New York Journal of Mathematics

New York J. Math. **20** (2014) 1063–1076.

Generalized hyperideals in locally associative left almost semihypergroups

Venus Amjad, Kostaq Hila and Faisal Yousafzai

Abstract. In this paper, we introduce the concept of a locally associative $\mathcal{LA}$-semihypergroup by generalizing the idea of a locally associative $\mathcal{LA}$-semigroup given in Mushtaq–Yusuf, 1979, and study an (m, n)-regular class of a locally associative $\mathcal{LA}$-semihypergroup. We give some examples to connect an $\mathcal{LA}$-semihypergroup with commutative hypergroups and commutative semihypergroups. We also characterize a locally associative $\mathcal{LA}$-semihypergroup H in terms of (m, n)-hyperideals and prove that if $R(L)$ is a 0-minimal right (left) hyperideal of H, then either $R^m \circ L^n = \{0\}$ or $R^m \circ L^n$ is a 0-minimal (m, n)-hyperideal of H for $m, n \geq 2$.

Contents

1. Introduction

A left almost semigroup ($\mathcal{LA}$-semigroup) is a groupoid $\mathcal{S}$ whose elements satisfy the following left invertive law $(ab)c = (cb)a$ for all $a, b, c \in \mathcal{S}$. This concept was first given by Kazim and Naseeruddin in 1972 [16]. In an $\mathcal{LA}$-semigroup, the medial law [16] $(ab)(cd) = (ac)(bd)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. An $\mathcal{LA}$-semigroup may or may not contain a left identity. The left identity of an $\mathcal{LA}$-semigroup allows us to introduce the inverses of elements in an $\mathcal{LA}$-semigroup. If an $\mathcal{LA}$-semigroup contains a left identity, then it is unique [20]. In an $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. By using medial law with left identity, we get $a(bc) = b(ac)$ for all $a, b, c \in \mathcal{S}$.

Received August 9, 2014.

2010 *Mathematics Subject Classification.* 20M10, 20N99.

Key words and phrases. $\mathcal{LA}$-semihypergroups, locally associative $\mathcal{LA}$-semihypergroups, pure left identity and (m, n)-hyperideals.

The third author is highly thankful to CAS-TWAS President's Fellowship.

ISSN 1076-9803/2014

An $\mathcal{LA}$-semigroup is a nonassociative and noncommutative algebraic structure midway between a groupoid and a commutative semigroup. This structure is closely related to a commutative semigroup; indeed if an $\mathcal{LA}$-semigroup contains a right identity, then it becomes a commutative semigroup [20]. The connection between a commutative inverse semigroup and an $\mathcal{LA}$-semigroup was established by Yousafzai et al. in [25] as follows: a commutative inverse semigroup $(\mathcal{S}, .)$ becomes an $\mathcal{LA}$-semigroup $(\mathcal{S}, *)$ under $a * b = ba^{-1}r^{-1}$ for all $a, b, r \in \mathcal{S}$. An $\mathcal{LA}$-semigroup $\mathcal{S}$ with a left identity becomes a semigroup under the binary operation "$\circ_e$" defined as follows: $x \circ_e y = (xe)y$ for all $x, y \in \mathcal{S}$ [25]. There are lot of results which have been added to the theory of an $\mathcal{LA}$-semigroup by Mushtaq, Kamran, Holgate, Jezek, Protic, Madad, Yousafzai and many other researchers. An $\mathcal{LA}$-semigroup is a generalization of a semigroup [20] and it has vast applications in semigroups, as well as in other branches of mathematics.

Hyperstructure theory was introduced in 1934, when F. Marty [19] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world (cf. [3, 4], [10]-[12], [22, 23], [29]). In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [3]. Many authors studied different aspects of semihypergroups, for instance, Corsini et al. [2], Davvaz et al. [5], Hila et al. [13, 15] and Leoreanu [18].

Recently, Hila et al. introduced the notion of $\mathcal{LA}$-semihypergroups [14]. They investigated several properties of hyperideals of $\mathcal{LA}$-semihypergroup and defined the topological space and study the topological structure of $\mathcal{LA}$-semihypergroups using hyperideal theory. In [24], Yaqoob et al. have characterized intra-regular $\mathcal{LA}$-semihypergroups by using the properties of their left and right hyperideals, and investigated some useful conditions for an $\mathcal{LA}$-semihypergroup to become an intra-regular $\mathcal{LA}$-semihypergroup. This nonassociative hyperstructure has been further explored by Yousafzai et al. in [26] and [27].

2. Preliminaries and examples

A map $\circ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ is called *hyperoperation* or *join operation* on the set $\mathcal{H}$, where $\mathcal{H}$ is a nonempty set and $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H})\backslash\{\emptyset\}$ denotes the set of all nonempty subsets of $\mathcal{H}$. A hypergroupoid is a set $\mathcal{H}$ together with a (binary) hyperoperation.

Let A and B be two nonempty subsets of $\mathcal{H}$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad a \circ A = \{a\} \circ A \text{ and } a \circ B = \{a\} \circ B.$$

A hypergroupoid $(\mathcal{H}, \circ)$ is called an $\mathcal{LA}$-semihypergroup [14] if

$$(x \circ y) \circ z = (z \circ y) \circ x$$

holds for all $x, y, z \in \mathcal{H}$. The law is called a left invertive law. A hypergroupoid $(\mathcal{H}, \circ)$ is called a right almost semihypergroup ($\mathcal{RA}$-semihypergroup) if $x \circ (y \circ z) = z \circ (y \circ x)$ holds for all $x, y, z \in \mathcal{H}$. The law is called a right invertive law. A hypergroupoid $(\mathcal{H}, \circ)$ is called an almost semihypergroup ($\mathcal{A}$-semihypergroup) if it is both an $\mathcal{LA}$-semihypergroup and an $\mathcal{RA}$-semihypergroup. Every $\mathcal{LA}$-semihypergroup satisfies the law

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w)$$

for all $w, x, y, z \in \mathcal{H}$. This law is known as medial law (cf. [14]).

Let $\mathcal{H}$ be an $\mathcal{LA}$-semihypergroup [24], then an element $e \in \mathcal{H}$ is called

(i) a left identity (resp. pure left identity) if for all $a \in \mathcal{H}$, $a \in e \circ a$ (resp. $a = e \circ a$);

(ii) a right identity (resp. pure right identity) if for all $a \in \mathcal{H}$, $a \in a \circ e$ (resp. $a = a \circ e$);

(iii) an identity (resp. pure identity) if for all $a \in \mathcal{H}$, $a \in e \circ a \cap a \circ e$ (resp. $a = e \circ a \cap a \circ e$).

We have shown in [26] that unlike an $\mathcal{LA}$-semigroup, an $\mathcal{LA}$-semihypergroup may have a right identity or an identity. This fact can lead us to the following major remark.

Remark 1. The right identity of an $\mathcal{LA}$-semihypergroup need not to be a left identity in general. An $\mathcal{LA}$-semihypergroup may have a left identity or a right identity or an identity. Moreover, an $\mathcal{LA}$-semihypergroup with a right identity need not be associative.

However an $\mathcal{LA}$-semihypergroup with pure right identity becomes a commutative semigroup with identity [26]. An $\mathcal{LA}$-semihypergroup with pure left identity e is called a *pure* $\mathcal{LA}$*-semihypergroup.* A pure $\mathcal{LA}$-semihypergroup $(\mathcal{H}, \circ)$ satisfies the following laws for all $w, x, y, z \in \mathcal{H}$:

$$(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x),$$

called a paramedial law, and

$$x \circ (y \circ z) = y \circ (x \circ z).$$

Example 1. Let $(\mathcal{H}, \circ)$ be an $\mathcal{LA}$-semihypergroup with pure left identity e. Define a binary hyperoperation $\hat{o}$ (e-sandwich hyperoperation) as follows:

$$a \ \hat{o} \ b = (a \circ e) \circ b \text{ for all } a, b \in \mathcal{H}.$$

Then $(\mathcal{H}, \hat{o})$ *becomes a commutative semihypergroup with pure identity.*

Example 2. Let $(H, \cdot, \leq)$ be any ordered $\mathcal{LA}$-semigroup [28]. If we define a hyperoperation $\circ$ on H by $x \circ y = \{z \in H : z \leq xy\} = (xy]$ for all $x, y \in H$, then it is easy to see that $(H, \circ)$ becomes an $\mathcal{LA}$-semihypergroup.

Example 3. An $\mathcal{A}$-semihypergroup $\mathcal{H}$ with pure left identity becomes an abelian hypergroup.

If there is an element 0 of an $\mathcal{LA}$-semihypergroup $(H, \circ)$ such that $x \circ 0 = 0 \circ x = 0 \ \forall \ x \in H$, we call 0 a *zero element* of H.

Example 4. Let us consider the following table for $H = \{a, b, c, d, e\}$ with a pure left identity d. It is easy to see that $(H, \circ)$ is a pure $\mathcal{LA}$-semihypergroup with a zero element a.

$\circ$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{a,e\}$	$\{a,e\}$	$\{a,c\}$	$\{a,e\}$
c	a	$\{a,e\}$	$\{a,e\}$	$\{a,b\}$	$\{a,e\}$
d	a	b	c	d	e
e	a	$\{a,e\}$	$\{a,e\}$	$\{a,e\}$	$\{a,e\}$

A subset A of an $\mathcal{LA}$-semihypergroup H is called a *left* (*right*) hyperideal of H if $H \circ A \subseteq A$ ($A \circ H \subseteq A$), and is called a *hyperideal* of H if it is both left and right hyperideal of H. A subset A of an $\mathcal{LA}$-semihypergroup H is called an $\mathcal{LA}$-*subsemihypergroup* of H if $A^2 \subseteq A$. A hyperideal A of an $\mathcal{LA}$-semihypergroup H with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only hyperideal of H properly contained in A.

An $\mathcal{LA}$-subsemihypergroup A of an $\mathcal{LA}$-semihypergroup H is said to be an (m, n)-*hyperideal* of H if $(A^m \circ H) \circ A^n \subseteq A$ where m, n are nonnegative integers such that $m, n \neq 0$. Here A^m or A^n are suppressed if $m = 0$ or $n = 0$, that is $A^0 \circ H = H$ or $H \circ A^0 = H$. Note that if $m = n = 1$, then an (m, n)-ideal A of an $\mathcal{LA}$-semihypergroup H is called a *bi-hyperideal* of H. If we take $m = 0$ or $n = 0$, then an (m, n)-hyperideal A of an $\mathcal{LA}$-semihypergroup H becomes a $(0, n)$-hyperideal or a $(m, 0)$-hyperideal of H, respectively.

An (m, n)-hyperideal A of an $\mathcal{LA}$-semihypergroup H with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only (m, n)-hyperideal of H properly contained in A.

An $\mathcal{LA}$-semihypergroup H with zero is said to be *nilpotent* if $H^l = \{0\}$ for some positive integer l.

Let m, n be nonnegative integers and H be an $\mathcal{LA}$-semihypergroup. We say that H is (m, n)-*regular* if for every element $a \in H$ there exists some $x \in H$ such that $a \in (a^m \circ x) \circ a^n$. Note that a^0 is defined as an operator element such that $a^0 \circ y = y$ and $z \circ a^0 = z$ for any $y, z \in H$.

An $\mathcal{LA}$-semihypergroup H is said to be *locally associative* $\mathcal{LA}$-semihypergroup if for all $a \in H$, $(a \circ a) \circ a = a \circ (a \circ a)$.

Example 5. Let $H = \{a, b, c, d\}$. Then the following multiplication table shows that $(H, \circ)$ is a locally associative $\mathcal{LA}$-semihypergroup with a zero element a.

$\circ$	a	b	c	d
a	a	a	a	a
b	a	$\{a,b,c,d\}$	$\{a,b,c\}$	$\{a,b,c\}$
c	a	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
d	a	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

Assume that H is a locally associative $\mathcal{LA}$-semihypergroup. Let us define $a^1 = a$, $a^{m+1} = a^m \circ a$ and $a^m = ((((a \circ a) \circ a) \circ a) \circ \cdots \circ a) \circ a = a^{m-1} \circ a$ for all $a \in H$ where $m \geq 1$. It is easy to see that $a^m = a^{m-1} \circ a = a \circ a^{m-1}$ for all $a \in H$ and $m \geq 2$ if H has a pure left identity. Also, we can show by induction, $(a \circ b)^m = a^m \circ b^m$ and $a^m \circ a^n = a^{m+n}$.

3. On locally associative $\mathcal{LA}$-semihypergroups

In this section, we characterize a pure locally associative $\mathcal{LA}$-semihypergroup in terms of (m, n)-hyperideals with the assumption that $m, n \geq 2$. If H is a pure locally associative $\mathcal{LA}$-semihypergroup, then it is easy to see that $H \circ A^m = A^m \circ H$ and $A^m \circ A^n = A^n \circ A^m$ for $m, n \geq 2$ such that $A^0 = e$ if occurs, where e is a pure left identity of H.

Lemma 1. *Let H be a pure $\mathcal{LA}$-semihypergroup. If R and L are the right and left hyperideals of H respectively, then $R \circ L$ is an (m, n)-hyperideal of H.*

Proof. Let R and L be the right and left hyperideals of H respectively, then

$$\begin{aligned}
\{(R \circ L)^m \circ H\} \circ (R \circ L)^n &= [(R^m \circ L^m) \circ H] \circ (R^n \circ L^n) \\
&= [(R^m \circ L^m) \circ R^n] \circ (H \circ L^n) \\
&= [(L^m \circ R^m) \circ R^n] \circ (H \circ L^n) \\
&= [(R^n \circ R^m) \circ L^m] \circ (H \circ L^n) \\
&= [(R^m \circ R^n) \circ L^m] \circ (H \circ L^n) \\
&= (R^{m+n} \circ L^m) \circ (H \circ L^n) \\
&= H \circ [(R^{m+n} \circ L^m) \circ L^n] \\
&= H \circ [(L^n \circ L^m) \circ R^{m+n}] \\
&= (H \circ H) \circ (L^{m+n} \circ R^{m+n}) \\
&= (H \circ L^{m+n}) \circ (H \circ R^{m+n}) \\
&= (R^{m+n} \circ H) \circ (L^{m+n} \circ H) \\
&= (H \circ R^{m+n}) \circ (H \circ L^{m+n}),
\end{aligned}$$

and

$$\begin{aligned}
&(H \circ R^{m+n}) \circ (H \circ L^{m+n}) = [H \circ (R^{m+n-1} \circ R)] \circ [(H \circ (L^{m+n-1} \circ L)] \\
&= [H \circ \{(R^{m+n-2} \circ R) \circ R\}] \circ [H \circ \{(L^{m+n-2} \circ L) \circ L\}] \\
&= [H \circ \{(R \circ R) \circ R^{m+n-2})] \circ [H \circ \{(L \circ L) \circ L^{m+n-2}\}] \\
&\subseteq [(H \circ H) \circ (R \circ R^{m+n-2})] \circ [(H \circ H) \circ (L \circ L^{m+n-2})] \\
&\subseteq [(H \circ R) \circ (H \circ R^{m+n-2})] \circ [(H \circ L) \circ (H \circ L^{m+n-2})] \\
&\subseteq [(R^{m+n-2} \circ H) \circ (R \circ H)] \circ [L \circ (H \circ L^{m+n-2})] \\
&\subseteq [(R^{m+n-2} \circ H) \circ R] \circ [H \circ (L \circ L^{m+n-2})] \\
&= [(R \circ H) \circ R^{m+n-2}] \circ (H \circ L^{m+n-1}) \\
&\subseteq (R \circ R^{m+n-2}) \circ (H \circ L^{m+n-1}) \\
&\subseteq (H \circ R^{m+n-1}) \circ (H \circ L^{m+n-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
[(R \circ L)^m \circ H] \circ (R \circ L)^n &\subseteq (H \circ R^{m+n}) \circ (H \circ L^{m+n}) \\
&\subseteq (H \circ R^{m+n-1}) \circ (H \circ L^{m+n-1}) \\
&\subseteq \cdots \subseteq (H \circ R) \circ (H \circ L) \\
&\subseteq [(H \circ H) \circ R] \circ L \\
&= [(R \circ H) \circ H] \circ L \subseteq R \circ L,
\end{aligned}$$

and also

$$\begin{aligned}
(R \circ L) \circ (R \circ L) &= (L \circ R) \circ (L \circ R) = [(L \circ R) \circ R] \circ L \\
&= [(R \circ R) \circ L] \circ L \subseteq [(R \circ H) \circ H] \circ L \subseteq R \circ L.
\end{aligned}$$

This shows that $R \circ L$ is an (m, n)-hyperideal of H. □

Theorem 1. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. If H has the property that it contains no nonzero nilpotent (m, n)-hyperideals and R (L) is a 0-minimal right (left) hyperideal of H, then either $R \circ L = \{0\}$ or $R \circ L$ is a 0-minimal (m, n)-hyperideal of H.*

Proof. Assume that R (L) is a 0-minimal right (left) hyperideal of H such that $R \circ L \neq \{0\}$, then by Lemma 1, $R \circ L$ is an (m, n)-hyperideal of H. Now we show that $R \circ L$ is a 0-minimal (m, n)-hyperideal of H. Let $\{0\} \neq M \subseteq R \circ L$ be an (m, n)-hyperideal of H. Note that since $R \circ L \subseteq R \cap L$, we have $M \subseteq R \cap L$. Hence $M \subseteq R$ and $M \subseteq L$. By hypothesis, $M^m \neq \{0\}$ and $M^n \neq \{0\}$. Since $\{0\} \neq H \circ M^m = M^m \circ H$, we have

$$\begin{aligned}
\{0\} \neq M^m \circ H &\subseteq R^m \circ H = (R^{m-1} \circ R) \circ H = (H \circ R) \circ R^{m-1} \\
&= (H \circ R) \circ (R^{m-2} \circ R) = (R \circ R^{m-2}) \circ (R \circ H) \\
&\subseteq (R \circ R^{m-2}) \circ R = R^m,
\end{aligned}$$

and

$$\begin{aligned}
R^m &\subseteq H \circ R^m = (H \circ H) \circ (R \circ R^{m-1}) = (R^{m-1} \circ R) \circ H \\
&= [(R^{m-2} \circ R) \circ R] \circ H = [(R \circ R) \circ R^{m-2}] \circ H \\
&= (H \circ R^{m-2}) \circ (R \circ R) \subseteq (H \circ R^{m-2}) \circ R \\
&= [(H \circ H) \circ (R^{m-3} \circ R)] \circ R \\
&= [(R \circ R^{m-3}) \circ (H \circ H)] \circ R \\
&= [(R \circ H) \circ (R^{m-3} \circ H)] \circ R \\
&\subseteq [R \circ (R^{m-3} \circ H)] \circ R \\
&= [R^{m-3} \circ (R \circ H)] \circ R \\
&\subseteq (R^{m-3} \circ R) \circ R = R^{m-1}.
\end{aligned}$$

Therefore $\{0\} \neq M^m \circ H \subseteq R^m \subseteq R^{m-1} \subseteq \cdots \subseteq R$. It is easy to see that $M^m \circ H$ is a right hyperideal of H. Thus $M^m \circ H = R$ since R is 0-minimal. Also

$$\begin{aligned}
\{0\} \neq H \circ M^n \subseteq H \circ L^n &= H \circ (L^{n-1} \circ L) \\
&= L^{n-1} \circ (H \circ L) \subseteq L^{n-1} \circ L = L^n,
\end{aligned}$$

and

$$\begin{aligned}
L^n &\subseteq H \circ L^n = (H \circ H) \circ (L \circ L^{n-1}) = (L^{n-1} \circ L) \circ H \\
&= [(L^{n-2} \circ L) \circ L] \circ H = (H \circ L) \circ (L^{n-2} \circ L) \\
&\subseteq L \circ (L^{n-2} \circ L) = L^{n-2} \circ (L \circ L) \subseteq L^{n-2} \circ L \\
&= L^{n-1} \subseteq \cdots \subseteq L,
\end{aligned}$$

therefore $\{0\} \neq H \circ M^n \subseteq L^n \subseteq L^{n-1} \subseteq \cdots \subseteq L$. It is easy to see that $H \circ M^n$ is a left hyperideal of H. Thus $H \circ M^n = L$ since L is 0-minimal. Therefore

$$\begin{aligned}
M &\subseteq R \circ L = (M^m \circ H) \circ (H \circ M^n) = (M^n \circ H) \circ (H \circ M^m) \\
&= [(H \circ M^m) \circ H] \circ M^n = [(H \circ M^m) \circ (H \circ H)] \circ M^n \\
&= [(H \circ (M^m \circ H)] \circ M^n = [(M^m \circ (H \circ H)] \circ M^n \\
&= (M^m \circ H) \circ M^n \subseteq M.
\end{aligned}$$

Thus $M = R \circ L$, which means that $R \circ L$ is a 0-minimal (m, n)-hyperideal of H. □

Theorem 2. *Let H be a pure $\mathcal{LA}$-semihypergroup. If R (L) is a 0-minimal right (left) hyperideal of H, then either $R^m \circ L^n = \{0\}$ or $R^m \circ L^n$ is a 0-minimal (m, n)-hyperideal of H.*

Proof. Assume that R (L) is a 0-minimal right (left) hyperideal of H such that $R^m \circ L^n \neq \{0\}$, then $R^m \neq \{0\}$ and $L^n \neq \{0\}$. Hence $\{0\} \neq R^m \subseteq R$

and $\{0\} \neq L^n \subseteq L$, which shows that $R^m = R$ and $L^n = L$ since R (L) is a 0-minimal right (left) hyperideal of H. Thus by Lemma 1, $R^m \circ L^n = R \circ L$ is an (m,n)-hyperideal of H. Now we show that $R^m \circ L^n$ is a 0-minimal (m,n)-hyperideal of H. Let $\{0\} \neq M \subseteq R^m \circ L^n = R \circ L \subseteq R \cap L$ be an (m,n)-hyperideal of H. Hence

$$\begin{aligned}\{0\} \neq H \circ M^2 &= (M \circ M) \circ (H \circ H) = (M \circ H) \circ (M \circ H)\\ &\subseteq (R \circ H) \circ (R \circ H) \subseteq R,\end{aligned}$$

and $\{0\} \neq H \circ M \subseteq H \circ L \subseteq L$. Thus

$$R = H \circ M^2 = (M \circ M) \circ (H \circ H) = (H \circ M) \circ M \subseteq H \circ M,$$

and $H \circ M = L$ since R (L) is a 0-minimal right (left) hyperideal of H. Therefore

$$\begin{aligned}M &\subseteq R^m \circ L^n \subseteq (H \circ M)^m \circ (H \circ M)^n = (H^m \circ M^m) \circ (H^n \circ M^n)\\ &= (H \circ H) \circ (M^m \circ M^n) = (M^n \circ M^m) \circ H = (H \circ M^m) \circ M^n\\ &= (M^m \circ H) \circ M^n \subseteq M.\end{aligned}$$

Thus $M = R^m \circ L^n$, which shows that $R^m \circ L^n$ is a 0-minimal (m,n)-hyperideal of H. $\square$

Theorem 3. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Assume that A is an (m,n)-hyperideal of H and B is an (m,n)-hyperideal of A such that B is idempotent. Then B is an (m,n)-hyperideal of H.*

Proof. It is trivial that B is an $\mathcal{LA}$-subsemihypergroup H. Secondly, since $(A^m \circ H) \circ A^n \subseteq A$ and $(B^m \circ A) \circ B^n \subseteq B$, then

$$\begin{aligned}(B^m \circ H) \circ B^n &= [(B^m \circ B^m) \circ H] \circ (B^n \circ B^n)\\ &= (B^n \circ B^n) \circ [(H \circ (B^m \circ B^m)]\\ &= [\{H \circ (B^m \circ B^m)\} \circ B^n] \circ B^n\\ &= [\{B^n \circ (B^m \circ B^m)\} \circ (H \circ H)] \circ B^n\\ &= [\{B^m \circ (B^n \circ B^m)\} \circ (H \circ H)] \circ B^n\\ &= [H \circ \{(B^n \circ B^m) \circ B^m\}] \circ B^n\\ &= [H \circ \{(B^n \circ B^m) \circ (B^{m-1} \circ B)\}] \circ B^n\\ &= [H \circ \{(B \circ B^{m-1}) \circ (B^m \circ B^n)\}] \circ B^n\\ &= [H \circ \{(B^m \circ (B^m \circ B^n)\}] \circ B^n\\ &= [B^m \circ \{(H \circ H) \circ (B^m \circ B^n)\}] \circ B^n\\ &= [B^m \circ \{(B^n \circ B^m) \circ (H \circ H)\}] \circ B^n\\ &= [B^m \circ \{(H \circ B^m) \circ B^n\}] \circ B^n\\ &= \big[B^m \circ [\{(H \circ H) \circ (B^{m-1} \circ B)\} \circ B^n]\big] \circ B^n\\ &= [B^m \circ \{(B^m \circ H) \circ B^n\}] \circ B^n\end{aligned}$$

$$\subseteq [B^m \circ \{(A^m \circ H) \circ A^n\}] \circ B^n$$
$$\subseteq (B^m \circ A) \circ B^n \subseteq B,$$

which shows that B is an (m,n)-hyperideal of H. □

Lemma 2. *Let $\langle a\rangle_{(m,n)} = (a^m \circ H) \circ a^n$, then $\langle a\rangle_{(m,n)}$ is an (m,n)-hyperideal of a pure $\mathcal{LA}$-semihypergroup H.*

Proof. Assume that H is a pure $\mathcal{LA}$-semihypergroup, then by using induction, it is easy to see that $\left(\langle a\rangle_{(m,n)}\right)^n \subseteq \langle a\rangle_{(m,n)}$. Also

$$\begin{aligned}
&\left(\{\langle a\rangle_{(m,n)}\}^m \circ H\right) \circ \{\langle a\rangle_{(m,n)}\}^n \\
&= [\{((a^m \circ H) \circ a^n)\}^m \circ H] \circ \{(a^m \circ H) \circ a^n\}^n \\
&= [\{(a^{mm} \circ H^m) \circ a^{mn}\} \circ H] \circ \{(a^{mn} \circ H^n) \circ a^{nn}\} \\
&= [a^{nn} \circ (a^{mn} \circ H^n)] \circ [H \circ \{(a^{mm} \circ H^m) \circ a^{mn}\}] \\
&= [[H \circ \{(a^{mm} \circ H^m) \circ a^{mn}\}] \circ (a^{mn} \circ H^n)] \circ a^{nn} \\
&= [a^{mn} \circ [[H \circ \{(a^{mm} \circ H^m) \circ a^{mn}\}] \circ H^n]] \circ a^{nn} \\
&\subseteq (a^{mn} \circ H) \circ a^{nn} = (a^{mn} \circ H^n) \circ a^{nn} \\
&= \{(a^m \circ H) \circ a^n\}^n \subseteq \left(\langle a\rangle_{(m,n)}\right)^n \subseteq \langle a\rangle_{(m,n)},
\end{aligned}$$

which shows that $\langle a\rangle_{(m,n)}$ is an (m,n)-hyperideal of H. □

Theorem 4. *Let H be a pure $\mathcal{LA}$-semihypergroup and $\langle a\rangle_{(m,n)}$ be an (m,n)-hyperideal of H. Then the following statements hold:*

(i) $\left(\langle a\rangle_{(1,0)}\right)^m \circ H = a^m \circ H$.

(ii) $H \circ \left(\langle a\rangle_{(0,1)}\right)^n = H \circ a^n$.

(iii) $\left[\left(\langle a\rangle_{(1,0)}\right)^m \circ H\right] \circ \left(\langle a\rangle_{(0,1)}\right)^n = (a^m \circ H) \circ a^n$.

Proof. (i). As $\langle a\rangle_{(1,0)} = a \circ H$, we have

$$\begin{aligned}
\left(\langle a\rangle_{(1,0)}\right)^m \circ H &= (a \circ H)^m \circ H \\
&= [(a \circ H)^{m-1} \circ (a \circ H)] \circ H \\
&= [H \circ (a \circ H)](a \circ H)^{m-1} \\
&= (a \circ H) \circ (a \circ H)^{m-1} \\
&= (a \circ H) \circ [(a \circ H)^{m-2} \circ (a \circ H)] \\
&= (a \circ H)^{m-2} \circ [(a \circ H) \circ (a \circ H)] \\
&= (a \circ H)^{m-2} \circ (a^2 \circ H) \\
&= \cdots = (a \circ H)^{m-(m-1)} \circ (a^{m-1} \circ H) \text{ [if } m \text{ is odd]} \\
&= \cdots = (a^{m-1} \circ H) \circ (a \circ H)^{m-(m-1)} \text{ [if } m \text{ is even]}
\end{aligned}$$

$$= a^m \circ H.$$

Analogously, we can prove (ii) and (iii). □

Corollary 1. *Let H be a pure $\mathcal{LA}$-semihypergroup and let $\langle a\rangle_{(m,n)}$ be an (m,n)-hyperideal of H. Then the following statements hold:*

(i) $\left(\langle a\rangle_{(1,0)}\right)^m \circ H = H \circ a^m$.

(ii) $H \circ \left(\langle a\rangle_{(0,1)}\right)^n = a^n \circ H$.

(iii) $\left[\left(\langle a\rangle_{(1,0)}\right)^m \circ H\right] \circ \left(\langle a\rangle_{(0,1)}\right)^n = (H \circ a^m) \circ (a^n \circ H)$.

Let $\mathfrak{L}_{(0,n)}$, $\mathfrak{R}_{(m,0)}$ and $\mathfrak{A}_{(m,n)}$ denote the sets of $(0,n)$-hyperideals, $(m,0)$-hyperideals and (m,n)-hyperideals of an $\mathcal{LA}$-semihypergroup H respectively.

Theorem 5. *If H is a pure $\mathcal{LA}$-semihypergroup, then the following statements hold*:

(i) *H is $(0,1)$-regular if and only if $\forall\ L \in \mathfrak{L}_{(0,1)}$, $L = H \circ L$.*

(ii) *H is $(2,0)$-regular if and only if $\forall\ R \in \mathfrak{R}_{(2,0)}$, $R = R^2 \circ H$ such that every R is semiprime.*

(iii) *H is $(0,2)$-regular if and only if $\forall\ U \in \mathfrak{A}_{(0,2)}$, $U = U^2 \circ H$ such that every U is semiprime.*

Proof. (i). Let H be $(0,1)$-regular, then for $a \in H$ there exists $x \in H$ such that $a \in x \circ a$. Since L is $(0,1)$-hyperideal, therefore $H \circ L \subseteq L$. Let $a \in L$, then $a \in x \circ a \in H \circ L \subseteq L$. Hence $L = H \circ L$. Converse is simple.

(ii). Let H be $(2,0)$-regular and R be $(2,0)$-hyperideal of H, then it is easy to see that $R = R^2 \circ H$. Now for $a \in H$ there exists $x \in H$ such that $a \in a^2 \circ x$. Let $a^2 \subseteq R$, then

$$a \in a^2 \circ x \subseteq R \circ H = (R^2 \circ H) \circ H = (H \circ H) \circ R^2 = R^2 \circ H = R,$$

which shows that every $(2,0)$-hyperideal is semiprime.

Conversely, let $R = R^2 \circ H$ for every $R \in \mathfrak{R}_{(2,0)}$. Since $H \circ a^2$ is a $(2,0)$-hyperideal of H such that $a^2 \subseteq H \circ a^2$, therefore $a \in H \circ a^2$. Thus

$$\begin{aligned} a \in H \circ a^2 &= (H \circ a^2)^2 \circ H = [(H \circ a^2) \circ (H \circ a^2)] \circ H \\ &= [(a^2 \circ H) \circ (a^2 \circ H)] \circ H = [a^2 \circ \{(a^2 \circ H) \circ H\}] \circ H \\ &= [a^2 \circ (H \circ a^2)] \circ H = [H \circ (H \circ a^2)] \circ a^2 \subseteq H \circ a^2 = a^2 \circ H, \end{aligned}$$

which implies that H is $(2,0)$-regular.

Analogously, we can prove (iii). □

The proof of the follwing is straightforward:

Lemma 3. *If H is a pure $\mathcal{LA}$-semihypergroup, then the following statements hold:*

(i) *If H is $(0,n)$-regular, then $\forall\ L \in \mathfrak{L}_{(0,n)}$, $L = H \circ L^n$.*

(ii) *If H is $(m,0)$-regular, then $\forall\ R \in \mathfrak{R}_{(m,0)}$, $R = R^m \circ H$.*

(iii) *If H is (m,n)-regular, then* $\forall\, U \in \mathfrak{A}_{(m,n)}$, $U = (U^m \circ H) \circ U^n$.

Corollary 2. *If H is a pure $\mathcal{LA}$-semihypergroup, then the following statements hold:*

(i) *If H is $(0,n)$-regular, then* $\forall\, L \in \mathfrak{L}_{(0,n)}$, $L = L^n \circ H$.
(ii) *If H is $(m,0)$-regular, then* $\forall\, R \in \mathfrak{R}_{(m,0)}$, $R = H \circ R^m$.
(iii) *If H is (m,n)-regular, then* $\forall\, U \in \mathfrak{A}_{(m,n)}$,

$$U = U^{m+n} \circ H = H \circ U^{m+n}.$$

The following is also straightforward:

Theorem 6. *Let H be a pure (m,n)-regular $\mathcal{LA}$-semihypergroup such that $m = n$. Then for every $R \in \mathfrak{R}_{(m,0)}$ and $L \in \mathfrak{L}_{(0,n)}$, $R \cap L = R^m \circ L \cap R \circ L^n$.*

Theorem 7. *Let H be a pure (m,n)-regular $\mathcal{LA}$-semihypergroup. If M (N) is a 0-minimal $(m,0)$-hyperideal ($(0,n)$-hyperideal) of H such that $M \circ N \subseteq M \cap N$, then either $M \circ N = \{0\}$ or $M \circ N$ is a 0-minimal (m,n)-hyperideal of H.*

Proof. Let M (N) be a 0-minimal $(m,0)$-hyperideal ($(0,n)$-hyperideal) of H. Let $O = M \circ N$, then clearly $O^2 \subseteq O$. Moreover

$$\begin{aligned}(O^m \circ H) \circ O^n &= [(M \circ N)^m \circ H] \circ (M \circ N)^n \\ &= [(M^m \circ N^m) \circ H] \circ (M^n \circ N^n) \\ &\subseteq [(M^m \circ H) \circ H] \circ (H \circ N^n) = (H \circ M^m) \circ (H \circ N^n) \\ &= (M^m \circ H) \circ (H \circ N^n) \subseteq M \circ N = O,\end{aligned}$$

which shows that O is an (m,n)-hyperideal of H. Let $\{0\} \neq P \subseteq O$ be a nonzero (m,n)-hyperideal of H. Since H is (m,n)-regular, therefore by using Lemma 3, we have

$$\begin{aligned}\{0\} \neq P &= (P^m \circ H) \circ P^n = [P^m \circ (H \circ H)] \circ P^n \\ &= [H \circ (P^m \circ H)] \circ P^n = [P^n \circ (P^m \circ H)] \circ (H \circ H) \\ &= (P^n \circ H) \circ [(P^m \circ H) \circ H] = (P^n \circ H) \circ (H \circ P^m) \\ &= (P^m \circ H) \circ (H \circ P^n).\end{aligned}$$

Hence $P^m \circ H \neq \{0\}$ and $H \circ P^n \neq \{0\}$. Further $P \subseteq O = M \circ N \subseteq M \cap N$ implies that $P \subseteq M$ and $P \subseteq N$. Therefore $\{0\} \neq P^m \circ H \subseteq M^m \circ H \subseteq M$ which shows that $P^m \circ H = M$ since M is 0-minimal. Likewise, we can show that $H \circ P^n = N$. Thus we have

$$\begin{aligned}P \subseteq O = M \circ N &= (P^m \circ H) \circ (H \circ P^n) = (P^n \circ H) \circ (H \circ P^m) \\ &= [(H \circ P^m) \circ (H \circ H)] \circ P^n = [H \circ (P^m \circ H)] \circ P^n \\ &= (P^m \circ H) \circ P^n \subseteq P.\end{aligned}$$

This means that $P = M \circ N$ and hence $M \circ N$ is 0-minimal. $\square$

Theorem 8. *Let H be a pure (m,n)-regular $\mathcal{LA}$-semihypergroup. If M (N) is a 0-minimal $(m,0)$-hyperideal ($(0,n)$-hyperideal) of H, then either $M\cap N=\{0\}$ or $M\cap N$ is a 0-minimal (m,n)-hyperideal of H.*

Proof. Once we prove that $M\cap N$ is an (m,n)-hyperideal of H, the rest of the proof is same as in Theorem 6. Let $O=M\cap N$, then it is easy to see that $O^2\subseteq O$. Moreover $(O^m\circ H)\circ O^n\subseteq (M^m\circ H)\circ N^n\subseteq M\circ N^n\subseteq H\circ N^n\subseteq N$. But, we also have

$$\begin{aligned}
(O^m\circ H)\circ O^n &\subseteq (M^m\circ H)\circ N^n=[M^m\circ(H\circ H)]\circ N^n\\
&=[H\circ(M^m\circ H)]\circ N^n=[N^n\circ(M^m\circ H)]\circ H\\
&=[M^m\circ(N^n\circ H)]\circ(H\circ H)\\
&=(M^m\circ H)\circ[(N^n\circ H)\circ H]\\
&=(M^m\circ H)\circ(H\circ N^n)\\
&=(M^m\circ H)\circ(N^n\circ H)\\
&=N^n\circ[(M^m\circ H)\circ H]\\
&=N^n\circ(H\circ M^m)=N^n\circ(M^m\circ H)\\
&=M^m\circ(N^n\circ H)=M^m\circ(H\circ N^n)\\
&\subseteq M^m\circ N\subseteq M^m\circ H\subseteq M.
\end{aligned}$$

Thus $(O^m\circ H)\circ O^n\subseteq M\cap N=O$ and therefore O is an (m,n)-hyperideal of H. □

4. Conclusions

Every $\mathcal{LA}$-semigroup can be considered as an $\mathcal{LA}$-semihypergroup but the converse is not true in general [27]. This leads us to the fact that an $\mathcal{LA}$-semihypergroup can be seen as the generalization of an $\mathcal{LA}$-semigroup. Thus the results of section 3 will generalize the results on an $\mathcal{LA}$-semigroup without hyper theory and the obtained results will give us the extension of the work carried out in [1] on (m,n)-ideals. Also if we consider the results of section 3 without hyper theory, then the obtained results will give us the extension of the work carried out in [21]. Finally if we take $m,n\geq 5$ in section 3, then all the results can be trivially followed for a pure $\mathcal{LA}$-semihypergroup without local associativity.

References

[1] Akram, Muhammad; Yaqoob, Naveed; Khan, Madad. On (m,n)-ideals in LA-semigroups. *Appl. Math. Sci. (Ruse)* **7** (2013), no. 41–44, 2187–2191. MR3039904.

[2] Bonansinga, Pia; Corsini, Piergiulio. On semihypergroup and hypergroup homomorphisms. *Boll. Un. Mat. Ital. B* (6) **1** (1982), no. 2, 717–727. MR0666599(83k:20087), Zbl 0511.20055,

[3] Corsini, Piergiulio. Prolegomena of hypergroup theory. Supplement to Riv. Mat. Pura Appl. *Aviani Editore, Tricesimo*, 1993. 215 pp. ISBN: 88-7772-025-5 MR1237639(94j:20083), Zbl 0785.20032.

[4] Corsini, Piergiulio; Leoreanu, Violeta. Applications of hyperstructure theory. Advances in Mathematics (Dordrecht), 5. *Kluwer Academic Publishers, Dordrecht*, 2003. xii+322 pp. ISBN: 1-4020-1222-5. MR1980853(2004e:08001), Zbl 1027.20051, doi: 10.1007/978-1-4757-3714-1

[5] Davvaz, Bijan. Some results on congruences on semihypergroups. *Bull. Malays. Math. Sci. Soc.* (2) **23** (2000), no. 1, 53–58. MR1815483, Zbl 1003.20060.

[6] Davvaz, B.; Dehghan Nezad, A.; Benvidi, A. Chain reactions as experimental examples of ternary algebraic hyperstructures. *MATCH Commun. Math. Comput. Chem.* **65** (2011), no. 2, 491–499. MR2663726(2012d:08003).

[7] Davvaz, B.; Dehghan Nezhad, A.; Benvidi, A. Chemical hyperalgebra: dismutation reactions. *MATCH Commun. Math. Comput. Chem.* **67** (2012), no. 1, 55–63. MR2920590.

[8] Davvaz, Bijan; Leoreanu-Fotea, Vileta. Hyperring theory and applications. *International Academic Press, USA*, 2007. iv, 328 pp. Zbl 1204.16033.

[9] Davvaz, B.; Nezhad, A. Dehghan. Dismutation reactions as experimental verifications of ternary algebraic hyperstructures. *MATCH Commun. Math. Comput. Chem.* **68** (2012), no. 2, 551–559. MR3025675.

[10] Davvaz, B.; Nezhad, A. Dehghan; Heidari, M. M. Inheritance examples of algebraic hyperstructures. *Inform. Sci.* **224** (2013), 180–187. MR3006238, Zbl 1293.08001, doi: 10.1016/j.ins.2012.10.023.

[11] Davvaz, B.; Santilli, R. M.; Vougiouklis, T. Studies of multi-valued hyperstructures for the characterization of matter-antimatter systems and their extension. *Algebras Groups Geom.* **28** (2011), no. 1, 105–116. MR2986357, Zbl 1231.20063.

[12] Ghadiri, M.; Davvaz, B.; Nekouian, R. H_v-semigroup structure on F_2-offspring of a gene pool. *Int. J. Biomath.* **5** (2012), no. 4, 1250011, 13 pp. MR2926076, Zbl 1280.92044, doi: 10.1142/S1793524511001520.

[13] Hila, Kostaq; Davvaz, Bijan; Naka, Krisanthi. On quasi-hyperideals in semihypergroups. *Comm. Algebra* **39** (2011), no. 11, 4183–4194. MR2855120(2012m:20120), Zbl 1253.20062, doi: 10.1080/00927872.2010.521932.

[14] Hila, Kostaq; Dine, Jani. On hyperideals in left almost semihypergroups. *ISRN Algebra* **2011** Art. ID 953124, 8 pp. MR3091439, Zbl 1227.20065, doi: 10.5402/2011/953124.

[15] Hila, Kostaq; Naka, Krisanthi. On pure hyperradical in semihypergroups. *Int. J. Math. Math. Sci.* **2012**, Art. ID 876919, 7 pp. MR2990875, Zbl 1257.20076, doi: 10.1155/2012/876919.

[16] Kazim, M. A.; Naseeruddin, Mohd. On almost semigroups. *Aligarh. Bull. Math.* **2** (1972), 1–7. MR0419644 (54 #7662), Zbl 0344.20049.

[17] Krgović, Dragica N. On 0-minimal $(0,2)$-bi-ideals of semigroups. *Publ. Inst. Math., Nouv. Sér.* **31(45)** (1982), 103–107. MR0710949 (84m:20068), Zbl 0514.20048.

[18] Leoreanu, Violeta. About the simplifiable cyclic semihypergroups. International Algebra Conference (Iaşi, 1998). *Ital. J. Pure Appl. Math.* **7** (2000), 69–76. MR1784548(2001g:20089), Zbl 0966.20035.

[19] Marty, F. Sur une généralisation de la notion de groupe. *Proceedings of the* 8^{th} *Congress des Mathematiciens Scandinaves* (Stockholm, Sweden, 1935). *Skand. Mat.-Kongr.* **8** (1935), 45–49. 45–49. Zbl 0012.05303.

[20] Mushtaq, Qaiser; Yusuf, S. M. On LA-semigroups. *Aligarh. Bull. Math.* **8** (1978), 65–70. MR0687711(84c:20086), Zbl 0509.20055.

[21] Mushtaq, Qaiser; Yusuf, S. M. On locally associative LA-semigroups. *J. Natur. Sci. Math.* **19** (1979), no. 1, 57–62. MR0596763(82a:20081), Zbl 0445.20033.

[22] Nezhad, A. Dehghan; Moosavi Nejad, S.M.; Nadjafikhah, M.; Davvaz, B. A physical example of algebraic hyperstructures: Leptons. *Indian J. Phys.* **86** (2012), no. 11, 1027–1032. doi: 10.1007/s12648-012-0151-x.

[23] Prenowitz, Walter; Jantosciak, James. Join geometries. A theory of convex sets and linear geometry. Undergraduate Texts in Mathematics. *Springer-Verlag, New York-Heidelberg*, 1979. xxi+534 pp. ISBN: 0-387-90340-2. MR0528635(80c:52001), Zbl 0421.52001, doi: 10.1007/978-1-4613-9438-9.

[24] Yaqoob, Naveed; Corsini, Piergiulio; Yousafzai, Faisal. On intra-regular left almost semihypergroups with pure left identity. *J. Math.* **2013**, Art. ID 510790, 10 pp. MR3101319, Zbl 1272.20064, arXiv:1211.5588, doi: 10.1155/2013/510790.

[25] Faisal; Khan, Asghar; Davvaz, Bijan. On fully regular $\mathcal{AG}$-groupoids. *Afr. Mat.* **25** (2014), no. 2, 449–459. MR3207031, Zbl 06366072, doi: 10.1007/s13370-012-0125-3.

[26] Yousafzai, F.; Corsini, P. Some characterization problems in LA-semihypergroups. *J. Algebra Number Theory, Adv. Appl.* **10** (2013), no. 1–2, 41–55.

[27] Yousafzai, Faisal; Hila, Kostaq; Corsini, Piergiulio; Zeb, Anwar. Existence of non-associative algebraic hyper-structures and related problems. To appear in *Afr. Mat.*, 2014. doi: 10.1007/s13370-014-0259-6.

[28] Yousafzai, Faisal; Khan, Asghar; Amjad, Venus; Zeb, Anwar. On fuzzy fully regular ordered AG-groupoids. *J. Intell. Fuzzy Systems* **26** (2014), no. 6, 2973–2982. doi: 10.3233/IFS-130963.

[29] Vougiouklis, Thomas. Hyperstructures and their representations. Hadronic Press Monographs in Mathematics. *Hadronic Press, Inc., Palm Harbor, FL*, 1994. vi+180 pp. ISBN: 0-911767-76-2. MR1270451(95h:20093), Zbl 0828.20076.

(Venus Amjad) Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan
venusmath@yahoo.com

(Kostaq Hila) Department of Mathematics & Computer Science, University of Gjirokastra, Gjirokastra, Albania
kostaq_hila@yahoo.com

(Faisal Yousafzai) School of Mathematical Sciences, University of Science and Technology of China, Hefei, China
yousafzaimath@gmail.com

This paper is available via http://nyjm.albany.edu/j/2014/20-52.html.