On the twistor space of a (co-)CR quaternionic manifold

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ABSTRACT. We characterise, in the setting of the Kodaira–Spencer deformation theory, the twistor spaces of (co-)CR quaternionic manifolds. As an application, we prove that, locally, the leaf space of any nowhere zero quaternionic vector field on a quaternionic manifold is endowed with a natural co-CR quaternionic structure.

Also, for any positive integers k and l, with kl even, we obtain the geometric objects whose twistorial counterparts are complex manifolds endowed with a conjugation without fixed points and which preserves an embedded Riemann sphere with normal bundle $l\mathcal{O}(k)$.

We apply these results to prove the existence of natural classes of co-CR quaternionic manifolds.

CONTENTS

Introduction

Twistor Theory is based on a nontrivial, and far from complete, dictionary between differential geometric and holomorphic objects. For example (see [5]), it is well-known that (up to a conjugation) any *anti-self-dual manifold* corresponds to a three-dimensional complex manifold endowed with a locally complete family of (embedded) Riemann spheres with normal bundle $2\mathcal{O}(1)$ (where $\mathcal{O}(1)$ is the dual of the tautological bundle over $\mathbb{C}P^1$). More generally, any *quaternionic manifold* (introduced in [15]) of dimension $4k$ corresponds to a complex manifold of dimension $2k+1$ endowed with a locally

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complete family of Riemann spheres with normal bundle $2k\mathcal{O}(1)$. Another such natural correspondence is given by the *three-dimensional Einstein–Weyl spaces* which, locally, correspond to complex surfaces endowed with locally complete families of Riemann spheres with normal bundle $\mathcal{O}(2)$ [5].

All these correspondences involve two steps:

(1) the construction of a *twistor space* for each of the given differential geometric structures, and

(2) the characterisation (among the complex manifolds endowed with a family of compact complex submanifolds) of the obtained twistor spaces.

In [12], it is shown that step (1) of all of the above mentioned correspondences are particular cases of the construction by which to any *co-CR quaternionic manifold* it is associated its twistor space.

In this paper, we provide the corresponding step (2) for a large class of co-CR quaternionic manifolds, thus showing that in order for a complex manifold Z to be the twistor space of a co-CR quaternionic manifold it is sufficient to be endowed with a locally-complete family $\mathcal{F}$ of Riemann spheres which (up to the restrictions imposed by a conjugation; see Theorem 3.1, below, for details) satisfies the following:

(a) For any $t \in \mathcal{F}$, its (holomorphic) normal bundle Nt is 'positive' and the exact sequence $0 \longrightarrow Tt \longrightarrow TZ|_t \longrightarrow Nt \longrightarrow 0$ splits.

(b) $\dim H^1(t, Tt \otimes N^*t)$ is independent of $t \in \mathcal{F}$.

Note that, by [12], condition (a) is, also, necessary for Z to be the twistor space of a co-CR quaternionic manifold, with $\mathcal{F}$ the corresponding family of twistor lines.

Consequently, up to a complexification, the 'Veronese webs' of [2] and the 'generalized hypercomplex structures' of [1] are particular classes of co-CR quaternionic structures. Furthermore, it follows that the 'bi-Hamiltonian structures' corresponding [2] to the former are obtained through a dimensional reduction of a hypercomplex structure.

As an application of Theorem 3.1, we prove (Corollary 3.3) that, locally, the leaf space of any nowhere zero quaternionic vector field on a quaternionic manifold is endowed with a natural co-CR quaternionic structure.

We, also, provide (Theorem 2.1) the corresponding step (2) for the dually flavoured *CR quaternionic manifolds*, introduced in [11].

Finally, a similar approach leads to (Corollary 4.3) a natural correspondence between the following classes, where k and l are positive integers:

(a) Complex manifolds endowed with a conjugation without fixed points and which preserves an embedded Riemann sphere with normal bundle $l\mathcal{O}(k)$.

(b) Quadruples (M, N, x, φ), with $x \in M \subseteq N$, N quaternionic, $M \subseteq N$ generic and of type (k, l) (see Definition 4.2), and $\varphi : N \to M$ a twistorial retraction of the inclusion $M \subseteq N$.

We apply these results to prove the existence of natural classes of co-CR quaternionic manifolds.

1. (Co-)CR quaternionic manifolds

In this section we recall, from [11] and [12] to which we refer for further details, the notions of CR and co-CR quaternionic manifolds.

Let $\mathbb{H}$ be the (unital) associative algebra of quaternions; note that, its automorphism group is $\mathrm{SO}(3,\mathbb{R})$ acting trivially on $\mathbb{R}$ and canonically on $\mathrm{Im}\mathbb{H}\,(=\mathbb{R}^3)$.

A *linear quaternionic structure* on a (real) vector space E is an equivalence class of morphisms of associative algebras from $\mathbb{H}$ to $\mathrm{End}(E)$, where two such morphisms σ_1 and σ_2 are equivalent if $\sigma_2 = \sigma_1 \circ a$, for some a in (the identity component of) $\mathrm{SO}(3,\mathbb{R})$.

Let E be a quaternionic vector space whose structure is given by the morphism $\sigma : \mathbb{H} \to \mathrm{End}(E)$. Then $Z = \sigma(S^2)$ depends only of the equivalence class of σ. Moreover, Z determines the linear quaternionic structure of E. Also, any $J \in Z$ is a linear complex structure on E which is called *admissible* (for the given linear quaternionic structure).

A *linear CR quaternionic structure* on a vector space U is a pair (E,ι), where E is a quaternionic vector space and $\iota : U \to E$ is an injective linear map such that, for any admissible linear complex structure J on E, we have $\mathrm{im}\,\iota + J(\mathrm{im}\,\iota) = E$.

By duality we obtain the notion of *linear co-CR quaternionic structure*.

Any quaternionic vector space E is isomorphic to $\mathbb{H}^k$, where $\dim E = 4k$, and the automorphism group of $\mathbb{H}^k$ is $\mathrm{Sp}(1) \cdot \mathrm{GL}(k,\mathbb{H})$.

The classification of the (co-)CR quaternionic vector spaces is much less trivial. It is based on a covariant functor from the category of CR quaternionic vector spaces to the category of holomorphic vector bundles over the Riemann spheres, which we next describe. Let (U, E, ι) be a CR quaternionic vector space and let Z be the space of admissible linear complex structures on E. For any $J \in Z$ denote $E^J = \ker(J+\mathrm{i})$. Then $E^{0,1} = \bigcup_{J \in Z}\{J\} \times E^J$ is a holomorphic vector subbundle of $Z \times E^{\mathbb{C}}$ (isomorphic to $2k\mathcal{O}(-1)$, where $\dim E = 4k$ and $\mathcal{O}(-1)$ is the tautological line bundle on $Z = \mathbb{C}P^1$). Furthermore, $\iota^{-1}\big(E^{0,1}\big)$ is a holomorphic vector subbundle of $Z \times U^{\mathbb{C}}$ which is called *the holomorphic vector bundle of* (U, E, ι) [11]. It follows that there exists a natural bijective correspondence between (isomorphism classes of) CR quaternionic vector spaces and holomorphic vector bundles, over the Riemann sphere, whose Birkhoff–Grothendieck decompositions contain only terms whose Chern numbers are at most -1.

A *quaternionic vector bundle* is a real vector bundle E with typical fibre $\mathbb{H}^k$ and structural group $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$. If E is a quaternionic vector bundle then the space Z of admissible linear complex structures on E is a (locally trivial) fibre bundle with typical fibre S^2 and structural group $\mathrm{SO}(3, \mathbb{R})$.

An *almost CR quaternionic structure* on a manifold M is a pair (E, ι), where E is a quaternionic vector bundle over M and $\iota : TM \to E$ is a vector bundles morphism such that (E_x, ι_x) defines a linear CR quaternionic structure on $T_x M$, for any $x \in M$.

By duality we obtain the notion of *almost co-CR quaternionic structure*.

Let (M, E, ι) be an almost CR quaternionic manifold. Suppose that E is endowed with a compatible connection ∇ and let $\pi : Z \to M$ be the bundle of admissible linear complex structures on E. Denote by $\mathcal{B}$ the complex distribution on Z whose fibre, at each $J \in Z$, is the horizontal lift, with respect to ∇, of $\iota^{-1}\big(\ker(J + \mathrm{i})\big)$. Then $\mathcal{C} = (\ker \mathrm{d}\pi)^{0,1} \oplus \mathcal{B}$ is an almost CR structure on Z. If $\mathcal{C}$ is integrable then (E, ι, ∇) is a *CR quaternionic structure* on M.

A *quaternionic manifold* is a CR quaternionic manifold (M, E, ι, ∇) for which ι is an isomorphism.

Let N be a quaternionic manifold. A submanifold $M \subseteq N$ is *generic* if $(TN|_M, \iota)$ is an almost CR quaternionic structure on M, where

$$\iota : TM \to TN|_M$$

is the inclusion; in particular, $\operatorname{codim} M \leq 2k - 1$, where $\dim N = 4k$. The terminology is justified by the fact that the set of real vector subspaces $U \subseteq \mathbb{H}^k$, of fixed codimension $l \leq 2k - 1$ and on which $\mathbb{H}^k$ induces a linear CR quaternionic structure, is open in the Grassmannian of real vector subspaces of codimension l of $\mathbb{H}^k$. Also, $M \subseteq N$ is generic if and only if, for any admissible local complex structure J on N, we have that M is a generic CR submanifold of (N, J).

Any hypersurface of a quaternionic manifold is generic, but this does not hold for higher codimensions as simple examples show (take, for example, $\mathbb{H}^k \times \mathbb{C}$ in $\mathbb{H}^{k+1}$).

Any generic submanifold of a quaternionic manifold inherits a natural CR quaternionic structure. Conversely, any real analytic CR quaternionic structure is obtained this way from a germ unique embedding into a quaternionic manifold.

Let (M, E, ρ) be an almost co-CR quaternionic manifold. Suppose that E is endowed with a compatible connection ∇ and let $\pi : Z \to M$ be the bundle of admissible linear complex structures on E. Denote by $\mathcal{B}$ the complex distribution on Z whose fibre, at each $J \in Z$, is the horizontal lift, with respect to ∇, of $\rho\big(\ker(J + \mathrm{i})\big)$. Then $\mathcal{C} = (\ker \mathrm{d}\pi)^{0,1} \oplus \mathcal{B}$ is an almost co-CR structure on Z; that is, $\mathcal{C} + \overline{\mathcal{C}} = T^{\mathbb{C}}Z$. If $\mathcal{C}$ is integrable (that is, its space of sections is closed under the usual bracket) then (E, ι, ρ) is a *co-CR*

quaternionic structure on M. Suppose, further, that $\mathcal{C} \cap \overline{\mathcal{C}} = (\ker \mathrm{d}\pi_Y)^{\mathbb{C}}$, where $\pi_Y : Z \to Y$ is a surjective submersion whose restriction to each fibre of π is injective, and with respect to which $\mathcal{C}$ is projectable (note that, the last condition is automatically satisfied if π_Y has connected fibres). Then Y endowed with $\mathrm{d}\pi_Y(\mathcal{C})$ is a complex manifold, called the *twistor space* of (M, E, ρ, ∇); in particular, π_Y restricted to each fibre of π is an injective holomorphic immersion.

2. On the twistor space of a CR quaternionic manifold

Recall [8] that a *smooth family of complex manifolds* is a surjective submersion $\pi : Z \to M$ whose domain is endowed with a CR structure $\mathcal{D}$ such that $\mathcal{D} \oplus \overline{\mathcal{D}} = (\ker \mathrm{d}\pi)^{\mathbb{C}}$. Any member of the family is a fibre t of Z endowed with the complex structure for which $T^{0,1}t = \mathcal{D}|_t$.

We shall denote by $\mathcal{S}$ the generating subsheaf of $T^{\mathbb{C}}Z/(\ker \mathrm{d}\pi)^{0,1}$ formed of the complex vector fields on Z which are projectable with respect to π and holomorphic when restricted to the fibres of Z. Note that, the exact sequence

$$0 \longrightarrow (\ker \mathrm{d}\pi)^{1,0} \longrightarrow \mathcal{S} \longrightarrow \pi^*(T^{\mathbb{C}}M) \longrightarrow 0$$

is the *fundamental sequence* [8, (4.1)].

Theorem 2.1. *Let* $\pi : Z \to M$ *be a smooth family of Riemann spheres. Suppose that* Z *is endowed with a CR structure* $\mathcal{C}$ *and an involutive diffeomorphism* τ *such that:*

(i) $\mathcal{C}$ *induces the given complex structure on each fibre of* Z.

(ii) *For each fibre* t *of* Z, *we have that* $(\mathcal{C}|_t)/T^{0,1}t$ *is a holomorphic subbundle of the restriction to* t *of* $T^{\mathbb{C}}Z/(\ker \mathrm{d}\pi)^{0,1}$ *such that the induced quotient of* $\left(T^{\mathbb{C}}Z/\mathcal{C}\right)|_t$ *through* $T^{1,0}t$ *is isomorphic to* $k\mathcal{O}(1)$, *for some positive integer* k.

(iii) τ *is anti-CR with respect to* $\mathcal{C}$, *preserves the fibres of* Z *and has no fixed points.*

Then there exists a CR quaternionic structure (E, ι, ∇) *on* M *whose twistor space is* $(Z, \mathcal{C})$*; moreover,* (E, ι) *is unique (up to isomorphisms) with these properties.*

Proof. We have that Z is a fibre bundle with typical fibre the Riemann sphere (apply [8, Theorem 6.3]). Furthermore, τ restricted to each fibre of Z is an involutive conjugation without fixed points and, thus, it is the antipodal map. Therefore Z is a sphere bundle with structural group $\mathrm{SO}(3)$.

Denote $\mathcal{B} = \mathcal{C}/(\ker \mathrm{d}\pi)^{0,1}$ and note that by using the fundamental sequence and (ii) we obtain an exact sequence

$$(2.1) \qquad 0 \longrightarrow \mathcal{B} \longrightarrow \pi^*(T^{\mathbb{C}}M) \longrightarrow \pi^*(T^{\mathbb{C}}M)/\mathcal{B} \longrightarrow 0$$

of complex vector bundles which are holomorphic when restricted to the fibres of Z. Furthermore, (ii), [14, Proposition 3.3], and [11, §3] imply that, at each $x \in M$, there exists a unique linear CR quaternionic structure

(E_x, ι_x) on $T_x M$ whose holomorphic vector bundle is the restriction of $\mathcal{B}$ to $\pi^{-1}(x)$.

To describe (E, ι) we use (2.1). Firstly, $E^{\mathbb{C}}$ is the direct image through π of $\pi^*(T^{\mathbb{C}}M)/\mathcal{B}$ (that is, for any open subset $U \subseteq M$, the space of sections of $E^{\mathbb{C}}|_U$ is the space of sections (which are holomorphic when restricted to the fibres of π) of the restriction to $\pi^{-1}(U)$ of $\pi^*(T^{\mathbb{C}}M)/\mathcal{B}$; the fact that $E^{\mathbb{C}}$ is a bundle is given by [9, Theorem 9]). Further, the long exact sequence of cohomology of (2.1) gives an injective complex linear map α between the spaces of sections of $\pi^*(T^{\mathbb{C}}M)$ and $\pi^*(T^{\mathbb{C}}M)/\mathcal{B}$, over any open set of the form $\pi^{-1}(U)$, where U is an open subset of M. Then $\iota|_U$ is the restriction of α to the space of sections of $\pi^*(T^{\mathbb{C}}M)|_U$ which intertwine the conjugation and the antipodal map.

Now, any connection on Z corresponds to a splitting of the fundamental sequence (cf. [8, Proposition 5.1]), which is invariant under the antipodal map. Obviously, such splittings exist but we want to obtain

$$\mathcal{S} = \mathscr{H} \oplus (\ker d\pi)^{1,0}$$

with $\mathcal{B} \subseteq \mathscr{H}$. To prove this, note that, instead of the fundamental sequence, we may use the exact sequence

$$(2.2) \qquad 0 \longrightarrow (\ker d\pi)^{1,0} \longrightarrow T^{\mathbb{C}}Z/(\ker d\pi)^{0,1} \longrightarrow \pi^*(T^{\mathbb{C}}M) \longrightarrow 0,$$

whilst, any splitting of (2.2) whose 'image' contains $\mathcal{B}$ corresponds to a splitting of

$$(2.3) \qquad 0 \longrightarrow (\ker d\pi)^{1,0} \longrightarrow T^{\mathbb{C}}Z/\mathcal{C} \longrightarrow \pi^*(T^{\mathbb{C}}M)/\mathcal{B} \longrightarrow 0.$$

But the restriction of (2.3) to each fibre $t \, (= \mathbb{C}P^1)$ of Z is

$$0 \longrightarrow \mathcal{O}(2) \longrightarrow \left(T^{\mathbb{C}}Z/\mathcal{C}\right)|_t \longrightarrow k\mathcal{O}(1) \longrightarrow 0.$$

Together with [9, Theorem 10], this quickly implies that there exists a connection on Z whose complexification contains $\mathcal{B}$, and the proof follows. $\quad\square$

Note that, by [11], the twistor space of any CR quaternionic manifold satisfies the conditions of Theorem 2.1.

Also, in Theorem 2.1, we have that $\mathcal{C}$ is a complex structure on Z if and only if $\dim M = 2k$. Then (i) and (ii) are equivalent to the fact that any fibre of Z is a complex submanifold and its (holomorphic) normal bundle is isomorphic to $k\mathcal{O}(1)$ (note that, k is necessarily even). Thus, Theorem 2.1 gives, in particular, the classical characterisation of the twistor space of any quaternionic manifold.

3. On the twistor space of a co-CR quaternionic manifold

Recall [8] (see [7]) that a family $\mathcal{F}$ of compact complex submanifolds of a complex manifold Z is *complex analytic* if there exist complex manifolds P and Q and holomorphic maps $\pi_Z : Q \to Z$, $\pi : Q \to P$, with π a proper

surjective submersion, such that $\mathcal{F} = \left\{ \pi_Z(\pi^{-1}(x)) \right\}_{x \in P}$ and π_Z restricted to each fibre of π is an injective immersion.

Theorem 3.1. *Let Z be a complex manifold endowed with a conjugation τ, without fixed points, and a locally complete family $\mathcal{F}$ of Riemann spheres.*

Then, locally, Z is the twistor space of a co-CR quaternionic manifold, for which $\mathcal{F}$ is the family of twistor lines, if the following conditions are satisfied:

- (i) *The Birkhoff–Grothendieck decomposition of the normal bundle Nt, of any $t \in \mathcal{F}$, contains only terms of Chern number at least 1.*
- (ii) *The exact sequence $0 \longrightarrow Tt \longrightarrow TZ|_t \longrightarrow Nt \longrightarrow 0$ splits, for any $t \in \mathcal{F}$, and $\dim H^1(t, Tt \otimes N^*t)$ is independent of $t \in \mathcal{F}$, where Tt and TZ are the holomorphic tangent bundles of t and Z, respectively.*
- (iii) *$\tau(\mathcal{F}) = \mathcal{F}$ and there exists $t_0 \in \mathcal{F}$ such that $\tau(t_0) = t_0$.*

Proof. Unless otherwise stated, all the objects and maps are assumed complex analytic; in particular, if P is a (complex) manifold then TP denotes its holomorphic tangent bundle.

By (i), [7], and [8, Theorem 6.3], (which is known to hold in the complex analytic setting, as well) the family $\mathcal{F}$ is given by a map

$$\pi_Z : Q \to Z,$$

where $\pi : Q \to P$ is a locally trivial fibre bundle with typical fibre $\mathbb{C}P^1$. Also, for any $x \in P$, there exists a natural isomorphism $T_x P = H^0(t_x, Nt_x)$, where $t_x = \pi_Z(\pi^{-1}(x))$. Furthermore, from (i) it follows quickly that π_Z is a submersion which, by passing to an open subset of Z, can be assumed surjective.

Let $\mathcal{B} = \ker d\pi_Z$. From the isomorphism between $\pi^*(TP)$ and the quotient of TQ through $\ker d\pi$, we obtain

$$(3.1) \qquad 0 \longrightarrow \ker d\pi \longrightarrow TQ/\mathcal{B} \longrightarrow \pi^*(TP)/\mathcal{B} \longrightarrow 0$$

which, for any $x \in P$, restricts to

$$(3.2) \qquad 0 \longrightarrow Tt_x \longrightarrow TZ|_{t_x} \longrightarrow Nt_x \longrightarrow 0,$$

and, consequently, gives

$$(3.3) \qquad 0 \longrightarrow \mathcal{B}|_{t_x} \longrightarrow t_x \times T_x P \to Nt_x \longrightarrow 0,$$

where we have identified $t_x = \pi^{-1}(x)$.

Condition (iii) gives that by passing, if necessary, to an open neighborhood of each point of P corresponding to a τ-invariant member of $\mathscr{F}$, we may suppose P be the complexification of a real-analytic submanifold M.

Now, similarly to the proof of Theorem 2.1, we obtain that M is endowed with an almost co-CR quaternionic structure for which $Q|_M$ is the bundle of admissible linear complex structures. Furthermore, any connection on $Q|_M$ whose complexification contains $\mathcal{B}|_{Q|_M}$ corresponds to a τ-invariant splitting

of the restriction to $Q|_M$ of (3.1) which by (ii) and [9, Theorem 10] exists, and the proof follows. □

Remark 3.2. Theorem 3.1 can be slightly extended, as follows.

(1) If in (i) of Theorem 3.1 we further assume that the Chern numbers are contained by $\{1, 2, 3\}$ then condition (ii) becomes superfluous.

(2) The conclusion of Theorem 3.1 still holds if we assume that Z contains a Riemann sphere t preserved by τ, which satisfies (ii), and (i), where for the latter the Chern numbers are contained by $\{k, k+1\}$, for some positive integer k (this is an immediate consequence of [7], [8, Theorems 6.3, 7.4], and Theorem 3.1).

Remark 3.2 shows, in particular, that Theorem 3.1 is a natural generalization of classical results (see [5], [6] and the references therein) on quaternionic and anti-self-dual manifolds, and three-dimensional Einstein–Weyl spaces.

It is well known that, locally, the leaf space of a nowhere zero conformal vector field, on an anti-self-dual manifold, is a (three-dimensional) Einstein–Weyl space (see [13] and the references therein). In higher dimensions, we have the following result.

Corollary 3.3. *Locally, the leaf space of any nowhere zero quaternionic vector field on a quaternionic manifold is a co-CR quaternionic manifold.*

Proof. Let V be a nowhere zero quaternionic vector field on a quaternionic manifold M whose orbits are the fibres of a submersion $\varphi : M \to N$. Then V lifts to a holomorphic vector field $\widetilde{V}$ on the twistor space Z_M of M (use, for example, [6]). Assume, for simplicity, that the one-dimensional holomorphic foliation generated by $\widetilde{V}$ is simple; that is, it is given by the fibers of a holomorphic submersion Φ from Z_M onto some complex manifold Z_N.

Then Φ maps the twistor lines on Z_M onto a complex analytic family of Riemann spheres on Z_N each of which has normal bundle $2k\mathcal{O}(1) \oplus \mathcal{O}(2)$, where $\dim M = 4(k+1)$. Furthermore, this family is parametrised by a complexification of N; consequently, it is locally complete (apply [7]).

By Remark 3.2(1), we have that Z_N is the twistor space of a co-CR quaternionic structure on N. □

Note that, in the proof of Corollary 3.3, the induced twistorial structure (of the co-CR quaternionic structure) on N is unique with the property that φ be a twistorial map (see [13] and [10] for the definition of twistorial structures and maps).

On the other hand, unlike the complex setting, the quaternionic distribution generated by a quaternionic vector field is not necessarily integrable. Indeed, if not, then any homogeneous quaternionic manifold would be locally isomorphic with the quaternionic projective space — a contradiction.

4. Another natural twistorial correspondence

Recall (see [3, p. 172] and note that the definitions extend easily to the complex analytic category) that two complex curves c_1 and c_2 on a complex manifold Z *have a contact of order k* at a point $x \in c_1 \cap c_2$ if for any holomorphic function f defined on some open neighborhood of x in Z we have that $f|_{c_1}$ vanishes up to the k-th order at x if and only if $f|_{c_2}$ vanishes up to the k-th order at x. Then a *k-jet of (complex) curves* on Z at x is an equivalence class of curves on Z which have a contact of order k at x. Further, the set $Y_k(Z)$ of all k-jets of curves at all the points of Z is, in a natural way, a complex manifold of dimension $kl + l + 1$, where $\dim_{\mathbb{C}} Z = l + 1$. Moreover, the canonical map from $Y_k(Z)$ onto Z is the projection of a locally trivial fibre space; in particular, $Y_0(Z) = Z$, whilst $Y_1(Z)$ is the projectivisation of the holomorphic tangent bundle of Z.

Theorem 4.1. *Let Z be a complex manifold endowed with a conjugation without fixed points and which preserves an embedded Riemann sphere $t \subseteq Z$ whose normal bundle is $l\,\mathcal{O}(k+1)$, with k and l positive integers.*

Then there exists a quaternionic manifold, of dimension $2l(k+1)$, whose twistor space is an open subset of $Y_k(Z)$, endowed with the conjugation induced by τ, and for which the canonical lift of t to $Y_k(Z)$ is a twistor line.

Proof. The conjugation on Z induces a conjugation on the normal bundle of the embedded Riemann sphere, covering the antipodal map. Hence, if k is even then, also, l must be even (see [14]).

Then, as in the proof of Theorem 3.1, we obtain on Z a locally complete family $\mathcal{F}$ of Riemann spheres given by holomorphic maps $\pi_Z : Q \to Z$ and $\pi : Q \to P$, where the former can be assumed a surjective submersion, whilst the latter is a locally trivial fibre bundle with typical fibre $\mathbb{C}P^1$. Also, by passing to an open subset, if necessary, P is the complexification of a real analytic manifold M.

Now, if we suitably blow up $k+1$ times Z at any point $z \in u$, of any $u \in \mathcal{F}$, we obtain a complex manifold $Z_{u,z}$ endowed with an embedded Riemann sphere with trivial normal bundle (of rank l). Hence, this is contained in an l-dimensional locally complete family $\mathcal{F}_{u,z}$ of Riemann spheres, embedded in $Z_{u,z}$, all of which are obtained as proper transforms of members of $\mathcal{F}$. Furthermore, $v \in \mathcal{F}$ transforms to a member of $\mathcal{F}_{u,z}$ if and only if u and v have a contact of order k at z.

Therefore π_Z factors into a holomorphic submersion with l-dimensional fibres from Q to $Y_k(Z)$ followed by the projection from $Y_k(Z)$ onto Z. Moreover, the normal bundle of the canonical lift to $Y_k(Z)$ of any member of $\mathcal{F}$ is isomorphic to $l(k+1)\mathcal{O}(1)$. This shows that an open subset of $Y_k(Z)$ is the twistor space of a quaternionic manifold N. Moreover, the projection from $Y_k(Z)$ onto Z corresponds to a twistorial retraction of the inclusion $M \subseteq N$ whose differential at each point is given by the cohomology sequence of the

exact sequence $0 \longrightarrow kl\mathcal{O} \longrightarrow l(k+1)\mathcal{O}(1) \longrightarrow l\mathcal{O}(k+1) \longrightarrow 0$; in particular, M is a generic submanifold of N (note that, the induced almost CR quaternionic structure on M can be also obtained by using the proof of Theorem 2.1, with Z replaced by $Q|_M$ endowed with the CR structure induced from Q). $\qquad\square$

We say (compare [3]) that two embeddings $\varphi : P \to Q$ and $\psi : P \to R$ define the same *embedding germ* if there exist open neighbourhoods U and V of $\varphi(P)$ and $\psi(P)$, respectively, and a diffeomorphism $\xi : U \to V$ such that $\psi = \xi \circ \varphi$; certainly, if Q and R are endowed with some geometric structure then we require ξ to preserve it.

Definition 4.2. Let (M, E, ι) be an almost CR quaternionic manifold and let k and l be positive integers. We say that (M, E, ι) is of *type* (k, l) if, for any $x \in M$, the holomorphic vector bundle of $(T_x M, E_x, \iota_x)$ is $l\mathcal{O}(-k)$.

Note that, if N is quaternionic and $M \subseteq N$ is generic of type (k, l) then $\dim M = l(k+1)$ and $\dim N = 2kl$; in particular, if $\dim N = 2(\dim M - 1)$ then the type of M is determined by its dimension.

Corollary 4.3. *There exists a natural correspondence between the following classes, where k and l are positive integers:*

(i) *complex manifolds endowed with a conjugation without fixed points and which preserves an embedded Riemann sphere with normal bundle $l\mathcal{O}(k)$;*

(ii) *quadruples (M, N, x, φ), with $x \in M \subseteq N$, N quaternionic, $M \subseteq N$ generic and of type (k, l), and $\varphi : N \to M$ a twistorial retraction of the inclusion $M \subseteq N$.*

Moreover, the correspondence is bijective if we pass to embedding germs.

Proof. How to pass from objects as in (i) to objects as in (ii) it is shown in the proof of Theorem 4.1.

Conversely, given a quadruple (M, N, x, φ) as in (ii) let $Z(N)$ be the twistor space of N. Then φ corresponds to a holomorphic submersion Φ from $Z(N)$ onto some complex manifold Z such that the family of twistor lines on $Z(N)$ is mapped by Φ into a family of Riemann spheres embedded into Z. Consequently, each member of this family has normal bundle isomorphic to $l\mathcal{O}(k)$. Furthermore, as φ is a retraction of the inclusion $M \subseteq N$, at least locally, the parameter space of this family is a complexification of M and therefore Z is also endowed with a conjugation τ. Now, let $t \subseteq Z$ be the image through Φ of the twistor line corresponding to x. Then (Z, τ, t) satisfies condition (i). $\qquad\square$

If in Theorem 4.1, we have $k = l = 1$, and if in Corollary 4.3, we have $k = l + 1 = 2$, then we obtain results of [5].

Example 4.4. Let n be a nonnegative even number and let S_n be the projectivisation of $\mathcal{O} \oplus \mathcal{O}(n)$. Then the conjugations of $\mathcal{O}$ and $\mathcal{O}(n)$ induce a conjugation τ of S_n covering the antipodal map.

Any meromorphic section of $\mathcal{O}(n)$ corresponds, up to a constant nonzero factor, to a divisor on $\mathbb{C}P^1$ (with m poles and $n + m$ zeros, for some natural number m). Furthermore, if the divisor is invariant under the antipodal map then the corresponding meromorphic section will intertwine the antipodal map and the conjugation of $\mathcal{O}(n)$. Let s be such a meromorphic section of $\mathcal{O}(n)$ with m poles (necessarily, m is even). Then the closure t of the image of the section of S_n induced by $(1, s)$ is preserved by τ. Also, the Chern number of its normal bundle is equal to $n + 2m$ (see [4, p. 519]). Thus, (S_n, τ, t) satisfies (i) of Corollary 4.3, with $k = n + 2m$ and $l = 1$.

If $n = 0$, in Example 4.4, then m can, also, be any odd natural number, by suitably changing the conjugation:

Example 4.5. Let σ be the conjugation on $S_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ given by the antipodal map, acting on each factor, and let Y_m be the space of $(2m - 1)$-jets of maps from $\mathbb{C}P^1$ to itself, where m is an odd natural number. On denoting by α and β the source and target projections, respectively, from Y_m onto $\mathbb{C}P^1$ then $(\alpha, \beta) : Y_m \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is the projection of a locally trivial fibre space with typical fibre the vector space of polynomials, in one (complex) variable, of degree at most $2m - 1$ and with zero constant term; in particular, $\dim_\mathbb{C} Y_m = 2m + 1$ (cf. [3]).

On associating to each jet $[\varphi] \in Y_m$, at $(x, y) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, the $(2m-1)$-jet of curves on $\mathbb{C}P^1 \times \mathbb{C}P^1$, at (x, y), given by the graph of φ, we see that Y_m is an open subset of the space of $(2m - 1)$-jets of curves on $\mathbb{C}P^1 \times \mathbb{C}P^1$. Therefore Y_m is the twistor space of a quaternionic manifold N_m of dimension $4m$. Note that, the twistor lines on Y_m are images of sections s of α such that $\beta \circ s : \mathbb{C}P^1 \to \mathbb{C}P^1$ has degree m. Furthermore, the corresponding generic submanifold $M_m \subseteq N_m$, of (ii) of Corollary 4.3, is just the space of holomorphic maps of degree m, from $\mathbb{C}P^1$ to itself, which commute with the antipodal map.

Note that, for all the generic submanifolds of (ii) of Corollary 4.3, given by Examples 4.4 and 4.5, condition (ii) of Theorem 3.1 is automatically satisfied such that the corresponding rational ruled surfaces are the twistor spaces of co-CR quaternionic manifolds; in fact, hyper co-CR manifolds (see [12] for the definition of the latter). Therefore suitable products of these manifolds provide examples covering all possible k and l in Corollary 4.3.

Example 4.6. Let $n \in \mathbb{N} \setminus \{0\}$ and let $\mathbb{R}^{n+3} \subseteq \mathbb{R}^{n+4}$ be embedded as a vector subspace; denote by $\ell \subseteq \mathbb{R}^{n+4}$ the orthogonal complement of $\mathbb{R}^{n+3}$, oriented so that the isomorphism $\mathbb{R}^{n+4} = \ell \oplus \mathbb{R}^{n+3}$ be orientation preserving.

Let M be the Grassmannian of three-dimensional oriented subspaces of $\mathbb{R}^{n+3}$, and let N be the Grassmannian of four-dimensional oriented subspaces of $\mathbb{R}^{n+4}$ which are not contained by $\mathbb{R}^{n+3}$. It is well known that

N is a quaternionic manifold (it is an open subset of a Wolf space). Also, M is both a CR quaternionic and a co-CR quaternionic manifold [12]. We embedd $M \subseteq N$ by $p \mapsto \ell \oplus p$, and define its retraction $\varphi : N \to M$, $q \mapsto q \cap \mathbb{R}^{n+3}$, where $q \cap \mathbb{R}^{n+3}$ is oriented so that the orthogonal decomposition $q = \ell_q \oplus \left(q \cap \mathbb{R}^{n+3}\right)$ be orientation preserving, where $\ell_q = q \cap \left(q \cap \mathbb{R}^{n+3}\right)^{\perp}$, oriented so that its positive unit vector is characterised by the fact that its scalar product with the positive unit vector of ℓ be positive. To show that φ is twistorial, let $\mathbb{C}^{n+4}$ be the complexification of $\mathbb{R}^{n+4}$. Then $Z(M) \subseteq \mathbb{C}P^{n+2}$ is the hyperquadric of isotropic directions in $\mathbb{C}^{n+3}$, whilst $Z(N)$ is the space of two-dimensional isotropic (complex) vector subspaces of $\mathbb{C}^{n+4}$ which are not contained by $\mathbb{C}^{n+3}$. Then φ corresponds to the holomorphic map $\Phi : Z(N) \to Z(M)$, $q \mapsto q \cap \mathbb{C}^{n+3}$.

Accordingly, the normal bundle of a twistor sphere in $Z(M)$ is $n\mathcal{O}(2)$.

Finally, note that M is a CR quaternionic submanifold, but not a co-CR quaternionic submanifold of N.

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