New York Journal of Mathematics

New York J. Math. **21** (2015) 1169–1177.

Robustly fiberwise minimal iterated function systems on the torus

H. Ebadizadeh, F. H. Ghane, A. Ehsani, M. Saleh and M. Zaj

Abstract. This work is devoted to the study of the strong minimality of a class of iterated function systems defined on the two dimensional torus T^2. This means that almost every orbital branch of each point is dense in the ambient space. Moreover, we prove that this property is robust under small perturbations of the generators.

Contents

1. Introduction

In this note, we commence the study of density of almost every orbital branch of minimal iterated function systems defined on the two dimensional torus T^2. An *iterated function system* is a semigroup generated by a collection $\mathcal{F} = \{f_0, \ldots, f_{k-1}\}$ of continuous self maps of a topological space X, denoted by $\mathrm{IFS}(X;\mathcal{F})$. The $\mathcal{F}$*-orbit* of $x \in X$ is the set of all points $y = h(x)$, for which $h \in \langle \mathcal{F} \rangle^+$.

An iterated function system $\mathrm{IFS}(X;\mathcal{F})$ with generators $\{f_0, \ldots, f_{k-1}\}$ is called *minimal* if each closed subset $A \subset X$ such that $f_i(A) \subset A$ for all $i = 0, \ldots, k-1$, is empty or coincides with X. This means that the $\mathcal{F}$-orbit of each $x \in X$ is dense in X.

Consider the *symbol space* Σ_+^k which is the set of one sided infinite words over the alphabet $\{0, \ldots, k-1\}$ equipped with the metric

$$d(\omega, \omega') = 2^{\min\{n; \omega_n \neq \omega'_n\}},$$

for each $\omega = (\omega_n)_{n=0}^{\infty}$, $\omega' = (\omega'_n)_{n=0}^{\infty} \in \Sigma_+^k$. Let σ be the left shift transformation on the symbol space Σ_+^k. We denote by ν^+ the Bernoulli measure on

Received March 11, 2015.

2010 *Mathematics Subject Classification.* Primary: 37E30; 37B10; Secondary: 37B05.

Key words and phrases. Minimal iterated function systems, fiberwisely minimal iterated function systems, locally contractible.

ISSN 1076-9803/2015

Σ_+^k. For an iterated function system IFS$(X;\mathcal{F})$ and $\omega = (\omega_n)_{n=0}^\infty \in \Sigma_+^k$, the ω-orbit of $x \in X$ under $\mathcal{F}$ is the sequence $(x_n)_{n=0}^\infty$ defined by $x_{n+1} = f_{\omega_n}(x_n)$ with $x_0 = x$. If $(\omega_n)_{n=0}^\infty$ is chosen according to some stochastic process, then $(x_n)_{n=0}^\infty$ is referred to as a *random orbit*. Such orbits are also referred to as a *chaos game*. An *orbital branch* corresponding to a sequence $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma_+^k$ is the set of compositions

$$f_\omega^n := f_{\omega_{n-1}} o \ldots o f_{\omega_0},\ n \in \mathbb{N}.$$

An iterated function system IFS$(X;\mathcal{F})$ is said to be *fiberwise minimal* if it satisfies the following: for each $x \in X$, there exists a subset $\Omega(x) \subset \Sigma_+^k$ of full measure such that for each $\omega \in \Omega(x)$, the ω-orbit of x under $\mathcal{F}$ is dense in X. In this context, the following question is interesting.

Question 1.1. *Under what conditions is an action of a semigroup on a compact manifold necessarily fiberwisely minimal in a persistent way?*

The main result of this work, Theorem A below, allows us to answer this question in the affirmative for certain iterated function systems defined on the torus T^2. Moreover, in our approach the fiberwise minimality persists under C^1-perturbations. Let us mention that some knowledge of robust minimality of iterated function systems is already provided, e.g., see [GHS10] and [HN14]. Also in [BFS14], the authors presented examples of fiberwisely minimal IFSs defined on the circle. One main novelty here is that in our construction the iterated function systems can be non-hyperbolic.

Definition 1.2. Suppose that $g : T^2 \to T^2$ is a map. We say that g is *locally contractible* at x, if there exists an open subset $U \subset T^2$ containing x satisfying $g(U) \cap U \neq \emptyset$, and

$$d_H(g^n(\mathrm{cl}(U)), x) \to 0,\ as\ n \to \infty,$$

where d_H is the Hausdorff metric defined on the space of all compact subsets of T^2.

Remark 1.3. Suppose that g is locally contractible at some point x. Then by the definition, for each $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$, one has $\mathrm{diam}(g^n(\mathrm{cl}(U)) < \varepsilon$. This fact implies that

$$d(g^n(x), g^n(y)) \to 0,\ as,\ n \to \infty,$$

for each $x, y \in \mathrm{cl}(U)$. Moreover, there exists $\eta > 0$ such that this convergence is uniform if $d(x, y) \leq \eta$, that is the restriction $g|_{\mathrm{cl}(U)}$ is an asymptotic contraction. Now, by Jachymski Theorem [J94], x is the unique contractive fixed point of g.

Definition 1.4. Suppose that $g : M \to M$ is a diffeomorphism on a compact surface M. We say that g is *generically locally contractible* at a point $x \in M$, if it is locally contractible at x and the real canonical form of $Dg(x)$ is not the identity matrix.

Now, we state the main result of this work.

Theorem A. *Suppose that* $g_1, g_2 \in \mathrm{Diff}^2(T^2)$ *such that* g_1 *is an irrational rotation and* g_2 *is an orientation preserving diffeomorphism which is generically locally contractible at some point* x*. Then there exists a* C^1*-neighborhood* $\mathcal{U}$ *of* (g_1, g_2) *such that any iterated function system* $\mathcal{F}$ *generated by any pair* $(f_1, f_2) \in \mathcal{U}$ *is forward minimal. In particular, the following statements hold:*

(a) $\mathcal{F}$ *is fiberwisely minimal.*
(b) $\mathcal{F}$ *possesses a dense subset of attracting periodic points.*

2. Proof of the main results

This section is devoted to the proof of the main result of this article. The starting point is Proposition 2.2 below, which gives sufficient conditions for existence a blending region. The notion of blending region is the main tool to produce robust minimal actions. This is used in [BR14], [BR15], [GHS10] and [HN14].

We begin by the definition of the notion.

Definition 2.1. Let M be a compact manifold. An open subset $B \subset M$ is said to be a *blending region* for a semigroup $\mathcal{F}$ of diffeomorphisms of M whenever there exist $f_1, \ldots, f_l \in \mathcal{F}$ and an open set $D \subset M$ with $\overline{B} \subset D$ and satisfying:

(a) $\overline{B} \subset f_1(B) \cup \ldots \cup f_l(B)$.
(b) $f_i : \overline{D} \to D$ is a contracting map, for $i = 1, \ldots, l$.

Proposition 2.2. Let $\mathcal{F}$ be an iterated function system defined on T^2. Assume that $\mathcal{F}$ contains an irrational rotation R. Also, suppose that there exists $f \in \mathcal{F}$ which admits a hyperbolic attracting fixed point x_0. Then $\mathcal{F}$ possesses a blending region.

Proof. Consider an iterated function system $\mathcal{F}$ and $R, f \in \mathcal{F}$ that satisfy the assumptions of the proposition. Suppose that $x_0 \in T^2$ is a hyperbolic attracting fixed point of f with the basin of attraction W. It is enough to take two open balls $B, D \subset W$ containing x_0 with $B \subset \overline{D} \subset W$. Since R is an irrational rotation, so the R-orbit of x_0 is dense. This fact and compactness of $\overline{B}$ imply that there exist $n_1, \ldots, n_l \in \mathbb{N}$ and large enough $k \in \mathbb{N}$, such that for $f_i := R^{n_i} \circ f^k$, $i = 1, \ldots, l$, one has that:

(i) $\overline{B} \subset f_1(B) \cup \cdots \cup f_l(B)$.
(ii) $f_i : \overline{D} \to D$ is a contracting map for each $i = 1, \ldots, l$.

In fact,

$$Df_i(x) = D(R^{n_i} \circ f^k)(x) = DR^{n_i}(f^k(x)).Df^k(x) = Id.Df^k(x) = Df^k(x),$$

where Id is the identity matrix. Therefore, f_i, $i = 1, \ldots, l$, are contracting maps on $\overline{D}$ which ensures that B is a blending region as we desired. □

From now on we fix $g_1, g_2 \in \mathrm{Diff}^2(T^2)$ that are, respectively, an irrational rotation and an orientation preserving diffeomorphism which is generically locally contractible at point x_0. First, we show that the action of semigroup $\mathcal{F}$ generated by $\{g_1, g_2\}$ is C^1-robustly minimal. This establishes the first statement of our main result.

Since g_2 is generically locally contractible at x_0, a straightforward computations shows that the real canonical form of the matrix $Dg_2(x_0)$ does not have the following form

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, one has two possibilities:

(i) x_0 is a hyperbolic attracting fixed point of g_2.

(ii) the matrix of $Dg_2(x_0)$ is similar to the following matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

where $0 < |\mu| < 1$.

For the case (i), Proposition 2.2 guarantees the existence of a blending region. So let us consider the case that $Dg_2(x_0)$ possesses a real canonical form as described in (ii). Suppose that g_2 is orientation preserving, so $\mu > 0$.

Now, we explain how one can bypass the difficulty caused by the non-hyperbolicity of the fixed point x_0. In fact, we provide a bounded distortion property for the iterates of g_2 over curves whose tangent space is contained in a center cone at each point.

Let us note that $Dg_2(x_0)$ admits a splitting $T_{x_0}(T^2) = E^s \oplus E^c$ such that the following conditions hold: there exists $0 < \lambda < 1$, for some choice of a Riemannian metric on T^2 which satisfies

$$\|Dg_2(x_0)|_{E^s}\| \leq \lambda, \quad \|Dg_2(x_0)|_{E^s}\|.\|Dg_2^{-1}(x_0)|_{E^c}\| \leq \lambda. \tag{1}$$

We extend the subbundles E^s and E^c continuously to some neighborhood V of x_0, that we denote by $\widetilde{E}^s$ and $\widetilde{E}^c$. Here, we do not require these extensions to be invariant under Dg_2.

For each $0 < \gamma < 1$, we define the center cone field $C^c_\gamma := (C^c_\gamma(x))_{x \in V}$ of width γ by

$$C^c_\gamma(x) = \{v_1 + v_2 \in \widetilde{E}^s_x \oplus \widetilde{E}^c_x : \|v_1\| \leq \gamma \|v_2\|\}. \tag{2}$$

Moreover, we define the stable cone field $C^s_\gamma := (C^s_\gamma(x))_{x \in V}$ of width γ in a similar way.

Fix $\gamma > 0$ and V small enough so that up to increasing $\lambda < 1$, the second inequality of (1) remains valid for any pair of vectors in the two cone fields:

$$\|Dg_2(x)v^s\|.\|Dg_2^{-1}(g_2(x))v^c\| \leq \lambda \|v^s\|.\|v^c\|, \tag{3}$$

for every $v^s \in C^s_\gamma(x)$, $v^u \in C^c_\gamma(x)$, and each point $x \in V \cap g_2^{-1}(V)$. Then, the center cone field is positively invariant: $Dg_2(x)C^c_\gamma(x) \subset C^c_\gamma(g_2(x))$ provided that x and $g_2(x)$ are contained in V. Indeed, according to (1)

$$Dg_2(x_0)C^c_\gamma(x_0) \subset C^c_{\lambda\gamma}(x_0) \subset C^c_\gamma(x_0),$$

and this extends to each $x \in V \cap g_2^{-1}(V)$, by continuity.

Let us recall the following definition.

Definition 2.3. The tangent bundle of an embedded submanifold $N \subset M$ is *Hölder continuous* if the mapping $x \mapsto T_xN$ defines a Hölder continuous section from N to the corresponding Grassmann bundle of M.

In the following, suppose that $I \subset V$ is a C^2 embedded arc of M which is tangent to the center cone field C^c_γ, this means that the tangent subspace to I at each point $x \in I$ is contained in the cone $C^c_\gamma(x)$. Then $g_2(I)$ is also tangent to the center cone field, if it is contained in V. We apply the argument used in Section 2 of [ABV00] to show that the tangent bundle of the iterates of the C^2-submanifold I, i.e., $g^n(I)$, $n \in \mathbb{N}$, are Hölder continuous (if they do not leave V) with uniform Hölder constant.

First, we recall the notion of Hölder variation of the tangent bundle in local coordinates, as follows: Let us take the exponential map on the embedded submanifold I. Suppose that $\delta_0 > 0$ is small enough so that the inverse of the exponential map exp_x is defined on the $B_{\delta_0}(x, I) := B_{\delta_0}(x) \cap I$, where $B_{\delta_0}(x)$ is the ball with the radius δ_0 and the center x in V. From now on we identify the neighborhood $B_{\delta_0}(x, I)$ of x in I with the corresponding neighborhood U_x of the origin in T_xI, through the local chart defined by exp_x^{-1}. So, x can be identified by $0 \in T_xI$. Now by reducing $\delta_0 > 0$, we may assume that $\widetilde{E}^s_x$ is contained in $C^s_\gamma(y)$ of each $y \in U_x$. Moreover, $C^c_\gamma(y) \cap \widetilde{E}^s_x = \{0\}$. Therefore , T_yI is parallel to the graph of a unique linear map

$$A_x(y) : T_xI \to \widetilde{E}^s_x.$$

For given constants $C > 0$ and $0 < \varepsilon \leq 1$, we say that the tangent bundle to I is (C, ε)-Hölder if

$$\|A_x(y)\| \leq C\rho_I(x, y)^\varepsilon \text{ for every } y \in I \cap U_x,\ x \in V, \tag{4}$$

where$\rho_I(x, y)$ denotes the distance from x to y along $I \cap U_x$, defined as a length of the geodesic connecting x to y inside $I \cap U_x$.

By the domination property (1) and the choice of V, there exist $\lambda' \in (\lambda, 1)$ and $\varepsilon \in (0, 1]$ such that

$$\|Dg_2(z)v^s\|.\|Dg_2^{-1}(g_2(z))v^c\|^{1+\varepsilon} \leq \lambda' < 1 \tag{5}$$

for each unit vectors $v^s \in C^s_\gamma(z)$ and $v^c \in C^c_\gamma(z)$ and $z \in V$. Now, by reducing δ_0 and increasing $\lambda' < 1$, (5) remains true if we replace z by any $y \in U_x$, $x \in V$.

In below, we fix ε and λ' and we define

$$\kappa(I) := \inf\{C > 0 : \text{the tangent bundle of } I \text{ is } (C, \varepsilon)\text{-Hölder}\}.$$

Note that since g_2 is locally contractible at x_0, the choice of V implies that $g_2^n(I) \subset V$, for all $n \geq 1$. So the following result follows by Proposition 2.2 and Corollary 2.4 of [ABV00].

Proposition 2.4. There exists $C_1 > 0$ for which the following hold:

(a) There is an integer $n_0 \geq 1$ such that $\kappa(g_2^n(I)) \leq C_1$, for each $n \geq n_0$.
(b) If $\kappa(I) \leq C_1$, then $\kappa(g_2^n(I)) \leq C_1$.
(c) If $\kappa(I) \leq C_1$, then the functions

$$J_k : g_2^k(I) \ni x \mapsto \log |\det(Dg_2|_{T_x(g_2^k(I))})|, \tag{6}$$

are (L, ε)-Hölder continuous with $L > 0$ depending only on C_1 and g_2.

Remark 2.5. If we consider the following condition

$$\|Dg_2(x_0)|_{E^s}\|.\|Dg_2^{-1}(x_0)|_{E^c}\|^i \leq \lambda, \tag{7}$$

for $i = 1, 2$, then we may take $\varepsilon = 1$ in the above argument which implies that $\kappa(I)$ possesses a bound on the curvature tensor of I. In particular, by the previous proposition, if I is C^2 then the curvature of all iterates $g_2^n(I)$, $n \geq 1$, is bounded by some constant that depends only on the curvature of I.

Now, since $\operatorname{diam}(g_2^n(V)) \to 0$, as $n \to \infty$, there exists $0 < \nu < 1$ such that for large enough n,

$$\operatorname{diam}(g_2^n(I)) < \nu \operatorname{diam}(I). \tag{8}$$

We fix n satisfying (8). Take $\widetilde{I} \subset I$ such that $\operatorname{dist}(x_0, \widetilde{I}) > 0$. Then by the Mean Value Theorem, there exists $0 < \alpha < 1$ such that

$$\|Dg_2^n(x)\| \leq \alpha, \text{ for each } x \in \widetilde{I}. \tag{9}$$

By continuity, we can choose a neighborhood W of $\widetilde{I}$ in V and $0 < \alpha < \alpha' < 1$ satisfying

$$\|Dg_2^n(x)\| \leq \alpha', \text{ for each } x \in W. \tag{10}$$

Let us take an open ball $W_0 \subset W$. By (10), $\operatorname{diam}(g_2^n(W_0)) \leq \alpha' \operatorname{diam}(W_0)$. Then there exists $k \geq 1$ so that $g_1^k(g_2^n(W_0)) \subset W_0$. Moreover,

$$\|D(g_1^k \circ g_2^n)|_{W_0}\| \leq \alpha',$$

according to (10). We set $h := g_1^k \circ g_2^n$ then, by the fixed point theorem, h possesses a unique hyperbolic attracting fixed point x_1. Now, Proposition 2.2 ensures the existence of a blending region in the case (ii).

Theorem 2.6. *Let $g_1, g_2 \in \operatorname{Diff}^1(T^2)$, respectively, be an irrational rotation and an orientation preserving diffeomorphism which is generically locally contractible at x_0. Then the action of semigroup generated by $\{g_1, g_2\}$ admits a blending region. In particular, it is C^1-robustly minimal.*

Proof. The previous argument implies that the IFS generated by (g_1, g_2) admits a hyperbolic attracting periodic point and hence it possesses a blending region B, according to Proposition 2.2. So, it is enough to prove that it acts C^1-robustly minimally on T^2.

Since B is an open subset of T^2 and g_1 is a minimal map on T^2, there exist positive integers $m_1, \ldots, m_s$ and $m'_1, \ldots, m'_t$ such that for $T_i := g_1^{m_i}$, $i = 1, \ldots, s$, and $S_j := g_1^{m'_j}$, $j = 1, \ldots, t$, it holds that

$$T^2 = \bigcup_{i=1}^{s} T_i(B) = \bigcup_{j=1}^{t} S_j^{-1}(B).$$

That is B has a cycle with respect to the semigroup action of $\langle g_1, g_2\rangle^+$. Now the proof follows by Corollary A of [BFMS15]. □

The first statement of Theorem A follows by Theorem 2.6.

Now, we are going to prove the density of an almost surely orbital branch. This statement is an immediate consequence of Corollary C of [BGMS15]. It is also an application of Theorem A of [SG15]. Indeed, the authors established the density of all almost fiberwise orbits for minimal IFSs which are defined on a compact metric space X by applying Theorem A (see Section 3 of [SG15] for more details). More precisely, they proved that for each $x \in X$, one can find a subset $\Omega(x) \subset \Sigma_+^k$ of probability 1 such that $\overline{\mathcal{O}(x,\omega)} = X$, for all $\omega \in \Omega(x)$.

This fact and Theorem 2.6 imply that the action of semigroup $\langle g_1, g_2\rangle^+$ is fiberwisely minimal. In particular, this property remains true under small C^1 perturbations of generators.

For completing the proof of the main result, it is enough to provide the density of hyperbolic attracting periodic points. The following auxiliary lemma is needed.

Lemma 2.7. *Consider an iterated function system $\mathcal{F} = (X; f_1, \ldots, f_N)$ which acts minimally on a compact metric space X. For every nonempty open set $U \subset X$ there exists $k \leq k_0 \in \mathbb{N}$ and $r = r(U) > 0$ such that for every ball $B \subset X$ of radius r, there exists a word $w = t_1 \ldots t_{t_k}$ on the alphabet $\{1, \ldots, N\}$ of length $k \leq k_0$ such that $f_{[w]}(B) \subset U$, where $f_{[w]} := f_{t_k} \circ \ldots \circ f_{t_1}$.*

Proof. Let $U \subset X$ be an open subset. Since the action of $\mathcal{F}$ on X is minimal, for each $x \in X$ there exist word $w(x)$ on alphabet $\{1, \ldots, N\}$ such that $f_{[w(x)]}(x) \in U$.

By continuity, there is a neighborhood V_x of x such that $f_{[w]}(V_x) \subset U$. Since X is compact, we can cover X by finitely many sets V_{x_i}. We take k_0 the maximum of the lengths of the words $w(x_i)$ and $r > 0$ the Lebesgue number of this covering.

Then, every ball $B \subset X$ of radius less than r is contained in some V_{x_i}, so there exists a word $w = t_1 \ldots t_{t_k}$ on the alphabet $\{1, \ldots, N\}$ of length $k \leq k_0$ such that $f_{[w]}(B) \subset U$. □

The argument used for Theorem 2.6 shows that there exists $h \in \langle g_1, g_2\rangle^+$ which possesses a hyperbolic attracting fixed point z. Assume that W is the basin of the attraction of z.

Let $x \in T^2$ and U be an open subset of T^2 containing x. Also let $W_1 \subset W$ be an open subset containing the fixed point z of h.

By applying Lemma 2.7 for W_1, there exist $r = r(W_1) > 0$ and $k_1 = k_1(W_1) \in \mathbb{N}$ such that for every ball $B \subset T^2$ of radius r, there exists a word $w = t_1 \ldots t_k$ on the alphabet $\{1, \ldots, 2\}$ of length $k \leq k_1$ such that $f_{[w]}(B) \subset W_1$.

Let $U(x)$ be the open ball of center x and radius r which is contained in U. Therefore, there exists a word $w(x)$ of length $k \leq k_1$ such that $f_{[w(x)]}(U(x)) \subset W_1$.

Now, we can apply Lemma 2.7 for $U(x)$, there exist $\rho = \rho(U(x)) > 0$ and $k_2 = k_2(U(x)) \in \mathbb{N}$ such that for every ball $B \subset T^2$ of radius ρ, there exist a word $w = s_1 \ldots s_l$ of length $l \leq k_2$ such that $f_{[w]}(B) \subset U(x)$.

Let $U(z)$ be the open ball of center z and radius ρ which is contained in W_1. So there exists a word $w(z)$ of length $l \leq K_2$ such that $f_{[w(z)]}(U(z)) \subset U(x)$.

Let λ be the minimum rate of contraction of $Dh(z)$, i.e., $\lambda = m(Dh(z))$, where $m(Dh(z)) = inf\{\|Dh(z)(v)\| : \|v\| = 1\}$. We set

$$L := max\{\|Dg_i(x)\| : \ x \in T^2, \ i = 1, 2\}.$$

Let us choose a positive integer n such that $\lambda^n L^{k_1+k_2} < 1$ and $h^r(W_1) \subset U(z)$. Also, we take $T := f_{[w(z)]} \circ h^n \circ f_{[w(x)]}$.

Then

$$T(U(x)) = f_{[w(z)]} \circ h^n \circ f_{[w(x)]}(U_x) \subset f_{[w(z)]}(h^n(W_1)) \subset f_{[w(z)]}(U(z)) \subset U(x).$$

Moreover, the choice of n shows that $\| T \| < 1$ on $U(x)$. These facts imply the existence of an attracting fixed point for T on $U(x) \subset U$, which is an attracting periodic point for the iterated function system $\langle g_1, g_2\rangle^+$. Moreover, this argument remains true for small perturbations of $\langle g_1, g_2\rangle^+$. This completes the proof of the main result.

Acknowledgments. We are grateful to the referee for valuable suggestions and useful comments.

References

[ABV00] Alves, José F.; Bonatti, Christian; Viana, Marcelo. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent.Math.* **140** (2000), no. 2, 351–398. MR1757000 (2001j:37063b), Zbl 0962.37012, arXiv:1403.2937, doi: 10.1007/s002220000057.

[AJS12] Arbieto, Alexander; Junqueira, André; Santiago, Bruno. On weakly hyperbolic iterated function systems. Preprint, 2012. arXiv:1211.1738.

[BFS14] Barrientos, Pablo G.; Fakhari, Abbas; Sarizadeh, Aliasghar. Density of fiberwise orbits in minimal iterated function systems on the circle. *Discrete*

Contin. Dyn. Syst. **34** (2014), no. 9, 3341–3352. MR3190981, Zbl 06375833, arXiv:1304.1445, doi: 10.3934/dcds.2014.34.3341.

[BFMS15] Barrientos, Pablo G.; Fakhari, Abbas; Malicet, Dominique; Sarizadeh, Ali. Expanding actions: minimality and ergodicity. Preprint, 2015. arXiv:1307.6054v2.

[BGMS15] Barrientos, Pablo G.; Ghane, Fatemeh H.; Malicet, Dominique; Sarizadeh, Ali. On the chaos game of iterated function systems. Preprint, 2015. arXiv:1506.07101v1.

[BR14] Barrientos, Pablo G.; Ki, Yuri; Raibekas, Artem. Symbolic blender-horseshoes and applications. *Nonlinearity* **27** (2014), no. 12, 2805–2839. MR3291132, Zbl 06383451, arXiv:1211.7088, doi: 10.1088/0951-7715/27/12/2805.

[BR15] Barrientos, Pablo G.; Raibekas, Artem. Dynamics of iterated function systems on the circle close to rotations. *Ergodic Theory Dynam. Systems* **35** (2015), no. 5, 1345–1368. MR3365725, Zbl 06495613, arXiv:1303.2521, doi: 10.1017/etds.2013.112.

[GHS10] Ghane, F. H.; Homburg, A. J.; Sarizadeh, A. C^1 robustly minimal iterated function systems. *Stoch. Dyn.* **10** (2010), no. 1, 155–160. MR2604683 (2011h:37029), Zbl 1183.37079, doi: 10.1142/S0219493710002899.

[HN14] Homburg, Ale Jan; Nassiri, Meysam. Robust minimality of iterated function systems with two generators. *Ergodic Theory Dynam. Systems* **34** (2014), no. 6, 1914–1929. MR3272778, Zbl 06379771, arXiv:1301.0400, doi: 10.1017/etds.2013.34.

[J94] Jachymski, J. An iff fixed point criterion for continuous self-mappings on a complete metric space. *Aequations Math.* **48** (1994), no. 2–3, 163–170. MR1295089 (96c:54069), Zbl 0805.47051, doi: 10.1007/BF01837990.

[SG15] Sarizadeh, Aliasghar; Ghane, Fatemeh H. Some stochastic property of topological action semigroups. Preprint, 2014. arXiv:1408.5763v1.

(H. Ebadizadeh) Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
ebadizadeh.hojjat@stu.um.ac.ir

(F. H. Ghane) Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
ghane@math.um.ac.ir

(A. Ehsani) Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
aza_ehsani@yahoo.com

(M. Saleh) Department of Mathematics, University of Neyshabur, Neyshabur, Iran
saleh@neyshabur.ac.ir

(M. Zaj) Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
zaj.marzie@yahoo.com

This paper is available via http://nyjm.albany.edu/j/2015/21-53.html.