New York Journal of Mathematics

New York J. Math. **20** (2014) 145–151.

# On the density of scattered translates of the general multiquadric in $C([a,b])$

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Abstract. This note concerns density properties of the general multiquadric, $(x^2+c^2)^{k-1/2}$, where $k$ is a fixed natural number. We establish that scattered translates of the general multiquadric are dense in $C([a,b])$, where $a$ and $b$ are finite. As a corollary, we show that scattered translated of the general multiquadric are dense in the function spaces $L^p([a,b])$, for $1 \leq p < \infty$.

Contents



## 1. Introduction

Approximation and interpolatory properties of the function

$$\phi(x) = (x^2+c^2)^{1/2},$$

called the multiquadric, have been investigated before, see for instance [1, 2, 5, 6]. These papers deal with integer or near-integer translates of the multiquadric. In [5], it was shown that continuous functions on a closed interval may be uniformly approximated by scattered translates of the multiquadric. We will improve the result found there, showing that the same is true for the $k^{th}$ order multiquadric, $\phi_k(x) = (x^2+c^2)^{k-1/2}$, where $k \in \mathbb{N}$. This family of general multiquadrics has also been studied, [3, 4], although the aims of those papers are a bit different than the present goal, since they consider divided differences of the general multiquadric.

This note is organized as follows. In the next section, various definitions and facts are collected. The third section contains the main theorem to

Received September 7, 2013.
2010 *Mathematics Subject Classification.* 41A30, 41A58.
*Key words and phrases.* Multiquadric approximation.

ISSN 1076-9803/2014

be proved, while the fourth section contains the details of the proof. Some general comments are collected in the final section.

## 2. Definitions and basic facts

We will need to know what "scattered" means. For our purposes, we have the following definition in mind.

**Definition 1.** A sequence of real numbers $\mathcal{X}$ is said to be *scattered* if:

(1) There exists $\delta > 0$ such that $\inf_{\substack{x,y\in\mathcal{X}\\ x\neq y}} |x-y| \geq \delta$.

(2) $\lim_{j\to-\infty} x_j = -\infty$ and $\lim_{j\to\infty} x_j = \infty$.

It's not hard to see that a scattered sequence must be countable. Take intervals of length $\delta/3$ centered at each point in $\mathcal{X}$, each of these intervals is disjoint and contains a rational number $r$. Letting a member of $\mathcal{X}$ correspond to the number $r$ which is in the same interval shows that the set $\mathcal{X}$ is at most countable. This allows us to index $\mathcal{X}$ with the integers.

Throughout the remainder of the paper we let $\mathcal{X} = \{x_j : j \in \mathbb{Z}\}$ be a fixed but otherwise arbitrary scattered sequence.

Part of the proof that we give later requires the following summation formula, which we state as a lemma.

**Lemma 1.** *For $N \in \mathbb{N}$, $0 \leq l \leq N$, and $p$ a polynomial of degree $l$. We have,*

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}p(j) = \begin{cases} 0 & 0 \leq l < N, \\ (-1)^N a_N \cdot N! & l = N, \end{cases}$$

*where $a_N$ is the leading coefficient of $p$.*

**Proof.** To see this we need only to use the binomial series expansion. For $N \in \mathbb{N}$, we have,

$$(1-x)^N = \sum_{j=0}^{N}(-1)^j\binom{N}{j}x^j.$$

Now we can differentiate $l$ times to yield

$$\left(\frac{d}{dx}\right)^l (1-x)^N = \sum_{j=0}^{N}(-1)^j\binom{N}{j}j(j-1)\cdots(j-l+1)x^{j-l}.$$

Since we can write an $l^{th}$ degree polynomial $p(j)$ as an appropriate linear combination of

$$\{1, j, j(j-1), j(j-1)(j-2), \ldots, j(j-1)\cdots(j-l+1)\},$$

all we must do to get the result is evaluate at $x = 1$. □

## 3. Main result

This section contains our theorem and an outline of the proof. We also give a corollary, which extends the main result.

**Theorem 1.** *Given $k \in \mathbb{N}$, a scattered sequence $\{x_j\}$, $\epsilon > 0$, and a continuous function $f : [a, b] \to \mathbb{R}$, we may find a sequence of coefficients $\{a_j\}_{j=1}^N$, such that*

$$\sup_{x\in[a,b]} \left| f(x) - \sum_{j=1}^{N} a_j \phi_k(x - x_j) \right| < \epsilon.$$

**Sketch of Proof.** The idea is to develop a Taylor expansion

$$\phi_k(x - y) = y^{2k-1} \sum_{j=0}^{\infty} \frac{A_{k,j}(x)}{y^j}. \tag{1}$$

Here, we will take $y >> 0$, so that the series converges. Then we show that the linear span of $\{A_{k,j}(x)\}$ contains $x^j$ for $j = 0, 1, 2, \ldots$. We then find coefficients to approximate an $n^{th}$ degree polynomial by using an appropriate Vandermonde matrix. Finally, since we may approximate polynomials, we appeal to the Stone–Weierstrass Theorem to finish the proof. □

This theorem, when combined with Hölder's Inequality allows us to replace the sup-norm above with the $L^p([a, b])$ norm. We state this in the following corollary.

**Corollary 1.** *Given $k \in \mathbb{N}$, a scattered sequence $\{x_j\}$, $\epsilon > 0$, and a continuous function $f : [a, b] \to \mathbb{R}$, then for $1 \le p < \infty$ we may find a sequence of coefficients $\{a_j\}_{j=1}^N$, such that*

$$\left\| f(x) - \sum_{j=1}^{N} a_j \phi_k(x - x_j) \right\|_{L^p([a,b])} < \epsilon.$$

**Remark.** The corollary above cannot be extended to the $p = \infty$ case. This follows from the continuity of each approximation and the fact that $C([a, b])$ is a closed subspace of $L^\infty([a, b])$.

## 4. Details

This section provides a rigorous justification for the outline of the proof. We begin with the Taylor expansion, which we recognize as the familiar binomial series. For $y >> 0$ we have,

$$
\begin{aligned}
&y^{-2k+1}\phi_k(x-y)\\
&=\sum_{n=0}^{\infty}\binom{k-\frac{1}{2}}{n}\sum_{j=0}^{n}\binom{n}{j}(-2x)^{n-j}\sum_{l=0}^{j}\binom{j}{l}x^{2(j-l)}c^{2l}y^{-(n+j)}\\
&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{k-\frac{1}{2}}{n}\binom{n}{j}\binom{j}{l}c^{2l}(-2)^{n-j}\frac{x^{j+n-2l}}{y^{n+j}}\\
&=\sum_{n=0}^{\infty}\sum_{j=n}^{2n}\sum_{l=0}^{j-n}\binom{k-\frac{1}{2}}{n}\binom{n}{j-n}\binom{j-n}{l}c^{2l}(-2)^{2n-j}\frac{x^{j-2l}}{y^{j}}\\
&=\sum_{j=0}^{\infty}\frac{(-1)^j}{y^j}\left\{\sum_{n=\lceil j/2\rceil}^{j}\sum_{l=0}^{j-n}\binom{k-\frac{1}{2}}{n}\binom{n}{j-n}\binom{j-n}{l}c^{2l}2^{2n-j}x^{j-2l}\right\}\\
&=\sum_{j=0}^{\infty}\frac{A_{k,j}(x)}{y^j}.
\end{aligned}
$$

In the fourth line we have re-indexed the sum, and in the fifth changed the order of summation. All of this hinges on the binomial series being absolutely convergent, but since we've assumed $y >> 0$, the argument of the binomial series will be close to 0. This gives us a formula for the polynomials $A_{k,j}(x)$:

$$
A_{k,j}(x)=(-1)^j\sum_{n=\lceil j/2\rceil}^{j}\sum_{l=0}^{j-n}\binom{k-\frac{1}{2}}{n}\binom{n}{j-n}\binom{j-n}{l}2^{2n-j}c^{2l}x^{j-2l}. \tag{2}
$$

We can glean lots of information from (2), for instance, $\deg(A_{k,j})$ has the same parity as $j$ and $\deg(A_{k,j})\le j$. We are interested in the leading term for $A_{k,j}$, for this will tell us the exact degree. We need only re-index and change the order of summation, since both sums are finite, there is no problem with convergence.

$$
\begin{aligned}
&\sum_{n=\lceil j/2\rceil}^{j}\sum_{l=0}^{j-n}\binom{k-\frac{1}{2}}{n}\binom{n}{j-n}\binom{j-n}{l}2^{2n-j}c^{2l}x^{j-2l}\\
&=\sum_{l=0}^{\lceil j/2\rceil}\sum_{n=\lceil j/2\rceil}^{j-l}\binom{k-\frac{1}{2}}{n}\binom{n}{j-n}\binom{j-n}{l}2^{2n-j}c^{2l}x^{j-2l}\\
&=\sum_{l=0}^{\lceil j/2\rceil}x^{j-2l}\left\{\sum_{n=\lceil j/2\rceil}^{j-l}\binom{k-\frac{1}{2}}{n}\binom{n}{j-n}\binom{j-n}{l}2^{2n-j}c^{2l}\right\}.
\end{aligned}
$$

To simplify notation, we write

$$
\begin{aligned}
(3)\qquad & A_{k,j}(x) \\
&= (-1)^j \sum_{l=0}^{\lceil j/2\rceil} c^{2l} x^{j-2l} \left\{ \sum_{n=\lceil j/2\rceil}^{j-l} \binom{k-\frac{1}{2}}{n} \binom{n}{j-n} \binom{j-n}{l} 2^{2n-j} \right\} \\
&= (-1)^j \sum_{l=0}^{\lceil j/2\rceil} a_{j-2l} x^{j-2l}.
\end{aligned}
$$

We are in position to state the following lemma.

**Lemma 2.** *Given $k \in \mathbb{N}$, and $j \geq 2k$, we have*

$$
a_{j-2l} = \begin{cases} 0 & 0 \leq l < k, \\ c^{2k}\binom{k-1/2}{k} & l = k. \end{cases}
$$

*Hence, the polynomial $A_{k,j}(x)$ is a polynomial of degree $j - 2k$.*

**Proof.** We need only find the sum in (3). To do this, we re-index:

$$
\begin{aligned}
&\sum_{n=\lceil j/2\rceil}^{j-l} \binom{k-1/2}{n} \binom{n}{j-n} \binom{j-n}{l} 2^{2n-j} \\
&= \sum_{n=0}^{j-\lceil j/2\rceil-l} \binom{k-1/2}{n+\lceil j/2\rceil} \binom{n+\lceil j/2\rceil}{j-\lceil j/2\rceil-n} \binom{j-\lceil j/2\rceil-n}{l} 2^{2n+2\lceil j/2\rceil-j}.
\end{aligned}
$$

Now we let $j = 2k + N$, for $N = 0, 1, 2, \ldots$, and we have two cases, the case that $N$ is even, and the case that $N$ is odd. Both cases being similar calculations, we will work the odd case here. By letting $N = 2m + 1$, we have

$$
\begin{aligned}
&\sum_{n=0}^{j-\lceil j/2\rceil-l} \binom{k-1/2}{n+\lceil j/2\rceil} \binom{n+\lceil j/2\rceil}{j-\lceil j/2\rceil-n} \binom{j-\lceil j/2\rceil-n}{l} 2^{2n+2\lceil j/2\rceil-j} \\
&= \sum_{n=0}^{k+m-l} \binom{k-1/2}{n+k+m+1} \binom{n+k+m+1}{k+m-n} \binom{k+m-n}{l} 2^{2n+1} \\
&= k!\binom{k-1/2}{k} \sum_{n=0}^{k+m-l} 2^{n-m}(-1)^{n+m+1} \frac{(2(n+m)+1)!!}{(2n+1)!l!(k+m-n-l)!} \\
&= \frac{(-1)^{m+1}k!}{l!(k+m-l)!}\binom{k-1/2}{k} \sum_{n=0}^{k+m-l} (-1)^n \binom{k+m-l}{n} \frac{(2(n+m)+1)!!}{2^m(2n+1)!!}.
\end{aligned}
$$

The last summand may be reduced by noting that

$$
\frac{(2(n+m)+1)!!}{2^m(2n+1)!!} = \frac{(2n+2m+1)(2n+2m-1)\cdots(2n+3)}{2^m}
$$

is a monic, $m^{th}$ degree polynomial in the variable $n$. Thus Lemma 1, gives us the result provided $k - l \geq 0$. This proves the case when N is odd, the even case is virtually the same computation. □

Now we choose a subset of $\mathcal{X}$ which allows us to recover $A_{k,2k+N}(x)$. Pick a set $\{y_j : 1 \leq j \leq 2k + N + 1\} \subset \mathcal{X}$ using the following conditions:

- $y_1 >> 0$.
- $y_j \geq 2y_{j-1}; \quad j = 2, 3, \ldots, 2k + N + 1$.

The modified $(2k + N + 1) \times (2k + N + 1)$ Vandermonde system

$$\sum_{j=1}^{2k+N+1} b_j y_j^l = \delta_{l,-N-1} \qquad l = 2k - 1, 2k - 2, \ldots, -N - 1$$

is invertible. The solution may be found by repeated use of Cramer's rule. We get

$$b_j = (-1)^{j+1} y_j^{N+1} \prod_{l=1, l\neq j}^{2k+N+1} \left[1 - \frac{y_j}{y_l}\right]^{-1}.$$

As a result of this, we have that the set of products

$$\left\{ b_j y_j^{-(N+1)} : j = 1, 2, \ldots, 2K + N + 1 \right\}$$

is uniformly bounded. Using these coefficients, we have

$$\sum_{j=1}^{2k+N+1} b_j \phi_k(x - y_j) = A_{k,N+2k}(x) + \mathcal{O}\left(\frac{1}{y_1}\right), \qquad a \leq x \leq b.$$

Since $y_1$ may be as large as we like and $A_{k,2k+N}(x)$ is an $N^{th}$ degree polynomial, to approximate continuous function $f$, we need only find a polynomial to approximate $f$ on $[a, b]$, then approximate the polynomial with the sum above.

## 5. Comments

In looking over the method provided here, it may seem that a large variety of functions may be substituted in place of the multiquadric without substantially changing the argument. We will provide some examples to illustrate that this is not the case. Since we use the Stone-Weierstrass theorem, we need the span of the expansion polynomials $A_j(x)$ to be the same as the span of the monomials $x^j$. In particular, finite expansions will not work. Even though $\phi(x) = (1 + x^2)^2$ shares many of the same properties as the general multiquadric, it does not share the same approximation property. A less trivial example is provided by $\phi(x) = (1+x^4)^{5/4}$, which seems to mimic the behavior of the multiquadric $(1 + x^2)^{5/2}$. Using the technique above *mutatis mutandis*, one can show that for $j > 5$ the leading term of $A_j(x)$ is not a multiple of $x^3, x^4, x^5$, or $x^6$, however, all other powers are represented. Hence this function does not enjoy the approximation property.

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