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On Banach spaces of universal disposition

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ABSTRACT. We present: i) an example of a Banach space of universal disposition that is not separably injective; ii) an example of a Banach space of universal disposition with respect to finite dimensional polyhedral spaces with the Separable Complementation Property; iii) a new type of space of universal disposition nonisomorphic to the previous existing ones.

CONTENTS

1. Introduction

The monograph [1] contains a study of separably injective spaces, among which one encounters two somewhat unexpected classes: ultrapowers of spaces of type $\mathscr{L}_{\infty,\lambda}$ and spaces of universal disposition with respect to the the class of separable spaces. Recall from [9] that given a class $\mathfrak{M}$ of Banach spaces the space U is said to be of almost universal disposition for $\mathfrak{M}$ if given and $\varepsilon > 0$, $A, B \in \mathfrak{M}$ and isometries $u : A \to U$ and $\imath : A \to B$ there is an ε-isometry $u' : B \to U$ such that $u = u'\imath$. The space U is said to be of universal disposition for $\mathfrak{M}$ (sometimes called of $\mathfrak{M}$-universal disposition) if the condition above also holds for $\varepsilon = 0$.

We are particularly interested in the classes $\mathfrak{M} = \mathfrak{S}$ of separable Banach spaces and $\mathfrak{M} = \mathfrak{F}$ of finite dimensional Banach spaces. Spaces of (almost) universal disposition for $\mathfrak{F}$ will simply be called spaces of (almost) universal disposition.

A Banach space E is said to be *separably injective* if for every separable Banach space X and each subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. In [1, Thm. 3.5] it is established that spaces of $\mathfrak{S}$-universal disposition are separably injective, as well as the ultrapowers of $\mathscr{L}_{\infty,\lambda}$-spaces [1, Thm. 4.4]. Which raises the question, not considered in [1], of whether spaces of universal

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disposition must also be separably injective. Our first result is to show that such is not the case.

2. A space of universal disposition that is not separably injective

To proceed with the example, we need to recall the construction of $\mathscr{F}^{\omega_1}(\mathbb{R})$, the only known example of space that is of universal disposition with respect to finite dimensional spaces but not with respect to separable spaces. Recall first the push-out construction; which, given an isometry $u : A \to B$ and an operator $t : A \to E$ will provides us with an extension of t through u at the cost of embedding E in a larger space as it is showed in the diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ E & \xrightarrow{u'} & \mathrm{PO} \end{array}$$

where $t'u = u't$. It is important to realize that u' is again an isometry and that t' is a contraction or an isometry if t is. Once a starting Banach space X has been fixed, the input data we need for our construction are:

- a class $\mathfrak{M}$ of Banach spaces;
- the family $\mathfrak{J}$ of all isometries acting between the elements of $\mathfrak{M}$;
- a family $\mathfrak{L}$ of norm one X-valued operators defined on elements of $\mathfrak{M}$.

For any operator $s : A \to B$, we establish $\mathrm{dom}(s) = A$ and $\mathrm{cod}(s) = B$. Notice that the codomain of an operator is usually larger than its range, and that the unique codomain of the elements of $\mathfrak{L}$ is X. Set $\Gamma = \{(u,t) \in \mathfrak{J} \times \mathfrak{L} : \mathrm{dom}\, u = \mathrm{dom}\, t\}$ and consider the Banach spaces of summable families $\ell_1(\Gamma, \mathrm{dom}\, u)$ and $\ell_1(\Gamma, \mathrm{cod}\, u)$. We have an obvious isometry

$$\oplus\mathfrak{J} : \ell_1(\Gamma, \mathrm{dom}\, u) \longrightarrow \ell_1(\Gamma, \mathrm{cod}\, u)$$

defined by $(x_{(u,t)})_{(u,t)\in\Gamma} \longmapsto (u(x_{(u,t)}))_{(u,t)\in\Gamma}$; and a contractive operator

$$\Sigma\mathfrak{L} : \ell_1(\Gamma, \mathrm{dom}\, u) \longrightarrow X,$$

given by $(x_{(u,t)})_{(u,t)\in\Gamma} \longmapsto \sum_{(u,t)\in\Gamma} t(x_{(u,t)})$. Observe that the notation is slightly imprecise since both $\oplus\mathfrak{J}$ and $\Sigma\mathfrak{L}$ depend on Γ. We can form their push-out diagram

$$\begin{array}{ccc} \ell_1(\Gamma, \mathrm{dom}\, u) & \xrightarrow{\oplus\mathfrak{J}} & \ell_1(\Gamma, \mathrm{cod}\, u) \\ {\scriptstyle \Sigma\mathfrak{L}}\downarrow & & \downarrow \\ E & \xrightarrow{\quad\imath\quad} & \mathrm{PO}\,. \end{array}$$

We obtain in this way an isometric enlargement of X such that for every $t : A \to X$ in $\mathfrak{L}$, the operator $\imath t$ can be extended to an operator $t' : B \to \mathrm{PO}$ through any embedding $u : A \to B$ in $\mathfrak{J}$ provided $\mathrm{dom}\, u = \mathrm{dom}\, t = A$. In the next step we keep the family $\mathfrak{J}$ of isometries, replace the starting space X by PO and $\mathfrak{L}$ by a family of norm one operators $\mathrm{dom}\, u \to \mathrm{PO}$, $u \in \mathfrak{J}$, and proceed again.

We start with $\mathscr{S}^0(X) = X$. The inductive step is as follows. Suppose we have constructed the directed system $(\mathscr{S}^\alpha(X))_{\alpha<\beta}$, including the corresponding linking maps $\imath_{(\alpha,\gamma)} : \mathscr{S}^\alpha(X) \longrightarrow \mathscr{S}^\gamma(X)$ for $\alpha < \gamma < \beta$. To define $\mathscr{S}^\beta(X)$ and the maps $\imath_{(\alpha,\beta)} : \mathscr{S}^\alpha(X) \longrightarrow \mathscr{S}^\beta(X)$ we consider separately two cases, as usual: if β is a limit ordinal, then we take $\mathscr{S}^\beta(X)$ as the direct limit of the system $(\mathscr{S}^\alpha(X))_{\alpha<\beta}$ and $\imath_{(\alpha,\beta)} : \mathscr{S}^\alpha(X) \longrightarrow \mathscr{S}^\beta(X)$ the natural inclusion map. Otherwise $\beta = \alpha + 1$ is a successor ordinal and we construct $\mathscr{S}^\beta(X)$ applying the push-out construction as above with the following data: $\mathscr{S}^\alpha(X)$ is the starting space, $\mathfrak{J}$ keeps being the set of all isometries acting between the elements of $\mathfrak{M}$ and $\mathfrak{L}_\alpha$ is the family of all isometries $t : S \to \mathscr{S}^\alpha(X)$, where $S \in \mathfrak{M}$.

We then set $\Gamma_\alpha = \{(u,t) \in \mathfrak{J} \times \mathfrak{L}_\alpha : \operatorname{dom} u = \operatorname{dom} t\}$ and make the push-out

$$\begin{array}{ccc} \ell_1(\Gamma_\alpha, \operatorname{dom} u) & \xrightarrow{\oplus \mathfrak{J}_\alpha} & \ell_1(\Gamma_\alpha, \operatorname{cod} u) \\ {\scriptstyle \Sigma\mathfrak{L}_\alpha}\big\downarrow & & \big\downarrow \\ \mathscr{S}^\alpha(X) & \longrightarrow & \mathrm{PO} \end{array} \tag{1}$$

thus obtaining $\mathscr{S}^{\alpha+1}(X) = \mathrm{PO}$. The embedding $\imath_{(\alpha,\beta)}$ is the lower arrow in the above diagram; by composition with $\imath_{(\alpha,\beta)}$ we get the embeddings

$$\imath_{(\gamma,\beta)} = \imath_{(\alpha,\beta)}\imath_{(\gamma,\alpha)},$$

for all $\gamma < \alpha$.

Set now as input data: $\mathfrak{M} = \mathfrak{F}$ the family of all finite dimensional spaces, $\mathfrak{J}$ the set of all isometries between elements of $\mathfrak{F}$ and $\mathfrak{L}$ all X-valued isometries defined on elements of $\mathfrak{F}$. Proceeding inductively until ω_1 we get the space $\mathscr{F}^{\omega_1}(X)$. This space is of universal disposition (cf. [1, Chapter 3]). We prove first a structure theorem.

Theorem 2.1. *The space $\mathscr{F}^{\omega_1}(\mathbb{R})$ is not separably injective.*

Proof. In [1, Thm. 3.23 (2)] it is proved that for all separable X, all the spaces $\mathscr{F}^{\omega_1}(X)$ are isometric; thus isometric to $\mathscr{F}^{\omega_1}(\mathbb{R})$.

Claim. *The space $C[0,1]$ is $(1+\varepsilon)$-complemented in $\mathscr{S}^\alpha(C[0,1])$ for all $\alpha < \omega_1$ and all $\varepsilon > 0$.*

Proof. Recall that a convex body is said to be a polyhedron if it is the convex hull of a finite set of points. A Banach space is said to be polyhedral if the unit ball of each of its finite dimensional subspaces is a polyhedron. It is a well-known fact that $C[0,1]$-valued operators defined on finite-dimensional polyhedral spaces can be extended with the same norm. A simple proof for this result can be derived from Kalman's theorem [10], which in turn can be easily proved by triangularisation. See also [11, 12] for the state-of-the-art about the problem of extension of $C(K)$-valued Lipschitz maps. Since every norm on a finite dimensional space can, for every $\varepsilon > 0$, be ε-approximated by a polyhedral norm, it follows that every norm one $C[0,1]$-valued operator defined on a finite dimensional Banach space can, for every $\varepsilon > 0$, be extended with norm at most $1+\varepsilon$.

Thus, if 1_C denotes the identity of $C[0,1]$ and $\varepsilon > 0$ has been fixed, as well as a limit ordinal $\beta < \omega_1$, pick a sequence $(\varepsilon_j)_{j<\beta}$ with all $\varepsilon_j > 0$ so that $\sum_j \varepsilon_j < \varepsilon$. If β is nonlimit, pick the sequence $(\varepsilon_j)_{j\leq\beta}$ so that each $\varepsilon_j > 0$ and $\sum_j \varepsilon_j < \varepsilon$. Now, all the elements in the composition $1_C \sum \mathfrak{L}$ are norm one finite rank operators that extend to $C[0,1]$-valued operators with norm at most $(1+\varepsilon_1)$; thus, $1_C \sum \mathfrak{L}$ extends to an operator $\ell_1(\Gamma, \operatorname{cod} u) \to C[0,1]$ with norm at most $(1+\varepsilon_1)$, hence to an operator $\mathscr{S}^1(C[0,1]) \to C[0,1]$ with norm at most $(1+\varepsilon_1)$. I.e., $C[0,1]$ is $(1+\varepsilon_1)$-complemented in $\mathscr{S}^1(C[0,1])$. Iterating the argument, one gets that $C[0,1]$ is actually $(1+\varepsilon)$-complemented in $\mathscr{S}^\beta(C[0,1])$. The Claim follows. □

Assume now that $\mathscr{F}^{\omega_1}(C[0,1])$ is separable injective. Let $S \to S'$ be an injective isometry between two separable spaces and let $\tau : S \to C[0,1]$ be an operator. As an operator $S \to C[0,1] \to \mathscr{F}^{\omega_1}(C[0,1])$, it can be extended to an operator

$$T : S' \to \mathscr{F}^{\omega_1}(C[0,1]).$$

However, the uncountable cofinality of ω_1 means that T actually has its range contained in some $\mathscr{S}^\beta(C[0,1])$. A composition with the projection

$$\mathscr{S}^\beta(C[0,1]) \to C[0,1]$$

provides an extension $S' \to C[0,1]$ of τ. In other words, $C[0,1]$ would be separably injective, which it is not. The proof of Theorem 2.1 is complete. □

The result above can be improved for separable Lindenstrauss spaces. Recall that a Banach space is called a Lindenstrauss space if it is an isometric predual of some $L_1(\mu)$.

Corollary 2.1. *Every separable Lindestrauss space is $(1+\varepsilon)$-complemented in $\mathscr{S}^\alpha(C[0,1])$ for all $\alpha < \omega_1$ and all $\varepsilon > 0$.*

Proof. Indeed, one can skip using Kalman's theorem and use instead the fact that norm one finite range operators with values on a Lindenstrauss space can be extended, for every $\varepsilon > 0$, with norm at most $1+\varepsilon$. □

Proposition 2.1. *The space $\mathscr{F}^{\omega_1}(\mathbb{R})$ contains 1-complemented copies of all isometric preduals of ℓ_1.*

Proof. Let X be an isometric ℓ_1-predual; it can therefore be renormed as follows to be a polyhedral Lindenstrauss space [8]: Let (e_n) be the canonical basis of $\ell_1 = X^*$ and let $(\varepsilon_n) \in c_0$ be a sequence of positive scalars. Set $\|x\| = \sup_n (1+\varepsilon_n)|e_n(x)|$. See [8] for details. A result of Lazar [13] shows that every compact operator with values on a polyhedral Lindenstrauss space can be extended to any separable superspace with the same norm. The paper [2] claims that the additional condition

$$(*) \qquad \forall x \in X; \|x\| = 1 \quad \dim\{f \in X^* : f(x) = 1\} < +\infty$$

has to be added to Lazar's result. But the renorming above mentioned satisfies condition $(*)$.

Consider the space $\mathscr{F}^{\omega_1}(X) = \mathscr{F}^{\omega_1}(\mathbb{R})$ and let 1_X be the identity of X. All the elements in the composition $1_C \sum \mathfrak{L}$ are norm one finite rank operators that extend to norm one X-valued operators; thus, $1_C \sum \mathfrak{L}$ extends to a norm one operator $\ell_1(\Gamma, \operatorname{cod} u) \to X$; hence, to a norm one operator $\mathscr{S}^1(X) \to X$. Iterating the argument, one gets that X is actually 1-complemented in $\mathscr{F}^{\omega_1}(X)$. □

3. Universal disposition and the Separable Complementation Property

As we have seen, the key point seems to be that c_0 is the only possible separable complemented subspace of a separably injective space, which means that there are few complemented separable subspaces in separably injective spaces. Let us consider the case of spaces of universal disposition. A property that somehow means the existence of many separable complemented subspaces is the so-called Separable Complementation Property (in short, SCP). Recall from [14], see also[6], that a Banach space X is said to have SCP if every separable subspace of X is contained in a separable subspace complemented in X. Recall from [1, Def. 2.25] that a Banach space X is said to be upper-c_0-saturated if every separable subspace is contained in a copy of c_0 contained in X. A few straightforward facts are:

Lemma 3.1. *A separably injective space with SCP is upper-c_0-saturated. In particular, it is c_0-saturated. A space of universal disposition with respect to separable spaces cannot have SCP.*

Proof.

(1) Indeed, every separable subspace must be contained in a separable separably injective subspace; i.e., in c_0.

(2) Observe that every copy of c_0 in a space with SCP must be complemented. But a space of universal disposition with respect to separable spaces must contain isometric copies of all spaces with density character $\aleph_1$ (see [1, Prop. 3.13 (2)]), which prevents them to have SCP since there are spaces with density character $\aleph_1$ containing uncomplemented copies of c_0: The simplest example being the space $C(\Delta)$ of continuous functions on the compact dyadic tree — the compact space having three types of points: the nodes of the dyadic tree, which are isolated points; the branches, each branch is the limit of its nodes, and the infinity point that appears by one-point compactification (see [4] for details). This space can be represented as a nontrivial twisted sum $0 \to c_0 \to C(\Delta) \to c_0(\aleph_1) \to 0$ and thus it is separably injective (see [1, Prop. 2.11]; also [3]). □

A different question is whether spaces of universal disposition can have SCP. Since, as we remarked in (2) above, every copy of c_0 in a space with SCP is complemented, picking a space E with density character $\aleph_1$ containing an uncomplemented copy of c_0 immediately yields a space $\mathscr{F}^{\omega_1}(E)$ that is of universal disposition, which has density character $\aleph_1$ and without SCP. We have not been able to settle the question of whether $\mathscr{F}^{\omega_1}(c_0)$ has SCP. We however show:

Lemma 3.2. *Under* CH*, every subspace of $\mathscr{F}^{\omega_1}(\mathbb{R})$ isomorphic to c_0 is complemented.*

Proof. Under CH, $\mathfrak{c} = \aleph_1$, which means that a set C of size $\mathfrak{c}$ can be written as a union $\bigcup_{\alpha<\omega_1} C_\alpha$ of countable sets C_α. Do this and write the set of all finite dimensional spaces as $\mathfrak{F} = \bigcup_{\alpha<\omega_1} \mathfrak{F}_\alpha$ and the set of all isometries between them as $\mathfrak{I} = \bigcup_{\alpha<\omega_1} \mathfrak{I}_\alpha$. Do the same with the set of isometric embeddings between elements of $\mathfrak{F}$ and $\mathbb{R}$, say $\mathfrak{L}^0 = \bigcup_{\alpha<\omega_1} \mathfrak{L}^0_\alpha$ and keep in mind that when $\mathscr{S}^\mu$ has been obtained one has to work with the set of isometric embeddings between elements of $\mathfrak{F}$ and $\mathscr{S}^\mu$, say $\mathfrak{L}^\mu = \bigcup_{\alpha<\omega_1} \mathfrak{L}^\mu_\alpha$. Now proceed with the construction of $\mathscr{F}^{\omega_1}(\mathbb{R})$ with the restriction that at step $\alpha < \omega_1$ one will only consider to make push-out the elements of $\mathfrak{F}_\alpha$ and $\mathfrak{I}_\alpha$ and $\cup_{\nu,\mu\leq\alpha}\mathfrak{L}^\mu_\nu$.

The immediate consequence of acting this way is that each space $\mathscr{S}^\alpha$ is separable.

Now, pick a λ-isomorphic copy of Y of c_0 inside $\mathscr{F}^{\omega_1}(\mathbb{R})$ and let $\phi : Y \to c_0$ be a λ-isomorphism. Since ω_1 has uncountable cofinality, there must be some $\alpha < \omega_1$ so that $Y \subset \mathscr{S}^\alpha$. By Sobczyk's theorem, ϕ can be extended to an operator $\Phi : \mathscr{S}^\alpha \to c_0$ with norm at most 2λ. Since c_0-valued finite-rank operators can be extended with the same norm everywhere (a well-known fact, see [1, Lemma 3.14] for details; it follows that the composition $\Phi\sum\mathfrak{L}_\alpha$ (see diagram (1)) can be extended to $\ell_1(\Gamma_\alpha, \operatorname{cod} u)$ with the same norm, hence to $\mathscr{S}^{\alpha+1}$. Proceeding inductively, one gets an extension $\widehat{\Phi} : \mathscr{F}^{\omega_1}(\mathbb{R}) \to c_0$ of Φ. The operator $\phi^{-1}\widehat{\Phi}$ is a projection onto Y. □

For smaller classes $\mathfrak{M}$ it is however possible to enjoy SCP. Let $\mathfrak{Pol}$ denote the class of finite dimensional polyhedral spaces.

Proposition 3.1. *There is a space of universal disposition for the class $\mathfrak{Pol}$ enjoying the SCP; precisely, such that every separable subspace is contained in a 1-complemented copy of $C[0,1]$.*

Proof. Set the controls of the device at: $\mathfrak{M} = \mathfrak{Pol}$; the starting space will be $C[0,1]$ and we will call $X_\alpha = C(B_{\mathscr{S}^{\alpha*}})$. Then proceed inductively. The first step is

$$\begin{array}{ccccc} \ell_1(\Gamma_\alpha, \operatorname{dom} u) & \xrightarrow{\oplus\mathfrak{I}_\alpha} & \ell_1(\Gamma_\alpha, \operatorname{cod} u) & & \\ {\scriptstyle \Sigma\mathfrak{L}_\alpha}\downarrow & & \downarrow & & \\ X & \longrightarrow & \mathrm{PO} & \xrightarrow{\delta} & C(B_{\mathrm{PO}^*}) \end{array} \tag{2}$$

where δ is the canonical isometric embedding. I.e., instead of replacing X by $\mathscr{S}^1$ at the first step, set $C(B_{\mathscr{S}^{1*}})$, and so on. Thus, step α will then be

$$\begin{array}{ccccc} \ell_1(\Gamma_\alpha, \operatorname{dom} u) & \xrightarrow{\oplus\mathfrak{I}_\alpha} & \ell_1(\Gamma_\alpha, \operatorname{cod} u) & & \\ {\scriptstyle \Sigma\mathfrak{L}_\alpha}\downarrow & & \downarrow & & \\ X_\alpha & \longrightarrow & \mathscr{S}^{\alpha+1} & \xrightarrow{\delta} & C(B_{\mathscr{S}^{\alpha+1*}}). \end{array} \tag{3}$$

Let us call $\mathscr{P}^{\omega_1}(C[0,1])$ the outcome of the device at ω_1.

To show that $\mathscr{P}^{\omega_1}(C[0,1])$ has the desired property, observe that every operator from a finite-dimensional polyhedral space into a $C(K)$-space can be extended with the same norm to any separable superspace. $\square$

Observe that the space $\mathscr{P}^{\omega_1}(C[0,1])$ is of almost universal disposition.

4. A new space of universal disposition

In [1, Section 6.6. Problem 16] is formulated the conjecture that there is a continuum of mutually nonisomorphic spaces of universal disposition having density character $\mathfrak{c}$. So far only two types are known: one is the space $\mathscr{F}^{\omega_1}(\mathbb{R})$; as for the other, one has to pick as family $\mathfrak{M}$ that of separable spaces, and as $\mathfrak{I}$ that of into isometries between separable spaces. With this choice and the same construction as for $\mathscr{F}^{\omega_1}(\mathbb{R})$ one gets the space $\mathcal{S}^{\omega_1}(\mathbb{R})$, which is of universal disposition for separable spaces and therefore separably injective. Actually, under CH it is isometric to the Fraissé limit in the category of separable spaces and into isometries [1, Prop. 3.3 and Thm. 3.23]. Here we construct a third type. Recall that $C(\Delta)$ represents here the space of continuous functions on the compact dyadic tree space as described above.

Proposition 4.1. *Under* CH*, for every separable* $C(K)$ *the space*

$$\mathscr{F}^{\omega_1}(C(\Delta)\oplus C(K))$$

is a space of universal disposition that is not isomorphic to either $\mathscr{F}^{\omega_1}(\mathbb{R})$ *or* $\mathcal{S}^{\omega_1}(\mathbb{R})$.

Proof. We settle first the case $C(K)\simeq c_0$, in which case $C(\Delta)\oplus C(K)\simeq C(\Delta)$. The space $\mathscr{F}^{\omega_1}(C(\Delta))$ is a space of universal disposition for exactly the same reason as $\mathscr{F}^{\omega_1}(\mathbb{R})$. It cannot be isomorphic to $\mathscr{F}^{\omega_1}(\mathbb{R})$ because it contains an uncomplemented copy of c_0. Let us show that it cannot be isomorphic to $\mathcal{S}^{\omega_1}(\mathbb{R})$ either:

From [7, Thm. 11] it follows that $C(\Delta)$ admits a polyhedral Lindenstrauss renorming with property $(*)$. Reasoning now as in Proposition 2.1, it follows that $C(\Delta)$ is 1-complemented in $\mathscr{F}^{\omega_1}(C(\Delta))$. Assume that $\mathscr{F}^{\omega_1}(C(\Delta))$ is isomorphic to $\mathcal{S}^{\omega_1}(\mathbb{R})$. This is a Grothendieck space [1, Thm 3.5 and Prop. 2.31] — operators into c_0 are weakly compact — hence $\mathscr{F}^{\omega_1}(C(\Delta))$ should also be; which is impossible since it contains a complemented copy of c_0 (the subspace of continuous functions on Δ with support contained in a given branch).

If $C(K)$ is not isomorphic to c_0 then it is not separably injective, by Zippin's theorem. The space $\mathscr{F}^{\omega_1}(C(\Delta)\oplus C(K))$ is a space of universal disposition not isomorphic to $\mathscr{F}^{\omega_1}(\mathbb{R})$ exactly as before. Let us show that it cannot be isomorphic to $\mathcal{S}^{\omega_1}(\mathbb{R})$ either:

Reasoning now as in the Claim of Theorem 2.1, $C(\Delta)\oplus C(K)$ is complemented in every $\mathscr{S}^{\alpha}(C(\Delta)\oplus C(K))$. Since $\mathcal{S}^{\omega_1}(\mathbb{R})$ is separably injective, an isomorphism between $\mathscr{F}^{\omega_1}(C(\Delta)\oplus C(K))$ and $\mathcal{S}^{\omega_1}(\mathbb{R})$ would imply that $C(\Delta)\oplus C(K)$ is separably injective; in particular $C(K)$ should be separably injective, which is not. $\square$

And, taking Corollary 2.1 into account one gets:

Corollary 4.1. *Under* CH*, for every separable Lindenstrauss space L,*

$$\mathscr{F}^{\omega_1}(C(\Delta)\oplus L)$$

is a space of universal disposition that is not isomorphic to either $\mathscr{F}^{\omega_1}(\mathbb{R})$ *or* $\mathcal{S}^{\omega_1}(\mathbb{R})$.

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