Singular *p*-adic transformations for Bernoulli product measures

Joanna Furno

ABSTRACT. Ergodic properties of *p*-adic transformations have been studied with respect to Haar measure. This paper extends the study of these properties to measures beyond Haar measure. Under these measures, coefficients do not appear in equal proportions. Adding a rational number that is not an integer then takes likely strings of coefficients to one of two unlikely strings of coefficients. It follows from this inequality that translation by a rational number other than an integer is singular with respect to these measures. Conjugation gives similar results for multiplication by rational numbers.

CONTENTS

1. Introduction and background

The *p*-adic numbers were originally developed by Kurt Hensel for use in number theory, with p denoting a fixed prime. Recently, the ergodic properties of many transformations on the *p*-adic numbers have been studied with respect to Haar measure—a natural measure that is translation-invariant. For example, [1–3, 12, 13] have explored the ergodicity of translation and multiplication maps with respect to Haar measure. Our goal is to examine the ergodic properties of transformations on the *p*-adic integers $\mathbb{Z}_p$ with respect to measures beyond Haar measure.

Let p be a prime number. We define the *p*-adic integers as a set of formal power series:

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \mathbb{Z} \text{ and } 0 \le a_i < p \text{ for all } i \ge 0 \right\}.$$

Addition and multiplication on $\mathbb{Z}_p$ are defined coordinatewise with carries. For nonzero $a \in \mathbb{Z}_p$, we define $\mathrm{ord}(a) = \min\{i : a_i \neq 0\}$ and use this function to define the p-adic absolute value on $\mathbb{Z}_p$ by

$$|a|_p = \begin{cases} 0 & \text{if } a = 0, \\ p^{-\mathrm{ord}(a)} & \text{if } a \neq 0. \end{cases}$$

The p-adic absolute value induces a metric that induces a topology with a basis that consists of the empty set and balls of the form

$$B_{p^{-n}}(a) = \left\{ x \in \mathbb{Z}_p : |x - a|_p \leq p^{-n} \right\},$$

where n is a nonnegative integer and $a \in \mathbb{Z}_p$. Such a ball is determined by the first n coordinates of a.

Let $(q_0, q_1, \ldots, q_{p-1})$ be a probability vector. For a given probability vector, let $q(i) = q_i$ for $0 \leq i < p - 1$. Then we define a probability measure μ on the Borel σ-algebra $\mathcal{B}$ by taking the measure of a ball to be

$$\mu\left(B_{p^{-n}}\left(\sum_{i=0}^{\infty} a_i p^i \right) \right) = \prod_{i=0}^{n-1} q(a_i)$$

for $n \geq 0$ and $\sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p$. We call μ an *independent and identically distributed (i.i.d.) Bernoulli measure*. On product spaces, i.i.d. Bernoulli measures are well-known to be invariant under Bernoulli shifts.

Although we have defined the p-adic integers as formal power series, we can identify certain series as natural integers or rational numbers. A p-adic integer with a finite expansion—one that ends in repeating zeros—is identified with an integer by summing the nonzero terms. The additive inverse—a negative integer—ends in repeating $(p-1)$'s. In general, the elements of $\mathbb{Q}$ in $\mathbb{Z}_p$ have coordinates that eventually repeat [18, 20].

A measurable transformation $T : X \to X$ is *nonsingular* with respect to a measure m on a σ-algebra $\mathcal{A}$ if, for all $A \in \mathcal{A}$, $m(A) = 0$ if and only if $m(T^{-1}A) = 0$. A measurable transformation $T : X \to X$ is *singular* with respect to a measure m on a σ-algebra $\mathcal{A}$ if this property fails to hold. In other words, T is singular with respect to m if there exists $A \in \mathcal{A}$ such that one of $m(A)$ or $m(T^{-1}A)$ is strictly positive and the other is zero.

Nonsingular systems are of great interest. They have been studied by many, for example, Alexandre Danilenko [4,5], Anthony Dooley [6–8] (along with coauthors), and Stanley Eigen and Arshag Hajian [9, 10]. This paper focuses on when translation maps

$$T_a : \mathbb{Z}_p \to \mathbb{Z}_p$$
$$x \mapsto x + a$$

and multiplication maps

$$M_a : \mathbb{Z}_p \to \mathbb{Z}_p$$
$$x \mapsto ax$$

are singular or nonsingular with respect to i.i.d. Bernoulli measures. In this situation, a surprising thing occurs:

Theorem 1.1. *Let $a \in \mathbb{Z}_p$ be a rational number that is not an integer. Let μ be an i.i.d. Bernoulli measure other than Haar measure. Then $T_a : \mathbb{Z}_p \to \mathbb{Z}_p$ is singular with respect to μ.*

Theorem 1.1 and composition of functions yield a similar result for multiplication maps:

Theorem 1.2. *Let μ be an i.i.d. Bernoulli measure on $\mathbb{Z}_p$ other than Haar measure, defined by a probability vector $(q_0, q_1, \ldots q_{p-1})$. If $a \in \mathbb{Z}_p^\times \setminus \{1, -1\}$ is a rational number, then the multiplication map $M_a : \mathbb{Z}_p \to \mathbb{Z}_p$ is singular with respect to μ. The multiplication map $M_{-1} : \mathbb{Z}_p \to \mathbb{Z}_p$ is nonsingular with respect to μ if and only if the probability vector is a palindrome.*

Remark 1.3. Sometimes singular measures appear naturally as limits of nonsingular measures, as occurs in [11, 16, 17]. These papers study a parametrized family of maps on the interval, where each map has a unique absolutely continuous invariant measure (a.c.i.m.). Each family contains a convergent sequence of parameters such that the associated a.c.i.m.'s converge to a singular measure, rather than the unique a.c.i.m. for the map with the limiting parameter. An analogous limit measure phenomenon occurs for translations with respect to i.i.d. Bernoulli measures.

We consider the parametrized family of translation maps $T_a : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by $T_a(x) = x + a$. The odometer or adding machine is well known to be nonsingular with respect to i.i.d. Bernoulli measures and occurs in this setting as T_1, translation by 1. From the fact that T_1 is nonsingular with respect to i.i.d. Bernoulli measures, the fact that translation by any integer is nonsingular with respect to i.i.d. Bernoulli measures on $\mathbb{Z}_p$ follows almost immediately from the definition of nonsingularity.

For $a = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p$, consider the partial sums $s_n = \sum_{i=0}^{n-1} a_i p^i$, a sequence in $\mathbb{Z}$. Then $s_n \to a$ as $n \to \infty$, from which it follows that $T_{s_n}(x)$ converges uniformly in x to $T_a(x)$. If a is a rational number but not an integer and if μ is an i.i.d. Bernoulli measure other than Haar measure, then Theorem 1.1 states that T_a is singular with respect to μ. Thus, we have an example of a convergent sequence of parameters s_n such that T_{s_n} is nonsingular with respect to μ for each n, but T_a is singular with respect μ for the limiting parameter a.

Let $a = \sum_{i=0}^{\infty} a_i p^i$ be a rational number. Since its coordinates eventually repeat, there exist integers l and r such that $a_{i+r} = a_i$ for all $i \geq l$. We note that l and r are not unique, although there are unique minimal choices for each. For a fixed rational number $a \in \mathbb{Z}_p$ and a fixed choice of l and r, we call $\sum_{i=0}^{l-1} a_i p^i$ the leading part of a, $c = \sum_{i=0}^{r-1} a_{l+i} p^i$ the repeating segment of a, and $R = \sum_{i=0}^{\infty} c p^{ri}$ the repeating part of a. The proof of Theorem 1.1 requires the following proposition:

Proposition 1.4. *Let $a \in \mathbb{Z}_p$ be a nonintegral rational number, and let μ be an i.i.d. Bernoulli measure other than Haar measure. Let r be the length of a repeating segment, and let R be the repeating part of a. Then there exists $N \in \mathbb{N}$ and $b \in \mathbb{Z}_p$ such that*

$$(1) \qquad \mu(B_{p^{-rN}}(b)) > \mu(T_R(B_{p^{-rN}}(b))) + \mu(T_{1+R}(B_{p^{-rN}}(b))).$$

Section 2 contains the proofs of Theorem 1.1 and Theorem 1.2. Section 3 contains proof of Proposition 1.4, as well as examples of the proof and concluding remarks.

The results in this paper are a part of the author's Ph.D. dissertation, completed under the supervision of Jane Hawkins at the University of North Carolina at Chapel Hill [14].

2. Proofs for theorems on singularity

The proof of Theorem 1.1 is done when we apply the Birkhoff Ergodic Theorem with an iterate of the shift and characteristic functions. The Birkhoff Ergodic Theorem is a classical result, the proof of which is in books such as [19, 21]. For a fixed prime p, the shift σ on $\mathbb{Z}_p$ acts by

$$\sigma \left(\sum_{i=0}^{\infty} x_i p^i \right) = \sum_{i=0}^{\infty} x_{i+1} p^i.$$

The shift σ is measure-preserving and totally ergodic with respect to i.i.d. Bernoulli measures. Thus, for all $n \in \mathbb{N}$ and for all i.i.d. Bernoulli measures μ, the iterate σ^n is measure-preserving and ergodic with respect to μ. The balls from Proposition 1.4 are used to define characteristic functions. We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Assuming that $a \in \mathbb{Z}_p$ is a nonintegral rational number and that μ is an i.i.d. Bernoulli measure other than Haar measure, our goal is to show that T_a is singular with respect to μ. Since T_a is invertible, we do this by finding a set X such that $\mu(X) > 0$, but $\mu(T_aX) = 0$. We fix a choice of l and r, so that the leading part of a is $\sum_{i=0}^{l-1} a_i p^i$ and the repeating part of a is $R = \sum_{i=0}^{\infty} (\sum_{j=0}^{r-1} a_{l+j} p^j) p^{ri}$.

Let $B = B_{p^{-rN}}(b)$ be the ball found in Proposition 1.4. Then $T_R(B)$ and $T_{1+R}(B)$ are disjoint balls of radius p^{-rN} centered at $b + R$ and $b + 1 + R$, respectively. Since B and $T_RB \cup T_{1+R}B$ are measurable sets and since σ is measure-preserving, the functions $\mathbb{1}_B \circ \sigma^l$ and $\mathbb{1}_{T_RB \cup T_{1+R}B} \circ \sigma^l$ are in $L^1(\mu)$. Since the shift σ is totally ergodic and measure-preserving with respect to the i.i.d. Bernoulli measure μ, the iterate σ^{rN} is ergodic and measure-preserving

with respect to μ. By the Birkhoff Ergodic Theorem, the sets

$$X = \left\{ x \in \mathbb{Z}_p : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_B(\sigma^{l+rNi}x) = \mu(B) \right\} \text{ and}$$

$$Y = \left\{ x \in \mathbb{Z}_p : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{T_R B \cup T_{1+R} B}(\sigma^{l+rNi}x) = \mu(T_R B \cup T_{1+R} B) \right\}$$

have full measure.

For $x \in X$, if $\sigma^{l+rNi}x \in B$, then there are two possibilities for $\sigma^{l+rNi}T_a x$. If adding a to x does not result in a carry after the $l + rNi - 1$st coordinate, then $\sigma^{l+rNi}T_a x \in T_R B$. If adding a to x does result in a carry after the $l + rNi - 1$st coordinate, then $\sigma^{l+rNi}T_a x \in T_{1+R} B$. In either case,

$$\sigma^{l+rNi}T_a x \in T_R B \cup T_{1+R} B.$$

This inclusion implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{T_R B \cup T_{1+R} B}(\sigma^{l+rNi}T_a x) \geq \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_B(\sigma^{l+rNi}x)$$

$$= \mu(B) > \mu(T_R B \cup T_{1+R} B).$$

Thus, $T_a(x)$ is not in Y. Since $T_a(X) \subset \mathbb{Z}_p \setminus Y$ and $\mu(Y) = 1$, it follows that $\mu(T_a(X)) = 0$. Since $\mu(X) = 1 > 0$ but $\mu(T_a X) = 0$, the translation T_a is singular with respect to μ. $\qquad \square$

Since the proof of Theorem 1.1 depends heavily on the repetitive structure of rational numbers, the proof does not generalize to $\mathbb{Z}_p \setminus \mathbb{Q}$. It is still an open question as to whether or not translation by $a \in \mathbb{Z}_p \setminus \mathbb{Q}$ is nonsingular with respect to i.i.d. Bernoulli measures other than Haar measure.

We define a transformation $P : \mathbb{Z}_p \to \mathbb{Z}_p$ by $(P(x))_i = p - 1 - x_i$. A probability vector $(q_0, q_1, \ldots q_{p-1})$ is a palindrome if $q(k) = q(p - 1 - k)$ for all $0 \leq k \leq p - 1$.

Proposition 2.1. *Let μ be an i.i.d. Bernoulli measure on $\mathbb{Z}_p$ defined by a probability vector $(q_0, q_1, \ldots q_{p-1})$. If the probability vector is a palindrome, then the transformation*

$$P : \mathbb{Z}_p \to \mathbb{Z}_p$$

$$\sum_{i=0}^{\infty} x_i p^i \mapsto \sum_{i=0}^{\infty} (p - 1 - x_i) p^i$$

preserves μ. If the probability vector is not a palindrome, then P is singular with respect to μ.

Proof. Since

$$P^2 \left(\sum_{i=0}^{\infty} x_i p^i \right) = P \left(\sum_{i=0}^{\infty} (p - 1 - x_i) p^i \right) = \sum_{i=0}^{\infty} x_i p^i,$$

we have $P^{-1} = P$. If the probability vector is a palindrome, then

$$q(k) = q(p - 1 - k)$$

for all $0 \leq k \leq p - 1$. On balls in $\mathbb{Z}_p$, we have

$$\mu\left(PB_{p^{-n}}\left(\sum_{i=0}^{\infty} a_i p^i\right)\right) = \prod_{i=0}^{n-1} q(p - 1 - a_i)$$

$$= \prod_{i=0}^{n-1} q(a_i) = \mu\left(B_{p^{-n}}\left(\sum_{i=0}^{\infty} a_i p^i\right)\right).$$

Since the collection of balls in $\mathbb{Z}_p$ form a semi-algebra that generates the Borel sets, the transformation P preserves μ.

If the probability vector is not a palindrome, then there exists an index k such that $q(k) \neq q(p - 1 - k)$. Without loss of generality, we suppose that $q(k) > q(p - 1 - k)$. Since $\mathbb{1}_{B_{p^{-1}}(k)}, \mathbb{1}_{B_{p^{-1}}(p-1-k)} \in L^1(\mu)$ and σ is ergodic and measure-preserving, the sets

$$X = \left\{x \in \mathbb{Z}_p : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{B_{p^{-1}}(k)}(\sigma^i x) = q(k)\right\},$$

$$Y = \left\{x \in \mathbb{Z}_p : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{B_{p^{-1}}(p-1-k)}(\sigma^i x) = q(p - 1 - k)\right\},$$

have full measure under μ, by the Birkhoff Ergodic Theorem. If $x \in X$, then $\sigma^i x \in B_{p^{-1}}(k)$ implies that $\sigma^i Px \in B_{p^{-1}}(p - 1 - k)$. Thus,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{B_{p^{-1}}(p-1-k)}(\sigma^i Px) \geq q(k) > q(p - 1 - k).$$

It follows that $P(X) \subset \mathbb{Z}_p \backslash Y$, so $\mu(P(X)) = 0$. Since $\mu(X) = 1$ but $\mu(P(X)) = 0$, the transformation P is singular with respect to μ. $\qquad\square$

Applying Theorem 1.1 and Proposition 2.1, we can use compositions involving multiplication maps, translations, and P to determine when a multiplication map is singular or nonsingular with respect to an i.i.d. Bernoulli measure. If μ is Haar measure and $a \in \mathbb{Z}_p^\times$, then M_a preserves Haar measure, as shown in [2, 3, 12, 13]. If M_a preserves Haar measure, then M_a is certainly nonsingular with respect to Haar measure. Also, if $a = 1$, then M_a is the identity map, which is nonsingular with respect to any measure. Theorem 1.2 addresses the singularity of other i.i.d. Bernoulli measures.

Proof of Theorem 1.2. Suppose that μ is not Haar measure and that $a \in \mathbb{Z}_p^\times \bigcap (\mathbb{Q} \backslash \{1, -1\})$. Since M_a is invertible, M_a is nonsingular with respect to μ if and only if $M_a^{-1} = M_{a^{-1}}$ is nonsingular with respect to μ. If a is an integer other than 1 or -1, then a^{-1} is not an integer. Thus, without loss of generality, we can assume that a is not an integer. Note that

$T_a = M_a \circ T_1 \circ M_a^{-1}$. By Theorem 1.1, T_a is singular with respect to μ, because μ is not Haar measure and a is a nonintegral rational number. On the other hand, the translation T_1 is nonsingular with respect to μ. Thus, M_a is singular with respect to μ.

Next, we examine the case of M_{-1}. If $x = \sum_{i=0}^{\infty} x_i p^i$, then

$$(Px + x)_i = p - 1 - x_i + x_i = p - 1$$

for all integers $i \geq 0$. Thus, we have $Px + x = -1$ for all $x \in \mathbb{Z}_p$, so $P(x) = -x - 1 = M_{-1} \circ T_1(x)$ for all $x \in \mathbb{Z}_p$. Since T_1 is nonsingular with respect to μ, the multiplication M_{-1} is nonsingular with respect to μ if and only if P is nonsingular with respect to μ. By Proposition 2.1, P is nonsingular with respect to μ if and only if the probability vector is a palindrome. $\qquad\square$

Although translation by a nonintegral rational number is singular with respect to every i.i.d. Bernoulli measure other than Haar measure, the translation T_a is nonsingular with respect to a related averaged measure. For a rational number $a \in \mathbb{Z}_p$, there exist integers r and s such that $a = r/s$. If r and s are relatively prime and $s > 0$, then we say that $a = r/s$ is in *reduced form*. Let μ be an i.i.d. Bernoulli measure on the Borel sigma-algebra $\mathcal{A}$. We define an averaged measure μ_a by $\mu_a(A) = (1/s) \sum_{i=0}^{s-1} \mu(T_a^{-i} A)$ for all $A \in \mathcal{A}$. Since r is an integer, T_r is nonsingular with respect to μ. Thus, for all $A \in \mathcal{A}$,

$$\mu_a(T_a^{-1}(A)) = \frac{1}{s} \left(\mu(T_r^{-1}(A)) + \sum_{i=1}^{s-1} \mu(T_a^{-i}(A)) \right),$$

so T_a is nonsingular with respect to μ_a.

If a is an integer, then it is shown in [14,15] that T_a is ergodic with respect to an i.i.d. product measure μ if and only if a is not divisible by p. This fact easily implies a more general statement for T_a with respect to μ_a.

Theorem 2.2. *Let $a = r/s \in \mathbb{Z}_p$ be a rational number in reduced form. Let μ be an i.i.d. Bernoulli measure on the Borel sigma-algebra $\mathcal{A}$. Define μ_a by $\mu_a(A) = (1/s) \sum_{i=0}^{s-1} \mu(T_a^{-i} A)$ for all $A \in \mathcal{A}$. Then T_a is ergodic with respect to μ_a if and only if $|a|_p = 1$.*

Proof. If $|a|_p < 1$, then $B_{p^{-1}}(0)$ is an invariant set for T_a. Now $T_a^{-i}(B_{p^{-1}}(0))$ is another ball of radius p^{-1}, so the measure $\mu(T_a^{-i}(B_{p^{-1}}(0)))$ is strictly between 0 and 1, for all $0 \leq i < s$. Thus $\mu_a(B_{p^{-1}}(0))$ is strictly between 0 and 1, so T_a is not ergodic with respect to μ_a.

If $|a|_p = 1$, then r is not divisible by p. Thus, μ is ergodic for T_r. If A is an invariant set for T_a, then $T_a^{-i}(A) = A$ for all $i \in \mathbb{Z}$. Since $T_a^{-s} = T_r^{-1}$, the set A is also invariant for T_r. By ergodicity, it follows that $\mu(A)$ is either

0 or 1. Moreover, $T_a^{-i}(A) = A$ implies that

$$\mu_a(A) = \frac{1}{s} \sum_{i=0}^{s-1} \mu(T_a^{-i}(A))$$

$$= \frac{1}{s} \sum_{i=0}^{s-1} \mu(A) = \mu(A).$$

Hence, $\mu_a(A)$ is either 0 or 1, so T_a is ergodic with respect to μ_a. $\qquad\square$

Let $a = r/s \in \mathbb{Z}_p$ be a rational number such that $|a|_p = 1$, and let μ be an i.i.d. Bernoulli measure. Since T_a is nonsingular and ergodic with respect to μ_a, we can investigate the orbit equivalence class of T_a with respect to μ_a, as is done for translation by integers with respect to i.i.d. Bernoulli measures in [14, 15]. The orbit equivalence class of T_a with respect to μ_a is still unknown for all examples other than those that reduce to known results— translation by an integer or with respect to Haar measure. We may ask whether the orbit equivalence class of T_a with respect to μ_a is related to the orbit equivalence class of T_r or T_s with respect to μ, and if so, how they are related.

3. Proof of Proposition 1.4

In this final section, we give the proof of Proposition 1.4, an example to illustrate parts of the proof, and some concluding remarks. Although Proposition 1.4 is stated in terms of measures of balls, the proof focuses on strings of coefficients determined by balls of a particular radius. Heuristically, we are trying to find a string of coefficients that is likely to occur often in $x \in \mathbb{Z}_p$ with respect to the i.i.d. Bernoulli measure μ, but the corresponding strings of coefficients in $T_a(x)$ are unlikely to occur with respect to μ.

Proof of Proposition 1.4. Let $a \in \mathbb{Z}_p$ be a nonintegral rational number, and let μ be an i.i.d. Bernoulli measure other than Haar measure. Let r be the length of a repeating segment c, and let $R = \sum_{i=0}^{\infty} cp^{ir}$ be the repeating part of a. We want to find $N \in \mathbb{N}$ and $b \in \mathbb{Z}_p$ such that

$$(2) \qquad \mu(B_{p^{-rN}}(b)) > \mu(T_R(B_{p^{-rN}}(b))) + \mu(T_{1+R}(B_{p^{-rN}}(b))).$$

We begin by considering balls of radius p^{-r} with center $x \in \mathbb{Z} \subset \mathbb{Z}_p$ such that $0 \le x < p^r$. Since translations are invertible isometries, we have $T_R(B_{p^{-r}}(x)) = B_{p^{-r}}(T_R(x))$. Since a ball of radius p^{-r} is determined by the first r coordinates of its center, we also have $B_{p^{-r}}(T_R(x)) = B_{p^{-r}}(c + x)$. Similarly, we have $T_{1+R}(B_{p^{-r}}(x)) = B_{p^{-r}}(1 + c + x)$. For a ball $B_{p^{-r}}(x)$ that has maximal measure among the balls of radius p^{-r}, we set

$$M = \mu(B_{p^{-r}}(x)),$$
$$m_0 = \mu(B_{p^{-r}}(c + x)), \text{ and}$$
$$m_1 = \mu(B_{p^{-r}}(1 + c + x)).$$

Using this notation, we define the following three conditions on the ball $B_{p^{-r}}(x)$:

(i) $M > m_0$, $M > m_1$, and $x = p^r - c - 1$,

(ii) $M > m_0$ and $0 \leq x < p^r - c - 1$, or

(iii) $M > m_1$ and $p^r - c - 1 < x < p^r$.

First, we show that if there is a ball of maximal measure satisfying one of these conditions, then we can find a ball satisfying (2). Next, we consider various cases for the measure μ, showing that in each case we can find at least one ball of maximal measure satisfying one of the three conditions.

If there is a ball of maximal measure $B_{p^{-r}}(x)$ that satisfies Condition (i), then we define $m = \max\{m_0, m_1\}$ and fix an integer

$$N > \log_{M/m} 2.$$

The ball $B = B_{p^{-rN}}(\sum_{i=0}^{N-1} x p^{ir})$ has measure

$$\mu(B) = \prod_{i=0}^{N-1} \mu(B_{p^{-r}}(x)) = M^N.$$

If $x = p^r - c - 1$, then $c + x = p^r - 1 < p^r$, so adding the first r coefficients of R to the first r coefficients of x does not result in a carry. Thus, each of the following groups of coefficients taken r at a time from $R + x$ are the same as the first group of r coefficients of $c + x$, so $\mu(T_R B) = m_0^N$. Similarly, we have $1 + c + x = p^r$, so adding the first r coefficients of $1 + R$ to the first r coefficients of x does result in a carry. Thus, each of the next groups of coefficients taken r at a time from $1 + R + x$ are the same as the first r coefficients of $1 + c + x$, so $\mu(T_{1+R} B) = m_1^N$. Finally, the choice of N implies that

$$\mu(B) = M^N > 2m^N$$
$$\geq \mu(T_R B) + \mu(T_{1+R}(B)),$$

so (2) is satisfied.

If there is a ball of maximal measure $B_{p^{-r}}(x)$ that satisfies Condition (ii), then we fix an integer

$$N > \log_{M/m_0} \frac{m_0 + m_1}{m_0}.$$

Again, the ball $B = B_{p^{-rN}}(\sum_{i=0}^{N-1} x p^{ir})$ has measure M^N. If $x < p^r - c - 1$, then $c + x < p^r$, so adding the first r coefficients of R to the first r coefficients of x does not result in a carry. Thus, it again follows that $\mu(T_R B) = m_0^N$. Similarly, we have $1 + c + x < p^r$, so adding the first r coefficients of $1 + R$ to the first r coefficients of x does not result in a carry. Thus, each of the following groups of coefficients taken r at a time from $1 + R + x$ are the same as the first r coefficients of $c + x$, so $\mu(T_{1+R} B) = m_1 m_0^{N-1}$. Finally,

the choice of N implies that

$$\mu(B) = M^N > \frac{m_0 + m_1}{m_0} m_0^N$$
$$= \mu(T_R B) + \mu(T_{1+R}(B)),$$

so (2) is satisfied.

A similar argument proves that Condition (iii) implies that (2) is satisfied. The only changes are switching m_0 and m_1, switching the defining inequalities for x, and observing that the additions do result in carries after each group of coefficients taken r at a time.

So far, we know that each of the three conditions on a ball of radius p^{-r} implies that we can find a ball, possibly of smaller radius, that satisfies (2). Now, we show that it is always possible find a ball of radius p^{-r} that satisfies one of the three conditions. We split the remainder of the proof into cases that depend on the measure μ. Since μ is not Haar measure, it is determined by a probability vector $(q_0, q_1, \ldots, q_{p-1})$ such that the weights q_i are not all equal. We let $Q = \max_i q_i$ be the largest weight. Either the probability vector that defines μ has a unique largest weight or it does not. If there is a unique largest weight, then a ball satisfying one of the three conditions has an explicit description. We now prove this case.

If there exists a unique largest weight, then there exists a weight q_j such that $q_j = Q$ and $q_i < q_j$ for all $i \neq j$. Then $B_{p^{-r}}(\sum_{i=0}^{r-1} j p^i)$ is the unique ball of radius p^{-r} that has maximal measure. If a is a positive integer or zero, then a ends in repeating zeros, which gives $R = 0$. If a is a negative integer, then a ends in repeating $p - 1$'s, which gives $R = -1$. By the assumption that a is not an integer, R is not equal to 0 or -1. Since R is not zero, $B_{p^{-r}}(R + \sum_{i=0}^{r-1} j p^i)$ is not equal to $B_{p^{-r}}(\sum_{i=0}^{r-1} j p^i)$. Uniqueness then implies that $M > m_0$. Similarly, since R is also not -1, $B_{p^{-r}}(1 + R + \sum_{i=0}^{r-1} j p^i)$ is not equal to $B_{p^{-r}}(\sum_{i=0}^{r-1} j p^i)$. Again, uniqueness implies that $M > m_1$. Thus, $B_{p^{-r}}(\sum_{i=0}^{r-1} j p^i)$ satisfies Condition (i) if $\sum_{i=0}^{r-1} j p^i = p^r - c - 1$, Condition (ii) if $\sum_{i=0}^{r-1} j p^i < p^r - c - 1$, or Condition (iii) if $\sum_{i=0}^{r-1} j p^i > p^r - c - 1$.

If $p = 2$ and μ is not Haar measure, then the two weights are not equal. Thus, there is a unique largest weight and the proof of the case $p = 2$ is complete. Thus, we can assume that $p \geq 3$ for the remainder of the proof of the proposition.

Let I be the set of indices such that $q(i) = Q$, the largest weight. Let k be the cardinality of I. If there is not a unique largest weight, then $k > 1$. Since μ is not Haar measure, we must also have $k < p$. Since we have k possibilities for maximal coefficients and since a ball of radius p^{-r} is determined by r coefficients, there are k^r distinct balls of radius p^{-r} of maximal measure. By not requiring that r is the minimal period, we can assume that $r \geq 2$. If $k \geq 2$ and $r \geq 2$, then $k^r \geq 2k$. Thus, either it is the case that

$$\mathcal{A}_0 = \left\{ B_{p^{-r}}(x) : 0 \leq x < p^r - c - 1 \right\}$$

contains at least k balls of maximal measure or it is the case that

$$\mathcal{A}_1 = \left\{ B_{p^{-r}}(x) : p^r - c - 1 < x < p^r - 1 \right\}$$

contains at least k balls of maximal measure. Any balls that satisfy Condition (ii) are in $\mathcal{A}_0$, and any balls that satisfy Condition (iii) are in $\mathcal{A}_1$.

Before we consider these two cases, we prove a fact about divisibility. For the collection $\mathcal{A}_i$, we suppose that for each $j \in I$ there exists a ball of maximal measure $B_{p^{-r}}(x_j)$ in $\mathcal{A}_i$ such that $x_j = j \mod p$ and

$$T_{i+c}(B_{p^{-r}}(x_j)) = B_{p^{-r}}(T_{i+c}(x_j))$$

has maximal measure. We define a group homomorphism $T_{i+c} \mod p$ on $\mathbb{F}_p$ by $j \mapsto j + i + c \mod p$. If a ball has maximal measure, then the first coordinate must also have maximal weight. Thus, the orbit of each $j \in I$ under $T_{i+c} \mod p$ is contained in I. Since $T_{i+c} \mod p$ is a group homomorphism of $\mathbb{F}_p$, the minimal period of each $j \in I$ divides p. Since μ is not Haar measure, I does not contain all indices. Hence, the minimal period is not p, so every $j \in I$ is a fixed point. Since $j + (i + c) = j \mod p$, it follows that p divides $i + c$.

The previous paragraph shows that if there are k maximal balls in $\mathcal{A}_i$ that map to maximal balls under T_{i+c}, such that every maximal index is equal modulo p to the center of one of these balls, then $i + c$ is divisible by p. For future reference, we give the contrapositive of this statement. For a collection of k maximal balls in $\mathcal{A}_i$ such that every maximal index is the first coordinate of the center of one of the balls, if $i + c$ is not divisible by p, then one of the balls of maximal measure in $\mathcal{A}_i$ does not map to another ball of maximal measure under T_{i+c}. With these observations, we proceed to prove the last two cases.

First, we show that if $\mathcal{A}_0$ contains all balls of maximal measure, then there exists a ball of maximal measure that satisfies Condition (ii). Since $T_c \mod p^r$ is a group homomorphism of the finite group $\mathbb{F}_{p^r}$, we can consider it as a group homomorphism on the balls of radius p^{-r}, which are determined by the first r coordinates of the centers. Thus, the balls are periodic under $T_c \mod p^r$, with periods that are divisible by p. Since p is prime and $1 < k < p$, the number of balls of maximal measure, k^r, is not divisible by p. Hence, there must be a cycle of balls that contains both a ball of maximal measure and a ball of smaller measure. In other words, there is a ball of maximal measure $B_{p^{-r}}(x)$ such that $T_c(B_{p^{-r}}(x)) = B_{p^{-r}}(c+x)$ does not have maximal measure. Since we assumed that all balls of maximal measure are in $\mathcal{A}_0$, it follows that the ball $B_{p^{-r}}(x)$ satisfies Condition (ii).

Next, we suppose that $\mathcal{A}_0$ contains at least k balls of maximal measure, but none of the balls satisfy Condition (ii). By the previous paragraph there exist balls of maximal measure $B_{p^{-r}}(x)$ such that $p^r - c - 1 \leq x < p^r$. We show that one of these balls must satisfy Condition (iii). If $\mathcal{A}_0$ contains at least k balls of maximal measure but none of them satisfy Condition (ii), then we have k maximal balls that map to maximal balls under T_c, such

that every maximal index is equal modulo p to the center of one of the balls. Thus, c is divisible by p.

We now argue that $B_{p^{-r}}(p^r - c - 1)$ cannot be the only ball of maximal measure with center x such that $p^r - c - 1 \leq x < p^r$. Since c is divisible by p, it follows that $x = p^r - c - 1 = p - 1 \mod p$. Thus, if $B_{p^{-r}}(p^r - c - 1)$ has maximal measure, then $p - 1$ must have maximal weight. Thus,

$$B_{p^{-r}}(p^r - 1) = B_{p^{-r}}\left(\sum_{i=0}^{r-1}(p-1)p^i\right)$$

also has maximal measure.

We have shown that whether or not $p - 1$ has maximal weight, there exists a ball of maximal measure in $\mathcal{A}_1$. Suppose that this ball has center $\sum_{i=0}^{r-1} x_i p^i$. For all $j \in I$, the ball with center $j + \sum_{i=1}^{r-1} x_i p^i$ will also have maximal measure. Since c is a multiple of p, if $p^r - c - 1 < \sum_{i=0}^{r-1} x_i p^i < p^r$, then it is also true that $p^r - c - 1 < j + \sum_{i=1}^{r-1} x_i p^i < p^r$. Thus, every maximal index is equal mod p to the center of a ball in $\mathcal{A}_1$. Since p divides c, it cannot divide $1 + c$. This implies that there must be a maximal ball $B_{p^{-r}}(x)$ such that $M > m_1$ and $p^r - c - 1 < x < p^r$, so we have satisfied Condition (iii).

If it is the case that $\mathcal{A}_1$ contains at least k balls of maximal measure, then the argument is similar to the case for $\mathcal{A}_0$. The only changes are switching $\mathcal{A}_0$ and $\mathcal{A}_1$, c and $c+1$, Conditions (ii) and (iii), and the defining inequalities for x. $\qquad\square$

We conclude with an example to illustrate parts of the proof of Proposition 1.4 and a discussion about extending the results in this paper.

Example 3.1. Let μ be the i.i.d. Bernoulli measure on $\mathbb{Z}_5$ that is defined by the probability vector $(3/14, 3/14, 1/7, 3/14, 3/14)$. We consider translation by $a = \sum(0 + 3 \cdot 5)5^{2n}$ and take the repeating segment $c = 0 + 3 \cdot 5 = 15$. In this example, we have $p^r - c - 1 = 25 - 15 - 1 = 9$, so balls in $\mathcal{A}_0$ have a center $0 \leq x < 9$ and balls in $\mathcal{A}_1$ have a center $9 < x < 25$. Each ball of maximal measure in $\mathcal{A}_0$ maps to another ball of maximal measure under T_{15}. As we expect from the proof of Proposition 1.4, the prime $p = 5$ divides $c = 15$ but not $c + 1 = 16$. Thus, there are balls of maximal measure in $\mathcal{A}_1$ that map to balls of smaller measure under T_{16}. One of these balls is $B_{5^{-2}}(1 + 3 \cdot 5)$. Since this ball satisfies Condition (iii), we take $N = 3 > \log_{4/3} 2$. Then the ball

$$B_{5^{-6}}(1 + 3 \cdot 5 + (1 + 3 \cdot 5)5^2 + (1 + 3 \cdot 5)5^4)$$

satisfies inequality (2).

The proof of Proposition 1.4 uses the fact that p is a prime, especially in the divisibility arguments toward the end of the proof. For a composite number g, it is possible to define $\mathbb{Z}_g$ as the set of formal power series in g

and to give it similar algebraic, topological, and measure-theoretic structures. Special cases of $a \in \mathbb{Z}_g$ can be found with properties that yield the required divisibility in the method of proof used for Proposition 1.4. However, there are also nonintegral rational numbers $a \in \mathbb{Z}_g$ that do not have these properties. Since the proof of Theorem 1.1 depends on Proposition 1.4, it is still unknown for some nonintegral rational numbers $a \in \mathbb{Z}_g$ whether or not $T_a : \mathbb{Z}_g \to \mathbb{Z}_g$ in singular with respect to i.i.d. Bernoulli measures other than Haar measure.

References

[1] ANASHIN, VLADIMIR. Ergodic transformations in the space of p-adic integers. p-adic mathematical physics, 3–24, AIP Conf. Proc., 826. *Amer. Inst. Phys., Melville, NY*, 2006. MR2258670(2007m:37015), Zbl 1152.37301, arXiv:math/0602083v1, doi: 10.1063/1.2193107.

[2] BRYK, JOHN; SILVA, CESAR E. Measurable dynamics of simple p-adic polynomials. *Amer. Math. Monthly* **112** (2005), no. 3, 212–232. MR2125384(2005m:37018), Zbl 1089.11066, doi: 10.2307/30037439.

[3] COELHO, ZAQUEU; PARRY, WILLIAM. Ergodicity of p-adic multiplications and the distribution of Fibonacci numbers. *Topology, ergodic theory, real algebraic geometry*, 51–70, Amer. Math. Soc. Transl. Ser. 2, 202. *Amer. Math. Soc., Providence, RI*, 2001. MR1819181(2002e:11103), Zbl 0985.11008.

[4] DANILENKO, ALEXANDRE I. Infinite rank one actions and nonsingular Chacon transformations. *Illinois J. Math.* **48** (2004), no. 3, 769–786. MR2114251(2006b:37009), Zbl 1069.37003.

[5] DANILENKO, ALEXANDRE I.; DEL JUNCO, ANDRÉS. Almost continuous orbit equivalence for non-singular homeomorphisms. *Israel J. Math.* **183** (2011), 165–188. MR2811157, Zbl 1236.37007, arXiv:0811.3917, doi: 10.1007/s11856-011-0046-0.

[6] DOOLEY, A. H.; HAMACHI, TOSHIHIRO. Nonsingular dynamical systems, Bratteli diagrams and Markov odometers. *Israel J. Math.* **138** (2003), 93–123. MR2031952(2004k:37005), Zbl 1061.37001, doi: 10.1007/BF02783421.

[7] DOOLEY, A. H.; HAWKINS, J.; RALSTON, D. Families of type III_0 ergodic transformations in distinct orbit equivalent classes. *Monatsh. Math.* **164** (2011), no. 4, 369–381. MR2861592(2012k:37013), Zbl 1246.37008, doi: 10.1007/s00605-010-0258-0.

[8] DOOLEY, ANTHONY H.; HAGIHARA, RIKA. Computing the critical dimensions of Bratteli–Vershik systems with multiple edges. *Ergodic Theory Dynam. Systems* **32** (2012), no. 1, 103–117. MR2873160, Zbl 1267.37007, doi: 10.1017/S0143385710000969.

[9] EIGEN, STANLEY; HAJIAN, ARSHAG. A characterization of exhaustive weakly wandering sequences for nonsingular transformations. *Comment. Math. Univ. St. Paul.* **36** (1987), no. 2, 227–233. MR919453(89a:28015), Zbl 0633.28005.

[10] EIGEN, S.; HAJIAN, A. Exhaustive weakly wandering sequences. *Indag. Math. (N.S.)* **18** (2007), no. 4, 527–538. MR2424311(2009b:37009), Zbl 1143.37005, doi: 10.1016/S0019-3577(07)80060-7.

[11] ESLAMI, PEYMAN; MISIUREWICZ, MICHAŁ. Singular limits of absolutely continuous invariant measures for families of transitive maps. *J. Difference Equ. Appl.* **18** (2012), no. 4, 739–750. MR2905294, Zbl 1251.37044, doi: 10.1080/10236198.2011.590480.

[12] FAN, AI-HUA; LI, MING-TIAN; YAO, JIA-YAN; ZHOU, DAN. p-adic affine dynamical systems and applications. *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 2, 129–134. MR2193660(2006k:37120), Zbl 1081.37003, doi: 10.1016/j.crma.2005.11.017.

[13] Fan, Ai-Hua; Li, Ming-Tian; Yao, Jia-Yan; Zhou, Dan. Strict ergodicity of affine p-adic dynamical systems on $\mathbb{Z}_p$. *Adv. Math.* **214** (2007), no. 2, 666–700. MR2349716(2008g:37038), Zbl 1121.37010, doi: 10.1016/j.aim.2007.03.003.

[14] Furno, Joanna. Ergodic theory of p-adic transformations. PhD thesis, University of North Carolina at Chapel Hill, 2013. 106 pp. ISBN: 978-1-303-10521-0. MR3153461.

[15] Furno, Joanna. Orbit equivalence of p-adic transformations and their iterates. To appear in *Monatsh. Math.* (2014). doi: 10.1007/s00605-014-0645-z.

[16] Keller, Gerhard. Stochastic stability in some chaotic dynamical systems. *Monatsh. Math.* **94** (1982), no. 4, 313–333. MR685377(84k:58130), Zbl 0496.58010, doi: 10.1007/BF01667385.

[17] Li, Zhenyang; Góra, Paweł; Boyarsky, Abraham; Proppe, Harald; Eslami, Peyman. Family of piecewise expanding maps having singular measure as a limit of ACIMs. *Ergodic Theory Dynam. Systems* **33** (2013), no. 1, 158–167. MR3009107, Zbl 1280.37035, doi: 10.1017/S0143385711000836.

[18] Mahler, Kurt. p-adic numbers and their functions. Second edition. Cambridge Tracts in Mathematics, 76. *Cambridge University Press, Cambridge-New York*, 1981. xi+320 pp. ISBN: 0-521-23102-7. MR644483(83i:12002), Zbl 0444.12013.

[19] Petersen, Karl. Ergodic Theory. Corrected reprint of the 1983 original. Cambridge Studies in Advanced Mathematics, 2. *Cambridge University Press, Cambridge,* 1989. xii+329 pp. ISBN: 0-521-38997-6. MR1073173(92c:28010), Zbl 0676.28008.

[20] Robert, Alain M. A course in p-adic analysis. Graduate Texts in Mathematics, 198. *Springer-Verlag, New York*, 2000. xvi+437 pp. ISBN: 0-387-98669-3. MR1760253(2001g:11182), Zbl 0947.11035, doi: 10.1007/978-1-4757-3254-2.

[21] Walters, Peter. An introduction to ergodic theory. Graduate Texts in Mathematics, 79. *Springer-Verlag, New York-Berlin*, 1982. ix+250 pp. ISBN: 0-387-90599-5. MR648108 (84e:28017), Zbl 0475.28009.

Department of Mathematics and Computer Science, Wesleyan University, 45 Wyllys Avenue, Middletown, CT 06459
jfurno@wesleyan.edu

This paper is available via `http://nyjm.albany.edu/j/2014/20-40.html`.