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A sequence of inclusions whose colimit is not a homotopy colimit

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Abstract. It is known that the homotopy colimit of a sequence of inclusions of T1 spaces is weakly equivalent with the actual colimit. We show that the assumption of T1 is essential by providing a counterexample for non-T1 spaces.

Contents

1. Introduction

It is well known that the canonical map from the homotopy colimit (telescope, [1]) of a sequence of inclusions

$$X_1 \subseteq \cdots \subseteq X_n \subseteq \dots$$

of T1 topological spaces to the actual colimit is a weak equivalence:

$$\text{hocolim}_n X_n \xrightarrow{\sim} \bigcup_n X_n. \tag{1}$$

The reason is simply that for any compact space K (using the covering definition, regardless of separation axioms), the image of a continuous map

$$f : K \to \bigcup_n X_n$$

is contained in one of the spaces X_n. This is easily seen as follows: Assume otherwise and pick points $s_n \in f(K) \smallsetminus X_n$. Let

$$S_m = \{s_m, s_{m+1}, \dots\}.$$

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Then $S_m \cap X_n$ is closed in X_n for each n, since the spaces are T1, and hence S_m is closed in $\bigcup X_n$. Hence, the sets $f^{-1}(S_m)$ are closed in K, $f^{-1}(S_m) \supseteq f^{-1}(S_{m+1})$, while

$$\bigcap_m f^{-1}(S_m) = f^{-1}\left(\bigcap_m S_m\right) = f^{-1}(\emptyset) = \emptyset.$$

This is a contradiction to K being compact.

The authors do not know an original reference for this simple argument, which however plays a key role in homotopy theory (cf. [2]). Clearly, the assumption that the spaces X_n are T1 is essential to the argument.

The first author noticed, however, that (1) also holds for so-called quasi-discrete spaces, which means spaces in which an intersection of any (possibly infinite) number of open sets is open. Finite spaces are examples of quasidiscrete spaces. Quasi-discrete spaces are, in some sense, the opposites of T1 spaces. For any T0 space X, there is a partial ordering on the set X where $x \leq y$ if and only if the closure of x contains y. For T1 spaces, this partial ordering is trivial. On the other hand, for quasidiscrete spaces, the ordering determines the topology completely: For a quasidiscrete space X, a subset $S \subseteq X$ is closed if and only if

$$x \in S,\ x \leq y \ \Rightarrow\ y \in S.$$

McCord [3] exhibited, for a quasi-discrete space X, a continuous map from the classifying space of the poset $(X_{\text{disc}}, \leq)$ (where X_{disc} denotes X with the discrete topology) to X which is a weak equivalence, and is functorial under inclusions. This implies (1).

The authors then began asking whether (1) is true for all topological spaces. Eventually, they found a counterexample, which is the subject of the present note. It remains an open problem if (1) is true for some reasonable separation axiom weaker than T1, such as TD spaces (a space is TD if every point x contains an open neighborhood U such that $U \smallsetminus \{x\}$ is open). While such follow-up questions may fall into the realm of curiosities, the example presented here is an important cautionary tale on the role of the T1 axiom in the foundations of homotopy theory.

2. The example

For $m \in \mathbf{N}$, let $Q_m \subseteq \mathbb{R}^2$ be defined as

$$(\{0\} \times [-1,1]) \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) : x \in \left(0, \frac{1}{m}\right) \right\},$$

with the subspace topology induced by $\mathbb{R}^2$. Observe that for $i \leq j$, $Q_j \subseteq Q_i$, and Q_j is open in Q_i. Also observe that

$$\bigcap_{i=1}^{\infty} Q_i = \{0\} \times [-1,1].$$

Let X be the set

$$\{x_1, x_2, \ldots, y\} \times Q_1$$

with topology generated by the basis

$$\begin{aligned}\beta = \{(\{x_k\} \times U) \cup (\{x_{k+1}, x_{k+2}, \ldots\} \times Q_k) \cup (\{y\} \times V) :\\ k \in \mathbb{N} \text{ and } V \subseteq U \subseteq Q_k \text{ and U and V are open in } Q_k\}\end{aligned}$$

We must check that β is actually a basis of topology.

Lemma 1. *β is closed under finite intersections.*

Proof. We consider two separate cases. In the first case, let

$$\begin{aligned}A &= (\{x_k\} \times U) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times V)\\ B &= (\{x_k\} \times W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times Z)\end{aligned}$$

for some $k \in \mathbf{N}$, and some open sets U, V, W, and Z such that $V \subseteq U \subseteq Q_k$ and $Z \subseteq W \subseteq Q_k$. Then

$$A \cap B = (\{x_k\} \times U \cap W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times V \cap Z)$$

$U \cap W$ and $V \cap Z$ are both open subsets of Q_k, and $(V \cap Z) \subseteq (U \cap W)$. Hence $A \cap B$ is in β.

In the second case, let

$$\begin{aligned}A &= (\{x_j\} \times U) \cup (\{x_{j+1}, \ldots\} \times Q_j) \cup (\{y\} \times V)\\ B &= (\{x_k\} \times W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times Z)\end{aligned}$$

for some $j, k \in \mathbb{N}$, and some open sets U, V, W, and Z such that $V \subseteq U \subseteq Q_j$ and $Z \subseteq W \subseteq Q_k$. Without loss of generality we may assume $j < k$. Then

$$\begin{aligned}A \cap B &= \big((\{x_{j+1}, \ldots\} \times Q_j) \cap (\{x_k\} \times W)\big)\\ &\quad \cup \big((\{x_{j+1}, \ldots\} \times Q_j) \cap (\{x_{k+1}, \ldots\} \times Q_k)\big)\\ &\quad \cup \big((\{y\} \times V) \cap (\{y\} \times Z)\big)\\ &= (\{x_k\} \times W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times Z \cap V),\end{aligned}$$

so $A \cap B$ is in β. □

Let X_n be the set $\{x_1, \ldots, x_n, y\} \times Q_1$, with the subspace topology inherited from X. Observe that X_n has basis

$$\begin{aligned}\beta_n = \{(\{x_k\} \times U) \cup (\{x_{k+1}, x_{k+2}, \ldots, x_n\} \times Q_k) \cup (\{y\} \times V) :\\ k \in \mathbb{N}, k \leq n \text{ and } V \subseteq U \subseteq Q_k \text{ and U and V are open in } Q_k\}\end{aligned}$$

Also observe that X_n has the subspace topology inherited from X_{n+1}.

Theorem 2. *$X = \bigcup_{n=1}^{\infty} X_n$, and X has the union topology, i.e., a subset $U \subseteq X$ is open if and only if for every n, $U \cap X_n$ is open in X_n.*

Proof. We want to show that a set U is open in X if and only if $U \cap X_n$ is open in X_n for all $n \in \mathbb{N}$.

One direction is trivial: if U is open in X, then by definition $U \cap X_n$ is open in X_n $n \in \mathbb{N}$.

Now suppose that $U \subseteq X$, and that $U \cap X_n$ is open in X_n for all $n \in \mathbb{N}$. Fix a point $q \in U$. We will exhibit an X-open neighborhood of q in U. Note that q is either of the form (x_k, z) or (y, z), for some $k \in \mathbb{N}$ and $z \in Q_1$, and these two cases need to be handled separately.

Case 1. Suppose $q = (x_k, z)$. Either $z \in Q_k$, or $z \in Q_1 \backslash Q_k$. These subcases again need to be handled separately.

Subcase 1a. Suppose $z \in Q_k$. Then for $m \geq k+1$, any basis element of β_m containing $q = (x_k, z)$ also contains $\{x_{k+1}, \ldots, x_m\} \times Q_k$. Furthermore, there exists an open neighborhood V of z such that $V \subset Q_k$ and $\{x_k\} \times V \subset U \cap X_k$. Thus U contains $(\{x_k\} \times V) \cup (\{x_{k+1}, \ldots\} \times Q_k)$, which is an open set in X containing q.

Subcase 1b. Suppose $z \notin Q_k$. Let $j < k$ be the unique integer such that $z \in Q_j \backslash Q_{j+1}$. Then for $m > k$, every element of β_m which contains (x_k, z) also contains $\{x_{j+1}, \ldots, x_m\} \times Q_j$. Hence U contains $\{x_{j+1}, \ldots\} \times Q_j$, which is an open set in X containing q.

Case 2. Suppose $q = (y, z)$. Again, we must distinguish two subcases:

Subcase 2a. $z \in \{0\} \times [-1, 1]$. By hypothesis $U \cap X_1$ is open in X_1, so it contains $\{y\} \times V$ for some V open in Q_1 and $z \in V$. Because V is open in Q_1, it must contain points not in $\{0\} \times [-1, 1]$. Thus, there exists a $k \in \mathbb{N}$ such that $V \subseteq Q_k$ but $V \nsubseteq Q_{k+1}$. Let t be a point in $V \cap (Q_k \backslash Q_{k+1})$. Now, for $m \geq k$, $U \cap X_m$ is open in X_m by hypothesis. But the definition of β_m implies that any open set in X_m containing (y, t) must also contain

$$\{x_{k+1}, \ldots, x_m\} \times Q_k.$$

Thus U contains

$$(\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times (V \cap Q_{k+1})),$$

which is open in X and contains q.

Subcase 2b. $z \in Q_k \backslash Q_{k+1}$ for some $k \in \mathbb{N}$. Then any element of β_k containing $q = (y, z)$ contains a set of the form

$$(\{x_k\} \times U) \cup (\{y\} \times V)$$

for some $V \subseteq U \subseteq Q_k$ open, $z \in V$. On the other hand, any element of β_m for $m \geq k$ which contains $q = (y, z)$ also contains

$$\{x_{k+1}, \ldots, x_m\} \times Q_k.$$

Hence, U contains

$$(\{x_k\} \times U) \cup (\{x_{k+1}, x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times V)$$

which is open in X and contains q. □

3. Proof that X is not the homotopy colimit of X_n

Theorem 3. *The weak equivalence* (1) *is false for the spaces X_n constructed in the previous section.*

The theorem is a consequence of the following two propositions.

Proposition 4. *There exists a continuous path in X from $(x_1, (0,0))$ to $(y, (0,0))$.*

Proof. Let $f : [0,1] \to X$ $f(0) = (x_1, (0,0))$, $f(t) = (x_{n+1}, (0,0))$, for $n \in (1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$, $n \in \mathbb{N}$, and $f(1) = (y, (0,0))$.

If U is open in X and $(x_n, (0,0)) \in U$, then for all $k \geq n, (x_k, (0,0)) \in U$. Hence, if $y \in U$, then $f^{-1}(U)$ is either $\{1\}$, all of $[0,1]$, or of the form

$$\left(1 - \frac{1}{n}, 1\right]$$

for some $n \in \mathbb{N}$. On the other hand, if $y \notin U$, then $f^{-1}(U)$, $[0,1)$, or of the form

$$\left(1 - \frac{1}{n}, 1\right)$$

for some $n \in \mathbb{N}$. Hence f is continuous. □

Proposition 5. *For $n \in \mathbb{N}$, there does not exist a continuous path in X_n from $(x_1, (0,0))$ to $(y, (0,0))$.*

This proposition will be proved in a sequence of lemmas.

Lemma 6. *Let A be the set $\{y\} \times \{0\} \times [-1,1]$. Then for $n \in \mathbb{N}$, A is closed in X_n.*

Proof. Observe that the complement of A in X_n is the union

$$\begin{aligned} &(\{x_1, \ldots, x_n\} \times Q_1) \cup (\{x_1, \ldots, x_n, y\} \times (Q_1 \backslash (\{0\} \times [-1,1]))) \\ &= (\{x_1\} \times Q_1) \cup (\{x_2, \ldots, x_n\} \times Q_1) \\ &\quad \cup (\{x_1, \ldots, x_n, y\} \times (Q_1 \backslash (\{0\} \times [-1,1]))). \end{aligned}$$

This is a basis element in X_n, hence A is closed in X_n. □

Lemma 7. *For all $n \in \mathbb{N}$, the space $\{y\} \times Q_1$ with the subspace topology inherited from X_n is homeomorphic to Q_1.*

Proof. Taking the intersection of each element of β_n with $\{y\} \times Q_1$ gives

$$\{\{y\} \times U : U \text{ is an open subset of } Q_1\}$$

as a basis for the inherited topology. Thus, the map sending (y, z) to z, for $z \in Q_1$, is trivially a homeomorphism from $\{y\} \times Q_1$ onto Q_1. □

Lemma 8. *For $n \in \mathbb{N}$, the set A is a path-component of X_n.*

Proof. The set $A = \{y\} \times \{0\} \times [-1, 1]$ is clearly path-connected. Now suppose $\omega : [0, 1] \to X_n$ is a path such that $\omega(0) = (y, (0, 0))$. We wish to show that $\omega([0, 1]) \subseteq A$. By Lemma 6, A is closed in Q_n, so $\omega^{-1}(A)$ is closed in $[-1, 1]$. Observe from the definition of β_n that $\{y\} \times Q_{n+1}$ is open in X_n. Thus $\omega^{-1}(\{y\} \times Q_{n+1})$ is an open subset of $[-1, 1]$, and hence a disjoint union of relatively open intervals. By Lemma 7, A is a path-component of $\{y\} \times Q_{n+1}$. Thus $\omega^{-1}(A)$ is a union of path components of $\omega^{-1}(\{y\} \times Q_{n+1})$. But the path-components of $\omega^{-1}(\{y\} \times Q_{n+1})$ are just disjoint relatively open intervals in $[0, 1]$, hence $\omega^{-1}(A)$ is also a disjoint union of relatively open intervals in $[0, 1]$, so $\omega^{-1}(A)$ is open in $[0, 1]$

By hypothesis, $\omega(0) \in A$, so $\omega^{-1}(A)$ is nonempty. Thus $\omega^{-1}(A)$ is a nonempty, closed, and open subset of $[-1, 1]$, hence it is all of $[-1, 1]$. □

Note that Proposition 5 is a formal consequence of Lemma 8.

Remark. The spaces X_n are, in fact, compact and hence they, and the space X, are compactly generated. Therefore, our example also applies to the version of the category of compactly generated spaces [4] without the T1 axiom.

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