of the form $\int_{\mathbb{R}^2} u(t_1, t_2) d\mu(\delta)$ (where $u(t_1, t_2)$ is a complex-valued function on $\mathbb{R}^2$ and $\mu(\delta)$ is a complex-valued function on $\mathfrak{B}(\mathbb{R}^2)$) may be understood as a limit of Riemann–Stieltjes type integral sums, if it exists. This means that we consider partitions of $\mathbb{R}^2$ by rectangles of the following form:

$$\delta_{n,k} := \{t_{1,n-1} < t_1 \leq t_{1,n}, \ t_{2,k-1} < t_2 \leq t_{2,k}\}, \qquad n, k \in \mathbb{Z},$$

and choose arbitrary points $(t_{1;n,k}, t_{2;n,k}) \in \delta_{n,k}$. The integral sum is defined by $\sum_{n,k} u(t_{1;n,k}, t_{2;n,k}) \mu(\delta_{n,k})$. The integral is a limit of integral sums as partitions become arbitrarily fine (i.e., the diameter of partitions tends to zero), if the limit exists, cf. [10, p. 307].

Fix arbitrary $h, g \in H$. From the definition of $\mu(\delta; h, g)$ it follows that $\mu(\delta; g, h) - \overline{\mu(\delta; h, g)} = \sum_{j=1}^{8} \alpha_j \mu_j(\delta)$, $\delta \in \mathfrak{B}(\mathbb{R}^2)$, where $\alpha_j \in \mathbb{C}$ and $\mu_j(\delta)$ are (nonnegative) finite measures on $\mathfrak{B}(\mathbb{R}^2)$ such that

$$\mu_j((0, 2\pi] \times (0, 2\pi]) = \mu_j(\mathbb{R}^2),$$

$j \in \overline{1,8}$. Namely, $\{\alpha_j\}_{j=1}^{8} = \{\frac{1}{4}, -\frac{1}{4}, \frac{i}{4}, -\frac{i}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{i}{4}, -\frac{i}{4}\}$,

$$\begin{aligned}
\{\mu_j\}_{j=1}^{8} = \{&\mu(\delta; g+h, g+h), \mu(\delta; g-h, g-h), \mu(\delta; g+ih, g+ih), \\
&\mu(\delta; g-ih, g-ih), \mu(\delta; h+g, h+g), \mu(\delta; h-g, h-g), \\
&\mu(\delta; h+ig, h+ig), \mu(\delta; h-ig, h-ig)\}.
\end{aligned}$$

Observe that

$$\mu_1 = \mu_5, \ \alpha_1 = -\alpha_5; \ \mu_2 = \mu_6, \ \alpha_2 = -\alpha_6; \ \mu_3 = \mu_8, \ \alpha_3 = -\alpha_8;$$

$$\mu_4 = \mu_7, \ \alpha_4 = -\alpha_7.$$

This follows from the representation (38) for each measure and the established in the previous section fact that the measure is uniquely determined from the representation of type (8). For example,

$$(R_{z_1,z_2}(g-ih), g-ih) = (R_{z_1,z_2}(h+ig), h+ig), \qquad z_1, z_2 \in \mathbb{T}_e,$$

and therefore $\mu_4 = \mu_7$. Consequently, we obtain the following relation:

$$(41) \qquad \mu(\delta; g, h) = \overline{\mu(\delta; h, g)}, \quad \delta \in \mathfrak{B}(\mathbb{R}^2), \ h, g \in H.$$

Choose arbitrary $\alpha, \beta \in \mathbb{C}$ and $h_1, h_2, g \in H$. By (40) we may write

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}}\right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}}\right) d\mu(\delta; \alpha h_1 + \beta h_2, g) \\
&= (R_{z_1,z_2}(\alpha h_1 + \beta h_2), g)_H = \alpha(R_{z_1,z_2} h_1, g)_H + \beta(R_{z_1,z_2} h_2, g)_H \\
&= \alpha \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}}\right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}}\right) d\mu(\delta; h_1, g) \\
&\quad + \beta \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}}\right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}}\right) d\mu(\delta; h_2, g),
\end{aligned}$$

$z_1, z_2 \in \mathbb{T}_e$. Therefore

$$\int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}} \right) \left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}} \right) d(\alpha \mu(\delta; h_1, g) + \beta \mu(\delta; h_2, g)$$
$$- \mu(\delta; \alpha h_1 + \beta h_2, g)) = 0,$$

$z_1, z_2 \in \mathbb{T}_e$. By Proposition 2.3 we obtain that

(42) $\qquad \mu(\delta; \alpha h_1 + \beta h_2, g) = \alpha \mu(\delta; h_1, g) + \beta \mu(\delta; h_2, g),$

$\delta \in \mathfrak{B}(\mathbb{R}^2), \ \alpha, \beta \in \mathbb{C}, \ h_1, h_2, g \in H$. Observe that

$$|\mu(\delta; h, h)| \leq \mu(\mathbb{R}^2; h, h) = \int_{\mathbb{R}^2} d\mu(\delta; h, h) = (R_{0,0}h, h)_H = \|h\|_H^2,$$

for all $\delta \in \mathfrak{B}(\mathbb{R}^2), \ h \in H$. Consequently, $\mu(\delta; h, g)$ is a sesquilinear (bilinear) functional with the norm less or equal to 1. In fact, we may apply Theorem from [1, p. 64] (the proof of this theorem is valid for finite-dimensional Hilbert spaces which are not ranked as Hilbert spaces in [1]). Therefore $\mu(\delta; h, g)$ admits the following representation:

(43) $\qquad \mu(\delta; h, g) = (E(\delta)h, g)_H, \quad \delta \in \mathfrak{B}(\mathbb{R}^2), \ h, g \in H,$

where $E(\delta)$ is a linear bounded operator on H: $\|E(\delta)\| \leq 1$. Observe that

$$(E(\delta)h, h)_H = \mu(\delta; h, h) \geq 0, \quad h \in H, \ \delta \in \mathfrak{B}(\mathbb{R}^2).$$

Therefore $E(\delta) \geq 0$, for all $\delta \in \mathfrak{B}(\mathbb{R}^2)$. Thus, we have

(44) $\qquad 0 \leq E(\delta) \leq E_H, \quad \delta \in \mathfrak{B}(\mathbb{R}^2).$

Notice that

$$(E(\emptyset)h, g)_H = \mu(\emptyset; h, g) = 0,$$
$$(E((0, 2\pi]^2)h, g)_H = \mu((0, 2\pi]^2; h, g) = \mu(\mathbb{R}^2; h, g)$$
$$= (R_{0,0}h, g)_H = (h, g)_H,$$

$h, g \in H$. Therefore

(45) $\qquad E(\emptyset) = 0, \quad E((0, 2\pi]^2) = E_H.$

For arbitrary $\delta_1, \delta_2 \in \mathfrak{B}(\mathbb{R}^2), \ \delta_1 \cap \delta_2 = \emptyset$, and $h, g \in H$, we may write:

$$(E(\delta_1 \cup \delta_2)h, g)_H = \mu(\delta_1 \cup \delta_2; h, g) = \mu(\delta_1; h, g) + \mu(\delta_2; h, g)$$
$$= (E(\delta_1)h, g)_H + (E(\delta_2)h, g)_H$$
$$= ((E(\delta_1) + E(\delta_2))h, g)_H,$$

and therefore

(46) $\qquad E(\delta_1 \cup \delta_2) = E(\delta_1) + E(\delta_2), \quad \delta_1, \delta_2 \in \mathfrak{B}(\mathbb{R}^2) : \ \delta_1 \cap \delta_2 = \emptyset.$

Denote

$$K = \{\delta \in \mathfrak{B}(\mathbb{R}^2) : \ \delta \subseteq (0, 2\pi]^2\}.$$

By Neumark's theorem [10, p. 499] we conclude that there exists a family $\{F(\delta)\}_{\delta \in K}$ of operators of the orthogonal projection in a Hilbert space $\widetilde{H} \supseteq H$ such that

$$(47) \qquad F(\emptyset) = 0, \quad F((0, 2\pi]^2) = E_{\widetilde{H}};$$

$$(48) \qquad F(\delta_1 \cap \delta_2) = F(\delta_1)F(\delta_2), \qquad \delta_1, \delta_2 \in K;$$

$$(49) \qquad F(\delta \cup \widehat{\delta}) = F(\delta) + F(\widehat{\delta}), \qquad \delta, \widehat{\delta} \in K : \ \delta \cap \widehat{\delta} = \emptyset;$$

$$(50) \qquad E(\delta) = P_H^{\widetilde{H}} F(\delta)|_H, \qquad \delta \in K.$$

Moreover, elements of the form $F(\delta)h$, $h \in H$, $\delta \in K$ determine $\widetilde{H}$.

Since μ is σ-additive, then by the latter property of F we conclude that F is weakly σ-additive. In fact, let $\delta = \cup_{k=1}^{\infty}\delta_k$, where $\delta, \delta_k \in K$ and $\delta_i \cap \delta_j = \emptyset$, $i, j \in \mathbb{N} : i \neq j$. For arbitrary $h, u \in H$ and $\widetilde{\delta}, \widehat{\delta} \in K$ we may write:

$$\left(\sum_{k=1}^{N} F(\delta_k) F(\widehat{\delta}) h, F(\widetilde{\delta}) u \right)_{\widetilde{H}} = \left(\sum_{k=1}^{N} F(\delta_k \cap \widetilde{\delta} \cap \widehat{\delta}) h, u \right)_{\widetilde{H}}$$

$$= \sum_{k=1}^{N} \left(E(\delta_k \cap \widetilde{\delta} \cap \widehat{\delta}) h, u \right)_H$$

$$= \sum_{k=1}^{N} \mu \left(\delta_k \cap \widetilde{\delta} \cap \widehat{\delta}; h, u \right)$$

$$\to_{N \to +\infty} \mu \left(\widetilde{\delta} \cap \widehat{\delta} \cap (\cup_{k=1}^{\infty} \delta_k); h, u \right)$$

$$= \left(E \left(\widetilde{\delta} \cap \widehat{\delta} \cap (\cup_{k=1}^{\infty} \delta_k) \right) h, u \right)_H$$

$$= \left(F \left(\widetilde{\delta} \cap \widehat{\delta} \cap (\cup_{k=1}^{\infty} \delta_k) \right) h, u \right)_{\widetilde{H}}$$

$$= \left(F \left(\cup_{k=1}^{\infty} \delta_k \right) F(\widehat{\delta}) h, F(\widetilde{\delta}) u \right)_{\widetilde{H}}.$$

By the linearity we conclude that

$$(S_N x, y)_{\widetilde{H}} \to_{N \to \infty} (S x, y)_{\widetilde{H}}, \qquad x, y \in L,$$

where

$$S_N := \sum_{k=1}^{N} F(\delta_k) = F\left(\cup_{k=1}^{N} \delta_k \right),$$
$$S := F\left(\cup_{k=1}^{\infty} \delta_k \right) = F(\delta),$$
$$L := \mathrm{Lin}\{F(\delta)h : \ h \in H, \delta \in K\}.$$

Choose arbitrary elements $h, g \in \widetilde{H}$. Since L is dense in $\widetilde{H}$, there exist elements $h_k, g_k \in L$ such that $\|h - h_k\| < \frac{1}{k}$, $\|g - g_k\| < \frac{1}{k}$, for all $k \in \mathbb{N}$.

Observe that

$$\left|((S_N - S)h, g)_{\widetilde{H}} - ((S_N - S)h_k, g_k)_{\widetilde{H}}\right|$$
$$\left|((S_N - S)h, g - g_k)_{\widetilde{H}} + ((S_N - S)(h - h_k), g_k)_{\widetilde{H}}\right|$$
$$\leq 2\|h\|\|g - g_k\| + 2\|h - h_k\|(\|g_k - g\| + \|g\|) \to_{k\to\infty} 0,$$

$(N \in \mathbb{N})$. For arbitrary $\varepsilon > 0$ we may choose $k \in \mathbb{N}$ such that

$$\left|((S_N - S)h, g)_{\widetilde{H}} - ((S_N - S)h_k, g_k)_{\widetilde{H}}\right| < \frac{\varepsilon}{2}.$$

There exists $\widehat{N} \in \mathbb{N}$ such that $N > \widehat{N}$ implies

$$\left|((S_N - S)h_k, g_k)_{\widetilde{H}}\right| < \frac{\varepsilon}{2}.$$

Then $\left|((S_N - S)h, g)_{\widetilde{H}}\right| < \varepsilon$. Therefore

$$(51) \qquad (S_N h, g)_{\widetilde{H}} \to_{N\to\infty} (Sh, g)_{\widetilde{H}}, \qquad h, g \in \widetilde{H}.$$

Define the following operator-valued functions:

$$(52) \quad F_{1,t} = F((0, t] \times (0, 2\pi]), \quad F_{2,t} = F((0, 2\pi] \times (0, t]), \qquad t \in [0, 2\pi].$$

For $t < 0$ we set $F_{1,t} = F_{2,t} = 0$, while for $t > 2\pi$ we set $F_{1,t} = F_{2,t} = E_{\widetilde{H}}$. Let us check that $\{F_{j,t}\}$ is a spectral family on $[0, 2\pi]$ such that $F_{j,0} = 0$; $j = 1, 2$. By (47) we see that $F_{j,0} = 0$, $F_{j,2\pi} = E_{\widetilde{H}}$, $j = 1, 2$. If $\lambda \leq \mu$, by (48) we may write

$$F_{1,\lambda} F_{1,\mu} = F((0, \lambda] \times (0, 2\pi]) F((0, \mu] \times (0, 2\pi])$$
$$= F((0, \lambda] \times (0, 2\pi]) = F_{1,\lambda},$$
$$F_{2,\lambda} F_{2,\mu} = F((0, 2\pi] \times (0, \lambda]) F((0, 2\pi] \times (0, \mu])$$
$$= F((0, 2\pi] \times (0, \lambda]) = F_{2,\lambda}.$$

It remains to check that $F_{j,t}$ is right-continuous $(j = 1, 2)$. For points $t \in (-\infty, 0) \cup [2\pi, +\infty)$ it is obvious. For arbitrary $t \in [0, 2\pi)$; $t_k \in [0, 2\pi)$: $t_k > t$, $k \in \mathbb{N}$; $\{t_k\}_1^\infty$ is decreasing and $t_k \to t$ as $k \to \infty$; and arbitrary $h, g \in \widetilde{H}$ we may write:

$$(53) \qquad ((F_{1,t_k} - F_{1,t})h, g)_{\widetilde{H}} = (F((t, t_k] \times (0, 2\pi])h, g)_{\widetilde{H}}$$
$$= (F(\cup_{n=1}^\infty ((t_{n+1}, t_n] \times (0, 2\pi]))h, g)_{\widetilde{H}}$$
$$- \left(F\left(\cup_{n=1}^{k-1}((t_{n+1}, t_n] \times (0, 2\pi])\right)h, g\right)_{\widetilde{H}}$$
$$\to_{k\to\infty} 0.$$

Here we used the weak σ-additivity of F. The monotone sequence of projections $\{F_{1,t_n}\}_{n=1}^\infty$ converges in the strong operator topology to a bounded operator. By (53) we conclude that this operator is $F_{1,t}$. If we would have $\lim_{u\to t+0} F_{1,u} h \neq F_{1,t} h$ for an element $h \in H$, then we could easily construct a sequence $\{t_k\}_{k=1}^\infty$ with above properties and satisfying $\|F_{1,t_k} h - F_{1,t} h\| > \varepsilon$ with some $\varepsilon > 0$. This contradiction shows that $F_{1,t}$ is right-continuous. For $F_{2,t}$ we may use similar arguments.

By (48) we may write

$$(54) \quad F_{1,u}F_{2,v} = F((0,u] \times (0,2\pi])F((0,2\pi] \times (0,v]) = F((0,u] \times (0,v])$$
$$= F((0,2\pi] \times (0,v])F((0,u] \times (0,2\pi]) = F_{2,v}F_{1,u},$$

$u,v \in [0,2\pi]$. Thus, $F_{1,u}$ and $F_{2,v}$ commute for all $u,v \in \mathbb{R}$. Set

$$(55) \qquad\qquad U_k = \int_0^{2\pi} e^{it} dF_{k,t}, \qquad k = 1,2.$$

Observe that U_1, U_2 are commuting unitary operators in $\widetilde{H}$. By (43), (50), (54) we may write

$$(56) \qquad \mu((a,b] \times (c,d];h,h) = (E((a,b] \times (c,d])h,h)_H$$
$$= (F((a,b] \times (c,d])h,h)_{\widetilde{H}}$$
$$= ((F_{1,b} - F_{1,a})(F_{2,d} - F_{2,c})h,h)_{\widetilde{H}},$$

$a,b,c,d \in [0,2\pi]:\ a < b,\ c < d,\ h \in H$. By (38) and (56) we conclude that

$$(57) \quad \left(P_H^{\widetilde{H}}(E_{\widetilde{H}} + z_1 U_1)(E_{\widetilde{H}} - z_1 U_1)^{-1}(E_{\widetilde{H}} + z_2 U_2)(E_{\widetilde{H}} - z_2 U_2)^{-1}\Big|_H h, h\right)_H$$
$$= \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}}\right)\left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}}\right) d(F_{1,t_1}F_{2,t_2}h,h)_{\widetilde{H}}$$
$$= \int_{\mathbb{R}^2} \left(\frac{1 + z_1 e^{it_1}}{1 - z_1 e^{it_1}}\right)\left(\frac{1 + z_2 e^{it_2}}{1 - z_2 e^{it_2}}\right) d\mu(\delta;h,h) = (R_{z_1,z_2}h,h)_H,$$

$z_1, z_2 \in \mathbb{T}_e,\ h \in H$. Consequently, R_{z_1,z_2} is a generalized resolvent of a pair of isometric operators $V_1 = V_2 = o_H$. Here $D(o_H) = \{0\}$, $o_H 0 = 0$. The proof of Theorem 3.1 is complete. $\qquad\square$

Proposition 3.2. *Let an operator-valued function R_{z_1,z_2} be given, which depends on complex parameters $z_1, z_2 \in \mathbb{T}_e$ and which values are linear bounded operators defined on a (whole) Hilbert space H. Let V_1, V_2 be closed isometric operators in H which satisfy relation (1). Suppose that conditions (1)–(3) of Theorem 3.1 are satisfied. Suppose that conditions (1)–(5) of Theorem 1.2 are satisfied with the choices $V = V_1$, $R_\zeta = \frac{1}{2}(E_H + R_{\zeta,0})$, and $V = V_2$, $R_\zeta = \frac{1}{2}(E_H + R_{0,\zeta})$. Then R_{z_1,z_2} is a generalized resolvent of a pair of isometric operators V_1, V_2.*

Proof. Since all conditions of Theorem 3.1 are satisfied, we can use the constructions from its proof. Thus, there exist commuting unitary operators U_1, U_2 in a Hilbert space $\widetilde{H} \supseteq H$ such that

$$(58) \quad R_{z_1,z_2}$$
$$= P_H^{\widetilde{H}}(E_{\widetilde{H}} + z_1 U_1)(E_{\widetilde{H}} - z_1 U_1)^{-1}(E_{\widetilde{H}} + z_2 U_2)(E_{\widetilde{H}} - z_2 U_2)^{-1}\Big|_H,$$

for $z_1, z_2 \in \mathbb{T}_e$. Then

$$(59) \qquad \frac{1}{2}\left(E_H + R_{\zeta,0}\right) = P_H^{\widetilde{H}}(E_{\widetilde{H}} - \zeta U_1)^{-1}\Big|_H, \quad \zeta \in \mathbb{T}_e;$$

$$(60) \qquad \frac{1}{2}\left(E_H + R_{0,\zeta}\right) = P_H^{\widetilde{H}}(E_{\widetilde{H}} - \zeta U_2)^{-1}\Big|_H, \quad \zeta \in \mathbb{T}_e;$$

$$(61) \quad \left(\frac{1}{2}\left(E_H + R_{\zeta,0}\right)h, g\right)_H = \int_{\mathbb{R}} \frac{1}{1 - \zeta e^{it}} d(F_{1,t}h, g)_{\widetilde{H}}, \quad \zeta \in \mathbb{T}_e, \ h, g \in H;$$

$$(62) \quad \left(\frac{1}{2}\left(E_H + R_{0,\zeta}\right)h, g\right)_H = \int_{\mathbb{R}} \frac{1}{1 - \zeta e^{it}} d(F_{2,t}h, g)_{\widetilde{H}}, \quad \zeta \in \mathbb{T}_e, \ h, g \in H.$$

Let us check that $U_1 \supseteq V_1$. Since conditions (1)–(5) of Theorem 1.2 are satisfied with the choice $V = V_1$, $R_\zeta = \frac{1}{2}\left(E_H + R_{\zeta,0}\right)$, then choosing an arbitrary $\zeta_0 \in \mathbb{D}\backslash\{0\}$ and $L := (E_H - \zeta_0 V)D(V)$, we conclude that conditions (1)–(5) of Theorem 1.1 are satisfied, see the proof of Theorem 2 in [3]. Thus, R_ζ is a generalized resolvent of a closed isometric operator in a Hilbert space H and therefore $R_{\zeta_0}^{-1}$ exists and is a bounded operator on H. Moreover, we have $D(V) = R_{\zeta_0} L$ (see the last formula on page 887 in [3]). By condition (1) of Theorem 1.2 we have $Vg = \frac{1}{\zeta_0}\left(E_H - R_{\zeta_0}^{-1}\right)g$, $g \in D(V)$.

Thus, we can apply constructions from the proof of Theorem 1 in [3, p. 880]. Notice that the above operator $V(= V_1)$ coincides with the operator U defined by (30) in [3]. By formula (26) in [3] we may write:

$$(63) \qquad \left(\frac{1}{2}\left(E_H + R_{\zeta,0}\right)h, g\right)_H = \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(E_t h, g)_H,$$

$\zeta \in \mathbb{T}_e, \ h, g \in H$. Comparing relations (61) and (63) we conclude that

$$(64) \quad \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d\left((E_t h, g)_H - (F_{1,t}h, g)_{\widetilde{H}}\right) = 0, \quad \zeta \in \mathbb{T}_e, \ h, g \in H.$$

Therefore (see considerations on page 882 in [3, p. 883])

$$(65) \qquad \int_0^{2\pi} e^{it} d(E_t h, g)_H = \int_0^{2\pi} e^{it} d(F_{1,t}h, g)_{\widetilde{H}}, \qquad h, g \in H.$$

Then (cf. [3, p. 886])

$$(Vh, g)_H = \int_0^{2\pi} e^{it} d(E_t h, g)_H = \int_0^{2\pi} e^{it} d(F_{1,t}h, g)_{\widetilde{H}} = (U_1 h, g)_{\widetilde{H}},$$

$$(66) \qquad\qquad\qquad\qquad h \in D(V), \ g \in H.$$

Therefore $Vh = P_H^{\widetilde{H}} U_1 h$, $h \in D(V)$. By $\|Vh\| = \|U_1 h\|$ we get $U_1 \supseteq V$. Relation $U_2 \supseteq V_2$ can be checked in the same manner. By (58) we see that R_{z_1,z_2} is a generalized resolvent of a pair V_1, V_2. $\qquad\qquad\qquad \square$

Theorem 3.3. *Let an operator-valued function R_{z_1,z_2} be given, which depends on complex parameters $z_1, z_2 \in \mathbb{T}_e$ and which values are linear bounded operators defined on a (whole) Hilbert space H. Let V_1, V_2 be closed isometric*

operators in H which satisfy relation (1). R_{z_1,z_2} is a generalized resolvent of a pair of isometric operators V_1, V_2 if an only if the following conditions are satisfied:

(1) *$R_{0,0} = E_H$.*

(2) *$R_{z_1,z_2}^* = R_{\frac{1}{\bar{z}_1},\frac{1}{\bar{z}_2}}$, $z_1, z_2 \in \mathbb{T}_e\backslash\{0\}$.*

(3) *For all $h \in H$, for the function $f(z_1, z_2) := (R_{z_1,z_2}h, h)_H$, $z_1, z_2 \in \mathbb{T}_e$, there exist limits:*

$$f(\infty, z_2) := \lim_{z_1\to\infty} f(z_1, z_2), \quad f(z_1, \infty) := \lim_{z_2\to\infty} f(z_1, z_2), \quad z_1, z_2 \in \mathbb{T}_e;$$
$$f(\infty, \infty) = \lim_{z_2\to\infty}\lim_{z_1\to\infty} f(z_1, z_2),$$

and the extended by these relations function $f(z_1, z_2)$, $z_1, z_2 \in \mathbb{T}_e \cup \{\infty\}$ belongs to H_2.

(4) *$\frac{1}{2}(E_H + R_{\zeta,0})(E_H - \zeta V_1)g = g$, for all $\zeta \in \mathbb{T}_e$, $g \in D(V_1)$.*

(5) *$\frac{1}{2}(E_H + R_{0,\zeta})(E_H - \zeta V_2)g = g$, for all $\zeta \in \mathbb{T}_e$, $g \in D(V_2)$.*

Proof. *Necessity.* The necessity of conditions (1)–(3) follows from Theorem 3.1. Repeating the arguments from the beginning of the proof of Proposition 3.2 we conclude that relations (59), (60) hold. By condition (1) of Theorem 1.2 with $V = V_1$, $R_\zeta = \frac{1}{2}(E_H + R_{\zeta,0})$, and $V = V_2$, $R_\zeta = \frac{1}{2}(E_H + R_{0,\zeta})$ it follows the validity of conditions (4), (5) of the present theorem, respectively.

Sufficiency. In order to apply Proposition 3.2 it is sufficient to check that conditions (1)–(5) of Theorem 1.2 for the choices $V = V_1$, $R_\zeta = \frac{1}{2}(E_H + R_{\zeta,0})$, and $V = V_2$, $R_\zeta = \frac{1}{2}(E_H + R_{0,\zeta})$ are satisfied. Condition (1) of Theorem 1.2 for these choices coincides with conditions (4), (5) of the present theorem. By Theorem 3.1 and considerations in its proof R_{z_1,z_2} is a generalized resolvent of $V_1 = V_2 = o_H$. Then relations (58), (59), (60) hold. By Theorem 1.2 for $V = o_H$ and the above-mentioned choices of R_ζ we obtain that conditions (3), (4), (5) of Theorem 1.2 are satisfied and they do not depend on V. The required condition (2) of Theorem 1.2 for $V = V_1$, $R_\zeta = \frac{1}{2}(E_H + R_{\zeta,0})$, and $V = V_2$, $R_\zeta = \frac{1}{2}(E_H + R_{0,\zeta})$ follows directly from condition (1) of the present theorem. $\square$

Observe that Theorem 3.3 characterizes a generalized resolvent *in terms of the original space H*. Of course, the existence of the generalized resolvent is equivalent to the affirmative answer on the Question in the Introduction. Thus, Theorem 3.3 gives some conditions for the affirmative answer. We formulate the corresponding result in the following corollary.

Corollary 3.4. *Let V_1, V_2 be closed isometric operators in H which satisfy relation (1). There exist a Hilbert space $\widetilde{H} \supseteq H$ and commuting unitary operators U_1, U_2 in $\widetilde{H}$, such that $U_1 \supseteq V_1$, $U_2 \supseteq V_2$, if and only if there exists an operator-valued function R_{z_1,z_2}, which depends on complex parameters*

$z_1, z_2 \in \mathbb{T}_e$, which values are linear bounded operators defined on a (whole) Hilbert space H, and which satisfies conditions (1)–(5) of Theorem 3.3.

More direct conditions will be obtained in the next section in a particular case of a unitary V_2.

4. The case of commuting isometric and unitary operators

In this section we shall show how Theorem 3.3 allows to parametrize generalized resolvents in the case of commuting isometric and unitary operators.

Let $V_1 = V$ be a closed isometric operator in a Hilbert space H, and $V_2 = U$ be a unitary operator in H. Suppose that relation (1) holds. In our case it takes the following form:

$$(67) \qquad\qquad VUh = UVh, \quad h \in (U^{-1}D(V)) \cap D(V).$$

Suppose that there exist a Hilbert space $\widetilde{H} \supseteq H$ and commuting unitary operators U_1, U_2 in $\widetilde{H}$, such that $U_1 \supseteq V$, $U_2 \supseteq U$. Consider the corresponding generalized resolvent of a pair V, U:

$$(68) \qquad \mathbf{R}_{z_1,z_2} = P_H^{\widetilde{H}} U_1(z_1) U_2(z_2)\Big|_H = P_H^{\widetilde{H}} U_1(z_1) U(z_2)$$
$$= P_H^{\widetilde{H}} U_1(z_1)|_H U(z_2) = (-E_H + 2\mathbf{R}_{z_1}(V))U(z_2),$$

$z_1, z_2 \in \mathbb{T}_e$, where $\mathbf{R}_{z_1}(V)$ is a generalized resolvent of the closed isometric operator V, which corresponds to the unitary extension U_1. On the other hand, we may write:

$$(69) \qquad \mathbf{R}_{z_1,z_2} = P_H^{\widetilde{H}} U_2(z_2) U_1(z_1)\Big|_H = P_H^{\widetilde{H}} U_2(z_2)|_H P_H^{\widetilde{H}} U_1(z_1)\Big|_H$$
$$= U(z_2)(-E_H + 2\mathbf{R}_{z_1}(V)),$$

$z_1, z_2 \in \mathbb{T}_e$. Comparing relations (68), (69) and simplifying we obtain that

$$(70) \qquad \mathbf{R}_{z_1}(V)(E_H - z_2 U)^{-1} = (E_H - z_2 U)^{-1}\mathbf{R}_{z_1}(V), \qquad z_1, z_2 \in \mathbb{T}_e.$$

Therefore

$$(71) \qquad\qquad U\mathbf{R}_{z_1}(V) = \mathbf{R}_{z_1}(V)U, \qquad z_1 \in \mathbb{T}_e.$$

By Chumakin's formula (6) we may write:

$$(72) \qquad\qquad \mathbf{R}_{z_1}(V) = [E_H - z_1(V \oplus \Phi_{z_1})]^{-1}, \qquad z_1 \in \mathbb{D},$$

where $\Phi_{z_1} \in \mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$. By (71) and (72) we obtain that

$$(73) \qquad\qquad (V \oplus \Phi_{z_1})U = U(V \oplus \Phi_{z_1}), \qquad z_1 \in \mathbb{D}.$$

Here the equality for the case $z_1 = 0$ follows by the analyticity of Φ_{z_1}.

Theorem 4.1. *Let V be a closed isometric operator in a Hilbert space H, and U be a unitary operator in H. Suppose that relation (67) holds. Let $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$ be the set of all functions Φ_{z_1} from*

$$\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$$

which satisfy relation (73). *Then the following statements hold:*

(i) *The set of all generalized resolvents of a pair* V, U *is nonempty if and only if* $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V)) \neq \emptyset.$

(ii) *Suppose that* $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V)) \neq \emptyset.$ *An arbitrary generalized resolvent of a pair* V, U *has the following form:*

$$\text{(74)} \qquad \mathbf{R}_{z_1,z_2} = (-E_H + 2\left[E_H - z_1(V \oplus \Phi_{z_1})\right]^{-1})U(z_2),$$

$z_1 \in \mathbb{D}$, $z_2 \in \mathbb{T}_e$, *where* $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$, *and*

$$\text{(75)} \qquad \mathbf{R}_{z_1,z_2} = \mathbf{R}^*_{\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}}, \quad z_1 \in \mathbb{D}_e, \ z_2 \in \mathbb{T}_e \backslash \{0\}.$$

On the other hand, an arbitrary $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$ *defines by relations* (74), (75) *a generalized resolvent of a pair* V, U. *Here, for* $z_1 \in \mathbb{D}_e$, $z_2 = 0$ *we define* $\mathbf{R}_{z_1,z_2}$ *by the weak continuity:* $\mathbf{R}_{z_1,0} = w. - \lim\limits_{z_2 \to 0} \mathbf{R}_{z_1,z_2}.$

Moreover, different functions from $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$ *give different generalized resolvents of a pair* V, U.

Proof. (i) If the set of all generalized resolvents of a pair V, U is nonempty, then by our considerations before the present theorem we see that

$$\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V)) \neq \emptyset.$$

On the other hand, suppose that $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V)) \neq \emptyset.$ Choose an arbitrary function $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$. Define a function $\mathbf{R}_{z_1,z_2}$ for $(z_1, z_2) \in (\mathbb{D} \times \mathbb{T}_e) \cup (\mathbb{D}_e \times (\mathbb{T}_e \backslash \{0\}))$ by relations (74), (75). Let $\mathbf{R}_{z_1}(V)$ be the generalized resolvent of V corresponding to Φ_{ζ_1} by Chumakin's formula. By (73) we obtain that relation (71) holds for $z_1 \in \mathbb{D}$. Therefore (71) holds for all $z_1 \in \mathbb{T}_e$, since the generalized resolvent $\mathbf{R}_\zeta(V)$ has the following property ([3]):

$$\text{(76)} \qquad \mathbf{R}^*_\zeta(V) = E_H - \mathbf{R}_{\frac{1}{\overline{\zeta}}}(V), \qquad \zeta \in \mathbb{T}_e \backslash \{0\}.$$

Consequently, relation (70) holds and we may write:

$$\text{(77)} \quad (-E_H + 2\mathbf{R}_{z_1}(V))U(z_2) = U(z_2)(-E_H + 2\mathbf{R}_{z_1}(V)), \quad z_1, z_2 \in \mathbb{T}_e.$$

By (77) and our definition of $\mathbf{R}_{z_1,z_2}$, for arbitrary $z_1 \in \mathbb{D}_e$, $z_2 \in \mathbb{T}_e \backslash \{0\}$ we may write:

$$\text{(78)} \qquad \mathbf{R}_{z_1,z_2} = \mathbf{R}^*_{\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}} = \left(U\left(\frac{1}{\overline{z_2}}\right)\right)^* \left(-E_H + 2\mathbf{R}_{\frac{1}{\overline{z_1}}}(V)\right)^*$$
$$= U(z_2)\left(-E_H + 2\mathbf{R}_{z_1}(V)\right) = \left(-E_H + 2\mathbf{R}_{z_1}(V)\right)U(z_2).$$

Thus, for all $(z_1, z_2) \in (\mathbb{D} \times \mathbb{T}_e) \cup (\mathbb{D}_e \times (\mathbb{T}_e \backslash \{0\}))$ we have the following representation:

$$\text{(79)} \qquad \mathbf{R}_{z_1,z_2} = (-E_H + 2\mathbf{R}_{z_1}(V))U(z_2).$$

For a fixed $z_1 \in \mathbb{D}_e$ by analyticity of $U(z_2)$ the following limit exists:

$$(80) \qquad w. - \lim_{z_2 \to 0} \mathbf{R}_{z_1, z_2} = (-E_H + 2\mathbf{R}_{z_1}(V)) U(0) =: \mathbf{R}_{z_1, 0}.$$

By (79), (80), (77) we see that

$$(81) \qquad \mathbf{R}_{z_1, z_2} = (-E_H + 2\mathbf{R}_{z_1}(V)) U(z_2) = U(z_2) (-E_H + 2\mathbf{R}_{z_1}(V)),$$

$z_1, z_2 \in \mathbb{T}_e$. Let us check that $\mathbf{R}_{z_1, z_2}$ is a generalized resolvent of a pair V, U by Theorem 3.3. The assumptions of Theorem 3.3 with $V_1 = V$, $V_2 = U$, $R_{z_1, z_2} = \mathbf{R}_{z_1, z_2}$ and H are satisfied. Condition (1) of Theorem 3.3 is satisfied, as well. By (81) for arbitrary $z_1, z_2 \in \mathbb{T}_e \backslash \{0\}$ we may write:

$$\mathbf{R}_{z_1, z_2}^* = (-E_H + 2\mathbf{R}_{z_1}(V))^* (U(z_2))^* = \left(-E_H + 2\mathbf{R}_{\frac{1}{\overline{z_1}}}(V) \right) U \left(\frac{1}{\overline{z_2}} \right)$$
$$= \mathbf{R}_{\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}}.$$

Thus, condition (2) of Theorem 3.3 is satisfied. By (81) we see that

$$\frac{1}{2} (E_H + \mathbf{R}_{\zeta, 0}) = \mathbf{R}_\zeta(V), \quad \frac{1}{2} (E_H + \mathbf{R}_{0, \zeta}) = (E_H - \zeta U)^{-1}, \quad \zeta \in \mathbb{T}_e.$$

Therefore condition (5) of Theorem 3.3 is trivial and condition (4) of Theorem 3.3 follows from the property (1) of Theorem 1.2.

It remains to check condition (3) of Theorem 3.3. Since $\mathbf{R}_\zeta(V)$ is a generalized resolvent of V, then there exists a unitary operator $Q \supseteq V$ in a Hilbert space $\mathbf{H} \supseteq H$ such that

$$\mathbf{R}_\zeta(V) = P_H^{\mathbf{H}}(E_{\mathbf{H}} - \zeta Q)^{-1} \big|_H, \qquad \zeta \in \mathbb{T}_e.$$

Then

$$-E_H + 2\mathbf{R}_{z_1}(V) = P_H^{\mathbf{H}} Q(z_1) \big|_H, \qquad z_1 \in \mathbb{T}_e.$$

Representation (81) takes the following form:

$$(82) \quad \mathbf{R}_{z_1, z_2} = \left(P_H^{\mathbf{H}} Q(z_1) \big|_H \right) U(z_2) = U(z_2) \left(P_H^{\mathbf{H}} Q(z_1) \big|_H \right), \quad z_1, z_2 \in \mathbb{T}_e.$$

Choose an arbitrary element $h \in H$. Set

$$f(z_1, z_2) := (\mathbf{R}_{z_1, z_2} h, h)_H,$$

$z_1, z_2 \in \mathbb{T}_e$. Then

$$(83) \qquad f(z_1, z_2) = (Q(z_1)(U(z_2)h), h)_{\mathbf{H}} = \left(U(z_2) \left(P_H^{\mathbf{H}}(Q(z_1)h) \right), h \right)_H,$$

where $z_1, z_2 \in \mathbb{T}_e$. Since operator-valued functions $Q(z)$ and $U(z)$ are analytic at ∞, we conclude that the limits in condition (3) of Theorem 3.3 exist. Moreover, the limit values $f(\infty, z_2)$, $f(z_1, \infty)$, $f(\infty, \infty)$ may be calculated by the formal substitution of ∞ in representations in (83) using $U(\infty) := -E_H$, $Q(\infty) := -E_{\mathbf{H}}$. Thus, we may use representation (83) for all values $z_1, z_2 \in \mathbb{T}_e \cup \{\infty\}$.

Let us check that $f(z_1, z_2)$ $(z_1, z_2 \in \mathbb{T}_e \cup \{\infty\})$ belongs to the class H_2. Holomorphy of $f(z_1, z_2)$ at (z_1, z_2), $z_1, z_2 \in \mathbb{T}_e \cup \{\infty\}$ follows from holomorphy of $Q(z)$ and $U(z)$ at all points $z \in \mathbb{T}_e \cup \{\infty\}$ and Hartogs's theorem.

By (11), (83) it follows that condition (a) in the definition of H_2 holds. Condition (c) in the definition of H_2 follows by relation (83).

Let us check condition (b) in the definition of H_2. Denote

$$W(z_1) = P_H^{\mathbf{H}} Q(z_1)\big|_H, \quad z_1 \in \mathbb{T}_e \cup \{\infty\}.$$

By (82) we see that

$$(84) \qquad W(z_1)U(z_2) = U(z_2)W(z_1), \quad z_1, z_2 \in \mathbb{T}_e \cup \{\infty\},$$

where the equality for infinite values of z_1 or z_2 holds trivially. By (83) we obtain that $f(z_1, z_2) = (U(z_2)W(z_1)h, h)_H$, $z_1, z_2 \in \mathbb{T}_e \cup \{\infty\}$. Choose arbitrary $z_1, z_2 \in \mathbb{D}$ and write (cf. [5, p. 531])

$$(85) \quad f(z_1, z_2) - f(\overline{z_1}^{-1}, z_2) - f(z_1, \overline{z_2}^{-1}) + f(\overline{z_1}^{-1}, \overline{z_2}^{-1})$$
$$= ((U(z_2) - U(\overline{z_2}^{-1}))(W(z_1) - W(\overline{z_1}^{-1}))h, h)_H.$$

By (12) it follows that operators $W(z_1) - W(\overline{z_1}^{-1})$, $U(z_2) - U(\overline{z_2}^{-1})$ are nonnegative bounded operators on H (for $z_1, z_2 = 0$ it is trivial). By (84) we see that operators $W(z_1) - W(\overline{z_1}^{-1})$ and $U(z_2) - U(\overline{z_2}^{-1})$ commute. Since the product of commuting bounded nonnegative operators is nonnegative, by (85) we conclude that condition (b) in the definition of H_2 holds. Consequently, $f(z_1, z_2) \in H_2$ and all conditions of Theorem 3.3 are satisfied. By Theorem 3.3 we obtain that $\mathbf{R}_{z_1, z_2}$ is a generalized resolvent of the pair V, U.

(ii) If $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V)) \neq \emptyset$, then by property (i) we see that the set of all generalized resolvents of a pair V, U is nonempty. Choose an arbitrary generalized resolvent $\mathbf{R}_{z_1, z_2}$ of a pair V, U. By our considerations before the present theorem we obtain that for $\mathbf{R}_{z_1, z_2}$ relation (74) holds. Relation (75) follows by property (2) in Theorem 3.3.

Choose an arbitrary function $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$. Repeating considerations in the proof of condition (i) we conclude that a function $\mathbf{R}_{z_1, z_2}$, defined by relations (74), (75), is a generalized resolvent of a pair V, U.

For different operator-valued functions Φ_{z_1}, $\widetilde{\Phi}_{z_1}$ from

$$\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$$

there correspond different generalized resolvents of a closed isometric operator V. Suppose that Φ_{z_1}, $\widetilde{\Phi}_{z_1}$ generate the same generalized resolvent $\mathbf{R}_{z_1, z_2}$ of a pair V, U. Writing relation (74) with Φ_{z_1} or $\widetilde{\Phi}_{z_1}$ and $z_2 = 0$ we obtain a contradiction. $\qquad\square$

Corollary 4.2. *Let V be a closed isometric operator in a Hilbert space H, and U be a unitary operator in H. Suppose that relation (67) holds. There exist a Hilbert space $\widetilde{H} \supseteq H$ and commuting unitary operators U_1, U_2 in $\widetilde{H}$, such that $U_1 \supseteq V$, $U_2 \supseteq U$ if and only if a set of all functions Φ_{z_1} from $\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$ which satisfy relation (73) is nonempty.*

Proof. It follows directly from the preceding theorem and the definition of the generalized resolvent. $\qquad\square$

In conditions of Theorem 4.1 we additionally suppose that

(86) $$UD(V) = D(V).$$

In this case condition (67) implies $VU = UV$. Condition (73) is equivalent to

(87) $$\Phi_{z_1} Ug = U\Phi_{z_1}g, \qquad g \in H \ominus D(V), \ z_1 \in \mathbb{D}.$$

Observe that the function $\Phi_{z_1} = 0$ belongs to $\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$ and satisfies (87). Thus, $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$ and therefore the set of generalized resolvents of V, U is nonempty.

Additionally suppose that H is separable and there exists a conjugation J on H such that

(88) $$UJ = JU^{-1}, \quad JD(V) = R(V).$$

Then

(89) $$J(H \ominus D(V)) = H \ominus R(V).$$

Denote

$$U_0 := U|_{H \ominus D(V)}.$$

By the Godič–Lucenko theorem ([4]) for the unitary operator U_0 there exists the following representation:

(90) $$U_0 = KL,$$

where K, L are two conjugations on a Hilbert space $H \ominus D(V)$. Set

$$\Theta = JK : \ H \ominus D(V) \to H \ominus R(V).$$

The operator Θ maps $H \ominus D(V)$ on the whole $H \ominus R(V)$ and $\Theta^{-1} = KJ$. By (88), (90) we obtain that

(91) $$\Theta Ug = U\Theta g, \qquad g \in H \ominus D(V).$$

Then

(92) $$U\Theta^{-1}f = \Theta^{-1}Uf, \qquad f \in H \ominus R(V).$$

Let Ψ_{z_1} be an arbitrary function from $\mathcal{S}(\mathbb{D}; H \ominus D(V), H \ominus D(V))$ such that

(93) $$\Psi_{z_1} U_0 = U_0 \Psi_{z_1}, \qquad z_1 \in \mathbb{D}.$$

Set

(94) $$\Phi_{z_1} = \Theta \Psi_{z_1}, \qquad z_1 \in \mathbb{D}.$$

Observe that Φ_{z_1} belongs to $\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$. By (91), (93) for arbitrary $g \in H \ominus D(V)$ and $z_1 \in \mathbb{D}$ we may write:

$$U\Phi_{z_1}g = U(\Theta(\Psi_{z_1}g)) = \Theta(U_0(\Psi_{z_1}g)) = \Theta(\Psi_{z_1}(U_0g)) = \Phi_{z_1}Ug.$$

Thus, Φ_{z_1} satisfies relation (87). Therefore $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$.

On the other hand, choose an arbitrary $\Phi_{z_1} \in \mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$. Then Φ_{z_1} belongs to $\mathcal{S}(\mathbb{D}; N_0(V), N_\infty(V))$ and satisfies relation (87). Set

$$(95) \qquad \Psi_{z_1} = \Theta^{-1}\Phi_{z_1}, \qquad z_1 \in \mathbb{D}.$$

Then relation (94) holds. Observe that $\Psi_{z_1} \in \mathcal{S}(\mathbb{D}; H \ominus D(V), H \ominus D(V))$. Fix an arbitrary $z_1 \in \mathbb{D}$. By (87), (92) for arbitrary $g \in H \ominus D(V)$ we may write:

$$\Psi_{z_1}U_0 g = \Theta^{-1}(\Phi_{z_1}(Ug)) = \Theta^{-1}(U(\Phi_{z_1}g));$$

$$U_0\Psi_{z_1}g = U(\Theta^{-1}(\Phi_{z_1}g)) = \Theta^{-1}(U(\Phi_{z_1}g)).$$

Therefore relation (93) holds.

Finally, we notice that by virtue of Cayley's transformation the case of commuting pairs of symmetric operators may be investigated. This will be done elsewhere.

Example 4.3. Let $H = \mathbb{C}^2$ be the two-dimensional space of complex vectors (of length 2), and the standard basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Denote $H_j = \mathrm{Lin}\{e_j\}$, $j = 1, 2$. Let a unitary operator U be given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and an isometric operator V with the domain $D(V) = H_1$ be defined by

$$V_1\alpha\vec{e}_1 = \alpha\vec{e}_1, \qquad \alpha \in \mathbb{C}.$$

Observe that

$$D(UV) \cap D(VU) = \{0\},$$

and therefore U and V commute. Define the corresponding partial isometry

$$V_0 = V \oplus 0.$$

Notice that

$$V_0U\vec{e}_2 = \vec{e}_1, \quad UV_0\vec{e}_2 = 0.$$

Thus, $V_0U \neq UV_0$.

It is easy to see that V_0 can be extended to E_H. Thus, there exist a generalized resolvent. It is readily checked that the set $\mathcal{S}_{V,U}(\mathbb{D}; N_0(V), N_\infty(V))$ consists of a unique function $\Phi_{z_1} = E|_{H_2}$, $z_1 \in \mathbb{D}$.

5. An application to moment problems

One of the most powerful applications of the theory of generalized resolvents is its application to interpolation problems. In this section we shall briefly describe an application of our results to a (rather typical) moment problem.

The (semi-)truncated two-dimensional trigonometric moment problem consists of finding a (nonnegative) measure $\mu(\delta)$, $\delta \in \mathfrak{B}(\mathbb{R}^2)$, $\mu((0, 2\pi]^2) = \mu(\mathbb{R}^2)$ such that

$$(96) \qquad \int e^{imt_1} e^{int_2} d\mu(t_1, t_2) = s_{m,n}, \qquad -M \le m \le M, \quad n \in \mathbb{Z},$$

where $\{s_{m,n}\}_{-M \le m \le M;\ n \in \mathbb{Z}}$ is a prescribed set of complex numbers (called moments), and M is a fixed positive integer.

Suppose that the moment problem (96) has a solution μ. Set

$$p(t_1, t_2) = \sum_{m=0}^{M} \sum_{n=-N}^{N} \alpha_{m,n} e^{imt_1} e^{int_2}, \qquad \alpha_{m,n} \in \mathbb{C};\ N \in \mathbb{N};\ t_1, t_2 \in \mathbb{R}.$$

Then

$$0 \le \int |p|^2 d\mu = \sum_{m,m'=0}^{M} \sum_{n,n'=-N}^{N} \alpha_{m,n} \overline{\alpha_{m',n'}} s_{m-m',n-n'}.$$

Moreover, by (96) we see that

$$(97) \qquad\qquad s_{-m,n} = \overline{s_{m,-n}}, \qquad 0 \le m \le M, \quad n \in \mathbb{Z}.$$

On the other hand, suppose that condition (97) holds and

$$(98) \qquad\qquad \sum_{m,m'=0}^{M} \sum_{n,n'=-N}^{N} \alpha_{m,n} \overline{\alpha_{m',n'}} s_{m-m',n-n'} \ge 0,$$

for all $\alpha_{m,n} \in \mathbb{C}$ and $N \in \mathbb{N}$. By the well-known lemma [5, Lemma 1] there exist a Hilbert space H and a sequence $\{x_{m,n}\}_{0 \le m \le M;\ n \in \mathbb{Z}}$ of elements of H, which span H, such that

$$(99) \qquad (x_{m,n}, x_{m',n'})_H = s_{m-m',n-n'}, \qquad 0 \le m, m' \le M, \quad n, n' \in \mathbb{Z}.$$

Consider the following operators:

$$(100)\ V_1 \sum_{0 \le m \le M-1;\ n \in \mathbb{Z}} a_{m,n} x_{m,n} = \sum_{0 \le m \le M-1;\ n \in \mathbb{Z}} a_{m,n} x_{m+1,n}, \quad a_{m,n} \in \mathbb{C},$$

$$(101) \qquad V_2 \sum_{0 \le m \le M;\ n \in \mathbb{Z}} b_{m,n} x_{m,n} = \sum_{0 \le m \le M;\ n \in \mathbb{Z}} b_{m,n} x_{m,n+1}, \quad b_{m,n} \in \mathbb{C},$$

where all but finite number of $a_{m,n}$, $b_{m,n}$ are zeros (this will be assumed in what follows in similar situations). It is not hard to verify that V_1, V_2 are well defined. For example, let us check that V_1 is well-defined. Suppose that

$$\sum_{0 \le m \le M-1;\ n \in \mathbb{Z}} a_{m,n} x_{m,n} = \sum_{0 \le m \le M-1;\ n \in \mathbb{Z}} a'_{m,n} x_{m,n}, \quad a_{m,n}, a'_{m,n} \in \mathbb{C}.$$

By (99) we may write

$$\left\| \sum_{0 \le m \le M-1;\ n \in \mathbb{Z}} (a_{m,n} - a'_{m,n})x_{m+1,n} \right\|^2$$

$$= \sum_{m,n,k,l} (a_{m,n} - a'_{m,n})\overline{(a_{k,l} - a'_{k,l})}(x_{m+1,n}, x_{k+1,l})$$

$$= \sum_{m,n,k,l} (a_{m,n} - a'_{m,n})\overline{(a_{k,l} - a'_{k,l})}(x_{m,n}, x_{k,l})$$

$$= \left\| \sum_{0 \le m \le M-1;\ n \in \mathbb{Z}} (a_{m,n} - a'_{m,n})x_{m,n} \right\|^2 = 0.$$

It is directly checked that V_1 and V_2 are linear and isometric. Moreover, they commute on $D(V_1)$. Set

$$V = \overline{V_1}, \quad U = \overline{V_2}.$$

Observe that U is unitary and V is a closed linear isometric operator. It is not hard to check that relation (67) holds. Moreover, relation (86) holds, as well. As it was shown after formula (86), the latter means that the set of all generalized resolvents of a pair V, U is nonempty. Thus, there exist a Hilbert space $\widetilde{H} \supseteq H$ and commuting unitary operators U_1, U_2 in $\widetilde{H}$, such that $U_1 \supseteq V$, $U_2 \supseteq U$. By the induction argument we get

$$x_{m,n} = U_1^m U_2^n x_{0,0}, \qquad 0 \le m \le M, \quad n \in \mathbb{Z}.$$

Therefore

$$s_{m,n} = (x_{m,n}, x_{0,0})_{\widetilde{H}} = (U_1^m U_2^n x_{0,0}, x_{0,0})_{\widetilde{H}}$$

$$= \int e^{imt_1} e^{int_2} d\widetilde{\mu}(t_1, t_2),$$

$0 \le m \le M$, $n \in \mathbb{Z}$. Here $\mathbf{E}_{t_1,t_2}$ is a (strongly right-continuous) spectral function of a pair of isometric operators V, U (corresponding to extensions U_1, U_2) and the "distribution" function $(\mathbf{E}_{t_1,t_2}h, h)_H$ defines a (nonnegative) finite measure $\widetilde{\mu}$ on $\mathfrak{B}(\mathbb{R}^2)$: $\widetilde{\mu}((0, 2\pi]^2]) = \widetilde{\mu}(\mathbb{R}^2)$.

Taking into account relation (97) we conclude that $\widetilde{\mu}$ is a solution of the moment problem (96). We have proved the following theorem.

Theorem 5.1. *Let the truncated two-dimensional trigonometric moment problem* (96) *be given. The moment problem has a solution if and only if conditions* (97)*,* (98) *hold.*

As we have seen, spectral functions of V, U generate solutions of the moment problem (96). Thus, we may use the corresponding generalized resolvents (all of them are parameterized by Theorem 4.1). We intend to discuss a description of all solutions in a separate paper.

References

[1] AHIEZER, NAUM I.; GLAZMAN, IZRAIL M. Teoriya lineĭnyh operatorov v gil'bertovom prostranstve. [The theory of linear operators in Hilbert space]. *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad*, 1950. 483 pp. MR0044034, Zbl 0041.22905.

[2] AROCENA, RODRIGO. On the extension problem for a class of translation invariant positive forms. *J. Operator Theory* **21** (1989), no. 2, 323–347. MR1023319, Zbl 0709.47002.

[3] ČUMAKIN M. E. Generalized resolvents of isometric operators. *Sibirsk. Mat. Ž* **8** (1967), 876–892. MR0217627, Zbl 0156.15202.

[4] GODIČ V. I., LUCENKO I. E. On the representation of a unitary operator as a product of two involutions. *Uspehi Mat. Nauk.* **20** (1965), no. 6, 64–65. MR0190750.

[5] KORÁNYI ADAM. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.* **101** (1961), 520–554. MR0136765, Zbl 0111.11501, doi: 10.2307/1993476.

[6] MARKELOV YU. S. An extension of a pair of isometric operators up to commuting. *Uchen. zapiski Ulyanovskogo pedinstituta* **21** (1967), no. 8, 205–213.

[7] MCGREGOR, JAMES L. Solvability criteria for certain N-dimensional moment problems. *J. Approx. Theory* **30** (1980), no. 4, 315–333. MR0616957, Zbl 0458.41025, doi: 10.1016/0021-9045(80)90034-9.

[8] MORÁN, MARÍA DOLORES. On commuting isometries. *J. Operator Theory* **24** (1990), 75–83. MR1086545, Zbl 0769.47004.

[9] MORÁN, MARÍA DOLORES. Unitary extensions of a system of commuting isometric operators. *Operator extensions, interpolation of functions and related topics* (Timişoara, 1992), 163–169, Oper. Theory Adv. Appl., 61. *Birkhäuser, Basel*, 1993. MR1246585, Zbl 0797.47003.

[10] RIESZ, FRÉDÉRIC; SZ.-NAGY, BÉLA. Vorlesungen über Funktionalanalysis. [Lectures on functional analysis.] "Mir", Moscow, 1979. (Russian edition).

[11] ŠTRAUS A.V. Generalized resolvents of symmetric operators. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **18** (1954), 51–86. MR0062954, Zbl 0055.10903.

[12] ZAGORODNYUK, SERGEY M. On commuting symmetric operators. *Methods Funct. Anal. Topology* **18** (2012), no. 2, 198–200. MR2978195, Zbl 1258.47037.

[13] ZAGORODNYUK, SERGEY M. Generalized resolvents of symmetric and isometric operators: the Shtraus approach. *Ann. Funct. Anal.* **4** (2013), no. 1, 175–285. MR3004219, Zbl 1290.47009, arXiv:1208.5530.

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