On subsets of S^n whose $(n+1)$-point subsets are contained in open hemispheres

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ABSTRACT. We investigate the nature of subsets of spheres which satisfy a tameness condition associated with the Bieri–Groves FP_m-conjecture. We find that there is a natural polyhedrality in the case of n-tame subsets of an $(n-1)$-sphere. In the case $n=3$ we establish a strong polyhedrality condition for certain maximal open 3-tame sets. Many examples are included.

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The Prologue

In the theory of modules over an abelian group Q and the theory of metabelian groups which surject onto Q with abelian kernel a certain invariant Σ^c plays an important role. This invariant is a closed subset of a sphere S^{n-1}. The sphere arises simply through the consideration of nonzero homomorphisms from Q to $\mathbb{R}$ up to positive scalar multiple: the sphere at infinity in a Euclidean topological vector space. The invariant itself depends on the module. The ideas were introduced in [4] where there is given an analysis of metabelian groups in terms of the invariant. It has wide significance, but the investigations of the present work were motivated by one particular conjecture known as the FP_m-conjecture: this is stated in [1] and we refer the reader to [2] for further background material on the invariant. The precise statement is then as follows:

Conjecture 0.1. *Let G be a finitely generated metabelian group with a normal abelian subgroup A such that $Q := G/A$ is free abelian of rank n. Let m be a natural number. Then G is of type FP_m if and only if every m-point subset of Σ^c is contained in an open hemisphere.*

The conjecture draws attention to the particular property of certain subsets of spheres referred to in the title of the paper and which we here refer to by the term *n-tameness*. When a subset of an $(n-1)$-sphere is n-tame, it turns out that there are always great $(n-2)$-spheres which do not meet the subset: this fact is significant in the group theoretical application because of a result showing that it implies certain subgroups are finitely generated (see [3], Lemma 5.1). We shall see below that this fact also underpins the abstract study of n-tame sets.

Therefore this is a paper only about the geometry and topology of n-tame sets. It is self-contained, requiring only elementary point-set topology and convex geometry. But the reader will better appreciate the paper for realizing this back story about metabelian groups.

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1. Statement of results; notion of maxtame set

Let S^{n-1} denote the unit sphere in Euclidean n-space $\mathbb{R}^n$. For a natural number m we use the terminology *m-point subset* to mean a nonempty subset of *at most m* points. We shall say that a subset V of S^{n-1} is said to be *m-tame* if every m-point subset of V is contained in some open hemisphere of S^{n-1}. We shall call a finite subset of $\mathbb{R}^n$ *balanceable* if its convex hull contains the origin, or equivalently if it is not contained in any open hemisphere. The terminology is chosen to harmonize with the notion of *balanced* that arises in tropical geometry (see for example [6]). In that case, a finite collection

of vectors is said to be balanced when there are positive weights assigned to each in such a way that the corresponding linear combination is zero. In our situation, we do not suppose *a priori* that any weights are assigned, but our balanceable sets are those for which a set of positive weights could be assigned to a nonempty subset to achieve a balanced collection. With this terminology, a subset V is m-tame if and only if V has no balanceable m-point subsets.

It is well known that every closed $(n+1)$-tame subset of S^{n-1} is contained in an open hemisphere. The same is true of open $(n+1)$-tame subsets. These two facts are covered by Lemma 2.3. As a consequence, the m-tameness condition is most interesting to study when $2 \le m \le n$. In this article we prove two theorems on polyhedrality when $m = n$ and illustrate them with examples and counterexamples in the case $n = 3$ (and $m = 2$ or 3). We use the following terminology.

Definition 1.1. A subset U of S^{n-1} is called *weakly maxtame* if U is open and n-tame, $\pi_0(U)$ is finite, and U is maximal amongst open n-tame sets with at most $|\pi_0(U)|$ components. A weakly maxtame set is called *maxtame* if it is maximal amongst all open n-tame subsets of S^{n-1}.

Our first theorem shows that polyhedrality is an intrinsic property of weakly maxtame sets.

Theorem A. *Let U be a weakly maxtame subset of S^{n-1}. Then there is a finite family $\mathcal{H}$ of open hemispheres of S^{n-1} such that:*

  (i) *Each component of U is an intersection of members of $\mathcal{H}$.*
  (ii) *The closure of each $H \in \mathcal{H}$ has a neighbourhood N such that*

$$U \cap H = U \cap N.$$

In particular, distinct components of U have disjoint closures.

We have the following conjecture which motivates the study of maxtame and weakly maxtame sets.

Conjecture B. *Let U be an n-tame subset of S^{n-1} which is either*

  (i) *closed; or*
  (ii) *open with finitely many connected components.*

Then U is contained in a maxtame subset.

The conjecture is elementary if $n \le 2$. We establish the conjecture for $n = 3$ (i.e., for 3-tame sets on a 2-sphere) and we also prove a weak version of the conjecture for arbitrary n.

Theorem B. *Let U be as in Conjecture* B. *Then:*

  (B1) *U is contained in a weakly maxtame subset.*
  (B2) *If $n \le 3$, U is contained in a maxtame subset.*

We go on to exhibit examples of maxtame sets on S^2. We shall also see that maximal open 2-tame subsets of S^2 need not be polyhedral even if they have a finite number of components: in fact there are nonpolyhedral examples having a single component or any other number of components.

When $n = 2$ a complete classification and description of (weakly) maxtame sets is easy to obtain and we also go on to describe their configuration space.

2. Embedding closed n-tame sets into polyhedral n-tame sets

In this section we shall prove the first part of Theorem B.

Theorem B1. *Let U be as in the conjecture. Then U is contained in a weakly maxtame subset.*

Let $\mathcal{T}(\ell, m, n)$ denote the set of open m-tame subsets of S^{n-1} having at most ℓ components, partially ordered by inclusion. Using Zorn's lemma it is immediate that $\mathcal{T}(\ell, m, n)$ has maximal elements. If U is any open m-tame set then U belongs to $\mathcal{T}(|\pi_0(U)|, m, n)$ and if $|\pi_0(U)|$ is finite then the maximal elements here are weakly maxtame sets containing U. This proves Theorem B1 in case (ii).

If U is closed and m-tame we shall show that U has an open m-tame neighbourhood U'. By compactness, the union U'' of some finite number of components of U' covers U and we may choose a weakly maxtame set containing U'' as above.

Therefore, to complete the proof of Theorem B1 it suffices to show that every closed m-tame subset is contained in an open m-tame subset. The following lemma is used in the proof. The conclusion we reach, Lemma 2.2(iii) below, is stronger than needed here but is used in its stronger form to prove the Key Lemma 3.1 below which plays a part in the proof of Theorem A.

Lemma 2.1. *Let $A_1, \ldots, A_m$ be closed subspaces of a compact metrizable space X. Suppose that B is a closed subspace of $\underbrace{X \times \cdots \times X}_{m}$ disjoint from $A_1 \times \cdots \times A_m$. Then there exist open sets $U_i \supseteq A_i$ and $V \supseteq B$ such that V is disjoint from $U_1 \times \cdots \times U_m$. Moreoever if all the A_i are equal then we may choose all U_i equal.*

Proof. Since X is compact and Hausdorff we can separate $A_1 \times \cdots \times A_m$ and B by a pair of open sets U and V, so it suffices to show that any neighbourhood U of $A_1 \times \cdots \times A_m$ contains an open neighbourhood of the form $U_1 \times \cdots \times U_m$ where $A_i \subseteq U_i \subseteq X$. Suppose for a contradiction that this is not the case and let U be an offending open neighbourhood of $A_1 \times \cdots \times A_m$. Let d be a metric on X matching its topology. For each natural number n let $W_{i,n} := \{(x \in X; \; d(x, A_i) < \frac{1}{n}\}$, and then choose an m-tuple $v_n \in W_{1,n} \times \cdots \times W_{m,n} \setminus U$ witnessing the assumption that U is

a bad set. By compactness the sequence (v_n) has a convergent subsequence whose limit belongs to $A_1 \times \cdots \times A_m \smallsetminus U$, a contradiction.

Therefore, for sufficiently large n, we have $W_{1,n} \times \cdots \times W_{m,n} \subseteq U$. If the A_i are all equal then for each n, the $W_{i,n}$ are all equal. $\square$

Lemma 2.2. *Let B be the set of ordered m-tuples $(x_1, \ldots, x_m)$ of*

$$\underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{m}$$

such that the set $\{x_1, \ldots, x_m\}$ is balanceable. Then the following hold:

(i) *B is closed.*
(ii) *A subset A of S^{n-1} is m-tame if and only if $\underbrace{A \times \cdots \times A}_{m}$ is disjoint from B.*
(iii) *If A is a closed m-tame subset of S^{n-1} then A has an open neighbourhood whose closure is also m-tame.*

Proof. (i) and (ii) are clear. To establish (iii), use Lemma 2.1 together with (i) to separate $A \times \cdots \times A$ and B by open sets, with the open neighbourhood of $A \times \cdots \times A$ being of the form $U \times \cdots \times U$ for some $U \supseteq A$. Then the closure $\overline{U \times \cdots \times U} = \bar{U} \times \cdots \times \bar{U}$ is again disjoint from B and U has the desired properties by (ii). $\square$

We note that if the set U in Theorem B1 has a finite number m of components then it follows from the proof that it is contained in a weakly maxtame set with at most m components.

As promised in the introduction, we include a proof of the basic and doubtless well known:

Lemma 2.3. *If U is either open or closed in S^{n-1} and is $(n+1)$-tame then U is contained in an open hemisphere.*

Proof. A classical result of Carathéodory states that the convex hull of a subspace of Euclidean space $\mathbb{R}^n$ is the union of the convex hulls of the $(n+1)$-point subsets, see ([5], Satz 9, end of the third paragraph).

Let U be $(n+1)$-tame in S^{n-1}. Let V be the convex hull of U in the ambient Euclidean space $\mathbb{R}^n$. The tameness condition ensures that none of the convex hulls of $(n+1)$-point subsets of U contain the origin. Therefore, by Carathéodory's theorem, V is a convex set in $\mathbb{R}^n$ not containing the origin. The closure $\bar{V}$ of V either does not contain the origin or contains the origin on its boundary. It follows that $\bar{V}$ is contained in a closed half-space with the origin on its boundary: this can be deduced from ([7],Theorem 11.5) for example, which states that a closed convex subset of $\mathbb{R}^n$ is the intersection of the closed affine half-spaces which contain it. If U is open then V is open and is contained in the interior open half-space. If U is closed then the origin does not belong to V and one can again deduce that V is contained in an open half-space with the origin on its boundary. $\square$

3. Polyhedrality of weakly maxtame sets

A subset of a sphere is called *polyhedral* if it is a finite union of finite intersections of hemispheres. The next result is our tool for establishing polyhedrality. For $v \in S^{n-1}$, we use the notation $H(v)$, $\bar{H}(v)$ to denote the open and closed hemispheres in S^{n-1} centred at v.

Lemma 3.1 (The Key Lemma). *Let $X \subseteq U$ be subsets of S^{n-1} such that U is either open or closed and n-tame, and X is closed and $(n+1)$-tame. Then there is an open hemisphere H containing X such that $U \cap \partial H = \varnothing$.*

Proof. Suppose first that U is closed. Consider the set $\mathcal{X}$ of subsets of U which contain X and which are contained in a closed hemisphere. We use Zorn's Lemma to show that $\mathcal{X}$ contains maximal elements. Lemma 2.3 shows that X is contained in a closed hemisphere and so $X \in \mathcal{X}$ and $\mathcal{X}$ is nonempty. Suppose now that $\mathcal{Y}$ is a totally ordered subset of $\mathcal{X}$. For any $Y \subseteq S^{n-1}$ let N_Y denote the set of $v \in S^{n-1}$ such that $Y \subseteq \bar{H}(v)$. Then N_Y is closed and for $Y \subseteq Y'$ we have $N_Y \supseteq N_{Y'}$. As Y runs through $\mathcal{Y}$, the N_Y are a family of nonempty closed sets with the finite intersection property. Therefore there exists $u \in \bigcap_{Y \in \mathcal{Y}} N_Y$. Now $\bigcup_{Y \in \mathcal{Y}} Y \subseteq \bar{H}(u)$ and hence totally ordered sets in $\mathcal{X}$ are bounded above.

Let Z be a maximal element of $\mathcal{X}$ and let H be a closed hemisphere which contains Z. We have $Z \subseteq H \cap U \in \mathcal{X}$ and so the maximality of Z guarantees that $Z = H \cap U$ and Z is closed. Consider the intersection $W := Z \cap \partial H$. Then W is an n-tame subset of the $(n-2)$-sphere ∂H and hence by Lemma 2.3 it is contained in an open hemisphere $K \subset \partial H$. Then ∂K is a great subsphere of codimension 2 in S^{n-1}. Since Z is closed, we can define a small rotation R fixing ∂K so that Z lies in the interior of the hemisphere $R(H)$. Maximality of Z now ensures that $U \cap \partial R(H) = \varnothing$. This argument is easiest to visualise in the case $n = 3$, but it works in any dimension $n \geq 2$, noting that if $n = 2$ then K is a point and its boundary is the empty set, which is the fixed set of a nontrivial rotation of the circle.

Suppose now that U is open. Using Urysohn's Lemma, we may choose a continuous function $f : S^{n-1} \to [0, 1]$ such that:

- f vanishes on $S^{n-1} \smallsetminus U$.
- f takes value 1 on X.
- f takes values in the open interval $(0, 1)$ on $U \smallsetminus X$.

For each natural number ℓ let U_ℓ denote the open set $f^{-1}\left(\left(\frac{1}{\ell+1}, 1\right]\right)$ and let V_ℓ denote the closed set $f^{-1}\left(\left[\frac{1}{\ell+1}, 1\right]\right)$. Then we have

$$X \subset U_1 \subset V_1 \subset U_2 \subset V_2 \subset \cdots$$

and $U = \bigcup_{\ell=1}^{\infty} V_\ell$. By the closed case already proved we know that

$$D_\ell = \{v \in S^{n-1}; \ X \subseteq H(v) \text{ and } V_\ell \cap \partial H(v) = \varnothing\}$$

is nonempty for each l. Therefore

$$E_\ell = \{v \in S^{n-1}; \ X \subseteq H(v) \text{ and } U_\ell \cap \partial H(v) = \varnothing\}$$

is nonempty for each l. The sets E_ℓ are closed and nested ($E_\ell \supseteq E_{\ell+1}$) so by compactness there is an x in their intersection. For this x we have $X \subseteq H(x)$ and $U_\ell \cap \partial H(x) = \varnothing$ for all ℓ. Since $\bigcup U_\ell = U$ the result follows. $\qquad \square$

Definition 3.2. Let U be an n-tame subset of S^{n-1}. We shall say that a point $x \in S^{n-1}$ is *addable (with respect to U)* if and only if $U \cup \{x\}$ is n-tame. We write U^+ for the set of all addable points.

We shall need to work with convex sets in spherical geometry. We adopt the convention that the spherical convex hull of $U \subseteq S^{n-1}$ is the projection from the origin to the sphere of the Euclidean convex hull.

Notation 3.3. Let U be a subset of S^{n-1}. We write $U[m]$ for the union of the spherical convex hulls of the m-point subsets of U. This is called the *m-hull of U*.

We note that an element of $U[m]$ is a point of S^{n-1} which can be expressed as a linear combination of an m-point subset of U with positive coefficients. We shall only need to employ this notation when the subset U is m-tame.

Lemma 3.4. *Let U be an n-tame subset of S^{n-1}. Then*

$$U^+ = S^{n-1} \smallsetminus -U[n-1].$$

Proof. If a point $x \in S^{n-1}$ belongs to a balanceable n-point subset $E \subseteq U \cup \{x\}$ then $-x$ belongs to the spherical hull of $E \smallsetminus \{x\}$ and this is contained in $U[n-1]$. Thus $x \in -U[n-1]$. The converse is equally straightforward. $\qquad \square$

Lemma 3.5. *Let U be an open n-tame subset of S^{n-1} and let H be an open hemisphere. Then $U \cup (H \cap \bar{U})$ is n-tame.*

Proof. Suppose not. Let F be a balanceable n-point subset of $U \cup (H \cap \bar{U})$. Set $E := F \smallsetminus U$. Since E is contained in the open hemisphere H it follows that at least one point u of F belongs to U. The balanceable condition tells us that there are $k \geq 1$ and positive scalars $\lambda_1, \ldots, \lambda_k$ with

$$u = -\frac{\lambda_1 u_1 + \cdots + \lambda_k u_k}{\|\lambda_1 u_1 + \cdots + \lambda_k u_k\|},$$

where $u_1, \ldots, u_k$ are distinct elements of $F \smallsetminus \{u\}$. This gives u as a continuous function of $u_1, \ldots, u_k$. Each of the points of E belongs to the closure of U. By nudging those u_i that belong to E back into U and adjusting the point u in the open set U via the formula above we then obtain a balanceable n-point subset of U, a contradiction. $\qquad \square$

It follows in particular that every point in the closure of U is addable.

The next lemma is not needed in what follows but illustrates a further way in which tame sets may be enlarged preserving the tame condition.

Lemma 3.6. *Let U be an n-tame subset of S^{n-1} and let C be a nonempty path-connected subset of U. Then $U \cup C[2]$ is n-tame.*

Proof. For each finite set F, write μF for the number of points not in U. For a contradiction, suppose that there is a balanceable n-point subset of $U \cup C[2]$. Let F be a choice of such a set with μF as small as possible. Clearly F contains at least one point w not in U. Then w belongs to $C[2]$ and we may choose two points u, v in C such that w lies on the short geodesic joining u to v. We note by tameness of U, u and v are not antipodal. Let $E := (F \smallsetminus \{w\}) \cup \{u, v\}$. We have $\mu E = \mu F - 1$ and therefore none of the n-point subsets of E are balanceable. It follows that E spans a simplex in Euclidean space which contains the origin in its interior. Let $p : [0, 1] \to C$ be a path from u to v. Consider the simplices $\Sigma(t)$ spanned by $E(t) := (F \smallsetminus \{w\}) \cup \{u, p(t)\}$. We have $E(1) = E$ while $E(0)$ is a face of E. It follows that for some t, $E(t)$ has a face which contains the origin. At this time, the vertices of this face form a balanceable set. However, $\mu E(t)$ remains always strictly less than μF because $p(t)$ is contained in U. This is a contradiction. $\qquad\square$

Applying the lemma iteratively we can replace $C[2]$ by its spherical hull.

Definition 3.7. Let U, V be closed polyhedral subsets of S^{n-1}. We shall say that U *is adjacent to* V if and only if $U \cap V$ has a codimension one component.

Proof of Theorem A. Let U be a weakly maxtame subset of S^{n-1}. Let F be a finite set of representatives of the components of U. We begin by finding a finite set $\mathcal{K}$ of open hemispheres such that:

- Every n-point subset of U is contained in some $K \in \mathcal{K}$.
- $U \cap \partial K = \varnothing$ for all $K \in \mathcal{K}$.

This is achieved by choosing one hemisphere $K = K(E)$ for each n-point subset E of F using Key Lemma 3.1. Then $\mathcal{K}$ has at most $\binom{|F|}{n}$ hemispheres. Given an arbitrary n-point subset G of U, let $E \subseteq F$ be the corresponding representatives. Connectivity ensures that $K(E)$ contains G and the bulleted conditions above are already satisfied.

Now consider the tiling of S^{n-1} produced by the great subspheres ∂K as K runs through $\mathcal{K}$. Let X denote the union of these subspheres. Let V denote the union of those components of the complement X^c which meet U. The same set F represents the components of V and the same set of hemispheres is witness to the fact that V is n-tame. Therefore, by maximality, $V = U$. It is now clear that each component is polyhedral.

Let $\mathcal{H}$ be the set of those members H of $\mathcal{K}$ with the property that some component of $\bar{U}$ is adjacent to ∂H. Part (i) of the theorem holds for $\mathcal{H}$.

Suppose now that $H \in \mathcal{H}$ as above. Let $B = B_r(p)$ denote the ball in S^{n-1} of radius r about the point p. By (i) and the adjacency condition there is some point $p \in \partial H$ and $r > 0$ with $B \cap H$ contained in U. We may assume

after making r sufficiently small that H is the only element of $\mathcal{H}$ whose boundary intersects B. Since U is weakly maxtame the union $U \cup B$ is not n-tame. Let $X_\mathcal{H}$ denote the union of the boundary great subspheres of the hemispheres in $\mathcal{H}$. By the polyhedral condition (i), if any point x in $(X_\mathcal{H})^c$ is addable with respect to U then so is every point in the connected component of x in $(X_\mathcal{H})^c$, and indeed this connected component may be added to U to obtain a larger n-tame set. Thus we see that none of the points in $B \smallsetminus \bar{H}$ are addable with respect to U, whereas $B \cap \bar{H} \subset \bar{U} \subset U^+$ (the last inclusion uses Lemma 3.5). Consider the antipodal point $-p$ with ball neighbourhood $-B$. By two applications of Lemma 3.4 we see that $(-B) \cap H \subset U[n-1]$ whereas $(-B) \smallsetminus H$ is disjoint from $U[n-1]$. Since we have established that U has finitely many polyhedral components it follows that $-p$ is in the spherical convex hull of $m < n$ components of $\bar{U}$, and moreover each of these components is contained in $\bar{H}$: if any of these components intersect H^c then the convex hull of all m components would intersect $(-B) \smallsetminus \bar{H}$. Choose a set $E \subset \partial H$ of minimal order m in the closure of $U \cap H$ whose convex hull contains $-p$. Then $\{p\} \cup E$ spans an m-simplex in ∂H containing the origin in its interior; we may replace p by $n - m$ points in $B \cap \partial H$ which together with E give a set F of points in ∂H which span an $(n-1)$-simplex containing the origin. Moreover each point in F is in the closure of a component of $U \cap H$. Let W be the union of these components of $U \cap H$; then $W[n-1] \cup \partial H$ contains a neighbourhood A of ∂H in H. Another application of Lemma 3.4 shows that the union $N = H \cup -A$ satisfies condition (ii) of the theorem.

Finally suppose that C_1 and C_2 are two connected components of U. Then C_1 is the intersection of finitely many open hemispheres by condition (i), and at least one of these hemispheres, call it H, does not contain C_2. By condition (ii) and the fact that C_2 is open and connected, we see that in fact C_2 is contained in the complement of a neighbourhood N of the closure of H, from which the last sentence of the theorem follows easily. $\qquad\square$

4. Maxtame sets on the 2-sphere: constructions and examples

We begin with the second part of Theorem B, an effective method for constructing maxtame sets on a 2-sphere from any weakly maxtame starting point. For clarity, here is a self-contained statement of the theorem. It is the case $n = 3$ of Conjecture B.

Theorem B2. *Let U be a 3-tame subset of S^2 which is either*

    (i) *closed; or*
    (ii) *open with finitely many connected components.*

Then U is contained in a maxtame subset.

In the proof we shall use the case $n = 3$ of the following result.

Lemma 4.1. *Let X be a subset of S^{n-1} which is n-tame but not $(n+1)$-tame. Then every path component of $S^{n-1} \smallsetminus X[n-1]$ is convex.*

Proof. This argument rests on the fact that every point of the sphere has a natural *rank with respect to X*. For $0 \le \ell \le n$ we say that $v \in S^{n-1}$ has rank ℓ if and only if v belongs to $X[\ell+1]$ but does not belong to $X[\ell]$. The hypothesis that X is not $(n+1)$-tame ensures that $X[n] = S^{n-1}$ and hence every point of the sphere has a rank ℓ in the range $0 \le \ell \le n-1$. The points of X itself are the points of rank 0. The set $S^{n-1} \setminus X[n-1]$ consists precisely of the points of rank $n-1$.

To prove the lemma we first fix a balanceable $(n+1)$-point subset F of X, i.e., a witness to the hypothesis that X is not $(n+1)$-tame. Since X is n-tame, each n-point subset of F has a spherical hull which is a face of a simplex triangulating S^{n-1}.

For any n-point subset E of X, the spherical convex hull Δ of E is a simplex properly contained in the sphere. Taking the points of F one by one, we may build up a simplicial triangulation of S^{n-1} in which Δ is a face. At each step, ignore points which lie in the spherical hull of the complex so far constructed. To add a point x of F when it does not belong to that hull we simply cone it off to those faces of the simplicial complex already constructed that are *visible* from x, in the sense that there are short geodesics from these faces to x which contain no points of the complex in their interior. At last there must come a point of F which, when added, results in a complex that covers the entire sphere since, with $E = \varnothing$, adding all points of F would achieve this.

Now suppose that v and w are points of rank $n-1$ in the same path component of $S^{n-1} \setminus X[n-1]$. If u is any point of $X[n-1]$ then the above argument shows that u lies in the $(n-2)$-skeleton of some triangulation of the sphere whose vertices belong to X. The $(n-1)$-faces of this triangulation are convex and are the path components of the complement of its $(n-2)$-skeleton. Therefore v and w both belong to the same convex component and hence the short geodesic between them lies within that component and so does not contain u. This shows that no point of $X[n-1]$ lies on the geodesic from v to w and the convexity follows. $\square$

The following is immediate on combining this with Lemma 3.4.

Corollary 4.2. *Let U be an n-tame subset of S^{n-1} which is not $(n+1)$-tame. Then each connected component of the set of addable points is convex.*

Remark 4.3. If U is open and $(n+1)$-tame then U is contained in an open hemisphere. In this case, the set of addable points is the union of all closed hemispheres containing U and it is equal to the complement of the spherical convex hull of $-U$. In particular the set of addable points of a nonempty $(n+1)$-tame set consists of a single component and is never convex.

Proof of Theorem B2. The reader will notice that much of the following reasoning could be applied in higher dimensions. However, at a critical point of the proof we have found the need to take advantage of the low dimension: we draw attention to this point when it arises.

By Lemma 3.4 if U is 3-tame in S^2 then the set U^+ of addable points is equal to $-U[2]^c$, the complement of the 2-hull of the antipodal set of U. If U is a union of convex spherical polygons, (for example, if U is weakly maxtame) then clearly U^+ is as well.

Using Theorem B1, we may assume that U is weakly maxtame and in particular that U is a disjoint union of convex spherical polygons. If U is 4-tame then U is contained in an open hemisphere and we are done. Therefore we may assume that there is a balanceable 4-point subset F. Then $F = \{a, b, c, d\}$ must consist of exactly 4 points and its convex hull in Euclidean space is a tetrahedron which contains the origin in its interior. Tameness guarantees that U is contained in the complement of $-F[2]$, the 2-hull of the antipodal set of F. We naturally regard $-F[2]$ as the 1-skeleton of a simplicial subdivision of S^2. The (open) faces $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ contain the set U^+ of all addable points. We now show how to replace U by a larger open set U_0 which is still polyhedral and n-tame so that $U \subseteq U_0 \subset U_0^+ \subseteq U^+$ and so that Φ_1 does not meet the interior of $U_0^+ \smallsetminus U_0$. Let $C_1, \ldots, C_\ell$ be an enumeration of the connected components of the interior of $U^+ \smallsetminus U$ that lie in Φ_1, recalling that both U and U^+ are finite unions of convex spherical polygons. Note that $U \cup C_1$ is n-tame, for if not then there would be a balanceable set containing at least one point of C_1. However, such a balanceable set can always be replaced by one that contains a single point of C_1 because C_1 is convex: a linear combination with positive weights of points in C_1 is an element of C_1. Now since C_1 consists entirely of addable points, this single point can be added without creating a balanceable set. Therefore we replace U by $U_{(1)} := U \cup C_1$. The set of addable points now becomes potentially smaller but remains polyhedral. Therefore when we turn to the component C_2 it maybe that the interior of $C_2' := C_2 \cap U_{(1)}^+$ is properly contained in C_2 and furthermore, C_2' might consist of more than one component.

At this point in the argument we need to use the hypothesis that $n = 3$. Therefore, for the remainder of the proof we shall make this assumption.

The set $U_{(2)} := U_{(1)} \cup (U_{(1)}^+ \cap C_2)$ is 3-tame. To see this suppose for a contradiction that there is a balanceable 3-point subset inside $U_{(2)}$. Then at least one of the 3 points must lie outside U_1 because $U_{(1)}$ is 3-tame. Moreover, if only one of the three points lies outside $U_{(2)}$ then there can be no contradiction because each one point is addable. If all three points lie outside of $U_{(2)}$ then we would have all three points lying on the face Φ_1 and each face is contained in an open hemisphere. So there must be exactly two points c, c' outside $U_{(2)}$ and one point u inside $U_{(2)}$. The balanceability condition means antipodal point $-u$ lies on the geodesic joining c to c' and therefore $-u$ lies in the convex set C_2. Since Φ_1 is contained in an open hemisphere and $-u$ lies in this face, it follows that u does not lie in Φ_1. In particular, u is not in the set C_1 and therefore u belongs to the original set U. However, every point of C_2 was addable with respect to U and so $-u$ is

addable, a contradiction. Thus $U_{(2)}$ is 3-tame. This part of the reasoning does not generalize straightforwardly to higher dimensions.

We now turn to C_3. The process of adding C_1 and then adding the splinters of C_2 that remained addable after adding C_1 may have splintered C_3 into many components, but the same argument allows us to add the interiors of these all in one step.

Continuing in this way, we may visit each of the sets $C_1, \ldots, C_\ell$ in turn, adding what can be added, preserving polyhedrality and arriving at a situation where the interior of the set of addable points has empty intersection with Φ_1. This new set may be labelled U_0.

We now turn to the next face Φ_2 of Δ. The process carried through for Φ_1 may have resulted in the number of components of addable points on Δ increasing dramatically. This has not mattered because it is only now that we enumerate the components and proceed as with the first face. Going on in this way, we can complete the process by visiting each face of Δ and arriving at a maxtame set as required. $\qquad\qquad\square$

We conclude this section with some examples of maxtame sets on S^2. The simplest example is shown in Figure 1. The construction of this example is simple and can be generalized in a number of ways. Any choice of 4 great circles in general position divides the sphere into 14 regions: 8 spherical triangles and 6 spherical quadrilaterals. There are two ways (antipodally related to each other) to choose 4 of the triangles in order to obtain a maxtame set. Figure 1 shows one of these choices. The example is the first in two infinite families of maxtame sets on S^2: see Figure 2 (Family A) and Figure 3 (Family B).
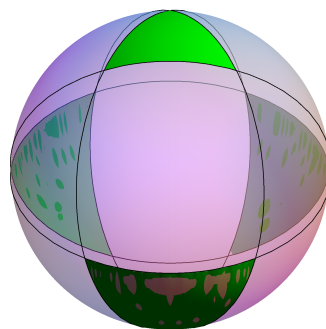

FIGURE 1. A 4-component maxtame set on S^2

Figure 5 shows these same sets in stereographic projection from the south pole.
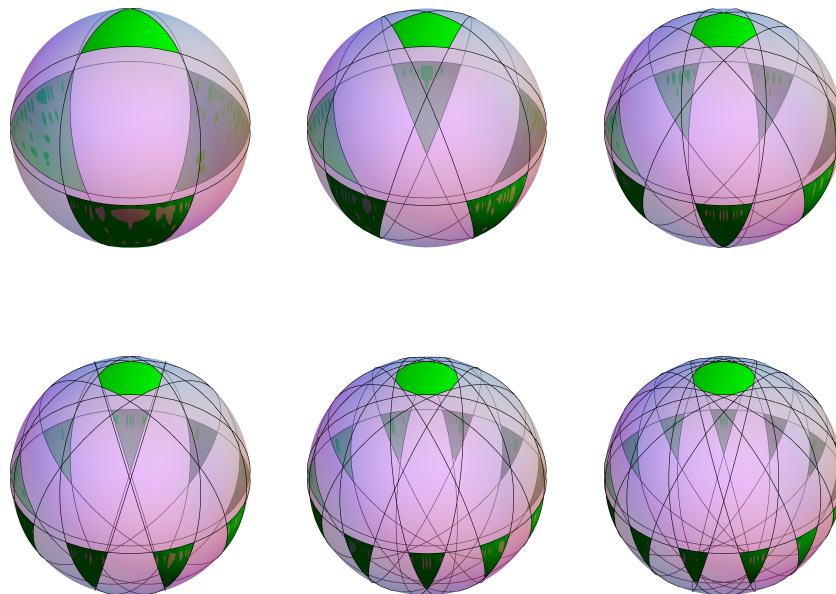
FIGURE 2. Family A: 3-dimensional rendition. The kth
member consists of a single $(2k+1)$-gon together with $2k+1$
triangles.

In both cases it can be checked straightforwardly that the shaded set U
satisfies the condition $U = -\bar{U}[2]^c$ so that the interior of the set of addable
points is exactly U. This allows us to deduce that all these sets are indeed
maxtame as claimed. The sets in Family A satisfy the stronger condition,
$\bar{U} = -U[2]^c$ meaning that the closure $\bar{U}$ is precisely the set of addable
points. In the case of Family B, there are some isolated addable points: in
particular, assuming that the set is drawn symmetrically around a north-
south axis then the north and south poles are addable. Being isolated, these
addable points do not allow the sets to be enlarged, i.e., they are already
maxtame. However it is possible to perturb these sets in order to produce
maxtame sets with additional regions but an otherwise similar appearance.

Checking that the sets in these figures are 3-tame and then maxtame
can be done using the great circles in the picture. For example, 9 great
circles are shown in Figure 6. If three points are chosen from the triangular
regions then one can see by inspection that at least one of those great circles
separates those points into a single hemisphere. Checking maxtameness
involves constructing the antipodal set $-U$ and then its 2-hull $-U[2]$. Again
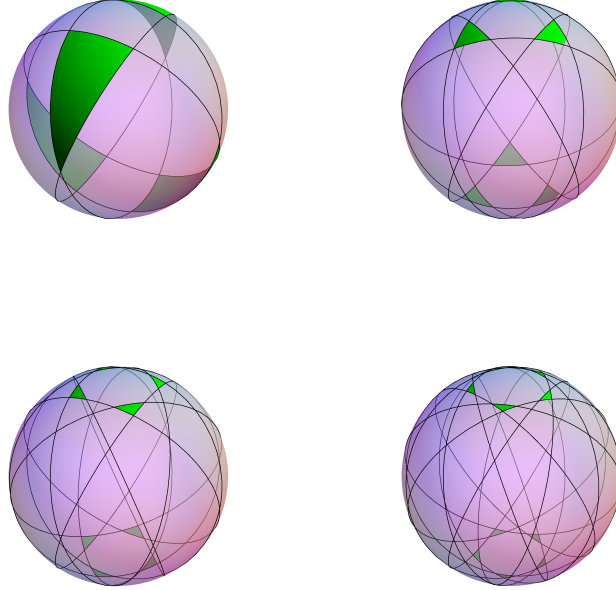this is easy to carry out by inspection of the diagrams.

FIGURE 3. Family B: 3-dimensional rendition.   The kth member consists of $2k + 2$ triangles.

Figure 6 shows how a perturbation of the 6-component member of Family B gives rise to a seventh addable region. Perturbations of the other sets in Family B may give rise to other interesting examples with an odd number of components. We shall show later that there are restrictions: in particular there are no maxtame sets on S^2 with 3 or 5 components.

5. Maxtame sets on S^1 and their configuration space

On the circle S^1 there are two maxtame sets associated with any choice of an odd number of distinct pairs of antipodal points as illustrated in Figure 4.
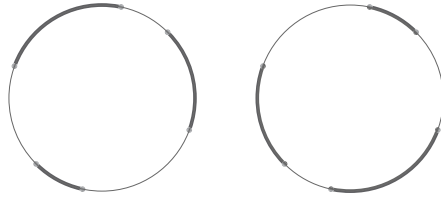

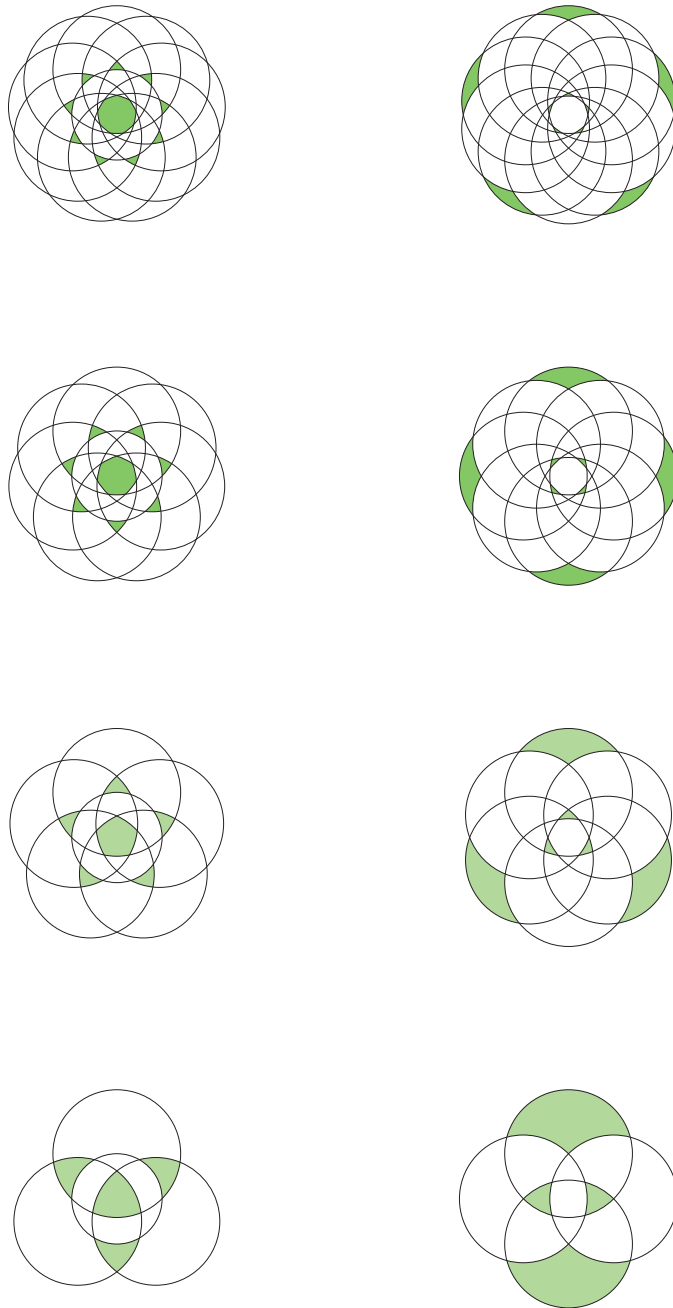FIGURE 4. Two maxtame sets on S^1 sharing the same 6-point boundary

FIGURE 5. Families A and B in stereographic projection

It is then interesting to consider the configuration space $\mathcal{S}$ of all such subsets. The double cover $S^1 \to \mathbb{RP}^1$ induces a map from $\mathcal{S}$ to the configuration space of odd numbers of points on the circle $\mathbb{RP}^1$, and $\mathcal{S}$ is a connected double cover of this space.
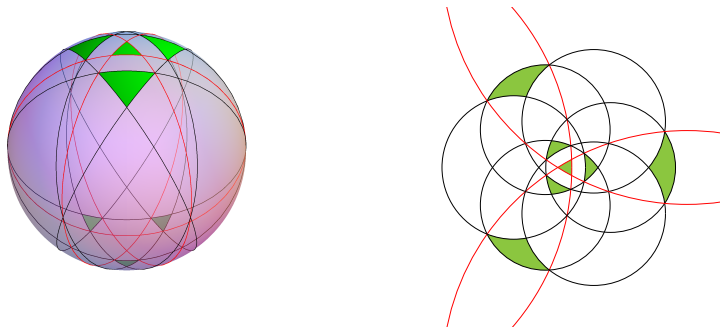

FIGURE 6. A maxtame set with 7 components

Lemma 5.1. *The configuration space of maxtame subsets of S^1 is homeomorphic to the infinite sphere S^∞.*

Proof. In view of the above remarks it suffices to show that the configuration space of odd numbers of points on $\mathbb{RP}^1$ is homeomorphic to $\mathbb{RP}^\infty$. Represent $\mathbb{RP}^1$ as $[0,1]/\{0 \sim 1\}$.

Let ℓ be an odd number. Then the configuration space C_ℓ of ℓ points is a quotient of $[0,1]^\ell$, where we quotient by the action of the symmetric group S_ℓ and also by the equivalence relation $(x_1, \ldots, x_\ell) \sim (y_1, \ldots, y_\ell)$ if and only if there exists indices $i \neq j$ such that $x_i = x_j$, $y_i = y_j$ and $x_k = y_k$ for $k \in \{1, \ldots, \ell\} \smallsetminus \{i, j\}$.

We almost get a fundamental domain for the S_n-action by taking

$$\{(x_1, \ldots, x_\ell);\ 0 \leq x_1 \leq x_2 \leq \ldots \leq x_\ell \leq 1\}.$$

This gives a polygonal subset B, homeomorphic to a ball (indeed it is an ℓ-simplex), of the unit cube. The face $x_1 = 0 \sim 1$ gets identified with the face $x_\ell = 1$ by the S_ℓ-action. The other faces all have $x_i = x_{i+1}$ and thus get "crushed" into the boundary equator separating the above two faces. Thus we have an ℓ-ball with its boundary points identified in antipodal pairs, i.e., $\mathbb{RP}^\ell$. Note that the subset with $x_\ell = 1$ is similarly homeomorphic to $\mathbb{RP}^{\ell-1}$ included into $\mathbb{RP}^\ell$ in the standard way. Thus the subset with $x_{\ell-1} = x_\ell = 1$ which is $C_{\ell-2}$ is $\mathbb{RP}^{\ell-2}$ included into $\mathbb{RP}^\ell$ in the standard way. Thus the union of all C_ℓ is $\mathbb{RP}^\infty$. $\square$

6. Maximal open 2-tame sets on S^1 with infinitely many components

We include two examples showing that the finiteness condition $|\pi_0| < 0$ cannot be removed from Definition 1.1 without severely altering the landscape. We'll view S^1 as the set of unit complex numbers.

Note that if U is maximal open and 2-tame on any sphere then

$$C := (U \cup -U)^c$$

is antipodally symmetric, closed and must have no interior points. In the first example we take the simplest of such sets, namely a symmetric pair of discrete sequences converging to limit points.

Example 6.1. *Let C denote the subset*

$$\{\pm e^{\pi i t}; \ t = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \cup \{\pm 1\}$$

of S^1. Then there is a maximal open 2-tame U such that $C = (U \cup -U)^c$.

Proof. Let $f : [0,1] \to S^1$ be the map $t \mapsto e^{\pi i t}$. Then we can take U to be the union

$$f(\tfrac{1}{2}, 1) \cup -f(\tfrac{1}{3}, \tfrac{1}{2}) \cup f(\tfrac{1}{4}, \tfrac{1}{3}) \cup -f(\tfrac{1}{5}, \tfrac{1}{4}) \cup f(\tfrac{1}{6}, \tfrac{1}{5}) \cup -f(\tfrac{1}{7}, \tfrac{1}{6}) \cup \cdots. \quad \square$$

Our second example involves a Cantor set.

Example 6.2. *We write all numbers in ternary instead of decimal. Thus $0.1\dot{1}$ denotes "zero point one recurring" and hence exactly a half. We consider the set C of points in the interval $[0, 11]$ (i.e., real numbers between zero and four) which can be expressed in ternary using only the digits 0 and 2. Note that recurring 2s are permitted here even though a number such as $0.220\dot{2}$ is equal to the expression 0.221 involving a 1. Let $p : [0, 11] \to S^1$ be defined by $t \mapsto e^{0.1\dot{1}\pi i t}$. Let Y be the set of intervals of ternary numbers of the form $(a_0 \cdot a_1 a_2 \ldots a_n 1, a_0 \cdot a_1 a_2 \ldots a_n 2)$ where the digits $a_i \in \{0, 2\}$ include an even number of 2s. Let U be the set $\bigcup_{I \in Y} p(I)$. Then U is maximal open and 2-tame and $U \cup -U$ has complement the Cantor set $p(C)$.*

7. Nonpolygonal maximal open 2-tame subsets of S^2

The polyhedrality associated with maxtame sets does not apply in general to maximal open k-tame sets in S^{n-1} when $k < n$ even if there are finitely many connected components. Figure 7(left) shows a maximal open 2-tame set obtained by starting with the open southern hemisphere, carving a flower shape, and placing the antipodal set in the northern hemisphere. Clearly any shape from the northern hemisphere can be used so long as it is the interior of its closure. Thus we see that polyhedrality fails for maximal open 2-tame sets on S^2 and that such sets exists with any finite number of components ≥ 1. As well as the obvious example with a single component, namely a

hemisphere, there are also exotic 1-component maximal open 2-tame sets such as the one illustrated in Figure 7(right).
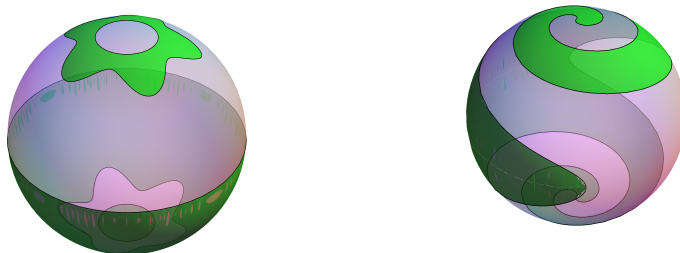
FIGURE 7. Nonpolyhedral maximal open 2-tame sets on S^2

8. Maxtame subsets of S^{n-1} with $n+1$ components

In this section we explain how the 3-component sets in Figure 4 and the 4-component sets in Figures 1, 2, 3, 5 belong to a family in which the nth members are maxtame $(n+1)$-component subsets of S^{n-1}.

Fix $n \geq 2$. Choose $n+1$ points $v_0, \ldots, v_n$ on the $(n-1)$-sphere such that their convex hull in $\mathbb{R}^n$ contains the origin as an interior point. For each i choose a closed hemisphere H_i which contains v_i in its interior but not containing any v_j with $j \neq i$. Let ∂_i denote the boundary of H_i. These boundaries carve the sphere into $2^{n+1} - 2$ regions in the following way. For each choice of $n+1$-tuple $(\varepsilon_0, \ldots, \varepsilon_n) \in \{\pm\}^{n+1}$ the intersection $\bigcap_i \varepsilon_i H_i$ is nonempty except in the two cases when all the ε_i are equal to each other. We use the shorthand notation $\varepsilon_0 \varepsilon_1 \ldots \varepsilon_n$ to denote the intersection.

Let U_i be the spherical n-simplex obtained by taking $\varepsilon_j = +$ for $j \neq i$ and $\varepsilon_i = -$. Let U denote the union of the U_i. This is the union of the regions cut out by the hemisphere boundaries which contain the vertices $v_0, \ldots, v_n$.

Lemma 8.1. *The interior of U is maxtame.*

Proof. In order to explain the argument we restrict to the case $n = 4$. The general case follows in the same way. Each 5-tuple in $\{\pm\}^5$ which contains at least one plus and at least one minus determines a nonempty intersection of hemispheres. The five 5-tuples $++++-, +++-+, ++-++, +-+++$ and $-++++$ represent the five spherical tetrahedral components of U. Each of these has four boundary simplices. Replacing a $+$ or a $-$ by the symbol ∂ in the ith coordinate to indicate intersection with the boundary ∂_i we see that the boundary of $++++-$ is made up of the four simplices $\partial+++-, +\partial++-, ++\partial+-, +++\partial-$. By replacing one of the plus symbols by a minus, we pass across a boundary to an adjacent region. For

example $+ + + + -$ and $+ + - + -$ are adjacent with common boundary $+ + \partial + -$. The antipodal set $-U$ is made up of the five spherical tetrahedra $- - - - +, \ldots, + - - - -$ described by the tuples with four minuses and one plus. Each of the components of $-U$ kisses every component of U save the antipodal component, that is to say, when two of these components meet, they meet in a single point. We also introduce the symbol $\pm$ into our tuples to indicate no restriction in the relevant coordinate. For example $+ + \pm - -$ denotes the intersection $H_0 \cap H_1 \cap -H_3 \cap -H_4$.

The fact that U is n-tame is immediate from the construction. To check that it is maxtame we appeal to Lemma 3.4. We need to know how to compute the convex hull of the union of any subset of the five antipodal simplices $\{- - - - +, \ldots, + - - - -\}$. The recipe is very simple. The convex hull consists of all regions $\varepsilon_0 \ldots \varepsilon_n$ in which ε_i takes one of the values occurring amongst the chosen simplices. In effect this removes any restriction from the coordinates where there are sign changes among the different chosen simplices. For example, the convex hull of

$$+ - - - - \cup - - + - -$$

is $\pm - \pm - -$ and the convex hull of

$$+ - - - - \cup - - + - - \cup - - - + -$$

is $\pm - \pm \pm -$. We illustrate this with the proof that

$$\mathrm{hull}(+ - - - - \cup - - + - - \cup - - - + -) = \pm - \pm \pm -.$$

The inclusion

$$\mathrm{hull}(+ - - - - \cup - - + - - \cup - - - + -) \subseteq \pm - \pm \pm -$$

is immediate because each of $+ - - - -$, $- - + - -$, $- - - + -$ is a subset of $\pm - \pm \pm -$ and $\pm - \pm \pm -$ is convex. Now observe that $\pm - \pm \pm -$ is an intersection of two hemispheres and in this case, on S^3, this intersection is a lens bounded by hemi-2-spheres. The lens sits in the circular boundary $\pm \partial \pm \pm \partial$ and each point in the lens resides on a unique great hemi-2-sphere with boundary this circle. Each of the three tetrahedra $+ - - - -$, $- - + - -$, $- - - + -$ shares one of its edges with a segment on the circle and in this way, the convex hull of the circle tetrahedra contains the circle and therefore all points of the lens.

It follows that the maxtame condition is satisfied: the closure of U is equal to $-U[4]^c$. $\square$

9. Proof that a maxtame subset of S^{n-1} cannot have $n + 2$ components

On S^2 we have seen no examples of sets with 2, 3, or 5 components. In the case of 2 and 3 the reason is a very easy consequence of Lemma 3.1.

Proposition 9.1. *Every maxtame subset of S^{n-1} with at most n components is a hemisphere.*

Proof. Let U be a maxtame set with $|\pi_0(U)| \leq n$. Let X be a finite set with exactly one point in each component. Then X is contained in an open hemisphere by n-tameness of U and therefore the Key Lemma 3.1 shows that there is a hemisphere H containing X whose boundary does not meet U. Connectivity of each component guarantees that U is contained in the interior of H. Since U is maxtame it follows that U equals the interior of H. $\qquad\square$

That there are no 5-component maxtame sets on S^2 also generalizes to higher dimensions.

Proposition 9.2. *Let U be an open n-tame subset of S^{n-1} with exactly $n + 2$ components. Then U is contained in a weakly maxtame set with at most $n + 1$ components.*

Proof. Choose representative points $x_1, \ldots, x_{n+2}$, taking one from each component of U. Let U_i be the component containing x_i. Set $X := \{x_1, \ldots, x_{n+2}\}$. By n-tameness, the simplex spanned by any n-point subset of X is contained in an open hemisphere. In particular, the convex hulls in $\mathbb{R}^n$ of the n-point subsets do not contain the origin. If the convex hull of X does not contain the origin then the Key Lemma 3.1 shows that X is contained in an open hemisphere which does not meet U and connectivity shows that U itself is contained in this open hemisphere. So we may as well assume that the convex hull of X does contain the origin. In that case, some $n + 1$ points of X have a convex hull which contains the origin and without loss of generality we assume that the simplex Δ spanned by the set $\{x_1, \ldots, x_{n+1}\}$ contains the origin within its interior. Imagine viewing Δ from the standpoint of x_{n+2}. We may assume that exactly i of the $(n-1)$-dimensional faces of Δ are visible from x_{n+2} and exactly $n + 1 - i$ of these faces are not visible. Coning each of the i visible faces of Δ off to x_{n+2} produces an n-simplex which does not contain the origin. Coning each of the $n - i$ nonvisible faces off to x_{n+2} produces n-simplices which do not overlap with each other and therefore at most one of these can contain the origin.

We may conclude from this first part of the argument that the simplex in $\mathbb{R}^n$ spanned by $n + 1$ points chosen from X contains the origin in at most 2 cases. Let Δ_i denote the simplex spanned by $X \smallsetminus \{x_i\}$, (so $\Delta = \Delta_{n+2}$). Without loss of generality we may assume that $\Delta_1, \ldots, \Delta_n$ do not contain the origin and that Δ_{n+1} and Δ_{n+2} both do contain the origin. Using the Key Lemma 3.1 we conclude that there are open hemispheres $H_1, \ldots, H_n$ whose boundaries do not meet U and such that $X \smallsetminus \{x_i\} \subset H_i$, and that there is a further open hemisphere H, again with boundary not meeting U, so that $X \smallsetminus \{x_{n+1}, x_{n+2}\} \subset H$. Connectivity and convexity now imply that we may replace $U_{n+1} \cup U_{n+2}$ by its convex hull and so enlarge U without violating its n-tameness. This procedure reduces the number of components

of U and the result follows from Theorem B1 together with the note following the proof of Lemma 2.2. $\qquad\square$

Thus we have the immediate corollary:

Corollary 9.3. *Every maxtame subset of S^{n-1} has 1, $n+1$, or $\geq n+3$ components.*

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