

# The second transpose of a derivation and weak amenability of the second dual Banach algebras

Massoud Amini, Morteza Essmaili  
and Mahmoud Filali

ABSTRACT. Let  $\mathcal{A}$  be a Banach algebra,  $\mathcal{A}^*$ ,  $\mathcal{A}^{**}$  and  $\mathcal{A}^{***}$  be its first, second and third dual, respectively. Let  $R : \mathcal{A}^{***} \rightarrow \mathcal{A}^*$  be the restriction map,  $J : \mathcal{A}^* \rightarrow \mathcal{A}^{***}$  be the canonical injection and  $\Lambda : \mathcal{A}^{***} \rightarrow \mathcal{A}^{***}$  be the composition of  $R$  and  $J$ . Let  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  be a continuous derivation and  $D'' : \mathcal{A}^{**} \rightarrow \mathcal{A}^{***}$  be its second transpose. We obtain a necessary and sufficient condition for  $\Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  to be a derivation. We apply this to prove some results on weak amenability of second dual Banach algebras.

## CONTENTS

1. Introduction	265
2. Some definitions	267
3. The second transpose of a derivation	268
4. Some examples	272
References	274

## 1. Introduction

The problem of a Banach algebra  $\mathcal{A}$  inheriting weak amenability from its second dual  $\mathcal{A}^{**}$  was originally studied by Ghahramani, Loy and Willis in [GLW96]. Then came [DaRV01], where Dales, Rodriguez and Velasco considered a continuous derivation  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  and studied conditions under which the second transpose  $D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  of  $D$  is again a derivation. They showed that  $D''$  is a derivation if and only if

$$D''(\mathcal{A}^{**}) \bullet \mathcal{A}^{**} \subseteq \mathcal{A}^*,$$

where “ $\bullet$ ” is the natural action of  $\mathcal{A}^{**}$  on  $\mathcal{A}^{***}$  as defined below in (2), see [DaRV01, Theorem 7.1]. As a consequence, if  $\mathcal{A}$  is Arens regular and every

Received April 27, 2015.

2010 *Mathematics Subject Classification*. Primary 46H20, Secondary 47B47.

*Key words and phrases*. Derivation, second dual of Banach algebras, weak amenability.

The first author was partly supported by a grant from IPM (No. 90430215).

derivation from  $\mathcal{A}$  into  $\mathcal{A}^*$  is weakly compact, then weak amenability passes from  $\mathcal{A}^{**}$  to  $\mathcal{A}$ , see [DaRV01, Corollary 7.5].

In [EF007, Theorem 2.2], Eshaghi and Filali obtained several affirmative results on this problem of inheritance of weak amenability including the one just mentioned. The arguments used in [EF007] do not use the criterion proved in [DaRV01, Theorem 7.1]. Moreover, the arguments used by Eshaghi and Filali yielded also in [EF007, Theorem 2.2] a short and a simple proof of the criterion proved in [DaRV01, Theorem 7.1]. It may be worthwhile to note that in most of these affirmative results, some conditions on regularity (such as  $\mathcal{A}$  is Arens regular, or  $D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$  or the topological center of  $\mathcal{A}^{**}$  being weakly amenable) were imposed every time. In [EF007, Theorem 2.4], the theorem was proved under the conditions that  $\mathcal{A}$  is a right ideal in  $\mathcal{A}^{**}$  and  $\mathcal{A}^{**}\mathcal{A} = \mathcal{A}^{**}$  and no Arens regularity was assumed. However, it turned out later on that in this situation  $\mathcal{A}$  must be Arens regular; see [EF07, Theorem 4.3].

In this note, we are mainly concerned with one of these affirmative results, which was proved in [GLW96, Theorem 2.3]. This theorem did not require any condition related to Arens regularity. It stated that if  $\mathcal{A}$  is a left ideal in  $\mathcal{A}^{**}$ , then weak amenability of the second dual  $\mathcal{A}^{**}$  is inherited by  $\mathcal{A}$ . The main ingredient used to prove this result is, that if under the above assumptions  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is a derivation, then  $\Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  is also a derivation, where  $\Lambda$  is the composition of the restriction map  $R : \mathcal{A}^{***} \rightarrow \mathcal{A}^*$  with the canonical injection  $J : \mathcal{A}^* \rightarrow \mathcal{A}^{***}$ . Again without assuming any condition on Arens regularity, it is claimed that the same method shows that if every derivation  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is weakly compact, then weak amenability of the second dual algebra  $\mathcal{A}^{**}$  is inherited by  $\mathcal{A}$  (see the remark after [GLW96, Theorem 2.3]). But there seems to be a gap in the passage proving that  $\Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  is also a derivation as we shall explain in this paper. So the claim in [GLW96, Theorem 2.3] and the remark following this theorem are still unknown.

Recently, some authors studied conditions under which the transpose of a dual-valued operator becomes a derivation, see for example [AP12] or [BaV11]. In [AP12, Proposition 1], Alejandro and Peña looked for conditions which guarantee that the first transpose of an  $\mathcal{A}^*$ -valued bounded linear map on  $\mathcal{A}$  is a bounded derivation on  $\mathcal{A}^{**}$ , where they safely assumed that  $\mathcal{A}$  is Arens regular. Unlike, Barootkoob and Vishki in [BaV11, Lemma 4(ii)], they obtained a condition for  $\Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  to be a derivation and claimed that the hypothesis of Arens regularity in [DaRV01, Corollary 7.5] is superfluous. It seems that part of the proof of [BaV11, Lemma 4(ii)] has the same gap as in the proof of [GLW96, Theorem 2.3] (see Remark 3.3). Many attempts to prove the theorem without the Banach algebra being Arens regular have failed so far even with the extra conditions that  $\mathcal{A}$  is a left ideal in  $\mathcal{A}^{**}$  or the derivations from  $\mathcal{A}$  to  $\mathcal{A}^*$  are weakly compact; and

a counter-example is still elusive even when these extra conditions are not assumed.

This paper is organized as follows. In Section 3, we shall obtain a necessary and sufficient condition for  $\Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  to be a derivation. With this, it will be easy to explain where the arguments used in [GLW96, Theorem 2.3] break down, and why it is more likely that  $\Lambda \circ D''$  is not a derivation in general (see Remark 3.3). Checking that  $\Lambda \circ D''$  is a derivation as well as passing weak amenability from  $\mathcal{A}^{**}$  to  $\mathcal{A}$  will be immediate under the extra condition on Arens regularity.

In Section 4, we will present examples showing that Arens regularity is not necessary for any of these statements to be true. When  $\mathcal{A}$  is the semigroup algebra  $\ell^1(\mathbb{Z}^+)$ , which is not Arens regular, we see that  $\Lambda \circ D''$  is a derivation and that the inclusion  $D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$  holds. The second example gives a non-Arens regular Banach algebra  $\mathcal{A}$  for which  $\mathcal{A}^{**}$  and  $\mathcal{A}$  are both weakly amenable. In the last example, we see that if  $G$  is a locally compact group which is not *SIN*, then there exist always derivations (even inner)  $D : L^1(G) \rightarrow L^\infty(G)$  such that  $\Lambda \circ D''$  is not a derivation. Note that in this last example,  $\mathcal{A}^{**}$  is not weakly amenable and  $\mathcal{A}$  is not a left ideal in  $\mathcal{A}^{**}$ .

### 2. Some definitions

Suppose that  $\mathcal{A}$  is a Banach algebra and  $E$  is a Banach  $\mathcal{A}$ -bimodule. A linear map  $D : \mathcal{A} \rightarrow E$  is a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}).$$

For each  $x \in E$ , the map  $D_x : \mathcal{A} \rightarrow E$  defined by

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A})$$

is a continuous derivation called the *inner derivation* induced by  $x$ . A Banach algebra  $\mathcal{A}$  is said to be *weakly amenable* if every continuous derivation from  $\mathcal{A}$  into the dual Banach  $\mathcal{A}$ -bimodule  $\mathcal{A}^*$  is inner, where the module actions on  $\mathcal{A}^*$  are defined by

$$(1) \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle \quad (a, b \in \mathcal{A}, f \in \mathcal{A}^*).$$

The first and second Arens products  $\square$  and  $\diamond$  on  $\mathcal{A}^{**}$  are defined by

$$\langle F \square G, f \rangle = \langle F, G \cdot f \rangle \quad \text{and} \quad \langle F \diamond G, f \rangle = \langle G, f \cdot F \rangle.$$

Here,

$$\langle G \cdot f, a \rangle = \langle G, f \cdot a \rangle \quad \text{and} \quad \langle f \cdot F, a \rangle = \langle F, a \cdot f \rangle,$$

for every  $F, G \in \mathcal{A}^{**}$ ,  $f \in \mathcal{A}^*$  and  $a \in \mathcal{A}$ . A Banach algebra  $\mathcal{A}$  is *Arens regular* if the first and second Arens products coincide in  $\mathcal{A}^{**}$ .

A linear functional  $f \in \mathcal{A}^*$  is *weakly almost periodic* if the map

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{A}^*, \\ a &\longrightarrow a \cdot f, \end{aligned}$$

is weakly compact. The space of weakly almost periodic functionals on  $\mathcal{A}$  is denoted by  $WAP(\mathcal{A})$ . It is shown by Pym [P65] that

$$WAP(\mathcal{A}) = \{f \in \mathcal{A}^* : \langle F \square G, f \rangle = \langle F \diamond G, f \rangle \text{ for every } F, G \in \mathcal{A}^{**}\}.$$

It follows that  $\mathcal{A}$  is Arens regular if and only if  $WAP(\mathcal{A}) = \mathcal{A}^*$ .

For the latest developments on  $WAP(\mathcal{A})$ , see [FNS15].

**Convention 2.1.** Throughout this paper, we regard  $\mathcal{A}^{***} := (\mathcal{A}^{**})^*$  as a Banach  $\mathcal{A}^{**}$ -bimodule with the following module actions:

$$(2) \quad \begin{aligned} \langle \psi \bullet F, G \rangle &= \langle \psi, F \square G \rangle, \\ \langle F \bullet \psi, G \rangle &= \langle \psi, G \square F \rangle, \end{aligned}$$

( $\psi \in \mathcal{A}^{***}, F, G \in \mathcal{A}^{**}$ ). Note that here the actions are defined as in (1), taking  $\mathcal{A}^{**}$  as the Banach algebra with the the first Arens product.

### 3. The second transpose of a derivation

Throughout the paper, a normed space is always identified with its canonical image in its second dual. In this way,  $\Lambda = R \circ J$  becomes simply  $R : \mathcal{A}^{***} \rightarrow \mathcal{A}^* \subseteq \mathcal{A}^{***}$ . Suppose that  $\mathcal{A}$  is a Banach algebra and  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is a continuous derivation. Then for each  $F, G \in \mathcal{A}^{**}$ ,

$$(3) \quad \begin{aligned} D''(F \square G) &= \lim_{\alpha} \lim_{\beta} D''(a_{\alpha} b_{\beta}) \\ &= \lim_{\alpha} \lim_{\beta} D''(a_{\alpha}) \cdot b_{\beta} + \lim_{\alpha} \lim_{\beta} a_{\alpha} \cdot D''(b_{\beta}) \\ &= D''(F) \bullet G + \lim_{\alpha} a_{\alpha} \bullet D''(G), \end{aligned}$$

where the above limits are taken with respect to the weak\*-topology in  $(\mathcal{A}^*)^{**}$  and

( $a_{\alpha}$ ) and ( $b_{\beta}$ ) are bounded nets in  $\mathcal{A}$ , weak\*-converging in  $\mathcal{A}^{**}$  to  $F$  and  $G$ , respectively.

We also have, using the same arguments as in [GLW96, Page 1495, lines 7-16] (these are correct),

$$\lim_{\alpha} \Lambda(a_{\alpha} \bullet \psi) = F \bullet \Lambda(\psi) \quad \text{for all } \psi \in \mathcal{A}^{***}.$$

Hence, we deduce from (3) that

$$\begin{aligned} \Lambda \circ D''(F \square G) &= \Lambda(D''(F) \bullet G) + \lim_{\alpha} \Lambda(a_{\alpha} \bullet D''(G)) \\ &= \Lambda(D''(F) \bullet G) + F \bullet (\Lambda \circ D''(G)). \end{aligned}$$

It follows that  $\Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  is a derivation if and only if, for each  $F$  and  $G \in \mathcal{A}^{**}$ ,

$$(4) \quad \Lambda(D''(F) \bullet G) = (\Lambda \circ D''(F)) \bullet G.$$

In the proofs of [BaV11, Lemma 4(ii)] and [GLW96, Theorem 2.3], the authors used the fact that (4) holds if and only if it holds on  $\mathcal{A}$ . The left-hand side of the identity is in  $\mathcal{A}^*$ , and everything would have been fine if

both sides were in  $\mathcal{A}^*$ . But there is no apparent reason for this to be so in general. The map  $(\Lambda \circ D''(F))$  being in  $\mathcal{A}^*$  does not make the map on right-hand side in  $\mathcal{A}^*$ ; we only know that it is in  $\mathcal{A}^{***}$ , and is given by the action of  $\mathcal{A}^{**}$  on  $\mathcal{A}^{***}$  as defined in (2). So, for the identity (4) to hold on  $\mathcal{A}^{**}$ , it is clear that further assumptions are required. In the following result, we obtain a necessary and sufficient condition for point  $\Lambda \circ D''$  to be a derivation.

**Theorem 3.1.** *Let  $\mathcal{A}$  be a Banach algebra,  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  be a continuous derivation, and let  $D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  be the second transpose of  $D$ . Then the following are equivalent:*

- (i)  $\Lambda \circ D''$  is a derivation.
- (ii) For every  $F, G \in \mathcal{A}^{**}$ ,  $\Lambda(D''(F) \bullet G) = (\Lambda \circ D''(F)) \bullet G$ .
- (iii) For every  $F, G \in \mathcal{A}^{**}$ ,  $\Lambda(D''(F) \bullet G) = (\Lambda \circ D''(F)) \bullet G$  on  $\mathcal{A}$  and

$$\Lambda \circ D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A}).$$

**Proof.** From the above argument, it follows that (i) and (ii) are equivalent.

(ii) $\Rightarrow$ (iii) It suffices to show that  $\Lambda \circ D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$ . Fix two elements  $F$  and  $G$  in  $\mathcal{A}^{**}$ . Let  $(H_\alpha)$  be a net in  $\mathcal{A}^{**}$  such that  $H_\alpha \rightarrow H$  in the weak\*-topology on  $\mathcal{A}^{**}$ . Then, by assumption, we have for each  $\alpha$ ,

$$\begin{aligned} \langle \Lambda(D''(F)), G \square H_\alpha \rangle &= \langle \Lambda(D''(F)) \bullet G, H_\alpha \rangle \\ &= \langle \Lambda(D''(F) \bullet G), H_\alpha \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_\alpha \langle \Lambda(D''(F)), G \square H_\alpha \rangle &= \lim_\alpha \langle \Lambda(D''(F) \bullet G), H_\alpha \rangle \\ &= \lim_\alpha \langle H_\alpha, R(D''(F) \bullet G) \rangle \\ &= \langle H, R(D''(F) \bullet G) \rangle \quad (\text{since } R(D''(F) \bullet G) \in \mathcal{A}^*) \\ &= \langle \Lambda(D''(F) \bullet G), H \rangle \\ &= \langle \Lambda(D''(F)) \bullet G, H \rangle \quad (\text{by assumption}) \\ &= \langle \Lambda(D''(F)), G \square H \rangle. \end{aligned}$$

Now using the fact that

$$WAP(\mathcal{A}) = \{f \in \mathcal{A}^* : \text{the map } H \rightarrow \langle G \square H, f \rangle \text{ is weak}^* \text{ continuous on } \mathcal{A}^{**} \text{ for each } G \in \mathcal{A}^{**}\},$$

we conclude that  $\Lambda(D''(F)) \in WAP(\mathcal{A})$ , as required.

(iii) $\Rightarrow$ (ii) By hypothesis,  $\Lambda(D''(F) \bullet G) = (\Lambda \circ D''(F)) \bullet G$  on  $\mathcal{A}$ . So, it suffices to show that  $(\Lambda \circ D''(F)) \bullet G \in \mathcal{A}^*$ . Again let  $(H_\alpha)$  be a net in  $\mathcal{A}^{**}$  such that  $H_\alpha \rightarrow H$  in the weak\*-topology on  $\mathcal{A}^{**}$ . Since  $\Lambda \circ D''(\mathcal{A}^{**}) \subseteq$

$WAP(\mathcal{A})$ , we have

$$\begin{aligned} \lim_{\alpha} \langle (\Lambda \circ D''(F)) \bullet G, H_{\alpha} \rangle &= \lim_{\alpha} \langle (\Lambda \circ D''(F)), G \square H_{\alpha} \rangle \\ &= \langle (\Lambda \circ D''(F)), G \square H \rangle \\ &= \langle (\Lambda \circ D''(F)) \bullet G, H \rangle. \end{aligned}$$

Therefore  $(\Lambda \circ D''(F)) \bullet G \in \mathcal{A}^*$ , and so the proof is complete.  $\square$

Before we prove next theorem, we check that the maps  $(\Lambda \circ D''(F)) \bullet G$  and  $\Lambda(D''(F)) \bullet G$  are equal on  $\mathcal{A}$  when either  $D$  is weakly compact, or  $\mathcal{A}$  is a left ideal in  $\mathcal{A}^{**}$ . In the first situation,  $D''(F) \in \mathcal{A}^*$ , and so

$$\Lambda(D''(F)) = D''(F)$$

for each  $F \in \mathcal{A}^{**}$ . In the second situation,

$$\langle \Lambda(D''(F)), G \square a \rangle = \langle D''(F), G \square a \rangle$$

for each  $F, G \in \mathcal{A}^{**}$  and  $a \in \mathcal{A}$ , since  $G \square a \in \mathcal{A}$ . Therefore, in both situations, we have for each  $F, G \in \mathcal{A}^{**}$  and  $a \in \mathcal{A}$ ,

$$\begin{aligned} (5) \quad \langle (\Lambda \circ D''(F)) \bullet G, a \rangle &= \langle D''(F), G \square a \rangle \\ &= \langle D''(F) \bullet G, a \rangle = \langle \Lambda(D''(F)) \bullet G, a \rangle, \end{aligned}$$

as required.

**Theorem 3.2.** *Let  $\mathcal{A}$  be a Banach algebra,  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  be a continuous derivation,  $D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^*$  be the second transpose of  $D$ , and suppose that  $D$  is weakly compact. Then the four following statements are equivalent.*

- (i)  $\Lambda \circ D''$  is a derivation.
- (ii)  $D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$ .
- (iii)  $D(\mathcal{A}) \subseteq WAP(\mathcal{A})$ .
- (iv)  $D''$  is a derivation.

**Proof.** (ii) $\Rightarrow$ (iii) is trivial, and (ii) $\Rightarrow$ (iv) follows as in the proof of [EF07, Theorem 2.1].

We prove (iv) $\Rightarrow$ (i). We check that under the given conditions, these two maps in (5) are actually equal as elements in  $\mathcal{A}^{***}$ . First, since  $D''$  is a derivation, we have  $D''(\mathcal{A}^{**}) \bullet \mathcal{A}^{**} \subseteq \mathcal{A}^*$  by [DaRV01, Theorem 7.1] or [EF07, Theorem 2.2]. So when  $D$  is weakly compact our claim is therefore straightforward, since  $(\Lambda \circ D''(F)) \bullet G = D''(F) \bullet G \in \mathcal{A}^*$ , and so both maps  $(\Lambda \circ D''(F)) \bullet G$  and  $\Lambda(D''(F)) \bullet G$  are in  $\mathcal{A}^*$ . They are therefore equal as elements in  $\mathcal{A}^{***}$ .

Using Theorem 3.1, we conclude that  $\Lambda \circ D''$  is a derivation.

We prove now the rest of the implications when  $D$  is weakly compact.

(i) $\Rightarrow$ (ii) Since  $\Lambda \circ D''$  is a derivation, it follows from Theorem 3.1 that

$$\Lambda \circ D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A}).$$

Now since  $D$  is weakly compact, we have

$$D''(\mathcal{A}^{**}) = \Lambda \circ D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A}).$$

(iii) $\Rightarrow$ (ii) Since  $D$  is weakly compact, we have for each  $F \in \mathcal{A}^{**}$ ,  $D''(F) = f$  for some  $f \in \mathcal{A}^*$ . Since  $D(\mathcal{A}) \subseteq WAP(\mathcal{A})$ , using the weak\*-continuity of  $D''$ , we conclude that  $f$  is in the weak closure of  $WAP(\mathcal{A})$ . Since  $WAP(\mathcal{A})$  is weakly closed in  $\mathcal{A}^*$ , we see that  $f \in WAP(\mathcal{A})$ , as required. To see that  $WAP(\mathcal{A})$  is weakly closed, let  $(f_\alpha)$  be a net in  $WAP(\mathcal{A})$  with a weak limit  $f$  in  $\mathcal{A}^*$ . Then, for each  $F, G \in \mathcal{A}^{**}$ , we have

$$\langle F \square G, f_\alpha \rangle \longrightarrow \langle F \square G, f \rangle, \quad \langle F \diamond G, f_\alpha \rangle \longrightarrow \langle F \diamond G, f \rangle.$$

Since  $\langle F \square G, f_\alpha \rangle = \langle F \diamond G, f_\alpha \rangle$  for each  $\alpha$ , we conclude that

$$\langle F \square G, f \rangle = \langle F \diamond G, f \rangle.$$

Thus,  $f \in WAP(\mathcal{A})$ . □

**Remark 3.3.** Let  $\mathcal{A}$  be a Banach algebra and  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  be a derivation. In the proof of [BaV11, Lemma 4(ii)] and [GLW96, Theorem 2.3], the authors used the fact that

$$\begin{aligned} (6) \quad \Lambda \circ D'' : \mathcal{A}^{**} \rightarrow (\mathcal{A}^{**})^* \text{ is a derivation} \\ \iff \text{for all } F, G \in \mathcal{A}^{**} \text{ and } a \in \mathcal{A}, \\ \langle \Lambda(D''(F) \bullet G), a \rangle = \langle (\Lambda \circ D''(F)) \bullet G, a \rangle. \end{aligned}$$

As explained at the beginning of this section, the latter identity needs not to hold for any  $a \in \mathcal{A}^*$ . Moreover, in the light of Theorems 3.1 and 3.2, the above statement implies the following fact, which points more towards the non-validity of this statement in general.

$$\text{A derivation } D : \mathcal{A} \rightarrow \mathcal{A}^* \text{ is weakly compact} \iff D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A}).$$

In other words,

$$(7) \quad D''(\mathcal{A}^{**}) \subseteq \mathcal{A}^* \iff D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A}).$$

To see this, when  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is weakly compact, then by (5) (or as in [GLW96, Theorem 2.3] when  $\mathcal{A}$  is a left ideal in  $\mathcal{A}^{**}$ ), we have

$$\langle \Lambda(D''(F) \bullet G), a \rangle = \langle (\Lambda \circ D''(F)) \bullet G, a \rangle \quad (F, G \in \mathcal{A}^{**}, a \in \mathcal{A}).$$

Accordingly, if (6) is true, then  $\Lambda \circ D''$  is a derivation, and so by Theorem 3.2,  $D''(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$ .

The converse is straightforward.

With the extra condition that  $\mathcal{A}$  is Arens regular, affirmative answers are now immediate with the help of Theorems 3.1 and 3.2.

**Corollary 3.4.** *Let  $\mathcal{A}$  be an Arens regular Banach algebra which is a left ideal in  $\mathcal{A}^{**}$ .*

- (i) *If  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is a continuous derivation, then  $\Lambda \circ D''$  is a derivation.*
- (ii) *If  $\mathcal{A}^{**}$  is weakly amenable, then  $\mathcal{A}$  is weakly amenable.*

**Proof.** (i) Since  $\mathcal{A}$  is a left ideal in  $\mathcal{A}^{**}$ , by the argument used in the proof of [GLW96, Theorem 2.3] (or by (5)), for each  $F, G \in \mathcal{A}^{**}$ , we have

$$\Lambda(D''(F) \bullet G) = (\Lambda \circ D''(F)) \bullet G$$

on  $\mathcal{A}$ . Since  $\mathcal{A}$  is Arens regular,

$$\Lambda \circ D''(\mathcal{A}^{**}) \subseteq \mathcal{A}^* = WAP(\mathcal{A}).$$

Thus, by Theorem 3.1,  $\Lambda \circ D''$  is a derivation.

(ii) Let  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  be a derivation. By (i),  $\Lambda \circ D''$  is a derivation. Since  $\mathcal{A}^{**}$  is weakly amenable, there is  $\psi \in \mathcal{A}^{***}$  such that

$$\Lambda \circ D''(F) = F \bullet \psi - \psi \bullet F \quad (F \in \mathcal{A}^{**}).$$

Therefore,  $D$  is the inner derivation induced by  $R(\psi)$ .  $\square$

A direct application of Theorem 3.2 gives an alternative proof for [DaRV01, Corollary 7.5].

**Corollary 3.5.** *Let  $\mathcal{A}$  be an Arens regular Banach algebra.*

- (i) *If  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is a weakly compact derivation, then  $\Lambda \circ D''$  is a derivation.*
- (ii) *If every continuous derivation  $D : \mathcal{A} \rightarrow \mathcal{A}^*$  is weakly compact and  $\mathcal{A}^{**}$  is weakly amenable, then  $\mathcal{A}$  is weakly amenable.*

#### 4. Some examples

Arens regularity seems to be an unavoidable condition to impose on  $\mathcal{A}$  to reach any affirmative answer related to the problem. But this is not necessary as the following examples show.

**Example 4.1.** First we look at statement (7). It is easy to see that this statement holds when  $\mathcal{A}$  is Arens regular. But the Arens regularity of  $\mathcal{A}$  is not necessary for it to hold. Recently, Choi and Heath characterized in [CH10, Theorem 2.6] weakly compact derivations from  $\ell^1(\mathbb{Z}^+)$  to its dual and they also showed in [CH10, Page 430] that every weakly compact derivation  $D : \ell^1(\mathbb{Z}^+) \rightarrow \ell^\infty(\mathbb{Z}^+)$  satisfies

$$D(\ell^1(\mathbb{Z}^+)) \subseteq c_0(\mathbb{Z}^+) \subseteq WAP(\ell^1(\mathbb{Z}^+)).$$

Now by Theorem 3.2, it follows that

$$D''(\ell^1(\mathbb{Z}^+)^{**}) \subseteq WAP(\ell^1(\mathbb{Z}^+)).$$

Thus (7) holds for  $\mathcal{A} = \ell^1(\mathbb{Z}^+)$ , though  $\mathcal{A}$  is not Arens regular.

But we do not know whether (7) is true in general.

**Example 4.2.** The following example shows, that Arens regularity of  $\mathcal{A}$  is not necessary in Corollary 3.4(ii).

Let  $S = (\mathbb{N}, \min)$  and let  $\beta S$  be the Stone–Čech compactification of  $S$ . Let  $c_0(S)$  be the  $C^*$ -algebra of functions on  $S$  vanishing at infinity and let

$$c_0^\perp(S) = \{\mu \in \ell^1(S)^{**} : \mu(f) = 0 \text{ for all } f \in c_0(S)\}.$$



As known,  $c_0(G)^\perp$  may be identified with the space  $M(\beta S \setminus S)$  of regular Borel measures on  $\beta S \setminus S$  and

$$\ell^1(S)^{**} = \ell^1(S) \oplus c_0^\perp(S).$$

It is easy to see that for each  $u, v \in \beta S \setminus S$  we have  $u \square v = u$ . Accordingly, for each  $\mu, \nu \in M(\beta S \setminus S)$ , we have  $\mu\nu = \nu(1)\mu$ . In other words, for each  $\mu, \nu \in M(\beta S \setminus S)$ , we have  $\mu\nu = \phi(\nu)\mu$ , where  $\phi$  is the augmentation character on  $M(\beta S \setminus S)$ . It follows that  $c_0(S)^\perp$  is a closed subalgebra of  $\ell^1(S)^{**}$ , which is weakly amenable by [DaLS10, Proposition 2.13]. Furthermore,  $\ell^1(S)$  is a two-sided ideal of  $\ell^1(S)^{**}$ , which is weakly amenable by [DaLS10, Example 10.10]. Therefore, by [DaLS10, Proposition 2.2 (vi)], we see that  $\ell^1(S)^{**}$  is weakly amenable.

Note that  $\ell^1(S)^{**}$  is not commutative, and so  $\ell^1(S)$  is not Arens regular. (In fact, the topological center of  $\ell^1(S)^{**}$  is  $\ell^1(S)$ .)

**Example 4.3.** Our next example shows, that unless  $G$  is a locally compact *SIN*-group, there are always (even inner) derivations  $D : L^1(G) \rightarrow L^\infty(G)$  for which  $\Lambda \circ D''$  is not a derivation. Note that here  $L^1(G)$  is always weakly amenable ([J91] or [DeG94]).

The proposition below holds also for weighted group algebras  $L^1(G, \omega)$ . For simplicity, it is presented here just for  $\omega = 1$ .

We recall first some necessary definitions. For a locally compact group  $G$ , we consider the Banach algebra  $L^1(G)$  under the convolution product. A function on  $G$  is *weakly almost periodic* when the set of all its left (equivalently, right) translates makes a relatively weakly compact subset in the  $C^*$ -algebra of bounded continuous functions on  $G$ . Let  $WAP(G)$  be the  $C^*$ -algebra of such functions.

A bounded function  $f$  on  $G$  is said to be *right uniformly continuous* when, for every  $\epsilon > 0$ , there exists a neighbourhood  $U$  of  $e$  such that

$$|f(s) - f(t)| < \epsilon \quad \text{whenever} \quad st^{-1} \in U.$$

The  $C^*$ -algebra of right uniformly continuous functions on  $G$  is denoted by  $LUC(G)$ . By analogy, we may also define the  $C^*$ -algebra  $RUC(G)$  of left uniformly continuous functions  $G$ . Since  $L^1(G)$  has a bounded approximate identity, Cohen–Hewitt factorization theorem (see [HR94, Theorem 32.22]) implies that  $L^\infty(G) \cdot L^1(G) = RUC(G)$  and  $L^1(G) \cdot L^\infty(G) = LUC(G)$ , see for example [Ü90, Proposition 3.3] or [HR79, 20.19]. We recall also that  $G$  is an *SIN*-group when there is a basis for the neighbourhoods of the identity in  $G$  consisting of sets  $U$  such that  $xUx^{-1} = U$ .

**Proposition 4.4.** *If for every continuous inner derivation*

$$D : L^1(G) \rightarrow L^\infty(G),$$

$\Lambda \circ D''$  *is a derivation, then  $G$  is a *SIN*-group.*

**Proof.** Suppose that  $f$  is an arbitrary element of  $L^\infty(G)$  and let

$$D_f : L^1(G) \longrightarrow L^\infty(G)$$

be the inner derivation induced by  $f$ . By hypothesis,  $\Lambda \circ D_f''$  is a derivation, hence by Theorem 3.1,  $(\Lambda \circ D_f'')(L^1(G)^{**}) \subseteq WAP(L^1(G))$ . In particular,

$$f \cdot g - g \cdot f \in WAP(L^1(G)) \quad \text{for all } g \in L^1(G).$$

On the other hand, as known (see for example [BeJM89]),

$$WAP(L^1(G)) = WAP(G) \subseteq LUC(G) \cap RUC(G).$$

Since  $f$  is arbitrary, it follows that for all  $f \in L^\infty(G)$  and  $g \in L^\infty(G)$ ,

$$f \cdot g - g \cdot f \in LUC(G) \cap RUC(G).$$

We conclude therefore that  $LUC(G) = RUC(G)$ , and so  $G$  is an *SIN*-group by [M90, Theorem 2].  $\square$

## References

- [AP12] ALEANDRO, MARÍA J.; PEÑA, CARLOS. On dual-valued operators on Banach algebras. *New York J. Math.* **18** (2012), 657–665. MR2991418, Zbl 1264.46036.
- [BaV11] BAROOTKOOB, S.; VISHKI, H. R. EBRAHIMI. Lifting derivations and  $n$ -weak amenability of the second dual of a Banach algebra. *Bull. Aust. Math. Soc.* **83** (2011), no. 1, 122–129. MR2765419, Zbl 1270.46042, arXiv:1007.1649, doi:10.1017/S0004972710001838.
- [BeJM89] BERGLUND, JOHN F.; JUNGHEHN, HUGO D.; MILNES, PAUL. Analysis on semigroups. Function spaces, compactifications, representations. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1989. xiv+334 pp. ISBN: 0-471-61208-1. MR0999922, Zbl 0727.22001.
- [CH10] CHOI, Y.; HEATH, M. J. Translation-finite sets and weakly compact derivations from  $\ell^1(\mathbb{Z}_+)$  to its dual. *Bull. London Math. Soc.* **42** (2010), no. 3, 429–440. MR2651938, Zbl 1204.43002, arXiv:0811.4432, doi:10.1112/blms/bdq003.
- [DaLS10] DALES, H. G.; LAU, A. T.-M.; STRAUSS, D. Banach algebras on semi-groups and their compactifications. *Mem. Amer. Math. Soc.* **205** (2010), no. 966, vi+165 pp. ISBN: 978-0-8218-4775-6. MR2650729, Zbl 1192.43001, doi:10.1090/S0065-9266-10-00595-8.
- [DaRV01] DALES, H. G.; RODRÍGUEZ-PALACIOS, A.; VELASCO, M. V. The second transpose of a derivation. *J. London Math. Soc. (2)* **64** (2001), no. 3, 707–721. MR1865558, Zbl 1023.46051, doi:10.1112/S0024610701002496.
- [DeG94] DESPIĆ, M.; GHAHRAMANI, F. Weak amenability of group algebras of locally compact groups. *Canad. Math. Bull.* **37** (1994), no. 2, 165–167. MR1275699, Zbl 0813.43001, doi:10.4153/CMB-1994-024-4.
- [EF07] ESHAGHI GORDJI, M.; FILALI, M. Arens regularity of module actions. *Studia Math.* **181** (2007), no. 3, 237–254. MR2322577, Zbl 1165.46024, doi:10.4064/sm181-3-3.
- [EF007] ESHAGHI GORDJI, M.; FILALI, M. Weak amenability of the second dual of a Banach algebra. *Studia. Math.* **182** (2007), no. 3, 205–213. MR2360627, Zbl 1135.46027, doi:10.4064/sm182-3-2.

- [FNS15] FILALI, M.; NEUFANG, M.; SANGANI MONFARED, M. Representations of Banach algebras subordinate to topologically introverted spaces. *Trans. Amer. Math. Soc.* **367** (2015), no. 11, 8033–8050. MR3391908, Zbl 06479456, doi: 10.1090/tran/6435.
- [GLW96] GHAHRAMANI, F.; LOY, R. J.; WILLIS, G. A. Amenability and weak amenability of second conjugate Banach algebras. *Proc. Amer. Math. Soc.* **124** (1996), no. 5, 1489–1497. MR1307520, Zbl 0851.46035, doi: 10.1090/S0002-9939-96-03177-2.
- [HR79] HEWITT, EDWIN; ROSS, KENNETH A. Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften, 115. *Springer-Verlag, Berlin-New York*, 1979. ix+519 pp. ISBN: 3-540-09434-2. MR0551496, Zbl 0837.43002, doi: 10.1007/978-1-4419-8638-2.
- [HR94] HEWITT, EDWIN; ROSS, KENNETH A. Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups. Die Grundlehren der mathematischen Wissenschaften, 152. *Springer-Verlag, New York-Berlin*, 1994 ix+771. MR0262773, Zbl 0830.43001.
- [J91] JOHNSON, B. E. Weak amenability of group algebras. *Bull. London Math. Soc.* **23** (1991), no. 3, 281–284. MR1123339, Zbl 0757.43002, doi: 10.1112/blms/23.3.281.
- [M90] MILNES, PAUL. Uniformity and uniformly continuous functions for locally compact groups. *Proc. Amer. Math. Soc.* **109** (1990), no. 2, 567–570. MR1023345, Zbl 0697.22005, doi: 10.2307/2048022.
- [P65] PYM, JOHN S. The convolution of functionals on spaces of bounded functions. *Proc. London Math. Soc.* (3) **15** (1965), 84–104. MR0173152, Zbl 0135.35503, doi: 10.1112/plms/s3-15.1.84.
- [Ü90] ÜLGER, A. Arens regularity sometimes implies the RNP. *Pacific J. Math.* **143** (1990), no. 2, 377–399. MR1051083, Zbl 0734.46032, doi: 10.2140/pjm.1990.143.377.

(Massoud Amini) DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, TARBIAT MODARES UNIVERSITY, 14115-134 TEHRAN, IRAN AND SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O.BOX 19395-5746, TEHRAN, IRAN  
 mamini@modares.ac.ir

(Morteza Essmaili) DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL AND COMPUTER SCIENCE, KHARAZMI UNIVERSITY, 50 TALEGHANI AVENUE, 15618 TEHRAN, IRAN  
 m.essmaili@khu.ac.ir

(Mahmoud Filali) DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF OULU, P.O. BOX 3000, OULU 90014, FINLAND  
 mahmoud.filali@oulu.fi

This paper is available via <http://nyjm.albany.edu/j/2016/22-12.html>.