New York Journal of Mathematics

New York J. Math. **20** (2014) 695–710.

Adjoining an identity to a finite filial ring

R. R. Andruszkiewicz and K. Pryszczepko

Abstract. The aim of this paper is to investigate the problem of embedding of filial ring into filial ring with an identity.

Contents

1. Introduction and preliminary results

All considered rings are associative. To denote that I is an ideal of a ring R we write $I \lhd R$. A ring R is called *filial* if $A \lhd B$ and $B \lhd R$ imply $A \lhd R$ for all subrings A, B of R. Filial rings were studied in many papers (cf. [APr09, FP04, FP05]).

Recall that, a subring A of a ring R is *n-accessible*, if $A = A_0 \lhd A_1 \lhd \cdots \lhd A_n = R$ for some subrings $A_0, A_1, \dots, A_n$ of R. Moreover A is said to be *precisely n-accessible* if it is not k-accessible in R for any positive integer $k < n$.

The problem of construction of precisely n-accessible subrings in a given ring plays a fundamental role in the theory of radicals. Such subrings are closely connected with the problem of the termination of Kurosh's chain (cf. [APu90, APu97, Hei68, Bei82, AS13]). In [APu97], it was proved that, if a commutative integral domain A is not filial, then it is possible to construct in A precisely n-accessible subrings for every natural number n. Using this fact one can construct Kurosh's chains (in the class of associative, not necessarily commutative rings) which terminate after an arbitrarily predetermined finite number of steps ([Bei82], [APu97]). These studies initiated by Beidar in [Bei82] and continued by Andruszkiewicz and Puczyłowski in [APu97], are the most valuable and recognizable in the theory of radicals. Application

Received December 21, 2013.

2010 *Mathematics Subject Classification.* 16D25, 13C05, 13B02.

Key words and phrases. Ideal, Filial ring.

ISSN 1076-9803/2014

of filial rings and their properties allowed to overcome enormous difficulties related to this subject.

A ring R is *strongly regular* if $a \in Ra^2$ for every $a \in R$. Obviously all strongly regular rings are von Neumann regular, and for commutative rings this two properties coincide. The class of all strongly regular rings $\mathbb{S}$ form a radical in the sense of Kurosh and Amitsur. One can easily check that every strongly regular ring is filial.

It is a well-known fact that any ring can be considered as an ideal in a ring with unity. The simplest way to embed a ring into a ring with an identity is to apply the Dorroh extension. A question for which rings R having some property there exists a unit ring also having the same property in which R is an ideal is important. In several papers, a few kinds of ring extensions connected with the above question, were studied (see for instance [FH64, Fun66]).

In [APr09], we have shown that every commutative torsion–free reduced filial ring is an ideal in some commutative torsion–free reduced filial ring with an identity. This nontrivial construction allowed us to obtain the classification theorem for reduced filial rings. At the conference "Radicals of rings and related topics" in Warsaw, Poland in 2009, the second author asked whether this was still true if all the above assumptions except filiality were dropped (cf. [P09]). In this note we answer this question in the negative. Namely, our main result can be stated as follows.

Theorem 1.1. *Every filial ring R such that $|R_p| \le p^3$ for all $p \in \mathbb{P}$, is an ideal in a filial ring S with an identity. Moreover, if the ring R is commutative (finite), then the ring S is also commutative (finite). Furthermore, for every $p \in \mathbb{P}$ there exists a commutative filial ring I with p^4 elements such that $p^2 I = 0$, which is not an ideal in any filial ring with an identity.*

In this paper we consider filial rings, not necessarily commutative. The class of these rings is poorly investigated and little is known about their structure. The obtained results and examples may be applied in the general ring theory.

Throughout the paper, $\mathbb{N}$ and $\mathbb{P}$ stand for the set of all positive integers and the set of all primes, respectively. The cardinality of the set X is denoted by $|X|$.

For a ring R, we denote by R^+ the additive group of R and for a prime p let $R_p = \{x \in R : p^k x = 0 \text{ for some } k \in \mathbb{N}\}$, $R(p) = \{x \in R : px = 0\}$. We write $o(x)$ for the order of an element x in the group R^+, and we say that the ring R is of bounded exponent if there exists $M \in \mathbb{N}$ such that $Mx = 0$ for every $x \in R$, otherwise we say that the ring R is of unbounded exponent. If R^+ is a p-group, then we say that R is a p-ring. For a subset S of R, we denote by $\langle S \rangle$, $[S]$, $a(R)$, $l_R(S)$, $(S)_R$ the subgroup of R^+ generated by S, the subring of R generated by S, the two-sided annihilator of S in R, the left annihilator of S in R and the ideal generated by S in R, respectively.

Remark 1.2. Let C be a commutative ring with an identity 1 and let A be a C-algebra. On the cartesian product $C^+ \times A^+$ we define a multiplication by the formula

$$(c_1, a_1)(c_2, a_2) = (c_1c_2, c_1a_2 + c_2a_1 + a_1a_2)$$

for all $c_1, c_2 \in C$, $a_1, a_2 \in A$. In this way we obtain a new C-algebra, which will be denoted by $C \boxplus A$. Instead of (c, a) we will write $c + a$. Identifying A with its image in $C \boxplus A$, via the embedding mapping $a \mapsto (0, a)$, we see that $A \lhd C \boxplus A$ and $(C \boxplus A)/A \cong C$. It is also clear that if A is commutative, then the algebra $C \boxplus A$ is commutative too. Moreover, if A possesses an identity, then $C \boxplus A \cong C \oplus A$. Note that every ring is a $\mathbb{Z}$-algebra in a natural way.

We start by recalling a well-known characterization of filial rings.

Lemma 1.3 ([APu88], Theorem 1)**.** *A ring S is filial if and only if*

$$(a)_S = (a)_S^2 + \langle a \rangle$$

for every $a \in S$.

The Andrunakievich Lemma implies that subidempotent rings (i.e., rings in which every ideal is idempotent) are filial (cf. [APu88]).

Lemma 1.4. *Let $I \lhd R$, R/I be a subidempotent ring, and for every $J \lhd I$, $J \lhd R$. If I is a filial ring then R is also a filial ring.*

Proof. Let $A \lhd B$ and $B \lhd R$. Then $(A + I)/I \lhd (B + I)/I \lhd R/I$ and $(A + I)/I \lhd R/I$, by filiality of R/I. Since $(A + I)/I = (A^2 + I)/I$, $A + I = A^2 + I$. Intersecting both sides with A we get, by modularity of the subgroup lattice in R^+, $A = A^2 + (I \cap A)$. Now, since $I \cap A \lhd I \cap B \lhd I$, so $I \cap A \lhd R$. Moreover, $RA \subseteq RA^2 + R(I \cap A) \subseteq BA + (I \cap A) \subseteq A$. The proof of the opposite inclusion $AR \subseteq A$ is similar. □

In what follows, β denotes the prime radical.

Proposition 1.5 ([FP05], Corollary 8)**.** *A β-radical ring is filial if and only if all its subrings are ideals.*

A ring in which all subrings are ideals is called an *H-ring*. Such rings have been studied by a number of authors, for instance Andrijanov [And66], [And67], Jones and Schäfer [JS57], Kruse [Kru68], [Kru64], Rédei [Red52.1], [Red52.2]. Especially, the results obtained by Andrijanov and Kruse give an advanced description of H-rings which are nil. We point out that their classification of H-rings is not complete and can be improved. This classification does not describe nil-H-rings up to isomorphism, and requires too many parameters to define these rings (cf. [And67], [Kru64]). Also Antipkin and Elizarov point out [AntE82], (page 461) these gaps.

If a nil ring R is both a p-ring and an H-ring, we shall say that R is a nil-H-p-ring. The following lemma was established by Kruse.

Lemma 1.6 ([Kru68], Lemma 2.7)**.** *If R is a nil-H-p-ring, then $a^3 \in \langle a^2 \rangle$ for every $a \in R$, and in particular $[a] = \langle a \rangle + \langle a^2 \rangle$.*

We start with some general, new observations concerning H-rings and filial rings. Notice that a ring R is an H-ring if and only if every subring of R generated by a single element is an ideal in R.

Proposition 1.7. *Let R be a nil-H-p-ring and let N be a ring such that $N^2 = pN = 0$. Then the ring $S = R \oplus N$ is an H-ring.*

Proof. Take any $s \in S$. Then $s = (a, x)$ for some $a \in R$ and $x \in N$. Clearly $s^2 = a^2$. Note that $ys = sy = 0 \in [s]$ for any $y \in N$, and by Lemma 1.6, for every $b \in R$, $ba = Ua + Va^2$ for some $U, V \in \mathbb{Z}$. If $p \nmid U$, then $a \in Ra$, and since R is a nil ring, so $a = 0$. Hence, we can assume that $p \mid U$. Since $pN = 0$, so $bs = Us + Vs^2 \in [s]$. Similarly, one can check that $sb \in [s]$. Consequently $[s] \lhd S$ and S is an H-ring. $\square$

A ring R is called *reduced* if it has no nonzero nilpotent elements.

Lemma 1.8. *If R is a commutative filial p-ring, which cannot be expressed as a direct sum of two nonzero ideals, then R is a nil ring or R has an identity.*

Proof. Suppose that $\beta(R) \neq R$. Then $R/\beta(R)$ is a nonzero commutative reduced filial p-ring. Hence $R/\beta(R)$ is a $\mathbb{Z}_p$-algebra, and by Theorem 4.1 of [FP04], $R/\beta(R)$ is a strongly regular ring. Hence, in the ring $R/\beta(R)$ there exists a nonzero idempotent, which can be lift to a nonzero idempotent $e \in R$. Thus $R = Re \oplus l_R(e)$, but $l_R(e) = 0$ and e is an identity of R. $\square$

Lemma 1.9. *Let A and B be p-rings such that A has an identity 1, $\beta(A) \neq 0$ and $\beta(B) \neq 0$. Then the ring $A \oplus B$ is not filial.*

Proof. By assumption, there exist $a \in A$ and $b \in B$ such that $a^2 = pa = 0$ and $b^2 = pb = 0$. Set $\alpha = (a, b)$. Then $\alpha^2 = 0$ and $p\alpha = 0$. Suppose that the ring $A \oplus B$ is filial. Then $\beta(A \oplus B) = \beta(A) \oplus \beta(B)$ is a filial ring. Hence, and by Proposition 1.5, $\beta(A \oplus B)$ is an H-ring. Since $\langle \alpha \rangle = [\alpha] \lhd \beta(A \oplus B) \lhd A \oplus B$, we have $\langle \alpha \rangle \lhd A \oplus B$. In particular $(a, 0) = (1, 0)(a, b) \in \langle \alpha \rangle$ and there exists $k \in \mathbb{Z}$ such that $(a, 0) = k(a, b)$. Now, $a = ka$ and $0 = kb$, so $p \mid k$ and this implies that $a = 0$, a contradiction. $\square$

Proposition 1.10. *If R is a filial commutative noetherian p-ring without identity, then $R = \mathbb{S}(R) \oplus \beta(R)$.*

Proof. Since the ring R is noetherian, so $R = R_1 \oplus R_2 \oplus \cdots \oplus R_s$ for some nonzero ideals $R_1, R_2, \ldots, R_s$ of R. Moreover, we can assume that each R_i cannot be expressed as a direct sum of two nonzero ideals. Applying Lemmas 1.8, 1.9 and Theorem 4.1 of [FP04] one can get that $R = \mathbb{S}(R) \oplus \beta(R)$. $\square$

Lemma 1.11. *Let R be a filial p-ring with an identity. Then $\beta(R) = \beta(R)(p) + p \cdot \langle 1 \rangle$, the group pR^+ is cyclic, and $y^2 \in (\beta(R)(p))^2 + \langle y \rangle$ for every $y \in \beta(R)$.*

Proof. Since R^+ is a p-group, so there exists $n \in \mathbb{N}$ such that $o(1) = p^n$ in R^+. Hence $p^n R = 0$ and $pR \subseteq \beta(R)$. By filiality of R and Proposition 1.5, $\beta(R)$ is an H-ring. Obviously, $\langle p \cdot 1\rangle = [p \cdot 1] \lhd \beta(R)$, so $\langle p \cdot 1\rangle \lhd R$ and $pR \subseteq p\langle 1\rangle$. Consequently the group pR^+ is cyclic and in particular $p\beta(R) \subseteq p\langle 1\rangle$.

If $p\beta(R) = \langle p \cdot 1\rangle$ then $p \cdot 1 = pi_0$ for some $i_0 \in \beta(R)$. Hence $pi_0 = i_0 \cdot pi_0$. Consequently $pi_0 \in \beta(R)(pi_0)$ and $pi_0 = 0$. Thus $p\beta(R) = 0$ and $\beta(R) = \beta(R)(p) + p \cdot \langle 1\rangle$.

Now, assume that $p\beta(R) \neq \langle p \cdot 1\rangle$. Then $p\beta(R) = \langle p^s \cdot 1\rangle$ for some positive integer $s > 1$. Take any $i \in \beta(R)$. Then $pi = k(p^s \cdot 1)$ for some $k \in \mathbb{Z}$. Hence $i - (kp^{s-1}) \cdot 1 \in \beta(R)(p)$, $(kp^{s-1}) \cdot 1 \in p \cdot \langle 1\rangle$ and $i = (i - (kp^{s-1}) \cdot 1) + (kp^{s-1}) \cdot 1$. This shows that $\beta(R) = \beta(R)(p) + p \cdot \langle 1\rangle$.

Fix any $y \in \beta(R)$. From the first part of the proof there exist $j \in \beta(R)(p)$ and $K \in \mathbb{Z}$ such that $y = j + Kp \cdot 1$. Next, $y^2 = j^2 + K^2p^2 \cdot 1 = j^2 + Kpy \in (\beta(R)(p))^2 + \langle y\rangle$ and the result follows. □

2. Almost null rings

We start with the following definition, which is due to R. L. Kruse.

Definition 2.1 ([Kru68], Definition 2.1)**.** A ring R is *almost null* if for every $a \in R$:

1. $a^3 = 0$.
2. $Ma^2 = 0$ for some square-free integer M which depends on a.
3. $aR \subseteq \langle a^2\rangle$ and $Ra \subseteq \langle a^2\rangle$.

These rings play an important role in the study of certain H-rings. Clearly, every almost null ring R is an H-ring such that $R^3 = 0$. Moreover, every homomorphic image and every subring of an almost null ring is almost null. Immediately, from the definition of an almost null ring, we have the following lemma.

Lemma 2.2. *Let I, J be subrings of a ring R such that $J^2 = IJ = JI = 0$. Then $I + J$ is an almost null ring if and only if I is an almost null ring.*

The proof of the next auxiliary proposition is straightforward.

Proposition 2.3. *Let R be a ring such that the group R^+ is torsion. Then:*

(i) *R is an almost null ring if and only if R_p is an almost null ring for every $p \in \mathbb{P}$.*
(ii) *R is an H-ring if and only if R_p is an H-ring for every $p \in \mathbb{P}$.*
(iii) *R is a filial ring if and only if R_p is a filial ring for every $p \in \mathbb{P}$.*

Proposition 2.4. *Let R be a nil-H-ring satisfying any of the following conditions:*

(i) *R is a p-ring and $ab \in \langle a^2\rangle \cap \langle b^2\rangle$ for any $a, b \in R$.*
(ii) *R is a p-ring of unbounded exponent.*
(iii) *There exists $x \in R^+$ such that $o(x) = \infty$.*

(iv) *$pR = 0$ for some prime p.*
(v) *$pR = p^2R$ for every prime p.*

Then R is an almost null ring.

Proof. (i). Take any $a \in R$. Let m be the smallest natural number such that $p^m a^2 = 0$. If $m > 1$, then $m-1 \in \mathbb{N}$ and $p^{m-1}a^2 \neq 0$, $p^m a^2 = 0$. Hence $(p^{m-1}a)^2 = 0$, but $0 \neq (p^{m-1}a)a \in \langle (p^{m-1}a)^2 \rangle = 0$, a contradiction. Thus $m = 1$ and $pa^2 = 0$. By Lemma 1.6, $a^3 = ka^2$ for some $k \in \mathbb{Z}$. If $p \nmid k$, then $a^2 \in Ra^2$, which gives $a^2 = 0$ and hence $a^3 = 0$. If $p \mid k$, then $ka^2 = 0$ and $a^3 = 0$, as well.

(ii). cf. Proposition 2.5 of [Kru68].

(iii). cf. Proposition 2.6 of [Kru68].

(iv). Follows from Corollary 2.3 of [FP04] and Theorem 3.3 of [FP09].

(v). Taking account of (iii) we may assume that the group R^+ is torsion. By Proposition 2.3, it is enough to prove that R_p is an almost null ring for every prime p. According to (ii), we need to consider only the case when R_p^+ is a group of bounded exponent, i.e., $p^s R_p = 0$ for some $s \in \mathbb{N}$. However, by the assumption, $p^s R_p = pR_p$, so $pR_p = 0$ and, by (iv), R_p is an almost null ring. $\square$

Proposition 2.5 ([Kru68], Proposition 2.10)**.** *Let S be a ring such that for some prime p, $pa^2 = 0$ for every $a \in S$. Then S is an almost null ring if and only if one of the following conditions is satisfied:*

(i) *$S^2 = 0$.*
(ii) *There exists $x \in S$ such that $x^2 \neq 0$, $px, x^2 \in a(S)$ and*
$$S = \langle x \rangle + a(S).$$
(iii) *There exist $x, y \in S$ such that $S = \langle x, y \rangle + a(S)$, $x^2 \neq 0$, $px^2 = 0$, $px, py, x^2 \in a(S)$, $y^2 = Ax^2$, $xy = F_1x^2$, $yx = F_2x^2$, where $A, F_1, F_2 \in \mathbb{Z}$ and the congruence*
$$X^2 + (F_1 + F_2)X + A \equiv 0 \pmod{p} \tag{2.1}$$
has no integer solution.

Remark 2.6.

(i) For rings S described in (i)–(iii) of Proposition 2.5, $\dim_{\mathbb{Z}_p} S/a(S)$ equals $0, 1, 2$, respectively. Hence, any ring described in the item (i) is not isomorphic to any ring described in items (ii) or (iii) and any ring described in the item (ii) is not isomorphic to any ring described in the item (iii).
(ii) Let $p \in \mathbb{P}$, $F_1, F_2, A \in \mathbb{Z}$ be as in Proposition 2.5. We now define a multiplication which will make the additive group
$$S^+ = \mathbb{Z}_p^+ \times \mathbb{Z}_p^+ \times \mathbb{Z}_p^+$$

into a ring. For generators $x = (1,0,0)$, $y = (0,1,0)$, $z = (0,0,1)$ of the group S^+, let

$$xz = zx = zy = yz = 0,\ x^2 = z,\ xy = F_1 z,\ yx = F_2 z,\ y^2 = Az.$$

It is not difficult to check that $a(bc) = (ab)c = 0$ for all $a, b, c \in S$, $S = \langle x, y\rangle + a(S)$, $o(x^2) = p$ in the group S^+ and $pS = 0$. Proposition 2.5 implies that S is an almost null ring.

The next proposition gives a useful example of nil-H-p-ring, which is not almost null.

Proposition 2.7 (cf. [And67])**.** *Let N be an almost null ring such that $p^m N = 0$ for some $m \in \mathbb{N}$. Let $R = [a] \oplus N$, where $o(a) = p^n$ for some positive integer $n > m$ and $a^2 = p^m a$. Then R is an H-ring. Moreover, R is an almost null ring if and only if $N^2 = 0$ and $n = m + 1$.*

Proof. Notice that $[a]$ is a nil ring because $a^3 = p^m a^2 = p^{2m} a$ and $p^n a = 0$. From the assumptions, it follows that R is a nil ring such that $p^n R = 0$. Let $r \in R$. Then there exist $k \in \mathbb{Z}$ and $x \in N$ such that $r = (ka, x)$. For $y \in N$, $yx = Ux^2$ for some $U \in \mathbb{Z}$, so

$$(0, y)r = (0, Ux^2) = U(0, x^2) = Ur^2 + (-U)kp^m r \in [r].$$

Moreover, $[a] = \langle a\rangle$ and $(a, 0)r = (ka^2, 0) = (kp^m a, 0) = p^m r \in [r]$. Similarly, one can show that $r(0, y) \in [r]$ and $r(a, 0) \in [r]$. Consequently, $[r] \lhd R$ and R is an H-ring.

Let R be an almost null ring. Then $[a]$ is also an almost null ring as a subring of R. Hence $0 = pa^2 = p^{m+1}a$ and $p^n \mid p^{m+1}$. But $m < n$, so $n = m + 1$. If $N^2 \neq 0$, then there exists $x_0 \in N$ such that $x_0^2 \neq 0$. Since $(0, x_0)(a, x_0) = (0, x_0^2)$ and $(a, x_0)^2 = (a^2, x_0^2) = (p^m a, x_0^2)$, so there exists $V \in \mathbb{Z}$ for which $(0, x_0^2) = V(p^m a, x_0^2)$. Thus $Vp^m a = 0$ and $Vx_0^2 = x_0^2$. But $o(a) = p^n > p^m$ and $Vp^m a = 0$, so $p \mid V$. Hence $px_0^2 = 0$, $Vx_0^2 = 0$ and $x_0^2 = 0$, a contradiction.

If $N^2 = 0$ and $[a]$ is an almost null ring, then $N \subseteq a(R)$ and R is an almost null ring by Lemma 2.2. $\square$

Below we give new important (see Theorem 1.1) example of an H-ring, which is not almost null.

Example 2.8. Let $I^+ = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \langle x_3\rangle$, where $o(x_1) = p^2$ and $o(x_2) = o(x_3) = p$. On I we define a multiplication on the generators x_1, x_2, x_3 of the additive group I^+ according to the following relations $x_1^2 = x_2$, $x_1x_3 = x_3x_1 = x_3^2 = px_1$, $x_ix_2 = x_2x_i = 0$ for $i = 1, 2, 3$.

One can easily check that $(xy)z = x(yz) = 0$ and $xy = yx$ for all $x, y, z \in I$. So I is an associative ring such that $I^3 = 0$. Take any $i \in I$. We claim that $[i] \lhd I$. It is enough to show that $x_1 i, x_3 i \in I$. Since $i^3 = 0$, so $[i] = \langle i\rangle + \langle i^2\rangle$. Moreover, $i = ax_1 + bx_2 + cx_3$ for some $a, b, c \in \mathbb{Z}$. Hence $i^2 = (c^2 + 2ac)px_1 + a^2x_2$, $x_1 i = 2cpx_1 + ax_2$, $x_3 i = (a + c)px_1$. Assume that $p \mid a$. Then $i^2 = c^2(px_1)$, $x_1 i = 2c(px_1)$, $x_3 i = c(px_1)$, and since $o(px_1) = p$, so $\langle c(px_1)\rangle = \langle c^2(px_1)\rangle$ and $x_1 i, x_3 i \in \langle i^2\rangle$. If $p \nmid a$, then there exists $V \in \mathbb{Z}$ such that $aV \equiv 1 \pmod p$. Hence $x_1 i = p(2cV - V^2(c^2+2c))i + Vi^2 \in [i]$ and $x_3 i = pV(a+c)i \in [i]$. Thus I is an H-ring, and consequently I is a filial ring.

Notice that I is not an almost null ring, since for instance $x_1x_3 = px_1 \notin \langle x_2\rangle = \langle x_1^2\rangle$.

The next theorem was established, without proof and in a different form, by Friedman in the short note [Fre60]. It is also deductible from Rédei's article [Red52.1].

Theorem 2.9. *All, up to an isomorphism, nonzero nil-H-p-rings generated by one element are:*

(i) $p^m\mathbb{Z}_{p^{m+n}}$ *for some* $m, n \in \mathbb{N}$, $m \leq n$.
(ii) $x\mathbb{Z}_p[x]/(x^3)$.
(iii) $x\mathbb{Z}_{p^m}[x]/(px^2, x^3)$, $m \in \mathbb{N}$, $m \geq 2$.
(iv) $x\mathbb{Z}_{p^{m+n}}[x]/(px^2 - p^m x, x^3 - p^{2m-2}x)$ *for some* $m, n \in \mathbb{N}$, $m \geq 2$.

By the above theorem and by Proposition 2.7, we get at once, the following proposition.

Proposition 2.10. *All, up to an isomorphism, almost null p-rings generated by a single element are rings of the form:*

(i) $p^m\mathbb{Z}_{p^{m+n}}$, $m, n \in \mathbb{N}$ *and* $n = m$ *or* $n = m+1$.
(ii) $x\mathbb{Z}_p[x]/(x^3)$.
(iii) $x\mathbb{Z}_{p^m}[x]/(px^2, x^3)$, $m \in \mathbb{N}$, $m \geq 2$.

3. The Dorroh extension of filial β-radical rings

By $Z(R)$ we denote the center of the ring R.

Theorem 3.1. *Let R be a β-radical ring. The following conditions are equivalent:*

(i) *The ring* $\mathbb{Z} \boxplus R$ *is filial.*
(ii) R *is an almost null ring such that* $pR = p^2R$ *for every* $p \in \mathbb{P}$.

Proof. Set $S = \mathbb{Z} \boxplus R$. Then $\beta(S) = R$ and $k \cdot 1 \in Z(S)$ for every $k \in \mathbb{Z}$.

(i)⇒(ii). The ring R is filial as an ideal of the filial ring S. By Proposition 1.5, R is a nil-H-ring. Take any $p \in \mathbb{P}$. By filiality of S and $p \cdot 1 \in Z(S)$ we get

$$p \cdot S = p^2 \cdot S + p \cdot \langle 1\rangle.$$

Therefore for every $a \in R$ there exist $x \in R$. $l_1, l_2 \in \mathbb{Z}$ such that

$$pa = p^2(l_1 \cdot 1 + x) + pl_2 \cdot 1.$$

Hence $(p^2 l_1 + pl_2) \cdot 1 = pa - p^2x \in R \cap \langle 1\rangle = \{0\}$ and $pa = p^2x \in p^2R$. Thus $pR = p^2R$ for every $p \in \mathbb{P}$. Finally, by Proposition 2.4(v), R is an almost null ring.

(ii)⇒(i). By the assumptions, it follows that $kR = k^2R$ for every $k \in \mathbb{Z}$. The ring R is an almost null ring, so $R^3 = 0$ and $aR = Ra = \langle a^2 \rangle$ for every $a \in R$. By Lemma 1.3, it is enough to prove that $(\alpha)_S \subseteq (\alpha)_S^2 + \langle \alpha \rangle$ for every $\alpha \in S$. But $\alpha = k + a$ for some $a \in R$ and $k \in \mathbb{Z}$. Because $k \cdot 1 \in Z(S)$, so $k^3 \cdot 1 = k^3 \cdot 1 + a^3 = \alpha(k^2 \cdot 1 - ka + a^2) \in (\alpha)_S$. Hence, $k^3R \subseteq (\alpha)_S$. Moreover $kR = k^2R$, so $kR \subseteq (\alpha)_S$. But $aR = Ra = \langle a^2 \rangle$ and $(\alpha)_S = R\alpha R + R\alpha + \alpha R + \langle \alpha \rangle = R(k+a)R + R(k+a) + (k+a)R + \langle \alpha \rangle$, so $RaR \subseteq R^3 = 0$ implies

$$(\alpha)_S = kR + \langle a^2 \rangle + \langle \alpha \rangle. \tag{3.1}$$

Next, $a^2 = \alpha^2 - ka - k\alpha$, so $a^2 \in (\alpha)_S^2 + kR + \langle \alpha \rangle$. Moreover,

$$\alpha^2 = k^2 \cdot 1 + (a^2 + 2ka),$$

so by (3.1) and $kR = k^2R$, we see that

$$(\alpha^2)_S = k^2R + \langle (a^2 + 2ka)^2 \rangle + \langle \alpha^2 \rangle = kR + \langle \alpha^2 \rangle.$$

Thus $kR \subseteq (\alpha)_S^2$. Moreover, $a^2 = \alpha^2 - ka - k\alpha$, so $a^2 \in (\alpha)_S^2 + \langle \alpha \rangle$, and by (3.1), we have $(\alpha)_S \subseteq (\alpha)_S^2 + \langle \alpha \rangle$. □

Example 3.2. Let $n \in \mathbb{N}$, $p \in \mathbb{P}$ and let N be an almost null ring such that $pN = 0$. Then N is a $\mathbb{Z}_{p^n}$-algebra with the multiplication

$$k \circ a = ka \ \text{ for } k \in \mathbb{Z}_{p^n}, a \in N.$$

Moreover, the function $f\colon \mathbb{Z} \boxplus N \to S = \mathbb{Z}_{p^n} \boxplus N$ given by the formula $f(k+a) = [k]_{p^n} + a$, where $[k]_{p^n}$ is the remainder of the division of k by p^n, is a surjective ring homomorphism. Hence, and by Theorem 3.1, the ring $\mathbb{Z}_{p^n} \boxplus N$ is filial.

Proposition 3.3. *Every almost null ring R is an ideal in a filial ring with an identity.*

Proof. Let $A = a(R)$ and denote by B^+ a divisible group in which A^+ is an essential subgroup. Denote by B the ring with zero multiplication on the additive group B^+. By Lemma 2.2, the ring $R \oplus B$ is an almost null and $I = \{(x,x) \ : \ x \in A\} \lhd R \oplus B$. Moreover, $I \subseteq a(R \oplus B)$. Let $S = (R \oplus B)/I$. Then $(R+I)/I \cong R/(R \cap I) \cong R$ and $(R+I)/I \lhd S$, so one can identify R with $(R+I)/I$. Moreover, S is an almost null ring as a homomorphic image of the almost null ring $R \oplus B$.

Note that $(0,b) + I \in a(S)$ for all $b \in B$. If $(r,b) + I \in a(S)$ for some $r \in R, b \in B$, then for every $y \in R$, $[(r,b)+I] \cdot [(y,0)+I] = (0,0)+I$. Hence $(ry, 0) \in I$, so $ry = 0$. This shows that $rR = 0$, and similarly $Rr = 0$. Thus $r \in a(R) = A$ and $(r,b) + I = [(r,r) + (0, b-r)] + I = (0, b-r) + I$. Hence $a(S) = \{(0,b) + I \ : \ b \in B\}$. Moreover, the function $b \mapsto (0,b) + I$ is a ring isomorphism from B onto $a(S)$. Hence the group $a(S)^+$ is divisible.

Take any $s \in S$ and any $p \in \mathbb{P}$. Then there exists a squarefree integer M such that $Ms^2 = 0$. Hence $(Ms)^2 = 0$, and directly by the definition of an almost null ring, $Ms \in a(S)$. Since $(p^2, M) \mid p$, so there exist $k, l \in \mathbb{Z}$ such that $p = kM + lp^2$. Thus $ps = k(Ms) + p^2(ls)$, and by divisibility of $a(S)^+$, $ps \in p^2S$. Hence, $pS = p^2S$. By Theorem 3.1, $\mathbb{Z} \boxplus S$ is a filial ring in which R is an ideal. □

Remark 3.4. By Lemma 1.11, the ring $\mathbb{Z}^0_{p^2} \oplus \mathbb{Z}^0_{p^2}$, where $\mathbb{Z}^0_{p^2}$ is the ring with zero multiplication on the additive group $\mathbb{Z}^+_{p^2}$, is not an ideal in any filial p-ring with an identity, but by Proposition 3.3, the ring $\mathbb{Z}^0_{p^2} \oplus \mathbb{Z}^0_{p^2}$ is an ideal in some filial ring with an identity.

4. Main Results

Lemma 4.1. *If a nil-H-p-ring I is an ideal in a filial p-ring R with an identity and $I(p)^2 \neq 0$, then $y^2 \in I(p)^2 + \langle y \rangle$ for every $y \in I$.*

Proof. Notice that $B = \beta(R)$ is a filial ring as an ideal of a filial ring R. By Proposition 1.5, $\beta(R)$ is an H-ring. From Corollary 4.6.9 of [KP69], I is nilpotent. Hence $I \subseteq B$. Since $I(p)^2 \neq 0$, so $B(p)^2 \neq 0$. By Proposition 2.4(iv), $B(p)$ is an almost null p-ring, so by Proposition 2.5, $|B(p)^2| = p$. Since $0 \neq I(p)^2 \subseteq B(p)^2$, so $I(p)^2 = B(p)^2$. Let $y \in I$. By Lemma 1.11, $y^2 \in B(p)^2 + \langle y \rangle$, so $y^2 \in I(p)^2 + \langle y \rangle$. □

Proposition 4.2. *Let R be a filial p-ring satisfying any of the following conditions:*

(i) *The group R^+ is cyclic.*
(ii) *$pR = 0$.*

Then R is an ideal in some filial p-ring with an identity.

Proof. (i). Let $R^+ = \langle a \rangle$ for some $a \in R$. Then $o(a) = p^s$ and $a^2 = p^r a$ for some $s \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$. Then $\langle a \rangle \cong p^r\mathbb{Z}_{p^{r+s}}$, but $p^r\mathbb{Z}_{p^{r+s}} \lhd \mathbb{Z}_{p^{r+s}}$, so R is an ideal in a filial p-ring with an identity.

(ii). Since R is a $\mathbb{Z}_p$-algebra, so by Remark 1.2, $\mathbb{Z}_p \boxplus R$ is a p-ring with an identity in which R is an ideal and such that $(\mathbb{Z}_p \boxplus R)/R \cong \mathbb{Z}_p$. From Lemma 1.4, it follows that $\mathbb{Z}_p \boxplus R$ is a filial ring. □

By Proposition 4.2, we get the following corollary.

Corollary 4.3. *If R is a filial p-ring such that $|R| \leq p^2$, then R is an ideal in some finite filial p-ring with an identity.*

Proposition 4.4. *Let R be a nil-H-p-ring generated by a single element and let N be a ring such that $N^2 = pN = 0$. Then the ring $R \oplus N$ is an ideal in some filial p-ring with an identity.*

Proof. By Theorem 2.9, R is isomorphic to one of the following rings:

(i) $p^m\mathbb{Z}_{p^{m+n}}$ for some $m, n \in \mathbb{N}$, $m \leq n$,
(ii) $x\mathbb{Z}_p[x]/(x^3)$, $p \in \mathbb{P}$,
(iii) $x\mathbb{Z}_{p^m}[x]/(px^2, x^3)$, $m \geq 2$, $p \in \mathbb{P}$,
(iv) $x\mathbb{Z}_{p^{m+n}}[x]/(px^2 - p^m x, x^3 - p^{2m-2}x)$ for some $m, n \in \mathbb{N}$, $m \geq 2$.

In case (i), $R \oplus N$ is an ideal in the ring with an identity $\mathbb{Z}_{p^{m+n}} \boxplus N$, which is filial by Example 3.2.

In case (ii), $p(R \oplus N) = 0$, so it is enough to use Proposition 4.2(ii).

In case (iii), let $S = \mathbb{Z}_{p^{2m-1}} \boxplus (y\mathbb{Z}_p[y]/(y^3) \oplus N)$. The ring S has an identity and S is filial by Example 3.2 and Lemma 2.2. One can check that the function $\varphi\colon x\mathbb{Z}_{p^m}[x]/I \oplus N \to S$ given by the formula

$$\varphi(\alpha X + \beta X^2 + b) = \alpha p^{m-1} + \beta p^{2m-2} + \alpha Y + \beta Y^2 + b,$$

where $X = x + (px^2, x^3)$, $Y = y + (y^3)$ for $\alpha, \beta \in \mathbb{Z}_{p^m}$ and $b \in N$ is an embedding of ring. Moreover $\operatorname{Im}\varphi \subseteq \beta(S)$, so $\operatorname{Im}\varphi \lhd S$.

In case (iv), let $T = \mathbb{Z}_{p^{2m+n-1}} \boxplus (y\mathbb{Z}_p[y]/(y^3) \oplus N)$. T is a filial ring by Example 3.2. It is easy to check that the function

$$\psi\colon x\mathbb{Z}_{p^{m+n}}[x]/(px^2 - p^m x, x^3 - p^{2m-2}x) \oplus N \to T$$

given by the formula

$$\psi(kX + b) = kp^{m-1} + kY + b$$

for $X = x + (px^2 - p^m x, x^3 - p^{2m-2}x)$, $Y = y + (y^3)$, $k \in \mathbb{Z}$, $b \in N$, is an embedding of rings. Moreover, $\operatorname{Im}\varphi \subseteq \beta(S)$, so $\operatorname{Im}\varphi \lhd S$. $\square$

Proposition 4.5. *Every filial p-ring R generated by one element is an ideal in some finite filial commutative p-ring with an identity.*

Proof. Without loss of generality we can assume that R has no identity. Let $R = [a]$ for some $a \in R$. Since R is commutative, so by assumptions and Proposition 1.10, it follows that $R \cong S \oplus N$, where N is a nonzero commutative nil-H-p-ring generated by one element and S is a finite direct sum of finite fields of characteristic p. From Proposition 4.4 and its proof there exists a finite commutative p-ring T with an identity such that $N \lhd T$. By Theorem 1.4, the ring $S \oplus T$ is filial. Moreover $S \oplus T$ is a commutative ring with an identity and $R \lhd S \oplus T$. $\square$

Theorem 4.6. *Every filial (commutative) p-ring R such that $|R| = p^3$ is an ideal in some finite filial (commutative) p-ring with an identity.*

Proof. Without loss of generality we can assume that R has no identity.

If the group R^+ is cyclic or $pR = 0$, then the thesis follows from Proposition 4.2. So, from now let $R^+ \cong \mathbb{Z}_{p^2}^+ \times \mathbb{Z}_p^+$. Assume that $R = A \oplus B$ for some nonzero ideals A, B of R. Without loss of generality we can assume that $A^+ \cong \mathbb{Z}_p$ and $B^+ \cong \mathbb{Z}_{p^2}^+$. Since the rings A and B are commutative, so we have the following cases:

(1) $R \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$.
(2) $R \cong \mathbb{Z}_p \times p\mathbb{Z}_{p^3}$.
(3) $R \cong \mathbb{Z}_p \times p^2\mathbb{Z}_{p^4}$.
(4) $R \cong p\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.
(5) $R \cong p^2\mathbb{Z}_{p^3} \times p\mathbb{Z}_{p^3}$.
(6) $R \cong p^3\mathbb{Z}_{p^4} \times p^2\mathbb{Z}_{p^4}$.

The ring described in (1) has an identity. The rings described in (2) and (3) are ideals in the filial rings $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ and $\mathbb{Z}_p \times \mathbb{Z}_{p^4}$, respectively (see Lemma 1.4). The ring described in (4) is not filial by Lemma 1.9. Finally, for rings described in (5) and (6) the thesis follows from Proposition 4.4.

From now on, we assume that the ring R cannot be written as a direct sum of two nonzero ideals. Obviously, there exist $x_1, x_2 \in R$ such that $R^+ = \langle x_1\rangle \oplus \langle x_2\rangle$, where $o(x_1) = p^2$, $o(x_2) = p$.

If R is a commutative ring which is not nil, then, by Proposition 1.10, $R \cong S \oplus N$, where $S \in \mathbb{S}$ and N is a nil ring. But R is indecomposable, so $N = 0$ and S is a field.

If R is a noncommutative ring which is not nil, then, by Theorem 3 of [AntE82], it follows that $R \cong \begin{pmatrix} \mathbb{Z}_{p^2} & p\mathbb{Z}_{p^2} \\ 0 & 0 \end{pmatrix}$ or $R \cong \begin{pmatrix} \mathbb{Z}_{p^2} & 0 \\ p\mathbb{Z}_{p^2} & 0 \end{pmatrix}$. In the first case

$$\left\langle \begin{pmatrix} p & p \\ 0 & 0 \end{pmatrix} \right\rangle \lhd \begin{pmatrix} p\mathbb{Z}_{p^2} & p\mathbb{Z}_{p^2} \\ 0 & 0 \end{pmatrix} \lhd \begin{pmatrix} \mathbb{Z}_{p^2} & p\mathbb{Z}_{p^2} \\ 0 & 0 \end{pmatrix},$$

but $\left\langle \begin{pmatrix} p & p \\ 0 & 0 \end{pmatrix} \right\rangle$ is not an ideal in $\begin{pmatrix} \mathbb{Z}_{p^2} & p\mathbb{Z}_{p^2} \\ 0 & 0 \end{pmatrix}$, so R is not filial. Similarly one can show that the ring $\begin{pmatrix} \mathbb{Z}_{p^2} & 0 \\ p\mathbb{Z}_{p^2} & 0 \end{pmatrix}$ is not filial.

It remains to consider the case when R is an indecomposable nil ring.

If $x_1^2 \notin \langle x_1\rangle$, then $|\langle x_1\rangle + \langle x_1^2\rangle| > p^2$. Hence $R = \langle x_1\rangle + \langle x_1^2\rangle$, so $R = [x_1]$ and the thesis follows directly from Proposition 4.5.

Assume that $x_1^2 \in \langle x_1\rangle$. Then there exists $t \in \mathbb{Z}$ such that $x_1^2 = tx_1$. If $p \nmid t$, then $x_1 \in Rx_1$ and since R is nil, so $x_1 = 0$. This is a contradiction. Therefore, $p \mid t$ and $x_1^2 \in \langle px_1\rangle$. Thus $[x_1] = \langle x_1\rangle$ and since R is an H-ring, so $\langle x_1\rangle \lhd R$. Next, $pR^2 = 0$ since $px_1^2 \in p\langle px_1\rangle = \{0\}$, and $p(x_2^2) = p(x_1x_2) = p(x_2x_1) = 0$ since $px_2 = 0$, and $R(p) = \langle px_1\rangle + \langle x_2\rangle$. Hence $x_1x_2, x_2x_1 \in \langle px_1\rangle$. Obviously $x_2^2 \in R(p)$, so there exist $k, l \in \mathbb{Z}$ such that $x_2^2 = k(px_1) + lx_2$. Then $x_2^3 = pkx_1x_2 + lx_2^2 = l(pkx_1 + lx_2)$ and for any $m \in \mathbb{N}$, $x_2^m = pL_mx_1 + l^{m-1}x_2$ for some $L_m \in \mathbb{Z}$. Since R is a nil ring, so $x_2^m = 0$ for some $m \in \mathbb{Z}$ and $l^{m-1}x_2 = 0$. Thus $p \mid l$ and $x_2^2 = pkx_1$.

From the above considerations we see that $x_1^2 = pax_1$, $x_1x_2 = pbx_1$, $x_2x_1 = pcx_1$, $x_2^2 = pdx_1$ for some $a, b, c, d \in \mathbb{Z}_p$.

If $d = 0$, then $x_2^2 = 0$ and $[x_2] = \langle x_2\rangle \lhd R$. In this case $R = \langle x_1\rangle \oplus \langle x_2\rangle$, which is a contradiction. Therefore, $d \neq 0$.

Now, assume that R is commutative. Theorem 2 of [AntE82] implies that there are only the following cases:

(1) $x_2^2 = px_1$, $x_1^2 = x_1x_2 = x_2x_1 = 0$,
(2) $x_2^2 = apx_1$ for some $0 \neq a \in \mathbb{Z}_p$, $x_1^2 = px_1$, $x_1x_2 = x_2x_1 = 0$.

In case (1), let $T = (\mathbb{Z}_{p^4} \boxplus y\mathbb{Z}_p[y]/(y^3))/(p^3 \cdot 1 - \overline{y}^2)$ where $\overline{y} = y + (y^3)$. The ring $\mathbb{Z}_{p^4} \boxplus y\mathbb{Z}_p[y]/(y^3)$ is filial by Example 3.2, so T is also a filial ring as a homomorphic image of a filial ring. Moreover, the function $\varphi\colon R \to T$ given by the formula $\varphi(kx_1+sx_2)=kp^2\cdot\overline{1}+sY$, where $k,s\in\mathbb{Z}$, $Y=\overline{y}+(p^3\cdot 1-\overline{y}^2)$ is an embedding of rings such that $\operatorname{Im}\varphi\subseteq\beta(T)$, so $\operatorname{Im}\varphi\lhd\beta(T)$.

In case (2), let $S=x\mathbb{Z}_p[x]/(x^3)$. Then S is an almost null ring by Proposition 2.5. The ring $\mathbb{Z}_{p^3}\boxplus S$ is filial by Example 3.2. Next, $T=(\mathbb{Z}_{p^3}\boxplus S)/I$, where $I=(ap^2\cdot 1-X^2)$, $X=x+(x^3)$ is also a filial ring as a homomorphic image of a filial ring. The function $\varphi\colon R\to T$ given by the formula $\varphi(kx_1+sx_2)=kp\cdot\overline{1}+sX$ is an embedding of rings such that $\operatorname{Im}\varphi\subseteq\beta(T)$, so $\operatorname{Im}\varphi\lhd\beta(T)$.

Now, assume that the ring R is not commutative. From Theorem 4 of [AntE82] we have only the following cases to consider:

(3) $x_2^2=x_2x_1=px_1$, $x_1^2=x_1x_2=0$,

(4) $x_2^2=apx_1$ for some $0\neq a\in\mathbb{Z}_p$, $x_1^2=x_1x_2=px_1$, $x_2x_1=0$.

In case (3), by Remark 2.6, there exists an almost null ring $S=[x,y]$ and $A\in\mathbb{Z}_p$ such that $o(x)=o(y)=o(x^2)=p$, $y^2=Ax^2$, $xy=0$, $yx=x^2$ and the congruence $X^2+X+A\equiv 0 \pmod p$ has no solutions. Let $T=(\mathbb{Z}_{p^3}\boxplus S)/I$, where $I=(p^2\cdot 1-x^2)$. The ring $\mathbb{Z}_{p^3}\boxplus S$ is filial by Example 3.2, so T is also a filial ring as a homomorphic image of a filial ring. The function $\varphi\colon R\to T$ given by $\varphi(kx_1+sx_2)=(kp\cdot 1+kx+sBy)+I$, where $k,s\in\mathbb{Z}$, $B\in\mathbb{Z}_p$ and $B\cdot A=1$, is an embedding of rings such that $\operatorname{Im}\varphi\subseteq\beta(T)$, so $\operatorname{Im}\varphi\lhd\beta(T)$.

In case (4), assume first that the congruence $X^2+X+a\equiv 0 \pmod p$ has no solutions. Then by Remark 2.6, there exists an almost null ring $S=[x,y]$ such that $o(x)=o(y)=o(x^2)=p$, $y^2=ax^2$, $yx=0$ and $xy=x^2$. Let $T=(\mathbb{Z}_{p^4}\boxplus S)/I$, where $I=(p^3\cdot 1-x^2)$. The ring $\mathbb{Z}_{p^4}\boxplus S$ is filial by Example 3.2, so T is also a filial ring as a homomorphic image of a filial ring. Moreover, the function $\varphi\colon R\to T$ given by the formula $\varphi(kx_1+sx_2)=(kp^2\cdot 1+kx+sy)+I$, where $k,s\in\mathbb{Z}$, is an embedding of rings such that $\operatorname{Im}\varphi\subseteq\beta(T)$, so $\operatorname{Im}\varphi\lhd\beta(T)$ and $\operatorname{Im}\varphi\lhd T$. Assume that there exists $b\in\mathbb{Z}_p$ such that $b^2+b+a=0$ in the field $\mathbb{Z}_p$. Since $a\neq 0$, so $p>2$. Hence, $\frac{1}{2}\in\mathbb{Z}_p$ and for $s\in\mathbb{Z}_p$ we have that $s^2+s=(s-\frac{1}{2})^2-\frac{1}{4}$. Therefore $\{s^2+s\ :\ s\in\mathbb{Z}_p\}$ has exactly $1+\frac{p-1}{2}<p$ elements. Thus, there exists $t\in\mathbb{Z}_p$ such that $s^2+s+at\neq 0$ for every $s\in\mathbb{Z}_p$. Moreover, $t\neq 0$ and $t\neq 1$, so $b^2+b+a=0$. By Remark 2.6, it follows that there exists an almost null ring $S=[x,y]$ such that $o(x)=o(x^2)=o(y)=p$, $y^2=atx^2$, $xy=x^2$, $yx=0$. The ring $\mathbb{Z}_{p^3}\boxplus S$ is filial by Example 3.2. Let $T=(\mathbb{Z}_{p^3}\boxplus S)/I$, where $I=(t(1-t)p^2\cdot 1-x^2)$. Then T is a filial ring and the function $\varphi\colon S\to\beta(T)$ given by the formula $\varphi(sx_1+kx_2)=(s(1-t)p\cdot 1+sx+kt^{-1}y)+I$ for $k,s\in\mathbb{Z}$ is an embedding of rings. $\square$

Lemma 4.7. *If a nil-H-p-ring I, which is not an almost null ring is an ideal in a filial ring with an identity, then I is an ideal in a filial p-ring with an identity.*

Proof. Let R be a filial ring with an identity such that $I\lhd R$. Without losing generality, we can assume that I is an essential ideal of R. Then

$\beta(R)$ is a filial ring as an ideal of R. By Corollary 8 of [FP05], $\beta(R)$ is an H-ring. From Corollary 4.6.9 of [KP69], we get that I is nilpotent. If the group $\beta(R)^+$ has an element of infinite order, then by Proposition 2.4(iii), $\beta(R)$ is an almost null ring. Hence I is an almost null ring as a subring of $\beta(R)$, a contradiction. Thus the group $\beta(R)^+$ is torsion. Moreover I is an essential ideal in R, so $\beta(R)^+$ is a p-group. Again, by Proposition 2.4(ii), the group $\beta(R)^+$ is of finite exponent. Hence there exists $n \in \mathbb{N}$ such that $p^n\beta(R)^+ = 0$. Moreover, $I \subseteq \beta(R)$, so $\beta(R)$ is an essential ideal of R.

If $p \cdot 1 \notin \beta(R)$, then $p \cdot 1 \in Z(R)$ implies that $p^m \cdot 1 \notin \beta(R)$ for every $m \in \mathbb{N}$. In particular $p^m R \neq 0$ for all $m \in \mathbb{N}$. By essentiality of $\beta(R)$ in R we have $p^{n+1}R \cap \beta(R) \neq 0$. Hence $p^{n+1}r \in \beta(R) \setminus \{0\}$ for some $r \in R$. Thus $(p(r)_R)^{n+1} \subseteq \beta(R)$, so $((p(r)_R + \beta(R))/\beta(R))^{n+1} = 0$ and $p(r)_R \subseteq \beta(R)$. Consequently $0 = p^n(p(r)_R) = p^{n+1}(r)_R$, so $p^{n+1}r = 0$, a contradiction. Hence $p \cdot 1 \in \beta(R)$ and there exists $s \in \mathbb{N}$ such that $(p \cdot 1)^s = 0$. Then $p^s R = 0$ and R is a p-ring. □

Now we prove Theorem 1.1 stated in Introduction.

Proof. The first part of the theorem is a direct consequence of Proposition 2.3, Corollary 4.3 and Theorem 4.6.

Let $p \in \mathbb{P}$ and let I be the ring described in Example 2.8. Then $p^2 I = 0$, $|I| = p^4$ and I is a commutative nil-H-ring, which is not an almost null. Next, $I(p)^+ = \langle px_1\rangle \oplus \langle x_2\rangle \oplus \langle x_3\rangle$, so $I(p)^2 = \langle px_1\rangle$. Hence $I(p)^2 + \langle x_1\rangle = \langle x_1\rangle$ and $x_1^2 \notin I(p)^2 + \langle x_1\rangle$. Thus, by Lemma 4.1 the ring I is not an ideal in any filial p-ring with an identity. By Lemma 4.7, I is not an ideal in any filial ring with an identity. □

From Theorem 1.1 it follows, that the 16-element ring is the smallest filial ring, which is not an ideal in any filial ring with an identity.

Acknowledgements. The authors are grateful to the referee and the editor for useful comments and advices.

References

[And66] Andrijanov, V. I. Mixed Hamiltonian nilrings. *Ural. Gos. Univ. Mat. Zap.* **5** (1966), 15–30. MR0201475 (34 # 1359).

[And67] Andrijanov, V. I. Periodic Hamiltonian rings. *Mat. Sb. (N.S.)* **74 (116)** (1967), 241–261. MR0217121 (36 # 212), Zbl 0192.37703, doi: 10.1070/SM1967v003n02ABEH002372.

[A03] Andruszkiewicz, R. R. The classification of integral domains in which the relation of being an ideal is transitive. *Comm. Algebra.* **31** (2003), no. 5, 2067–2093. MR1976268 (2004c:13035), Zbl 1075.13009, doi: 10.1081/AGB-120018987.

[APr09] Andruszkiewicz, R. R.; Pryszczepko, K. A classification of commutative reduced filial rings. *Comm. Algebra.* **37** (2009), no. 11, 3820–3826. MR2573220 (2011b:16009), Zbl 1180.13004, doi: 10.1080/00927870802545703.

[APr10] Andruszkiewicz, R. R.; Pryszczepko, K. On commutative reduced filial rings. *Bull. Aust. Math. Soc.* **81** (2010), no. 2, 310–316. MR2609112 (2011e:16001), Zbl 1209.13009, doi: 10.1017/S0004972709000847.

[APr12] Andruszkiewicz, R. R.; Pryszczepko, K. The classification of commutative Noetherian, filial rings with identity. *Comm. Algebra.* **40** (2012), no. 5, 1690–1703. MR2924477, Zbl 1253.13003, doi: 10.1080/00927872.2011.554933.

[APu88] Andruszkiewicz, R.; Puczyłowski, E. R. On filial rings. *Portugal. Math.* **45** (1988), no. 2, 139–149. MR0952532 (89j:16032), Zbl 0653.16024.

[APu90] Andruszkiewicz, R. R.; Puczyłowski, E. R. Kurosh's chains of associative rings. *Glasgow Math. J.* **32** (1990), no. 1, 67–69. MR1045086 (91c:16017), Zbl 0695.16004, doi: 10.1017/S001708950000906X.

[APu97] Andruszkiewicz, R. R.; Puczyłowski, E. R. Accessible subrings and Kurosh's chains of associative rings. *Algebra Colloq.* **4** (1997), no. 1, 79–88. MR1440025 (98d:16034), Zbl 0876.16015, doi: 10.1017/S1446788713000268.

[AS13] Andruszkiewicz, Ryszard R.; Sobolewska, Magdalena. Accessible subrings and Kurosh's chains of associative rings. *J. Aust. Math. Soc.* **95** (2013), no. 2, 145–157. MR3142352, Zbl 06250152, doi: 10.1017/S1446788713000268.

[AntE82] Antipkin, V. G.; Elizarov, V. P. Rings of order p^3. *Sibirsk. Mat. Zh.* **23** (1982), no. 4, 9–18, 219. MR0668331 (84d:16025), Zbl 0497.16009, doi: 10.1007/BF00968650.

[Bei82] Beĭdar, K. I. A chain of Kurosh may have an arbitrary finite length. *Czechoslovak Math. J.* **32 (107)** (1982), no. 3, 418–422. MR0669784 (83m:16007), Zbl 0503.16005.

[FP04] Filipowicz, M.; Puczyłowski, E. R. Left filial rings. *Algebra Colloq.* **11** (2004), no. 3, 335–344. MR2081192 (2005d:16006), Zbl 1079.16001.

[FP05] Filipowicz, M.; Puczyłowski, E. R. On filial and left filial rings. *Publ. Math. Debrecen* **66** (2005), no. 3–4, 257–267. MR2137768 (2006d:16005), Zbl 1081.16001.

[FP09] Filipowicz, M.; Puczyłowski, E. R. The structure of left filial algebras over a field. *Taiwanese J. Math.* **13** (2009), no. 3, 1017–1029. MR2526355 (2010f:16005), Zbl 1183.16001.

[Fre60] Freĭdman. Letter to the editors (with regard to an article by M. Šperling). *Mat. Sb.* **52 (94)** (1960), 915–916. MR0122836 (23 # A168).

[FH64] Fuchs, L.; Halperin, I. On the imbedding of a regular ring in a regular ring with identity. *Fund. Math.* **54** (1964), 285–290. MR0166211 (29 # 3488), Zbl 0143.05401.

[Fun66] Funayama, Nenosuke. Imbedding a regular ring with identity. *Nagoya Math. J.* **27** (1966), 61–64. MR0195885 (33 # 4081), Zbl 0143.05402.

[GW04] Gardner, B. J.; Wiegandt, R. Radical theory of rings. Monographs and Textbooks in Pure and Applied Mathematics, 261. *Marcel Dekker, Inc., New York*, 2004. xii+387 pp. ISBN: 0-8247-5033-0. MR2015465 (2004m:16031), Zbl 1034.16025, doi: 10.1201/9780203913352.

[Hei68] Heinicke, A. G. A note on lower radical constructions for associative rings. *Canad. Math. Bull.* **11** (1968), 23–30. MR0241463 (39 # 2803), Zbl 0155.35903, doi: 10.4153/CMB-1968-003-x.

[JS57] Jones, A.; Schäffer, J. J. Concerning the structure of certain rings. *Bol. Fac. Ingen. Agrimens. Montevideo.* **6** (1957/1958), 327–335. MR0101874 (21 # 680), Zbl 0084.03704.

[Kru64] Kruse, Robert Leroy. Rings with periodic additive group in which all subrings are ideals. Thesis (Ph.D.) - California Institute of Technology. *ProQuest LLC, Ann Arbor, MI*,(1964). 56 pp. MR2614404, `http://resolver.caltech.edu/CaltechETD:etd-09302002-085204`.

[Kru68] Kruse, Robert L. Rings in which all subrings are ideals. I. *Canad. J. Math.* **20** (1968), 862–871. MR0227207 (37 # 2792), Zbl 0162.04501, doi: 10.4153/CJM-1968-083-1.

[KP69] Kruse, Robert L.; Price, David T. Nilpotent rings. *Gordon and Breach Science Publishers, New York-London-Paris*, 1969. viii+127 pp. MR0266956 (42 # 1858), Zbl 0198.36102.

[P09] Pryszczepko, K. Radicals in classification of commutative reduced filial rings. *Workshop: Radicals of rings and related topics* (Stefan Banach International Center, Warsaw, 2009). http://aragorn.pb.bialystok.pl/~piotrgr/BanachCenter/meeting.html.

[Red52.1] Rédei, L. Vollidealringe im weiteren Sinn. I. *Acta Math. Acad. Sci. Hungar.* **3** (1952), 243–268. MR0054569 (14,941a), Zbl 0049.02101, doi: 10.1007/BF02027824.

[Red52.2] Rédei, L. Die Vollidealringe. *Monatsh. Math.* **56** (1952), 89–95. MR0049162 (14,127c), Zbl 0047.26603, doi: 10.1007/BF01412826.

[Sza81] Szász, F. A. Radicals of rings. *John Wiley & Sons, Ltd., Chichester*, 1981. 287 pp. ISBN: 0-471-27583-2. MR0636787 (84a:16012), Zbl 0461.16009.

(R. R. Andruszkiewicz) Institute of Mathematics, University of Białystok, 15-267 Białystok, Akademicka 2, Poland
randrusz@math.uwb.edu.pl

(K. Pryszczepko) Institute of Mathematics, University of Białystok, 15-267 Białystok, Akademicka 2, Poland
karolp@math.uwb.edu.pl

This paper is available via http://nyjm.albany.edu/j/2014/20-34.html.