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Extending Chang's construction to the category of m-zeroids and some category containing the category of Abelian ℓ-groups with strong unit

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Abstract. In this note we prove that it is impossible to extend the natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit described by C. C. Chang, 1958, and Cignoli & Mundici, 1997, to a natural equivalence between the category of m-zeroids and some category containing the category of Abelian ℓ-groups with strong unit.

Contents

1. Introduction

In 1958, C. C. Chang [3] showed that there is a natural equivalence between the category of totally ordered MV-algebras and the category of totally-ordered Abelian ℓ-groups with strong unit. In 1997, Cignoli & Mundici [4] generalized Chang's construction to show that there is a natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit with a functor commonly called Γ. It is of interest to determine if this functor can be extended to more general classes of algebraic structures.

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To that end, in 2001, Esteva & Godo [6] introduced a spectrum of monoidal logic systems which included involutive monoidal t-norm based logic systems, IMTL-algebras for short, and then proceeded in 2007 (see [7]) to use the IMTL-algebras to attempt a generalization of the Γ functor established by Cignoli & Mundici. They succeeded in defining a functor that is a categorical equivalence between the subcategories of *linearly-ordered* objects from the categories of lattice-ordered partially-associative Abelian groupoids with strong associative unit and IMTL-algebras. This categorical equivalence cannot be extended (yet) beyond these linearly-ordered subcategories to the corresponding non-linearly-ordered categories.

Using the fact that an m-zeroid is a generalization of an MV-algebra (see [2]), this paper attempts a generalization of Cignoli & Mundici's Γ functor to a natural equivalence between the category of m-zeroids and some category containing the category of Abelian ℓ-groups with strong unit. This is an attempted generalization of the functor Esteva & Godo [7] developed in their paper, since it is easy to see that the varieties of IMTL-algebras and m-zeroids are term equivalent (see Remark 1). The generalization being made here involves the other side of the functor: the subcategory of linearly-ordered partially-associative Abelian groupoids with strong associative unit considered by Esteva & Godo will be generalized to some category containing the category of Abelian ℓ-groups with strong unit.

In this note we prove the following result:

Main Theorem. *There does not exist an extension of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit presented by Chang* [3] *and Cignoli & Mundici* [4] *to the category of m-zeroids and some category containing the category of Abelian ℓ-groups with strong unit.*

We begin in Section 2 by providing definitions for the algebraic structures, namely the m-zeroid, MV-algebra, Abelian ℓ-group, Abelian ℓ-monoid, and the dual Abelian ℓ-monoid. Section 3 summarizes the natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit presented by Chang and Cignoli & Mundici, and then considers an extension of this natural equivalence to a natural equivalence between the category of m-zeroids and the category of dual Abelian ℓ-monoids with cancellative unit.

2. Definitions

Since C. C. Chang [3] first defined the algebraic system called an *MV-algebra*, numerous generalizations for an MV-algebra have been proposed, including the structure called an *m-zeroid*. For the purposes of this paper a combination of the definitions given by F. Paoli [8] and Cignoli & Mundici [4] for the m-zeroid and the MV-algebra will be used.

Definition 1. An *m-zeroid* is a structure $\boldsymbol{Z} = \langle Z, +, -, 0, \leq \rangle$ such that:

(Z1) $x + y = y + x$;
(Z2) $x + (y + z) = (x + y) + z$;
(Z3) $\langle Z, \leq \rangle$ is a lattice;
(Z4) $-(-x) = x$;
(Z5) $x + 0 = 0$;
(Z6) $x + -x = 0$;
(Z7) $x \leq y$ iff $0 = -x + y$;
(Z8) $x + (y \vee z) = (x + y) \vee (x + z)$.

Definition 2. An *MV-algebra* is an m-zeroid that also satisfies:

(Z9) $-(-x + y) + y = -(-y + x) + x$.

It should be noted that both (Z3) and (Z7) can be expressed equationally strictly in terms of the algebraic operations $+$ and $-$. Thus the class of m-zeroids forms a variety and the class of MV-algebras forms a variety. As a consequence, a homomorphism between two m-zeroids, or a homomorphism between two MV-algebras, will be defined as usual—as a function that preserves the operations. However, in general, (Z3) and (Z7) will be sufficient and their equational representations will not be used.

For both the m-zeroid and the MV-algebra, the following binary operation will be defined in order to reduce the notation where possible:

$$x \cdot y = -(-x + -y).$$

The following is then an immediate consequence of property (Z4):

$$x + y = -(-x \cdot -y).$$

Note that for an m-zeroid $x \vee y$ can be written in terms of the other operations, including $\wedge$. However, in an MV-algebra

$$-(-x + y) + y = -(-y + x) + x,$$

so the join can be written in terms of only the operations $+$ and $-$ as follows:

$$x \vee y = -(-x + y) + y = -(-y + x) + x.$$

Similarly for the meet:

$$x \wedge y = -[-(x + -y) + -y] = -[-(y + -x) + -x].$$

In both structures, -0 plays the role of the identity element, and 0 plays the role of "collector of the opposites", as Paoli [8] calls it. Also note that in both m-zeroids and MV-algebras, -0 represents a universal lower bound and 0 a universal upper bound.

Remark 1. We could denote an IMTL-algebra as $\langle Z, \star, \rightarrow, 1, \leq \rangle$, with 0 as the identity element, where the operations of the IMTL-algebra correspond

to those of the m-zeroid defined above as follows:

$$\begin{aligned} 1 &\iff 0 \\ 0 &\iff -0 \\ x \to 0 &\iff -x \\ x \star y &\iff -(-x + -y) \\ x \to y &\iff -x + y \\ ((x \to 0) \star (y \to 0)) \to 0 &\iff x + y \end{aligned}$$

and where the lattice operations remain identical for both structures. As noted in [7], by adding the *divisibility condition* $x \vee y = (x \to y) \to y$, the IMTL-algebra will become an MV-algebra. This divisibility condition is equivalent to adding (Z9) to an m-zeroid to create an MV-algebra, since:

$$x \vee y = (x \to y) \to y \iff x \vee y = -(-x + y) + y$$

and since $x \vee y = y \vee x$, we have $-(-x + y) + y = -(-y + x) + x$.

Definition 3. An *Abelian ℓ-group* is a structure $\boldsymbol{G} = \langle G, +, -, 0, \leq \rangle$ such that:

(G1) $x + y = y + x$;
(G2) $x + (y + z) = (x + y) + z$;
(G3) $\langle G, \leq \rangle$ is a lattice;
(G4) $-(-x) = x$;
(G5) $x + 0 = x$;
(G6) $x + -x = 0$;
(G7) $x \leq y$ iff $0 \leq -x + y$;
(G8) $x + (y \vee z) = (x + y) \vee (x + z)$.

An Abelian ℓ-group is simply a group in which a partial ordering has been placed on the elements of the group such that the order forms a lattice (G3) *and* that the group operation preserves the order, an easy consequence of (G8): if $y \leq z$, then $x + z = x + (y \vee z) = (x + y) \vee (x + z)$, so $x + y \leq x + z$. As with m-zeroids and MV-algebras, (G3) and (G7) can be expressed using equations, and so the class of Abelian ℓ-groups forms a variety.

One of the fundamental properties of Abelian ℓ-groups is the following, proven in Birkhoff [1]:

Lemma 1. *Except for the trivial case $G = \{0\}$, an Abelian ℓ-group has no universal bounds.*

Definition 4. An element, $u \geq 0$, of an Abelian ℓ-group is a *strong unit* if for all $x \in \boldsymbol{G}$ there exists an integer $n \geq 1$ such that $nu \geq x$.

For any given Abelian ℓ-group, if a strong unit exists, it is not necessarily unique. Not every Abelian ℓ-group will have a strong unit.

Definition 5. An *Abelian ℓ-monoid* is a structure $\boldsymbol{M} = \langle M, +, 0, \leq \rangle$ such that:

(M1) $x + y = y + x$;

(M2) $x + (y + z) = (x + y) + z$;
(M3) $\langle M, \leq \rangle$ is a lattice;
(M4) $x + 0 = x$;
(M5) $x + (y \vee z) = (x + y) \vee (x + z)$.

Definition 6. A *dual Abelian ℓ-monoid* is a structure $\boldsymbol{M'} = \langle M', +, 0, \leq \rangle$ such that:

(M′1) $x + y = y + x$;
(M′2) $x + (y + z) = (x + y) + z$;
(M′3) $\langle M', \leq \rangle$ is a lattice;
(M′4) $x + 0 = x$;
(M′5) $x + (y \wedge z) = (x + y) \wedge (x + z)$.

Remark 2. It should be noted that these two definitions are not equivalent. In an Abelian ℓ-group,

$$x + (y \vee z) = (x + y) \vee (x + z) \quad \Longleftrightarrow \quad x + (y \wedge z) = (x + y) \wedge (x + z).$$

This is *not* true in an Abelian ℓ-monoid. Isotonicity of the Abelian ℓ-monoid is a consequence of (M5). Similarly, isotonicity for the dual Abelian ℓ-monoid is a consequence of (M′5). The definition of an Abelian ℓ-monoid has been taken from Birkhoff [1]; the author has introduced the concept of a dual Abelian ℓ-monoid. And lastly, note that the dual of an Abelian ℓ-monoid is a dual Abelian ℓ-monoid.

There are significant differences between Abelian ℓ-monoids, dual Abelian ℓ-monoids, and Abelian ℓ-groups. Both Abelian ℓ-monoids and dual Abelian ℓ-monoids can have universal upper bounds and universal lower bounds, unlike Abelian ℓ-groups (see Lemma 1). In fact, it is possible for them to have both a universal upper bound and a universal lower bound.

Another difference between Abelian ℓ-groups and Abelian ℓ-monoids deals with the dual lattice. If L is a lattice, then the *dual lattice* L' is found by simply inverting the lattice, so that meets become joins and joins become meets.

Similarly, if w is a statement involving meets and joins, then the *dual statement* w' is the same statement but with the meets and joins interchanged. (This concept is described in Darnel [5].)

Lemma 2. *The dual of an Abelian ℓ-group is an Abelian ℓ-group.*

This is stated in Birkhoff [1] and is true since in an Abelian ℓ-group both statements (M5) and (M′5) hold. However, the same cannot be said for either Abelian ℓ-monoids nor dual Abelian ℓ-monoids.

3. Extension of Chang's construction

First consider the construction of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit presented by Chang [3] and refined by Cignoli & Mundici [4]. Lemmas 3,

4, 5, 6, 7, and 8, and Theorems 1 and 2, have all been proven in Cignoli & Mundici [4].

Let $\mathcal{MV}$ be the category of MV-algebras and let $\mathcal{A}_1$ be the category of Abelian ℓ-groups with strong unit. Let $\boldsymbol{G} = \langle G, +, -, 0, \leq, u\rangle$ be an Abelian ℓ-group with a specified strong unit u.

Definition 7. The *unit interval* $[0, u]$ of $\boldsymbol{G} = \langle G, +, -, 0, \leq, u\rangle$ is defined by:

$$[0, u] = \{x \in G \mid 0 \leq x \leq u\}.$$

Now define $\boldsymbol{\Phi}(\boldsymbol{G}, u) = \langle [0, u], \oplus, \neg, u, \leq\rangle$ such that for each $x, y \in [0, u]$,

$$\neg x = u - x,$$
$$x \oplus y = u \wedge (x + y).$$

As done with m-zeroids earlier, to shorten the notation, define:

$$x \odot y = \neg(\neg x \oplus \neg y).$$

At this point we have two different symbols for binary addition, namely $+$ and $\oplus$, used for a multitude of structures. $\oplus$ will be used when the interval structure of the corresponding algebra needs to be emphasized, but it should be noted that the two symbols are interchangeable.

Lemma 3. *$\boldsymbol{\Phi}(\boldsymbol{G}, u)$ is an MV-algebra in which the natural lattice-order of $\boldsymbol{\Phi}(\boldsymbol{G}, u)$ agrees with the order of $[0, u]$ inherited from $\boldsymbol{G}$ by restriction.*

A *unital ℓ-group homomorphism* will be a group homomorphism:

$$f : \langle G, +^{\boldsymbol{G}}, -^{\boldsymbol{G}}, 0^{\boldsymbol{G}}, \leq^{\boldsymbol{G}}, u\rangle \longrightarrow \langle H, +^{\boldsymbol{H}}, -^{\boldsymbol{H}}, 0^{\boldsymbol{H}}, \leq^{\boldsymbol{H}}, v\rangle$$

in $\mathcal{A}_1$ that also satisfies:

$$f(x \vee y) = f(x) \vee f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y),$$
$$f(u) = v.$$

For every unital ℓ-group homomorphism f, let $\boldsymbol{\Phi}_f = f|_{[0,u]}$ be the restriction of f to the interval $[0, u]$. Then:

Lemma 4. *$\boldsymbol{\Phi}$ is a functor from $\mathcal{A}_1$ into $\mathcal{MV}$.*

The functor necessary for the other direction is a little more complicated. Every MV-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, 0, \leq\rangle$ must give rise to an Abelian ℓ-group with strong unit. The Abelian ℓ-group with strong unit corresponding to $\boldsymbol{A}$ will be denoted $\boldsymbol{G}_{\boldsymbol{A}}$. The construction from $\boldsymbol{A}$ to $\boldsymbol{G}_{\boldsymbol{A}}$ is described in detail by Cignoli & Mundici [4] and begins with the following definition:

Definition 8. For every MV-algebra $\boldsymbol{A}$, a sequence $\langle a\rangle = (a_1, a_2, \ldots)$ of elements of $\boldsymbol{A}$ is *good* if and only if for each $i = 1, 2, \ldots$, $a_i \oplus a_{i+1} = a_i$, and there exists an integer $n \geq 0$ such that $a_r = -0$ for all $r > n$.

As in Cignoli & Mundici's paper, the sequence

$$\langle a\rangle = (a_1, a_2, \ldots, a_n, -0, -0, -0, \ldots)$$

shall be denoted as simply $\langle a\rangle = (a_1, a_2, \ldots, a_n)$. This implies that:

$$(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n, -0^m)$$

for any arbitrary m-tuple of -0s. However the sequence $(0^m, a_1, a_2, \ldots, a_n)$ is distinct from the sequence $(a_1, a_2, \ldots, a_n)$.

Definition 9. Let $\boldsymbol{A}$ be an MV-algebra. Let $\langle a\rangle = (a_1, a_2, \ldots, a_n)$ and $\langle b\rangle = (b_1, b_2, \ldots b_m)$ be good sequences in $\boldsymbol{A}$. Define the *sum* $\langle c\rangle = \langle a\rangle + \langle b\rangle$ to be the sequence $\langle c\rangle = (c_1, c_2, \ldots)$, where for all $i = 1, 2, \ldots$,

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus (a_{i-2} \odot b_2) \oplus \cdots \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

Note that when $p > n$ and $q > m$, $a_p = b_q = -0$ and so $c_j = -0$ when $j > n + m$. In fact:

Lemma 5. *For any MV-algebra $\boldsymbol{A}$, the sum of two good sequences of $\boldsymbol{A}$ is a good sequence of $\boldsymbol{A}$.*

Now consider the set of all good sequences $M_{\boldsymbol{A}}$ on an MV-algebra $\boldsymbol{A}$, equipped with the sum operation defined in Definition 9 and identity (-0). This structure satisfies cancellation:

Lemma 6. *$M_{\boldsymbol{A}}$ is an Abelian monoid called the* enveloping monoid *of $\boldsymbol{A}$ and satisfies* cancellation*: For any good sequences $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$, if $\langle a\rangle + \langle b\rangle = \langle a\rangle + \langle c\rangle$, then $\langle b\rangle = \langle c\rangle$.*

The Abelian group $G_{\boldsymbol{A}}$ is the group of quotients of this enveloping monoid $M_{\boldsymbol{A}}$.

Lemma 7. *$G_{\boldsymbol{A}}$ is an Abelian group called the* enveloping group *of $\boldsymbol{A}$.*

To complete the construction, a translation invariant lattice structure is defined for both $M_{\boldsymbol{A}}$ and $G_{\boldsymbol{A}}$. This lattice structure arises naturally from the underlying lattice structure of $\boldsymbol{A}$ (see Cignoli & Mundici [4]). Once the translation invariant lattice structure is established, the Abelian group $G_{\boldsymbol{A}}$ becomes an Abelian ℓ-group. This Abelian ℓ-group $\boldsymbol{G}_{\boldsymbol{A}} = \langle G_{\boldsymbol{A}}, +, -, -0, \prec\rangle$ is called the *Chang ℓ-group of $\boldsymbol{A}$*. It contains a strong unit, namely $u_{\boldsymbol{A}} = [(0), (-0)]$, and so is an Abelian ℓ-group with strong unit.

Now, define an *mz-homomorphism* to be a function:

$$f : \langle A, +^{\mathbf{A}}, -^{\mathbf{A}}, 0^{\mathbf{A}}, \leq^{\mathbf{A}}\rangle \longrightarrow \langle B, +^{\mathbf{B}}, -^{\mathbf{B}}, 0^{\mathbf{B}}, \leq^{\mathbf{B}}\rangle$$

in $\mathcal{MV}$ that also satisfies:

$$\begin{aligned} f(x +^{\mathbf{A}} y) &= f(x) +^{\mathbf{B}} f(y) \\ f(-^{\mathbf{A}} x) &= -^{\mathbf{B}} f(x) \\ f(x \vee y) &= f(x) \vee f(y) \\ f(x \wedge y) &= f(x) \wedge f(y) \\ f(0^{\mathbf{A}}) &= 0^{\mathbf{B}}. \end{aligned}$$

Then if $\langle a \rangle = (a_1, a_2, \dots)$ is a good sequence of $\boldsymbol{A}$, then $(f(a_1), f(a_2), \dots)$ is a good sequence of $\boldsymbol{B}$. Let $f^*(\langle a \rangle) = (f(a_1), f(a_2), \dots)$. Note that $f^* : M_{\boldsymbol{A}} \longrightarrow M_{\boldsymbol{B}}$ satisfies:

$$\begin{aligned} f^*(\langle a \rangle +^{\boldsymbol{A}} \langle b \rangle) &= f^*(\langle a \rangle) +^{\boldsymbol{B}} f^*(\langle b \rangle) \\ f^*(\langle a \rangle \wedge \langle b \rangle) &= f^*(\langle a \rangle) \wedge f^*(\langle b \rangle) \\ f^*(\langle a \rangle \vee \langle b \rangle) &= f^*(\langle a \rangle) \vee f^*(\langle b \rangle). \end{aligned}$$

This is proven in Cignoli & Mundici [4]. Lastly, define

$$f^{\circledast}([\langle a \rangle, \langle b \rangle]) = [f^*(\langle a \rangle), f^*(\langle b \rangle)]$$

and let $u_{\boldsymbol{A}}$ and $u_{\boldsymbol{B}}$ be the strong units of $\boldsymbol{G_A}$ and $\boldsymbol{G_B}$ respectively as defined above. Then $f^{\circledast}$ is a unital ℓ-group homomorphism from $\langle \boldsymbol{G_A}, u_{\boldsymbol{A}} \rangle$ into $\langle \boldsymbol{G_B}, u_{\boldsymbol{B}} \rangle$. Let $\boldsymbol{\Psi}(\boldsymbol{A}) = \langle \boldsymbol{G_A}, u_{\boldsymbol{A}} \rangle$ and $\boldsymbol{\Psi}_f = f^{\circledast}$. Then:

Lemma 8. *$\boldsymbol{\Psi}$ is a functor from $\mathcal{MV}$ into $\mathcal{A}_1$.*

The two functors $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ together will yield that:

Theorem 1. *There exists a natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit.*

One of the more interesting properties of the construction of the Abelian ℓ-group presented by Chang and Cignoli & Mundici is the following:

Theorem 2. *Let $\boldsymbol{A}$ be an MV-algebra and let $u_{\boldsymbol{A}}$ be the strong unit of $\boldsymbol{G_A}$ defined as above. Let $\varphi_{\boldsymbol{A}} : \boldsymbol{A} \longrightarrow \boldsymbol{\Phi}(\boldsymbol{G_A}, u_{\boldsymbol{A}}) \subseteq \boldsymbol{G_A}$ be defined by:*

$$\varphi(a) = [(a), (-0)]$$

for all $a \in \boldsymbol{A}$. Then $\varphi_{\boldsymbol{A}}$ isomorphically maps $\boldsymbol{A}$ onto $\boldsymbol{\Phi}(\boldsymbol{G_A}, u_{\boldsymbol{A}})$.

This finishes the construction of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit presented in Cignoli & Mundici [4]. The question now is whether or not this natural equivalence can be extended to a natural equivalence between the category of m-zeroids and some category containing the category of Abelian ℓ-groups with strong unit. The natural instinct is to try to extend the natural equivalence to the category of m-zeroids and the category of Abelian ℓ-groups. Another possibility is to extend the natural equivalence to the category of m-zeroids and the category of Abelian ℓ-groups with some type

of unit that is a generalization of the strong unit. For our purposes, a more general category will be considered.

There are two properties that it seems reasonable, and desirable, for a natural equivalence to retain for this extension: cancellation of the specialized unit and an equivalent of Theorem 2 for the m-zeroids. The requirement of cancellation of the specialized unit is necessary for the construction of the enveloping monoid. And in order for the natural equivalence to be an *extension* of the construction above, the m-zeroid should map isomorphically onto some interval of the dual Abelian ℓ-monoid. In fact, if the extension is to work properly and u is the specialized unit in the dual Abelian ℓ-monoid, then the m-zeroid should map isomorphically onto the interval $[0, u]$. In an attempt to retain these properties, consider an extension of the natural equivalence to a natural equivalence between the category of m-zeroids and the category of dual Abelian ℓ-monoids.

To begin, consider the specialized unit:

Definition 10. Let $\boldsymbol{L} = \langle L, +, 0, \leq \rangle$ be an Abelian ℓ-monoid (or a dual Abelian ℓ-monoid). A non-zero element $u \geq 0$ in $\boldsymbol{L}$ is a *cancellative unit* if for all $x, y \in \boldsymbol{L}$, $x + u = y + u$ implies $x = y$.

Now, let $\mathcal{MZ}$ be the category of m-zeroids and let $\mathcal{M}_1$ be the category of dual Abelian ℓ-monoids with cancellative unit. If there is to be a natural equivalence with the required properties above, then for every m-zeroid $\boldsymbol{Z} = \langle Z, \oplus, \neg, 0, \leq \rangle$ there must be a corresponding dual Abelian ℓ-monoid $\boldsymbol{M} = \langle M, +, -, 0, \leq \rangle$ with cancellative unit u such that $\boldsymbol{Z}$ gets mapped isomorphically onto $[0, u]$ in $\boldsymbol{M}$.

Theorem 3. *Let $\boldsymbol{Z} = \langle Z, \oplus, \neg, 0, \leq \rangle$ be an m-zeroid. Let $\boldsymbol{M} = \langle M, +, 0, \leq \rangle$ be a dual Abelian ℓ-monoid with cancellative unit u. If $\varphi : \boldsymbol{Z} \longrightarrow \boldsymbol{M}$ is an injective mapping such that for all $x, y \in \boldsymbol{Z}$:*

i) $\varphi(0) = u$,
ii) $\varphi(\neg 0) = 0$,
iii) $\varphi(x) + \varphi(\neg x) = u$, *and*
iv) $\varphi(x \oplus y) = u \wedge (\varphi(x) + \varphi(y))$,

then $\boldsymbol{Z}$ is an MV-algebra.

Proof. Since $\boldsymbol{Z}$ is already an m-zeroid, the only property that needs to be proven is $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. For convenience in the notation, if $x \in \boldsymbol{Z}$, its corresponding element $\varphi(x)$ will be denoted as simply x in $\boldsymbol{M}$. Similarly, $\varphi(\boldsymbol{Z})$ will simply be denoted $\boldsymbol{Z}$. Also recall that the symbols $+$ and $\oplus$ are interchangeable, but $\oplus$ will be used when the interval structure of $\boldsymbol{Z}$ needs to be emphasized.

To prove $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$, first consider the following:

For all $a, b \in \boldsymbol{Z} \subset \boldsymbol{M}$,

$$\begin{aligned} a + b + (\neg a \oplus \neg b) &= (a + b) + [u \wedge (\neg a + \neg b)] \\ &= [(a + b) + u] \wedge [(a + b) + (\neg a + \neg b)] \\ &= [(a + b) + u] \wedge [(a + \neg a) + (b + \neg b)] \\ &= [(a + b) + u] \wedge [u + u] \\ &= [(a + b) \wedge u] + u \\ &= (a \oplus b) + u. \end{aligned}$$

Thus $a + b + (\neg a \oplus \neg b) = (a \oplus b) + u$.

Now let $x, y \in \boldsymbol{Z} \subset \boldsymbol{M}$. Then:

$$\begin{aligned} x + (\neg x \oplus y) + u &= x + (\neg x) + y + (x \oplus \neg y) \\ &= u + y + (x \oplus \neg y). \end{aligned}$$

Thus, using cancellation,

$$x + (\neg x \oplus y) = y + (x \oplus \neg y).$$

Adding $\neg(\neg x \oplus y) + \neg(x \oplus \neg y)$ to both sides yields:

$$\begin{aligned} x + (\neg x \oplus y) + \neg(\neg x \oplus y) + \neg(x \oplus \neg y) \\ = y + (x \oplus \neg y) + \neg(\neg x \oplus y) + \neg(x \oplus \neg y). \end{aligned}$$

This implies

$$x + u + \neg(x \oplus \neg y) = y + u + \neg(\neg x \oplus y),$$

which in turn implies

$$x + \neg(x \oplus \neg y) = y + \neg(\neg x \oplus y).$$

Now, since $x \oplus \neg y \geq x$, $\neg(x \oplus \neg y) \leq \neg x$ and so $x + \neg(x \oplus \neg y) \leq u$. Thus, $x + \neg(x \oplus \neg y) = x \oplus \neg(x \oplus \neg y)$. Similarly, $y + \neg(\neg x \oplus y) = y \oplus \neg(\neg x \oplus y)$. Therefore:

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

and so $\boldsymbol{Z}$ is an MV-algebra. □

Thus if there exists an isomorphic image of an m-zeroid $\boldsymbol{Z}$ in a dual Abelian ℓ-monoid with a cancellative unit $\boldsymbol{M}$, the structure imposed on it by $\boldsymbol{M}$ forces $\boldsymbol{Z}$ to be an MV-algebra. This leads to the following corollary:

Corollary 1. *There does not exist an extension of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit presented by Chang* [3] *and Cignoli & Mundici* [4] *to the category of m-zeroids and some category containing the category of Abelian ℓ-groups with strong unit.*

Proof. Suppose such an extension were possible. Let Ω be the functor from $\mathcal{MZ}$ to $\mathcal{M}_1$, and let Δ be the functor from $\mathcal{M}_1$ to $\mathcal{MZ}$. Let $\boldsymbol{Z}$ be an m-zeroid which is *not* an MV-algebra. By the natural equivalence, $\boldsymbol{Z}$ corresponds to a dual Abelian ℓ-monoid $\Omega(\boldsymbol{Z})$ with cancellative unit u. Since the natural equivalence is an extension, every m-zeroid $\boldsymbol{Z}$ must be mapped isomorphically to $\Delta(\Omega(\boldsymbol{Z}), u) \subset \Omega(\boldsymbol{Z})$. But by Theorem 3, $\Delta(\Omega(\boldsymbol{Z}), u)$ is an MV-algebra, and thus $\boldsymbol{Z}$ is an MV-algebra, which is a contradiction. Thus the natural equivalence cannot be extended. □

4. Conclusion

Even though the result is negative, Theorem 3 is more general than the negative result stated in Remark 3 by Esteva & Godo [7]. However, their success in extending Cignoli & Mundici's Γ functor to a functor that is a categorical equivalence between the subcategories of *linearly-ordered* objects from the categories of lattice-ordered partially-associative Abelian groupoids with strong associative unit and IMTL-algebras indicates that further study needs to be made of IMTL-algebras and these lattice-ordered partially-associative Abelian groupoids.

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