New York Journal of Mathematics

New York J. Math. **23** (2017) 351–364.

Hopf algebra structures and tensor products for group algebras

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Abstract. The modular group algebra of an elementary abelian p-group is isomorphic to the restricted enveloping algebra of a commutative restricted Lie algebra. The different ways of regarding this algebra result in different Hopf algebra structures that determine cup products on cohomology of modules. However, it is proved in this paper that the products with elements of the polynomial subring of the cohomology ring generated by the Bocksteins of the degree one elements are independent of the choice of these coalgebra structures.

Contents

1. Introduction

This paper concerns the group algebra kE of an elementary abelian p-group E of order p^r over a field k of characteristic p. This algebra has a natural coalgebra structure $kE \to kE \otimes_k kE$ given by $g \mapsto g \otimes g$ for each g in E. On the other hand, if $E = \langle g_1, \dots, g_r \rangle$, a change of variables $x_i = g_i - 1$ realizes kE as a truncated polynomial ring $k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)$. This is isomorphic to the restricted enveloping algebra of the restricted p-Lie algebra k^r with trivial bracket and p-power operation. Again, there is a natural Hopf

Received December 15, 2016.

2010 *Mathematics Subject Classification.* 20J06 (primary), 20C20.

Key words and phrases. Coproduct, elementary abelian group, Hochschild cohomology, Hopf algebra, tensor product of modules.

JFC was partially supported by NSA grant H98230-15-1-0007 and by Simons Foundation grant 054813-01. SBI was partially supported by NSF grant DMS-1503044. JFC would like to thank the University of Utah for kind hospitality during his visit when the work on this paper was started.

ISSN 1076-9803/2017

algebra structure, this time given by the map $x_i \mapsto x_i \otimes 1 + 1 \otimes x_i$. The two coalgebra structures are not the same and they define different tensor products on kE-modules as well as different actions of the cohomology ring $H^*(E,k) \cong \operatorname{Ext}^*_{kE}(k,k)$ on $\operatorname{Ext}^*_{kE}(M,N)$ for kE-modules M, N.

The differences in the Hopf structures have shown up in several works. For example, Avrunin and Scott [1] exploited a change in the coalgebra structure to prove a conjecture of the first author [3] that the rank variety and the support variety of a kE-module are homeomorphic. In [6, 10] the authors define bundles on projective space using modules of constant Jordan type and the Lie coalgebra map. The construction is not available with the group coalgebra map. Both of these works used the fact that with the Lie algebra structure there is an abundance of sub-Hopf algebras generated by units in the algebra. The immediate motivation for this paper is the desire to make efficient use of categorical equivalences and functors relating commutative algebra and group representation theory; see [7]. The fact that the Hopf algebra structures differ has been an obstruction to this end.

For any Hopf algebra A over k and A-module M, the cohomology ring $\operatorname{Ext}^*_A(M,M)$ is a module over the cohomology ring $\operatorname{Ext}^*_A(k,k)$. The action is given by a homomorphism of rings

$$\theta^M \colon \operatorname{Ext}^*_A(k,k) \longrightarrow \operatorname{Ext}^*_A(M,M)$$

described as follows: take a homogeneous element ζ of $\operatorname{Ext}^n_A(k,k)$, regard it as a length-$n$ exact sequence beginning and ending in the trivial module k, then tensor over k with M. The image is the class of that sequence. The map, in general, depends on the coalgebra structure. The primary result of this paper is that for the group algebra of an elementary abelian group the dependence is not so bad.

Specifically, for E an elementary abelian p-group, if S is the polynomial subring of $\mathrm{H}^*(E,k)$ generated by the Bocksteins of the degree one elements, then the restriction of θ^M to S is the same for both the group and the Lie coalgebras structures on kE. As a direct corollary one gets that for $\zeta \in S$ and L_ζ the kE-module introduced in [4], the isomorphism class of $L_\zeta \otimes_k M$ does not depend on which of the two Hopf algebra structures is used to define the action on the tensor product.

A key input in our work is the fact, proved by Pevtsova and Witherspoon [12], that for any Hopf algebra A, the map θ^M factors through the Hochschild cohomology ring $\mathrm{HH}^*(A/k;A)$. The advantage gained by this observation is that the first map, to $\mathrm{HH}^*(A/k;A)$, depends on the coalgebra structure and not on M, while the second depends on M and not on the choice of coalgebra structures. So it is sufficient to show that, for $A = kE$, the first map is the same on the elements of S regardless of the coalgebra. This is accomplished by a straightforward calculation using the fact that E is a direct product of cyclic groups.

Section 2 of the paper is devoted to preliminaries on Hopf algebras and cohomology, mainly a detailed proof of the factorization of θ^M discussed above. Basic facts about the cohomology of elementary abelian p-groups are recalled in Section 3, while Section 4 presents a proof of the main theorem. Results on the tensor products of L_ζ modules are presented in Section 5.

2. Hopf algebras and cohomology

This section concerns the cohomology of modules over Hopf algebras. The main result is Theorem 2.7, due to Pevtsova and Witherspoon [12, Lemma 13]. We present a detailed proof because the constructions of the maps involved in the statement of the result are of critical importance in the next section.

Let k denote a field and A a Hopf algebra over k, with unit $\eta\colon k \to A$, coalgebra map $\Delta\colon A \to A \otimes_k A$, and counit $\varepsilon\colon A \to k$. We assume that A has an antipode σ, that is to say, σ is the inverse of the identity on A, under the convolution product. We adapt Sweedler's notation and write

$$\Delta(\alpha) = \sum_{(\alpha)} \alpha_1 \otimes \alpha_2 \quad \text{for } \alpha \in A.$$

Unless stated to the contrary, the term "module" means a left module.

Construction 2.1. Let M be an A-module. Recall that for each A-module X, there is a structure of an A-module on $X \otimes_k M$ induced by the diagonal:

$$\alpha \cdot (x \otimes m) = \sum_{(\alpha)} \alpha_1 x \otimes \alpha_2 m.$$

The assignment $X \mapsto X \otimes_k M$ defines an additive functor, that we denote θ^M_Δ, on the category of A-modules, and has the following properties.

(1) The natural map $M \to k \otimes_k M = \theta^M_\Delta(k)$ that sends m to $1 \otimes m$ is an isomorphism of left A-modules.
(2) When M is projective, so is the A-module $\theta^M_\Delta(X) = X \otimes_k M$.

These are standard computations. It follows that there is an induced homomorphism of graded k-algebras:

$$\Theta^M_\Delta\colon \operatorname{Ext}^*_A(k,k) \longrightarrow \operatorname{Ext}^*_A(M,M)\,. \tag{2.2}$$

The notation is intended to emphasize the fact that the map depends on the coalgebra structure on A.

We write A^{e} for the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ of A. Since k is a field, the Hochschild cohomology of A as a k-algebra can be introduced as

$$\operatorname{HH}^*(A/k; A) = \operatorname{Ext}^*_{A^{\mathrm{e}}}(A, A)\,.$$

An A^{e}-module is the same thing as a left-right A-bimodule. In particular, A is naturally an A^e-module, with action defined by $(\alpha \otimes \beta) \cdot a = \alpha a \beta$.

Construction 2.3. Given an A-module M and an A^{e}-module Y, there is a residual A-module structure on $Y \otimes_A M$, defined by

$$\alpha \cdot (y \otimes m) = (\alpha y) \otimes m\,,$$

for $\alpha \in A$, $y \in Y$ and $m \in M$. The assignment $Y \mapsto Y \otimes_A M$ is an additive functor, denoted ψ^M, from A^{e}-modules to A-modules. The next assertions are immediate.

(1) The natural map $\psi^M(A) = A \otimes_A M \to M$ that sends $a \otimes m$ to am, is an isomorphism of A-modules.
(2) When P is a projective A^{e}-module, the A-module $\psi^M(P)$ is projective.

It follows that ψ^M induces a homomorphism of graded k-algebras:

$$\Psi^M\colon \operatorname{HH}^*(A/k;A) \longrightarrow \operatorname{Ext}^*_A(M,M)\,. \tag{2.4}$$

Note that this map is entirely independent of the coalgebra structure on A.

Construction 2.5. Let X be an A-module. Then $X \otimes_k A$ has a structure of an A-module induced by the diagonal Δ. It also has a right A-module action induced by the right action of A on itself. In short, $X \otimes_k A$ is a left A^{e}-module, with action determined by

$$(\alpha \otimes \beta) \cdot (x \otimes a) = \sum_{(\alpha)} \alpha_1 x \otimes \alpha_2 a \beta.$$

The assignment $X \mapsto X \otimes_k A$ defines an additive functor, that we denote ϕ_Δ, from A-modules to A^{e}-modules. This has the following properties.

(1) The natural isomorphism $A \xrightarrow{\cong} k \otimes_k A = \phi_\Delta(k)$, mapping $a \to 1 \otimes a$, is one of A^{e}-modules, where the A^{e}-action on A is the usual one.
(2) The A^{e}-linear map $\iota\colon A^{\mathrm{e}} \to \phi_\Delta(A) = A \otimes_k A$ where $1 \otimes 1$ maps to $1 \otimes 1$, is an isomorphism, with inverse defined by the assignment

$$\alpha \otimes \beta \mapsto \sum_{(\alpha)} \alpha_1 \otimes \sigma(\alpha_2)\beta\,.$$

In particular, $\phi_\Delta(A)$ is a free A^{e}-module of rank one, and $\phi_\Delta(P)$ is projective whenever P is a projective A-module.

Statement (1) is readily verified, given that $\varepsilon\colon A \to k$ is the counit of the coalgebra structure on A; that is to say, for any $\alpha \in A$, one has

$$\sum_{(\alpha)} \varepsilon(\alpha_1)\alpha_2 = \alpha\,.$$

As to (2), since the map ι is A^{e}-linear, by construction, it suffices to verify that its composition with the given map (henceforth denoted ι^{-1}, in anticipation) is the identity. Moreover, ι^{-1} is evidently a homomorphism of right A-modules, and since

$$\iota(\alpha \otimes 1) = (\alpha \otimes 1) \cdot (1 \otimes 1) = \sum_{(\alpha)} \alpha_1 \otimes \alpha_2$$

it suffices to verify that ι^{-1} maps the term on the right to $\alpha \otimes 1$, for any $\alpha \in A$. To this end, recall that Δ is coassociative, so that

$$\sum_{(\alpha)} \sum_{(\alpha_1)} \alpha_{11} \otimes \alpha_{12} \otimes \alpha_2 = \sum_{(\alpha)} \sum_{(\alpha_2)} \alpha_1 \otimes \alpha_{21} \otimes \alpha_{22}.$$

This explains the second of the following equalities.

$$\begin{aligned} \iota^{-1}\left(\sum_{(\alpha)} \alpha_1 \otimes \alpha_2\right) &= \sum_{(\alpha)} \sum_{(\alpha_1)} \alpha_{11} \otimes \sigma(\alpha_{12})\alpha_2 \\ &= \sum_{(\alpha)} \sum_{(\alpha_2)} \alpha_1 \otimes \sigma(\alpha_{21})\alpha_{22} \\ &= \sum_{(\alpha)} \alpha_1 \otimes \varepsilon(\alpha_2) \\ &= \left(\sum_{(\alpha)} \alpha_1 \varepsilon(\alpha_2)\right) \otimes 1 \\ &= \alpha \otimes 1. \end{aligned}$$

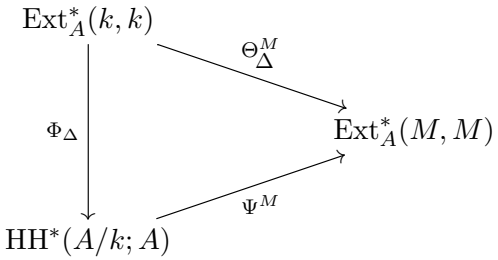
The third equality is a consequence of the definition of the antipode and the last equality holds because ε is the counit of the coalgebra structure on A.

Given properties (1) and (2) of ϕ_Δ, it is immediate that it induces a homomorphism of graded k-algebras:

$$\Phi_\Delta \colon \operatorname{Ext}_A^*(k,k) \longrightarrow \operatorname{HH}^*(A/k;A)\,. \tag{2.6}$$

The result below, proved by Pevtsova and Witherspoon [12], links the three homomorphisms, (2.2), (2.4), and (2.6), constructed above.

Theorem 2.7. *Let A be a Hopf algebra over k. For each A-module M, the following diagram of graded k-algebras commutes.*



Proof. Let X be an A-module. Using the description of the A-action on $X \otimes_k M$ and the A^{e}-action on $\phi_\Delta(X)$, it is a direct verification that the canonical bijection

$$X \otimes_k M \longrightarrow (X \otimes_k A) \otimes_A M = \phi_\Delta(X) \otimes_A M \quad \text{where } x \otimes m \mapsto x \otimes 1 \otimes m$$

is compatible with the A-module structures. It yields an isomorphism of functors $\theta^M_\Delta \cong \psi^M\phi_\Delta$ on the category of A-modules. Since ϕ_Δ take projectives to projectives, it follows that there is an equality of induced functors. This is the stated result. □

Definition 2.8. Let k be a field and A a k-algebra. In what follows we say that A-modules M and N are *stably isomorphic* if there exist projective A-modules P and Q such that $M \oplus P \cong N \oplus Q$.

Let P_* be a projective resolution of an A-module M. For any integer $d \geq 0$, the image of the boundary map $\partial\colon P_d \to P_{d-1}$ is independent of the choice of P, up to a stable isomorphism. We denote it $\Omega^d(M)$, and call it a dth syzygy module of M.

Fix an element ζ in $\operatorname{Ext}^d_A(k,k)$ and a dth syzygy module $\Omega^d(k)$. Then ζ is represented by a homomorphism on $\Omega^d(k)$, that we also call ζ. So we get an exact sequence of A-modules:

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^d(k) \xrightarrow{\zeta} k \longrightarrow 0. \tag{2.9}$$

That is, the module L_ζ is defined to be the kernel of map ζ on $\Omega^d(k)$. Up to a stable isomorphism, this is independent of the choice of a syzygy module.

Given a k-algebra A, we say that a map $\Delta\colon A \to A \otimes_k A$ *induces a Hopf structure on* A if there exists a Hopf algebra structure on A (and this includes an antipode) with Δ as the comultiplication. For ease of comprehension, given a coalgebra map Δ and A-modules X, M, the A-module defined on the vector space $X \otimes_k M$ using the Hopf structure Δ is denoted

$$\Delta(X \otimes_k M).$$

This is precisely the module $\theta^M_\Delta(X)$ defined in Construction 2.1.

Corollary 2.10. *Let $\Delta_1, \Delta_2\colon A \to A \otimes_k A$ be maps inducing Hopf algebra structures on A. If $\zeta \in \operatorname{Ext}^d_A(k,k)$ is such that $\Phi_{\Delta_1}(\zeta) = \Phi_{\Delta_2}(\zeta)$, then for each A-module M, the A-modules $\Delta_1(L_\zeta \otimes_k M)$ and $\Delta_2(L_\zeta \otimes_k M)$ are stably isomorphic.*

Proof. Let P_* be a projective resolution of k. For $i = 1, 2$ the complex $\theta^M_{\Delta_i}(P_*)$ is a projective resolution of $\theta^M_{\Delta_i}(k) \cong M$. Thus, the A-modules $\theta^M_{\Delta_i}(\Omega^d(k))$ and $\Omega^d(M)$ are stably isomorphic. Therefore, the exact sequence (2.9) induces an exact sequence of A-modules

$$0 \longrightarrow \theta^M_{\Delta_i}(L_\zeta) \longrightarrow \Omega^d(M) \oplus (\text{proj}) \xrightarrow{\Theta^M_{\Delta_i}(\zeta)} M \longrightarrow 0$$

where $\Theta^M_{\Delta_i}$ is the map (2.2). Since $\Phi_{\Delta_1}(\zeta) = \Phi_{\Delta_2}(\zeta)$, by hypothesis, Theorem 2.7 implies $\Theta^M_{\Delta_1}(\zeta) = \Theta^M_{\Delta_2}(\zeta)$. This yields the desired result. □

Remark 2.11. Assume that the k-algebra A is finite dimensional. Then finitely generated modules over A admit minimal projective resolutions, and

hence syzygy modules are well-defined, up to isomorphism of A-modules. What is more, each $\zeta \in \operatorname{Ext}_A^d(k,k)$ is represented by a *unique* homomorphism $\Omega^d(k) \to k$, and then setting L_ζ to be its kernel pins down the latter, up to isomorphism. In the same vein, $\Omega^d(k) \otimes_k M \cong \Omega^d(M)$, so we get a well-defined module $L_\zeta \otimes_k M$.

It follows from the argument in Corollary 2.10 that if $\Phi_{\Delta_1}(\zeta) = \Phi_{\Delta_2}(\zeta)$ the A-modules $\Delta_1(L_\zeta \otimes_k M)$ and $\Delta_2(L_\zeta \otimes_k M)$ are in fact isomorphic.

3. Cohomology of elementary abelian p-groups

In this section, we set notation and review some facts about the cohomology, and Hochschild cohomology, of elementary abelian p-groups; see [8, Section 4.5] and [11] for details. Throughout k will be a field of positive characteristic p.

Let $E := \langle g \rangle$ be a cyclic group of order p. Setting $x = g - 1$ we may write $A := kE$, the group algebra of E over k, as a truncated polynomial ring $A \cong k[x]/(x^p)$. Consider the complex of projective A-modules

$$P_*\colon \qquad \cdots \longrightarrow A \xrightarrow{x} A \xrightarrow{x^{p-1}} A \xrightarrow{x} A \longrightarrow 0\,, \tag{3.1}$$

that is nonzero in degrees ≥ 0. The augmentation $\varepsilon\colon P_* \to k$, that maps P_i to zero for $i > 0$ and is the canonical surjection for $i = 0$, is a morphism of complexes, and a quasi-isomorphism; thus (P_*, ε) is a minimal projective resolution of k, over A.

Let $E := \langle g_1, \dots, g_r \rangle$ be an elementary abelian group of order p^r. For each integer $i = 1, \dots, r$, set $A_i := k[x_i]/(x_i^p)$. Then $A := A_1 \otimes_k \cdots \otimes_k A_r$ is the group algebra of E, where $x_i = g_i - 1$ for each i. With $(P_*^{(i)}, \varepsilon_i)$ the projective A_i-resolution of k, from (3.1), the complex

$$(P_*, \varepsilon) := (P_*^{(1)} \otimes_k \cdots \otimes_k P_*^{(r)}, \varepsilon_1 \otimes \cdots \otimes \varepsilon_r)\,. \tag{3.2}$$

is a projective A-resolution of k. Set

$$P_{j_1,\cdots,j_r} := P_{j_1}^{(1)} \otimes_k \cdots \otimes_k P_{j_r}^{(r)} = A_1 \otimes_k \cdots \otimes_k A_r = A$$

and let $\theta_{j_1,\dots,j_r}\colon P_* \to k$ be the map whose restriction to $P_{j_1,\dots,j_r}$ is the augmentation $A \to k$ and whose restriction to $P_{\ell_1,\dots,\ell_r}$ is zero if $j_i \neq \ell_i$ for some i. Let $\hat{\eta}_i := \theta_{j_1,\dots,j_r}$ where $j_i = 1$ and $j_\ell = 0$ for $\ell \neq i$. Let $\hat{\zeta}_i := \theta_{j_1,\dots,j_r}$ where $j_i = 2$ and $j_\ell = 0$ for $\ell \neq i$.

The cohomology ring of A has the form

$$\operatorname{Ext}_A^*(k,k) = \begin{cases} k[\eta_1, \dots, \eta_r] & \text{if } p = 2, \\ \Lambda(\eta_1, \dots, \eta_r) \otimes_k k[\zeta_1, \dots, \zeta_r] & \text{otherwise,} \end{cases}$$

where each η_i is represented by the cocycle $\hat{\eta}_i$ and each ζ_i is represented by $\hat{\zeta}_i$. Here η_i is in degree 1 and ζ_i is in degree 2. As the resolution (P_*, ε) is minimal, η_i and ζ_i are uniquely represented by $\hat{\eta}_i$ and $\hat{\zeta}_i$, respectively.

When $p = 2$, let $\zeta_i = \eta_i^2$ for $i = 1, \dots, r$. Let S be the polynomial subring of $\operatorname{Ext}_A^*(k,k)$ generated by the ζ_i's, so that

$$S = k[\zeta_1, \dots, \zeta_r]\,. \tag{3.3}$$

The Bockstein map is an operation on cohomology that raises degrees by one. If $p = 2$ it coincides with the Steenrod square. For each i, the Bockstein of the cohomology class η_i is the class ζ_i. Thus, when $k = \mathbb{F}_p$, the subring S is the subring generated by the images of the degree one classes under the Bockstein map.

Now consider the Hochschild cohomology. As before, set $A := k[x]/(x^p)$. The enveloping algebra A^{e} is a truncated polynomial ring in variables

$$y := x \otimes 1 \quad \text{and} \quad z := 1 \otimes x,$$

so that $A^{\mathrm{e}} = k[y,z]/(y^p, z^p)$. The A^{e} action on A is defined by the surjection $\mu\colon A^{\mathrm{e}} \to A$ that maps y and z to x. Thus $A^{\mathrm{e}} \cong A[y-z]/(y-z)^p$. The kernel of μ is the ideal $(y-z)$ and the minimal projective resolution of A as an A^{e}-module has the form:

$$Q_*\colon \qquad \dots \longrightarrow A^{\mathrm{e}} \xrightarrow{y-z} A^{\mathrm{e}} \xrightarrow{(y-z)^{p-1}} A^{\mathrm{e}} \xrightarrow{y-z} A^{\mathrm{e}} \longrightarrow 0 \tag{3.4}$$

with canonical augmentation $Q_* \to A$, also denoted μ.

Let $A := k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)$ and $A_i := k[x_i]/(x_i^p)$. With $(Q_*^{(i)}, \mu_i)$ the projective ${A_i}^{\mathrm{e}}$-resolution of A_i from (3.4), the complex

$$(Q_*, \mu) := (Q_*^{(1)} \otimes_k \cdots \otimes_k Q_*^{(r)}, \mu_1 \otimes \cdots \otimes \mu_r) \tag{3.5}$$

is a projective A^{e}-resolution of A. Set

$$Q_{j_1,\cdots,j_r} := Q_{j_1}^{(1)} \otimes_k \cdots \otimes_k Q_{j_r}^{(r)} = {A_1}^{\mathrm{e}} \otimes_k \cdots \otimes_k {A_r}^{\mathrm{e}} \cong A^{\mathrm{e}}\,.$$

Let $\sigma_{j_1,\dots,j_r}\colon Q_* \to A$ be the map whose restriction to $Q_{j_1,\dots,j_r}$ is the canonical map $A^{\mathrm{e}} \to A$ and whose restriction to $Q_{\ell_1,\dots,\ell_r}$ is zero if $j_i \neq \ell_i$ for some i. Let $\hat{\delta}_i := \sigma_{j_1,\dots,j_r}$ where $j_i = 1$ and $j_\ell = 0$ for $\ell \neq i$ and $\hat{\chi}_i := \sigma_{j_1,\dots,j_r}$ where $j_i = 2$ and $j_\ell = 0$ for $\ell \neq i$.

The Hochschild cohomology ring of A over k has the form

$$\operatorname{Ext}_{A^{\mathrm{e}}}^*(A,A) = \begin{cases} A[\delta_1, \dots, \delta_r] & \text{if } p = 2, \\ \Lambda_A(\delta_1, \dots, \delta_r) \otimes_A A[\chi_1, \dots, \chi_r] & \text{otherwise,} \end{cases}$$

with δ_i and χ_i the cohomology classes of $\hat{\delta}_i$ and $\hat{\chi}_i$ respectively.

4. Changing the coalgebra structure on kE

Let k be a field of positive characteristic p and set

$$A = k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)\,.$$

There are two often-used coalgebra structures on A making it a Hopf algebra.

The first comes from viewing A as the group algebra of an elementary abelian p-group, say $\langle g_1, \dots, g_c \rangle$ with base field of characteristic p. Then A has comultiplication $\Delta_{\mathrm{Gr}} \colon A \to A \otimes A$ given by $g \mapsto g \otimes g$; equivalently,

$$\Delta_{\mathrm{Gr}}(x_i) = x_i \otimes 1 + x_i \otimes x_i + 1 \otimes x_i \,.$$

The antipode is the homomorphism of k-algebras (note that A is commutative) induced by the map $g_i \mapsto g_i^{-1}$, which translates to

$$\sigma_{\mathrm{Gr}}(x_i) = (1 + x_i)^{-1} - 1 = -x_i + x_i^2 - \cdots + x_i^{p-1}.$$

The other coalgebra structure on A comes from viewing it as the restricted enveloping algebra of the restricted p-Lie algebra k^r, with trivial bracket and p-power operation. Then the comultiplication $\Delta_{\mathrm{Lie}} \colon A \to A \otimes A$ is given by

$$\Delta_{\mathrm{Lie}}(x_i) = x_i \otimes 1 + 1 \otimes x_i \,.$$

The antipode is the homomorphism of k-algebras $A \to A$ defined by

$$\sigma_{\mathrm{Lie}}(x_i) = -x_i.$$

The different coalgebra structures induce different actions of $\operatorname{Ext}_A^*(k,k)$ on the cohomology of modules; see Example 5.4. However, the actions do agree on the subalgebra generated by the Bocksteins of the degree one elements. This is the content of Theorem 4.4. A key step in its proof is an explicit computation of the map Φ_Δ from Construction 2.5 for the different coalgebra structures. In view of the computations recalled in Section 3, this amounts to describing the maps

$$\Phi_{\Delta_{\mathrm{Gr}}}, \Phi_{\Delta_{\mathrm{Lie}}} \colon k[\eta_1, \dots, \eta_r, \zeta_1, \dots, \zeta_r] \longrightarrow A[\delta_1, \dots, \delta_r, \chi_1, \dots, \chi_r]$$

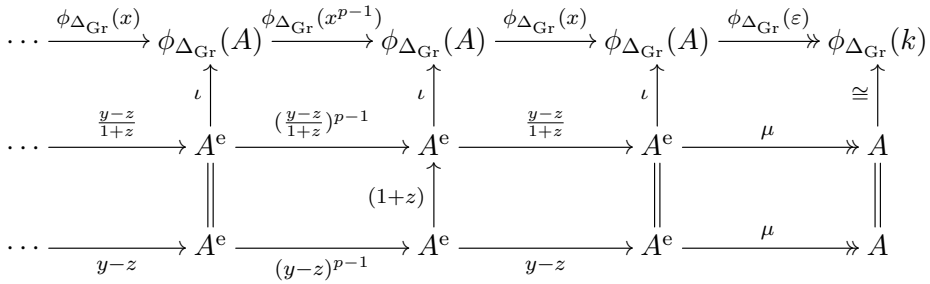
from the cohomology of A to its Hochschild cohomology.

Theorem 4.1. *With the Hopf algebra structure on A induced by Δ_{Gr} and σ_{Gr}, the homomorphism $\Phi_{\Delta_{\mathrm{Gr}}} \colon \operatorname{Ext}_A^*(k,k) \to \operatorname{Ext}_{A^{\mathrm{e}}}^*(A,A)$ of k-algebras is given by*

$$\Phi_{\Delta_{\mathrm{Gr}}}(\eta_i) = (1 + x_i)\delta_i \quad \textit{and} \quad \Phi_{\Delta_{\mathrm{Gr}}}(\zeta_i) = \chi_i \quad \textit{for } i = 1, \dots, r.$$

Proof. We first verify the result for $r = 1$; then $A = k[x]/(x^p)$. In what follows we use the maps $\phi_{\Delta_{\mathrm{Gr}}}$ and ι, and their properties, from Construction 2.5 without comment. Let P_* be the minimal projective resolution of k over A from (3.1) and Q_* the minimal projective resolution of A over A^{e} from (3.4). Applying $\phi_{\Delta_{\mathrm{Gr}}}$ to P_* yields a projective resolution of A over A^{e}. This gives the top row in the following commutative diagram of complexes

of A^{e}-modules:



The bottom row is the augmentation of the minimal projective resolution (3.4) of A over A^{e}. It is clear that the lower part of the diagram is commutative. As to the upper part, the commutativity of the square on the top right corner is clear. For the next square, we note that $\phi_{\Delta_{\mathrm{Gr}}}(x)$ is the map that takes $1\otimes 1$ to $x\otimes 1$ in $\phi_{\Delta_{\mathrm{Gr}}}(A)$. However, this is not multiplication by the element $y = x\otimes 1$ in A^{e}; see Construction 2.5. Instead, we have that

$$y(1\otimes 1) = x\otimes 1 + x\otimes x + 1\otimes x \quad \text{and} \quad z(1\otimes 1) = 1\otimes x$$

in $\phi_{\Delta_{\mathrm{Gr}}}(A)$. Hence, one has

$$(y-z)(1\otimes 1) = (1+z)(x\otimes 1)$$

and $\phi_{\Delta_{\mathrm{Gr}}}(x)$ is multiplication by $(y-z)/(1+z)$ as asserted. Likewise, $\phi_{\Delta_{\mathrm{Gr}}}(x^{p-1})$ is multiplication by $((y-z)/(1+z))^{p-1}$.

It is clear from the construction that the cocycles $\hat{\eta}$ and $\hat{\zeta}$, from $P_* \to k$, are mapped to the cocycles $(1+z)\hat{\delta}$ and $\hat{\chi}$, respectively, from $Q_* \to A$. This yields the desired result. For later use we denote

$$\kappa\colon Q_* \longrightarrow \phi_{\Delta_{\mathrm{Gr}}}(P_*) \tag{4.2}$$

the morphism of complexes of A^{e}-modules constructed above.

Assume $r \geq 2$. Let P_* be the resolution of k over A, and let Q_* be the resolution of A over A^{e}. The tensor product, over k, of the morphisms $\kappa^{(i)}\colon Q_*^{(i)} \to \phi_{\Delta_{\mathrm{Gr}}}(P_*^{(i)})$ from (4.2) yields a morphism

$$\kappa := \kappa^{(1)} \otimes_k \cdots \otimes_k \kappa^{(r)}\colon Q_* \longrightarrow \phi_{\Delta_{\mathrm{Gr}}}(P_*)\,,$$

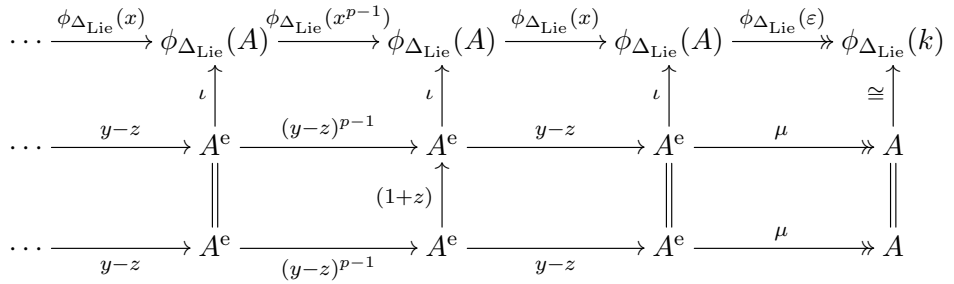
of complexes of A^{e}-modules that lifts the isomorphism $A \cong \phi_{\Delta_{\mathrm{Gr}}}(k)$. Once again, it is evident, by inspection, that the cocycles $\hat{\eta}_i$ and $\hat{\zeta}_i$ are mapped to the cocycles $(1+z)\hat{\delta}_i$ and $\hat{\chi}_i$, respectively. □

An analogous argument gives also the next result.

Theorem 4.3. *With the Hopf algebra structure on A induced by Δ_{Lie} and σ_{Lie}, the homomorphism $\Phi_{\Delta_{\mathrm{Lie}}}\colon \operatorname{Ext}_A^*(k,k) \to \operatorname{Ext}_{A^{\mathrm{e}}}^*(A,A)$ of k-algebras is given by*

$$\Phi_{\Delta_{\mathrm{Lie}}}(\eta_i) = \delta_i \quad \textit{and} \quad \Phi_{\Delta_{\mathrm{Lie}}}(\zeta_i) = \chi_i \quad \textit{for } i = 1,\dots,r.$$

Proof. The key point, as in the proof of the preceding theorem, is to verify that one has a commutative diagram of complexes of A^{e}-modules:



This proof of the commutativity is similar to that of the previous case. □

The next result is direct consequence of the preceding computations.

Theorem 4.4. *Let $A = k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)$, with k a field of positive characteristic p. For any A-module M, the homomorphisms*

$$\Theta^M_{\Delta_{\mathrm{Gr}}}, \Theta^M_{\Delta_{\mathrm{Lie}}} \colon \operatorname{Ext}^*_A(k,k) \longrightarrow \operatorname{Ext}^*_A(M,M)$$

defined in (2.2) using the coalgebra maps Δ_{Gr} and Δ_{Lie}, respectively, coincide on the subring $S = k[\zeta_1, \dots, \zeta_r]$ of $\operatorname{Ext}^*_A(k,k)$ defined in (3.3). □

Remark 4.5. In ongoing work, in collaboration with Luchezar L. Avramov, we have established a version of the preceding theorem for more general finite dimensional commutative algebras; the techniques required are rather more involved and will be presented elsewhere. This raises that possibility that such a result holds for any finite dimensional commutative Hopf algebra.

5. Tensor products of L_ζ-modules

As in Section 4, let k be a field of positive characteristic p and set

$$A = k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)\,.$$

Let S be the subalgebra of $\operatorname{Ext}^*_A(k,k)$ identified in (3.3). We investigate the circumstances under which the tensor products of L_ζ modules (see Definition 2.8) are independent of the Hopf algebra structures on A described in Section 4. The main result is as follows; see the paragraph preceding Corollary 2.10 for notation.

Theorem 5.1. *Let ζ be a homogeneous element of S. For any A-module M, there is an isomorphism $\Delta_{\mathrm{Gr}}(L_\zeta \otimes_k M) \cong \Delta_{\mathrm{Lie}}(L_\zeta \otimes_k M)$ of A-modules.*

Proof. The result is immediate from Corollary 2.10 and Theorem 4.4. □

Corollary 5.2. *Suppose that $\zeta_1, \dots, \zeta_n$ are homogeneous elements of positive degree in $\operatorname{Ext}^*_A(k,k)$. If all but one of $\zeta_1, \dots, \zeta_n$ is in the subring S, then there is an isomorphism of A-modules*

$$\Delta_{\mathrm{Gr}}(L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_n}) \cong \Delta_{\mathrm{Lie}}(L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_n})\,.$$

Proof. Without loss of generality, it may be assumed that $\zeta_1, \dots, \zeta_{n-1}$ are in S. The proof is by a backwards induction on n, the base case $n = 1$ being a tautology. The induction hypothesis yields the second isomorphism below

$$\begin{aligned}\Delta_{\mathrm{Gr}}(L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_n}) &\cong \Delta_{\mathrm{Gr}}(L_{\zeta_1} \otimes_k \Delta_{\mathrm{Gr}}(L_{\zeta_2} \otimes_k \cdots \otimes_k L_{\zeta_n})) \\ &\cong \Delta_{\mathrm{Gr}}(L_{\zeta_1} \otimes_k \Delta_{\mathrm{Lie}}(L_{\zeta_2} \otimes_k \cdots \otimes_k L_{\zeta_n})) \\ &\cong \Delta_{\mathrm{Lie}}(L_{\zeta_1} \otimes_k \Delta_{\mathrm{Lie}}(L_{\zeta_2} \otimes_k \cdots \otimes_k L_{\zeta_n})) \\ &\cong \Delta_{\mathrm{Lie}}(L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_n}).\end{aligned}$$

The third one is by Theorem 5.1, and the other two are standard. □

Remark 5.3. The modules L_ζ have some remarkable properties. Under certain circumstances, the annihilator in $\mathrm{H}^*(E,k)$ of $\mathrm{Ext}^*_{kE}(L_\zeta, L_\zeta)$ is the ideal generated by ζ. This happens, for example, if $p > 2$ and n is even [5]. In general, the annihilator of the cohomology of L_ζ depends on the choice of the coalgebra structure as we see in Example 5.4. The sequence (2.9) has a translation

$$\mathcal{E}_\zeta: \quad 0 \longrightarrow \Omega^1(k) \longrightarrow L_\zeta \oplus Q \longrightarrow \Omega^d(k) \longrightarrow 0$$

where Q is the projective cover of k. The translated sequence represents the cohomology class ζ in $\mathrm{Ext}^1_{kE}(\Omega^d(k), \Omega^1(k)) \cong \mathrm{Ext}^d_{kE}(k,k)$. Thus ζ is in the annihilator of $\mathrm{Ext}^*_{kE}(M,M)$ for a module M, if and only if $\mathcal{E}_\zeta \otimes_k M$ splits. This is equivalent to the requirement that there is a stable isomorphism

$$L_\zeta \otimes_k M \;\cong\; \Omega^d(M) \oplus \Omega^1(M).$$

The following example, noted already in [3, 9], shows that the conclusion of Theorem 5.1 may fail if ζ is not in S.

Example 5.4. Let k be a field of characteristic 2 and E an elementary abelian group of order 4; thus $\mathrm{H}(E,k) = k[\eta_1, \eta_2]$; see Section 3. Set

$$\zeta = \eta_1 - \alpha\eta_2 \in \mathrm{H}^1(E,k)$$

where $\alpha \in k$ with $\alpha \neq 0$ or 1. The module L_ζ has a k-basis consisting of elements u, v such that $x_1 u = \alpha x_2 u$ and $x_1 v = x_2 v$. Using the Lie coalgebra structure, we can compute that $L_\zeta \otimes_k L_\zeta$ is isomorphic to a direct sum of two copies of L_ζ generated by $u \otimes u$ and $u \otimes v$. However, with the group coalgebra structure, $L_\zeta \otimes_k L_\zeta$ is indecomposable. Indeed, under this structure, one has

$$x_1(u \otimes u) = \alpha(u \otimes v) + \alpha(v \otimes u) + \alpha^2(v \otimes v) = x_2(u \otimes u) + \alpha^2(v \otimes v).$$

and the last term that makes it impossible to decompose $L_\zeta \otimes_k L_\zeta$.

Remark 5.5. Computer calculations using Magma [2] give evidence that Corollary 5.2 might have a strong converse. In one experiment, two random elements γ_1 and γ_2 were chosen in $\mathrm{H}^4(E,k)$ with E an elementary abelian group of order 8 and k the field with 8 elements. The tensor product of modules L_{γ_1} and L_{γ_2} was taken using both of the coalgebra structures and the two results were compared. This operation was performed several times

and in every case, the two tensor products were isomorphic if and only if one of the two chosen cohomology elements was in S. The same experiment was performed taking two elements in degree two of an elementary abelian group of order 27 over a field of order 9, with the same result.

6. An equality of varieties

Let E be an elementary abelian p-group and k an algebraically closed field of characteristic p. In the paper [1], Avrunin and Scott prove a conjecture of the first author (see [3]) asserting the equivalence of the support variety of a module with a rank variety for that same module. For notation, let $kE = k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)$. Let M be a kE-module. The support variety $V_G(M)$ of M is the closed subset of $\operatorname{Proj} \mathrm{H}^*(G,k)$ consisting of all homogeneous prime ideals that contain the annihilator $J(M)$ in $\mathrm{H}^*(E,k)$ of the cohomology ring $\operatorname{Ext}^*_{kE}(M,M)$. The rank variety of M, denoted $V_G^r(M)$ is the set of all points $[\alpha_1, \dots, \alpha_r]$ in $\mathbb{P}^{r-1}$ such that $\alpha^*(M)$ is not a free module. Here $\alpha\colon k[t]/(t^p) \to kE$ is given by $\alpha(t) = \alpha_1 x_1 + \cdots + \alpha_r x_r$ and $\alpha^*(M)$ is the restriction of M to a $k[t]/(t^P)$-module along the map α.

The conjecture states that for $\alpha \in \mathbb{P}^{r-1}$, $\alpha \in V_G^r(M)$ if and only if $\alpha^*(J(M)) = \{0\}$, where $\alpha^*(J(M))$ is the restriction of the ideal to the cohomology ring of $k[t]/(t^p)$ along α. This all makes sense because α is a flat embedding. The most difficult part is the proof of the assertion that if $\alpha^*(J(M)) = \{0\}$, then $\alpha^*(M)$ is a free module over $k[t]/(t^p)$. The proof by Avrunin and Scott uses a spectral sequence argument under the assumption that kE has the coalgebra structure of the restricted enveloping algebra of an elementary Lie algebra. In this case any such α is a map of Hopf algebras and this point is important in the proof. The other key step is their proof that the variety is independent of the coalgebra structure.

This last step is an easy consequence of Theorem 4.4. The point is to restrict to the subring S. The annihilator in S of $\operatorname{Ext}^*_{kE}(M,M)$ is $S \cap J(M)$. Moreover $V_G(J(M)) = V_G(S \cap J(M))$. While the ideal $J(M)$ depends on the coalgebra structure, the ideal $J(M) \cap S$ does not, by Theorem 4.4.

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This paper is available via http://nyjm.albany.edu/j/2017/23-17.html.