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From Stinespring dilation to Sz.-Nagy dilation on the symmetrized bidisc and operator models

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Abstract. We provide an explicit normal distinguished boundary dilation to a pair of commuting operators (S, P) having the closed symmetrized bidisc Γ as a spectral set. This is called Sz.-Nagy dilation of (S, P). The operator pair that dilates (S, P) is obtained by an application of Stinespring dilation of (S, P) given by Agler and Young. We further prove that the dilation is minimal and the dilation space is no bigger than the dilation space of the minimal unitary dilation of the contraction P. We also describe model space and model operators for such a pair (S, P).

Contents

1. Introduction

The *closed symmetrized bidisc* and its *distinguished boundary*, denoted by Γ and $b\Gamma$ respectively, are defined by

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \le 1, |z_2| \le 1\} \subseteq \mathbb{C}^2,$$
$$b\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| = |z_2| = 1\} \subseteq \Gamma.$$

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Clearly, the points of Γ and $b\Gamma$ are the symmetrization of the points of the closed bidisc $\overline{\mathbb{D}}^2$ and the torus $\mathbb{T}^2$ respectively, where the symmetrization map is the following:

$$\pi : \mathbb{C}^2 \to \mathbb{C}^2, \quad (z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2).$$

Function theory, hyperbolic geometry and operator theory related to the set Γ have been well studied over past three decades (e.g., [3, 4, 5, 6, 7, 8, 11, 12, 16]).

Definition 1.1. A pair of commuting operators (S, P), defined on a Hilbert space $\mathcal{H}$, that has Γ as a spectral set is called a Γ-*contraction*, i.e., the joint spectrum $\sigma(S, P) \subseteq \Gamma$ and

$$\|f(S, P)\| \leq \sup_{(z_1,z_2)\in\Gamma} |f(z_1, z_2)|,$$

for all rational functions f with poles off Γ.

By virtue of polynomial convexity of Γ, the definition can be made more precise by omitting the condition on joint spectrum and by replacing rational functions by polynomials. It is clear from the definition that if (S, P) is a Γ-contraction then so is (S^*, P^*) and $\|S\| \leq 2,\ \|P\| \leq 1$.

A commuting d-tuple of operators $\underline{T} = (T_1, T_2, \dots, T_d)$ for which a particular subset of $\mathbb{C}^d$ is a spectral set, has been studied for a long time and many important results have been obtained (see [17]). Let $W \subseteq \mathbb{C}^d$ be a spectral set for $\underline{T} = (T_1, T_2, \dots, T_d)$. A *normal* bW*-dilation* of $\underline{T}$ is a commuting d-tuple of normal operators $\underline{N} = (N_1, \dots, N_d)$ defined on a larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that the joint spectrum $\sigma(\underline{N}) \subseteq bW$ and $q(\underline{T}) = P_{\mathcal{H}} q(\underline{N})\big|_{\mathcal{H}}$, for any polynomial q in d-variables $z_1, \dots, z_d$. A celebrated theorem of Arveson states that W is a complete spectral set for $\underline{T}$ if and only if $\underline{T}$ has a normal bW-dilation, (Theorem 1.2.2 and its corollary, [10]). Therefore, a necessary condition for $\underline{T}$ to have a normal bW-dilation is that W be a spectral set for $\underline{T}$. Sufficiency has been investigated for several domains in several contexts, and it has been shown to have a positive answer when $W = \overline{\mathbb{D}}$ [18], when W is an annulus [1], when $W = \overline{\mathbb{D}^2}$ [9] and when $W = \Gamma$ [3]. Also we have failure of rational dilation on a triply connected domain in $\mathbb{C}$ [2, 14].

The main aim of this paper is to construct an explicit normal $b\Gamma$-dilation to a Γ-contraction (S, P), which we call Sz.-Nagy dilation of (S, P). As a consequence we obtain a concrete functional model for (S, P). The principal source of inspiration is the following dilation theorem which will be called Stinespring dilation of (S, P).

Theorem 1.2 (Agler and Young, [3])**.** *Let (S, P) be a pair of commuting operators on a Hilbert space $\mathcal{H}$ such that the joint spectrum $\sigma(S, P) \subseteq \Gamma$. The following are equivalent:*

(1) *(S, P) is a Γ-contraction.*

(2) $\rho(\alpha S,\, \alpha^2 P) \geq 0$, *for all* $\alpha \in \mathbb{D}$, *where*

$$\rho(S, P) = 2(I - P^*P) - (S - S^*P) - (S^* - P^*S).$$

(3) *For every matrix polynomial* f *in two variables*

$$\|f(S, P)\| \leq \sup_{z \in \Gamma} \|f(z)\|.$$

(4) *There exist Hilbert spaces* $\mathcal{H}_-$, $\mathcal{H}_+$ *and commuting normal operators* $\tilde{S}, \tilde{P}$ *on* $\mathcal{K} = \mathcal{H}_- \oplus \mathcal{H} \oplus \mathcal{H}_+$ *such that the joint spectrum* $\sigma(\tilde{S}, \tilde{P})$ *is contained in the distinguished boundary of* Γ *and* $\tilde{S}, \tilde{P}$ *are expressible by operators matrices of the form*

$$\tilde{S} = \begin{bmatrix} * & * & * \\ 0 & S & * \\ 0 & 0 & * \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} * & * & * \\ 0 & P & * \\ 0 & 0 & * \end{bmatrix}$$

with respect to the orthogonal decomposition $\mathcal{K} = \mathcal{H}_- \oplus \mathcal{H} \oplus \mathcal{H}_+$.

The reason of calling it Stinespring dilation is that part (4) of the above theorem was obtained by an application of Stinespring's theorem ([17]).

In Theorem 4.3, which is the main result of this paper, we provide such Hilbert spaces $\mathcal{H}_-$, $\mathcal{H}_+$ and such operators $\tilde{S}, \tilde{P}$ explicitly. Indeed, the dilation space $\mathcal{K}\,(= \mathcal{H}_- \oplus \mathcal{H} \oplus \mathcal{H}_+)$ can be chosen to be $l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$ which is same as the dilation space of the minimal unitary dilation of P and the operator $\tilde{P}$ can precisely be the minimal unitary dilation of P. Here $\mathcal{D}_P = \overline{\operatorname{Ran}}\, D_P$, where $D_P = (I - P^*P)^{\frac{1}{2}}$. In order to construct an operator for $\tilde{S}$, i.e., to remove the stars from the matrix of $\tilde{S}$, we need a couple of operators F, F_* which turned out to be the unique solutions to the operator equations

$$\begin{aligned} S - S^*P &= D_P X D_P, & X &\in \mathcal{L}(\mathcal{D}_P), \\ S^* - SP^* &= D_{P^*} X_* D_{P^*}, & X_* &\in \mathcal{L}(\mathcal{D}_{P^*}), \end{aligned} \tag{1.1}$$

respectively (Theorem 3.3). Such an operator equation (1.1) was solved in [11] (Theorem 4.1 in [11]) independently to get a Γ-isometric dilation of a Γ-contraction (Theorem 4.3 in [11]) but it was not a normal $b\Gamma$-dilation. The unique operators F and F_* were called the *fundamental operators* of the Γ-contractions (S, P) and (S^*, P^*) respectively. The fundamental operators of (S, P) and (S^*, P^*) play the key role in the construction of the operator that works for $\tilde{S}$. Since the dilation space is precisely the space of minimal unitary dilation of P, the dilation naturally becomes minimal. This is somewhat surprising because it is a dilation in several variables.

As the title of the paper indicates, we obtain Sz.-Nagy dilation of a Γ-contraction (S, P) from its Stinespring dilation in the sense that we obtain the key ingredient in the dilation, the fundamental operator, as a consequence of Stinespring dilation. Indeed, Theorem 1.2 leads to the following model for Γ-contractions (Theorem 3.2 in [6]).

Theorem 1.3 (Agler and Young [6])**.** *Let (S,P) be a Γ-contraction on a Hilbert space $\mathcal{H}$. There exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a Γ-co-isometry $(S^\flat, P^\flat)$ on $\mathcal{K}$ and an orthogonal decomposition $\mathcal{K}_1 \oplus \mathcal{K}_2$ of $\mathcal{K}$ such that:*

(i) *$\mathcal{H}$ is a common invariant subspace of $S^\flat$ and $P^\flat$, and*

$$S = S^\flat|_{\mathcal{H}}, \quad P = P^\flat|_{\mathcal{H}}.$$

(ii) *$\mathcal{K}_1$ and $\mathcal{K}_2$ reduce both $S^\flat$ and $P^\flat$.*

(iii) *$(S^\flat|_{\mathcal{K}_1}, P^\flat|_{\mathcal{K}_1})$ is a Γ-unitary.*

(iv) *There exists a Hilbert space E and an operator A on E such that $\omega(A) \leq 1$ and $(S^\flat|_{\mathcal{K}_2}, P^\flat|_{\mathcal{K}_2})$ is unitarily equivalent to (T_ψ, T_z) acting on $H^2(E)$, where $\psi \in \mathcal{L}(E)$ is given by $\psi(z) = A^* + A\bar{z}, \quad z \in \bar{\mathbb{D}}$.*

In Section 3, we establish the existence and uniqueness of fundamental operator F of (S,P) (Theorem 3.3) by an application of Theorem 1.3. Moreover, we show that the numerical radius of F is not greater than 1.

In Section 5, we describe a functional model for Γ-contractions (Theorem 5.3), which can be treated as a concrete formulation of the model given as Theorem 1.3 above. We specify the model space and model operators. Also a model is provided for a *pure Γ-isometry* $(\hat{S}, \hat{P})$ in terms of Toeplitz operators (T_φ, T_z) defined on the vectorial Hardy space $H^2(\mathcal{D}_{\hat{P}^*})$, where the multiplier function is given as $\varphi(z) = \hat{F}_*^* + \hat{F}_* z$, $\hat{F}_*$ being the fundamental operator of $(\hat{S}, \hat{P})$. This model is obtained independently in a simpler way without an application of the functional model for pure Γ-contractions (see Theorem 3.1 in [12]). Let us mention that the class of pure Γ-isometries parallels the class of pure isometries in one variable operator theory.

In Section 2, we recall some preliminary results from the literature of Γ-contraction and these results will be used in sequel.

2. Preliminary results on Γ-contractions

In the literature of Γ-contraction, [3, 4, 5, 6], there are special classes of Γ-contractions like Γ-unitaries, Γ-isometries, Γ-co-isometries which are analogous to unitaries, isometries and co-isometries of single variable operator theory.

Definition 2.1. A commuting pair (S,P), defined on a Hilbert space $\mathcal{H}$, is called a Γ-*unitary* if S and P are normal operators and $\sigma(S,P)$ is contained in the distinguished boundary $b\Gamma$.

Definition 2.2. A commuting pair (S,P) is called a Γ-*isometry* if it the restriction of Γ-unitary to a joint invariant subspace, i.e., a Γ-isometry is a pair of commuting operators which can be extended to a Γ-unitary.

Definition 2.3. A Γ-*co-isometry* is the adjoint of a Γ-isometry, i.e., (S,P) is a Γ-co-isometry if (S^*, P^*) is a Γ-isometry.

Definition 2.4. A Γ-isometry (S, P) is said to be *pure* if P is a pure isometry. A *pure* Γ-*co-isometry* is the adjoint of a pure Γ-isometry.

Definition 2.5. Let (S, P) be a Γ-contraction on a Hilbert space $\mathcal{H}$. A commuting pair (T, V) defined on $\mathcal{K}$ is said to be a Γ-isometric (or Γ-unitary) extension if $\mathcal{H} \subseteq \mathcal{K}$, (T, V) is a Γ-isometry (or a Γ-unitary) and $T|_{\mathcal{H}} = S$, $V|_{\mathcal{H}} = P$.

We are now going to state some useful results on Γ-contractions without proofs because the proofs are either routine or could be found out in [3] and [6].

Proposition 2.6. *If T_1, T_2 are commuting contractions then their symmetrization $(T_1 + T_2, T_1T_2)$ is a Γ-contraction.*

Note that, all Γ-contractions do not arise as a symmetrization of two contractions. The following result characterizes the Γ-contractions which can be obtained as a symmetrization of two commuting contractions.

Lemma 2.7 ([6])**.** *Let (S, P) be a Γ-contraction. Then*

$$(S, P) = (T_1 + T_2, T_1T_2)$$

for a pair of commuting operators T_1, T_2 if and only if $S^2 - 4P$ has a square root that commutes with both S and P.

Here is a set of characterizations for Γ-unitaries.

Theorem 2.8 ([6])**.** *Let (S, P) be a pair of commuting operators defined on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(1) *(S, P) is a Γ-unitary.*
(2) *There exist commuting unitary operators U_1 and U_2 on $\mathcal{H}$ such that*

$$S = U_1 + U_2, \quad P = U_1U_2.$$

(3) *P is unitary, $S = S^*P$, and $r(S) \leq 2$, where $r(S)$ is the spectral radius of S.*

We now present a structure theorem for the class of Γ-isometries.

Theorem 2.9 ([6])**.** *Let S, P be commuting operators on a Hilbert space $\mathcal{H}$. The following statements are all equivalent:*

(1) *(S, P) is a Γ-isometry.*
(2) *If P has Wold-decomposition with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $P|_{\mathcal{H}_1}$ is unitary and $P|_{\mathcal{H}_2}$ is pure isometry then $\mathcal{H}_1$, $\mathcal{H}_2$ reduce S also and $(S|_{\mathcal{H}_1}, P|_{\mathcal{H}_1})$ is a Γ-unitary and $(S|_{\mathcal{H}_2}, P|_{\mathcal{H}_2})$ is a pure Γ-isometry.*
(3) *P is an isometry , $S = S^*P$ and $r(S) \leq 2$.*

3. The fundamental operator of a Γ-contraction

Let us recall that the *numerical radius* of an operator T on a Hilbert space $\mathcal{H}$ is defined by

$$\omega(T) = \sup\{|\langle Tx, x\rangle| \ : \ \|x\|_{\mathcal{H}} \leq 1\}.$$

It is well known that

$$r(T) \leq \omega(T) \leq \|T\| \quad \text{and} \quad \frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \tag{3.1}$$

where $r(T)$ is the spectral radius of T. The following is an interesting result about the numerical radius of an operator and this will be used in this section.

Lemma 3.1. *The numerical radius of an operator X is not greater than 1 if and only if Re $\beta X \leq I$ for all complex numbers β of modulus 1.*

Proof. It is obvious that $\omega(X) \leq 1$ implies that Re $\beta X \leq I$ for all $\beta \in \mathbb{T}$. We prove the other way. By hypothesis, $\langle \text{Re } \beta X h, h\rangle \leq 1$ for all $h \in \mathcal{H}$ with $\|h\| \leq 1$ and for all $\beta \in \mathbb{T}$. Note that $\langle \text{Re } \beta X h, h\rangle = \text{Re } \beta\langle Xh, h\rangle$. Write $\langle Xh, h\rangle = e^{i\varphi_h}|\langle Xh, h\rangle|$ for some $\varphi_h \in \mathbb{R}$, and then choose $\beta = e^{-i\varphi_h}$. Then we get $|\langle Xh, h\rangle| \leq 1$ and this holds for each $h \in \mathcal{H}$ with $\|h\| \leq 1$. Hence done. □

We are going to prove the existence and uniqueness of solution to the operator equation

$$S - S^*P = D_P X D_P, \quad X \in \mathcal{L}(\mathcal{D}_P)$$

by an application of a famous result due to Douglas, Muhly and Pearcy. Let us again mention here that the same operator equation has been solved in [11] (Theorem 4.2) independently by using operator Fejer–Riesz Theorem. Here is the famous result of Douglas, Muhly and Pearcy.

Proposition 3.2 (Douglas, Muhly and Pearcy, [13]). *For $i = 1, 2$, let T_i be a contraction on a Hilbert space $\mathcal{H}_i$, and let X be an operator mapping $\mathcal{H}_2$ into $\mathcal{H}_1$. A necessary and sufficient condition that the operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by the matrix*

$$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

be a contraction is that there exist a contraction C mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ such that

$$X = \sqrt{I_{\mathcal{H}_1} - T_1 T_1^*}\, C \sqrt{I_{\mathcal{H}_2} - T_2^* T_2}.$$

Theorem 3.3 (Existence and Uniqueness). *For a Γ-contraction (S, P) defined on $\mathcal{H}$, the operator equation*

$$S - S^*P = D_P X D_P$$

has a unique solution F in $\mathcal{L}(\mathcal{D}_P)$ and $\omega(F) \leq 1$.

Proof. By Theorem 1.3, there is a Γ-co-isometry (T,V) on a larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is a joint invariant subspace of T and V and

$$S = T|_{\mathcal{H}},\ P = V|_{\mathcal{H}}.$$

Also $\mathcal{K}$ has orthogonal decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ and

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \quad \text{on } \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$$

such that (T_1, V_1) is a Γ-unitary and there is a Hilbert space E and a unitary $U_1 : \mathcal{K}_2 \to H^2(E)$ such that

$$T_2^* = U_1^* T_\varphi U_1,\ V_2^* = U_1^* T_z U_1,$$

where $\varphi(z) = A + A^* z$ for some $A \in \mathcal{B}(E)$ with numerical radius of A being not greater than 1. Clearly $T_2 = U_1^* T_\varphi^* U_1$ and $V_2 = U_1^* T_z^* U_1$. Again $H^2(E)$ can be identified with $l^2(E)$ and consequently the operator pair (T_φ, T_z) can be identified with (M_φ, M_z), where M_φ and M_z are defined on $l^2(E)$ in the following way:

$$M_\varphi = \begin{bmatrix} A & 0 & 0 & \cdots \\ A^* & A & 0 & \cdots \\ 0 & A^* & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad M_z = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore we can say that there is a unitary $U : \mathcal{K}_2 \to l^2(E)$ such that $T_2 = U^* M_\varphi^* U,$ and $V_2 = U^* M_z^* U$. Now

$$\begin{aligned}
T_2 - T_2^* V_2 &= U^* \begin{bmatrix} A & A^* & 0 & \cdots \\ 0 & A & A^* & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U \\
&\quad - U^* \begin{bmatrix} A^* & 0 & 0 & \ldots \\ A & A^* & 0 & \cdots \\ 0 & A & A^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & I & 0 & \cdots \\ 0 & 0 & I & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U \\
&= U^* \begin{bmatrix} A & A^* & 0 & \cdots \\ 0 & A & A^* & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U - U^* \begin{bmatrix} 0 & A^* & 0 & \cdots \\ 0 & A & A^* & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U \\
&= U^* \begin{bmatrix} A & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U.
\end{aligned}$$

Also

$$D_{V_2}^2 = I - V_2^* V_2 = U^*(I - M_z M_z^*)U = U^* \begin{bmatrix} I & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U.$$

It is merely said that $D_{V_2}^2 = D_{V_2}$ and therefore if we set

$$X = U^* \begin{bmatrix} A^* & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U$$

then $X \in \mathcal{L}(\mathcal{D}_{V_2})$ and $T_2 - T_2^* V_2 = D_{V_2} X D_{V_2}$. Since (T_1, V_1) is a Γ-unitary, Theorem 2.8 guarantees that $T_1 = T_1^* V_1$. Therefore,

$$T - T^* V = \begin{bmatrix} T_1 - T_1^* V_1 & 0 \\ 0 & T_2 - T_2^* V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & T_2 - T_2^* V_2 \end{bmatrix}.$$

Also

$$D_V^2 = \begin{bmatrix} I_{\mathcal{K}_1} - V_1^* V_1 & 0 \\ 0 & I_{\mathcal{K}_2} - V_2^* V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{K}_2} - V_2^* V_2 \end{bmatrix}.$$

Therefore, $\mathcal{D}_V = \mathcal{D}_{V_2}$ and X satisfies the relation $T - T^* V = D_V X D_V$. Also

$$\|T - T^* V\| = \|D_V X D_V\| \le \|X\| \le 2, \qquad \text{by relation (3.1) as } \omega(X) \le 1.$$

Now consider the matrix

$$J = \begin{bmatrix} V^* & \dfrac{T - T^* V}{2} \\ 0 & V \end{bmatrix}$$

defined on $\mathcal{K} \oplus \mathcal{K}$. Since

$$\frac{T - T^* V}{2} = D_V \frac{X}{2} D_V = (I - V^* V)^{\frac{1}{2}} \frac{X}{2} (I - V^* V)^{\frac{1}{2}},$$

where $\dfrac{T - T^* V}{2}$ and $\dfrac{X}{2}$ are contractions, by Proposition 3.2, the matrix J is a contraction. Again let us consider another matrix J_H defined on $\mathcal{H} \oplus \mathcal{H}$ by

$$J_H = \begin{bmatrix} P_{\mathcal{H}} V^* |_{\mathcal{H}} & P_{\mathcal{H}} \left(\dfrac{T - T^* V}{2} \right) |_{\mathcal{H}} \\ 0 & P_{\mathcal{H}} V |_{\mathcal{H}} \end{bmatrix}.$$

Since (T, V) is Γ-co-isometric extension of (S, P), we have that

$$J_H = \begin{bmatrix} P^* & \dfrac{S - S^* P}{2} \\ 0 & P \end{bmatrix}.$$

For

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H},$$

we have that

$$
\begin{aligned}
\left\| J_H \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|^2 &= \left\| \begin{bmatrix} P_{\mathcal{H}} V^* h_1 + P_{\mathcal{H}} \left(\dfrac{T - T^* V}{2} \right) h_2 \\ P_{\mathcal{H}} V h_2 \end{bmatrix} \right\|^2 \\
&= \left\| P_{\mathcal{H}} \left(V^* h_1 + \frac{T - T^* V}{2} h_2 \right) \right\|^2 + \| P_{\mathcal{H}} V h_2 \|^2 \\
&\leq \left\| V^* h_1 + \frac{T - T^* V}{2} h_2 \right\|^2 + \| V h_2 \|^2, \text{ since } P_{\mathcal{H}} \text{ is a projection} \\
&\leq \left\| \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|^2, \text{ since } J \text{ is a contraction.}
\end{aligned}
$$

Therefore, J_H is a contraction. Applying Proposition 3.2 again we get an operator $F \in \mathcal{L}(\mathcal{H})$ such that $\dfrac{F}{2}$ is a contraction and that

$$\frac{S - S^* P}{2} = D_P \frac{F}{2} D_P.$$

Obviously the domain of F can be specified to be $\mathcal{D}_P \subseteq \mathcal{H}$. Hence

$$S - S^* P = D_P F D_P$$

where $F \in \mathcal{L}(\mathcal{D}_P)$ and the existence of the fundamental operator of (S, P) is guaranteed.

For uniqueness let there be two such solutions F and F_1. Then

$$D_P \tilde{F} D_P = 0, \quad \text{where } \tilde{F} = F - F_1 \in \mathcal{L}(\mathcal{D}_P).$$

Then

$$\langle \tilde{F} D_P h, D_P h' \rangle = \langle D_P \tilde{F} D_P h, h' \rangle = 0$$

which shows that $\tilde{F} = 0$ and hence $F = F_1$.

To show that the numerical radius of F is not greater than 1, note that $\rho(\alpha S, \alpha^2 P) \geq 0$, for all $\alpha \in \mathbb{D}$, by Theorem 1.2 and the inequality can be extended by continuity to all points in $\overline{\mathbb{D}}$. Therefore, in particular for $\beta \in \mathbb{T}$, we have

$$D_P^2 \geq \operatorname{Re} \beta (S - S^* P) = \operatorname{Re} \beta (D_P F D_P)$$

which implies that

$$D_P (I_{\mathcal{D}_P} - \operatorname{Re} \beta F) D_P \geq 0.$$

Therefore,

$$\langle (I_{\mathcal{D}_P} - \operatorname{Re} (\beta F)) D_P h, D_P h \rangle = \langle D_P (I_{\mathcal{D}_P} - \operatorname{Re} (\beta F)) D_P h, h \rangle \geq 0$$

and consequently we obtain

$$\operatorname{Re} \beta F \leq I_{\mathcal{D}_P}, \text{ for all } \beta \in \mathbb{T}.$$

Thus, by Lemma 3.1, the numerical radius of F is not greater than 1. □

Remark 3.4. *The fundamental operator of a Γ-isometry or a Γ-unitary (S, P) is the zero operator because $S = S^* P$ in this case.*

The following result is obvious and a proof to this can be found in [12].

Proposition 3.5. *Let (S,P) and (S_1,P_1) be two Γ-contractions on a Hilbert space $\mathcal{H}$ and let F and F_1 be their fundamental operators respectively. If (S,P) and (S_1,P_1) are unitarily equivalent then so are F and F_1.*

4. Geometric construction of normal dilation

In this section, we present an explicit construction of a normal $b\Gamma$-dilation, i.e., a Γ-unitary dilation of a Γ-contraction. In the literature, the Γ-unitary and Γ-isometric dilation of a Γ-contraction are defined in the following way:

Definition 4.1. Let (S,P) be a Γ-contraction on a Hilbert space $\mathcal{H}$. A pair of commuting operators (T,U) defined on a Hilbert space $\mathcal{K}\supseteq\mathcal{H}$ is said to be a Γ-*unitary dilation* of (S,P) if (T,U) is a Γ-unitary and

$$P_{\mathcal{H}}(T^mU^n)|_{\mathcal{H}}=S^mP^n,\quad n=0,1,2,\dots.$$

Moreover, the dilation will be called *minimal* if

$$\mathcal{K}=\overline{\text{span}}\{T^mU^nh:\ h\in\mathcal{H},\ m,n=0,\pm1,\pm2,\dots\},$$

where T^{-m},U^{-n} for positive integers m,n are defined as T^{*m} and U^{*n} respectively. A Γ-*isometric dilation* of a Γ-contraction is defined in a similar way where the word Γ-unitary is replaced by Γ-isometry. But when we talk about minimality of such a Γ-isometric dilation, the powers of the dilation operators will run over nonnegative integers only.

In the dilation theory of a single contraction ([18]), it is a notable fact that if V is the minimal isometric dilation of a contraction T, then V^* is a co-isometric extension of P. The other way is also true, i.e., if V is a co-isometric extension of T, then V^* is an isometric dilation of T^*. Here we shall see that an analogue holds for a Γ-contraction.

Proposition 4.2. *Let (T,V) on $\mathcal{K}\supseteq\mathcal{H}$ be a Γ-isometric dilation of a Γ-contraction (S,P). If (T,V) is minimal, then (T^*,V^*) is a Γ-co-isometric extension of (S^*,P^*). Conversely, if (T^*,V^*) is a Γ-co-isometric extension of (S^*,P^*) then (T,V) is a Γ-isometric dilation of (S,P).*

Proof. We first prove that $SP_{\mathcal{H}}=P_{\mathcal{H}}T$ and $PP_{\mathcal{H}}=P_{\mathcal{H}}V$, where

$$P_{\mathcal{H}}:\mathcal{K}\to\mathcal{H}$$

is orthogonal projection onto $\mathcal{H}$. Clearly

$$\mathcal{K}=\overline{\text{span}}\{T^mV^nh\ :\ h\in\mathcal{H}\text{ and }m,n\in\mathbb{N}\cup\{0\}\}.$$

Now for $h\in\mathcal{H}$ we have that

$$\begin{aligned}SP_{\mathcal{H}}(T^mV^nh)&=S(S^mP^nh)=S^{m+1}P^nh\\&=P_{\mathcal{H}}(T^{m+1}V^nh)=P_{\mathcal{H}}T(T^mV^nh).\end{aligned}$$

Thus we have that $SP_{\mathcal{H}}=P_{\mathcal{H}}T$ and similarly we can prove that

$$PP_{\mathcal{H}}=P_{\mathcal{H}}V.$$

Also for $h \in \mathcal{H}$ and $k \in \mathcal{K}$ we have that

$$\begin{aligned}\langle S^*h, k\rangle = \langle P_{\mathcal{H}}S^*h, k\rangle = \langle S^*h, P_{\mathcal{H}}k\rangle = \langle h, SP_{\mathcal{H}}k\rangle &= \langle h, P_{\mathcal{H}}Tk\rangle \\ &= \langle T^*h, k\rangle.\end{aligned}$$

Hence $S^* = T^*|_{\mathcal{H}}$. Similarly $P^* = V^*|_{\mathcal{H}}$. The converse part is obvious. □

Now we present the geometric construction of Sz.-Nagy dilation of a Γ-contraction.

Theorem 4.3. *Let (S, P) be a Γ-contraction defined on a Hilbert space $\mathcal{H}$. Let F and F_* be the fundamental operators of (S, P) and its adjoint (S^*, P^*) respectively. Let*

$$\begin{aligned}\mathcal{K}_0 &= \cdots \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots \\ &= l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*}).\end{aligned}$$

Consider the operator pair (T_0, U_0) defined on $\mathcal{K}_0$ by

$$\begin{aligned}&T_0(\dots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots) \\ &= (\dots, Fh_{-2} + F^*h_{-1}, Fh_{-1} + F^*D_Ph_0 - F^*P^*h_1, \\ &\qquad \underbrace{Sh_0 + D_{P^*}F_*h_1}, F_*^*h_1 + F_*h_2, F_*^*h_2 + F_*h_3, \dots) \\ &\quad \cdot U_0(\cdots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots) \\ &= (\dots, h_{-2}, h_{-1}, D_Ph_0 - P^*h_1, \underbrace{Ph_0 + D_{P^*}h_1}, h_2, h_3 \dots),\end{aligned}$$

where the 0-th position of a vector in $\mathcal{K}_0$ has been indicated by an under brace. Then (T_0, U_0) is a minimal Γ-unitary dilation of (S, P).

Proof. The matrices of T_0 with respect to the orthogonal decompositions $l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$ and $\cdots \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots$ of $\mathcal{K}_0$ and the matrix of U_0 with respect to the decomposition

$$\cdots \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots$$

are the following:

$$T_0 = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & S & A_4 \\ 0 & 0 & A_5 \end{bmatrix}$$

$$
(4.1) \qquad = \left[\begin{array}{cccc|c|cccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & F & F^* & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & F & F^* & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & F & F^*D_P & -F^*P^* & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & S & D_{P^*}F_* & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & 0 & F_*^* & F_* & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & F_*^* & F_* & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & F_*^* & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right],
$$

$$
(4.2) \qquad U_0 = \left[\begin{array}{cccc|c|cccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & I & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & I & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & D_P & -P^* & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & P & D_{P^*} & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & 0 & 0 & I & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & I & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right].
$$

The dilation space $\mathcal{K}_0$ is obviously the minimal unitary dilation space of the contraction P and clearly the operator U_0 is the minimal unitary dilation of P. The space $\mathcal{H}$ can be embedded inside $\mathcal{K}_0$ by the canonical map $h \mapsto (\dots, 0, 0, \underbrace{h}, 0, 0, \dots)$. The adjoints of T_0 and U_0 are defined in the following way:

$$
\begin{aligned}
T_0^*(\dots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots) \\
= (\dots, Fh_{-3} + F^*h_{-2}, Fh_{-2} + F^*h_{-1}, \underbrace{D_PFh_{-1} + S^*h_0}, \\
- PFh_{-1} + F_*^*D_{P^*}h_0 + F_*h_1, F_*^*h_1 + F_*h_2, \dots),
\end{aligned}
$$

$$
\begin{aligned}
U_0^*(\dots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots) \\
= (\dots, h_{-3}, h_{-2}, \underbrace{D_Ph_{-1} + P^*h_0}, -Ph_1, D_{P^*}h_0, h_1 \dots).
\end{aligned}
$$

To prove (T_0, U_0) to be a minimal Γ-unitary dilation of (S, P) we have to show the following:

(i) (T_0, U_0) is a Γ-unitary.
(ii) (T_0, U_0) dilates (S, P).
(iii) The dilation (T_0, U_0) is minimal.

For proving (T_0, U_0) to be a Γ-unitary one needs to verify, by virtue of Theorem 2.8(3), the following:

$$T_0U_0 = U_0T_0, \quad T_0 = T_0^*U_0 \quad \text{and} \quad r(T_0) \le 2.$$

Now

$$\begin{aligned}
&T_0U_0(\dots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots)\\
&= T_0(\dots, h_{-2}, h_{-1}, D_Ph_0 - P^*h_1, \underbrace{Ph_0 + D_{P^*}h_1}, h_2, h_3 \dots)\\
&= (\dots, Fh_{-1} + F^*D_Ph_0 - F^*P^*h_1,\\
&\qquad (FD_P + F^*D_PP)h_0 + (-FP^* + F^*D_PD_{P^*})h_1 - F^*P^*h_2,\\
&\qquad \underbrace{SPh_0 + SD_{P^*}h_1 + D_{P^*}F_*h_2}, F_*^*h_2 + F_*h_3, F_*^*h_3 + F_*h_4, \dots).
\end{aligned}$$

Also

$$\begin{aligned}
&U_0T_0(\dots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots)\\
&= U_0(\dots, Fh_{-2} + F^*h_{-1}, Fh_{-1} + F^*D_Ph_0 - F^*P^*h_1,\\
&\qquad \underbrace{Sh_0 + D_{P^*}F_*h_1}, F_*^*h_1 + F_*h_2, F_*^*h_2 + F_*h_3, \dots)\\
&= (\dots, Fh_{-1} + F^*D_Ph_0 - F^*P^*h_1, D_PSh_0 + (D_PD_{P^*}F_* - P^*F_*^*)h_1\\
&\qquad - P^*F_*h_2, \underbrace{PSh_0 + (PD_{P^*}F_* + D_{P^*}F_*^*)h_1 + D_{P^*}F_*h_2},\\
&\qquad\quad F_*^*h_2 + F_*h_3, F_*^*h_3 + F_*h_4, \dots).
\end{aligned}$$

In order to prove $T_0U_0 = U_0T_0$ we have to prove the following things:

(a_1) $D_PS = FD_P + F^*D_PP$.
(a_2) $D_PD_{P^*}F_* - P^*F_*^* = -FP^* + F^*D_PD_{P^*}$.
(a_3) $SD_{P^*} = D_{P^*}F_*^* + PD_{P^*}F_*$.
(a_4) $F^*P^* = P^*F_*$.

(a_1). Let $J = FD_P + F^*D_PP - D_PS$. Then J is an operator from $\mathcal{H}$ to $\mathcal{D}_P$. Since F is the solution of $S - S^*P = D_PXD_P$ we have that

$$\begin{aligned}
D_PJ &= D_PFD_P + D_PF^*D_PP - D_P^2S\\
&= (S - S^*P) + (S^* - P^*S)P + (I - P^*P)S\\
&= 0.
\end{aligned}$$

Clearly $\langle Jh, D_Ph_1\rangle = \langle D_PJh, h_1\rangle = 0$ for all $h, h_1 \in \mathcal{H}$ and hence $J = 0$ which proves (a_1).

(a_2). It is enough to show that $FP^* - P^*F_*^* = F^*D_PD_{P^*} - D_PD_{P^*}F_*$, where each side is defined from $\mathcal{D}_{P^*}$ to $\mathcal{D}_P$.

$$\begin{aligned}
&D_P(FP^* - P^*F_*^*)D_{P^*}\\
&= (D_PFD_P)P^* - P^*(D_{P^*}F_*^*D_{P^*}), \text{ using the relation } PD_P = D_{P^*}P\\
&= (S - S^*P)P^* - P^*(S^* - SP^*)^*\\
&= SP^* - S^*PP^* - P^*S + P^*PS
\end{aligned}$$

$$= (S^* - P^*S)(I - PP^*) - (I - P^*P)(S^* - SP^*)$$
$$= (S^* - P^*S)D_{P^*}^2 - D_P^2(S^* - SP^*)$$
$$= (D_PF^*D_P)D_{P^*}^2 - D_P^2(D_{P^*}F_*D_{P^*}).$$

For a proof of $PD_P = D_{P^*}P$ one can see Chapter I of [18]. Hence (a_2) is proved.

(a_3). Setting $J_1 = D_{P^*}F_*^* + PD_{P^*}F_* - SD_{P^*}$ which maps $\mathcal{D}_{P^*}$ into $\mathcal{H}$ and using the same argument as in the proof of (a_1), we can obtain $J_1D_{P^*} = 0$ which proves (a_3).

(a_4). This follows from the fact that $PF = F_*^*P|_{\mathcal{D}_P}$.

Proof of $PF = F_*^*P|_{\mathcal{D}_P}$. For $D_Ph \in \mathcal{D}_P$ and $D_{P^*}h' \in \mathcal{D}_{P^*}$, we have that

$$\begin{aligned}
\langle PFD_Ph, D_{P^*}h'\rangle &= \langle D_{P^*}PFD_Ph, h'\rangle \\
&= \langle PD_PFD_Ph, h'\rangle \\
&= \langle P(S - S^*P)h, h'\rangle \\
&= \langle (S - PS^*)Ph, h'\rangle \\
&= \langle D_{P^*}{F_*}^*D_{P^*}Ph, h'\rangle, \quad (\text{since } S^* - SP^* = D_{P^*}F_*D_{P^*}) \\
&= \langle {F_*}^*PD_Ph, D_{P^*}h'\rangle. \qquad \square
\end{aligned}$$

Therefore $T_0U_0 = U_0T_0$.

We now show that $T_0 = T_0^*U_0$. We have that

$$\begin{aligned}
&T_0^*U_0(\dots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \dots) \\
&= T_0^*(\dots, h_{-2}, h_{-1}, D_Ph_0 - P^*h_1, \underbrace{Ph_0 + D_{P^*}h_1}, h_2, h_3 \dots) \\
&= (\dots, Fh_{-2} + F^*h_{-1}, Fh_{-1} + F^*D_Ph_0 - F^*P^*h_1, \\
&\quad \underbrace{(D_PFD_P + S^*P)h_0 + (-D_PFP^* + S^*D_{P^*})h_1}, \\
&\quad (-PFD_P + F_*^*D_{P^*}P)h_0 + (PFP^* + F_*^*D_{P^*}^2)h_1 + F_*h_2, \\
&\quad F_*^*h_2 + F_*h_3, \dots).
\end{aligned}$$

Since F is the fundamental operator of (S, P) we have $S = S^*P + D_PFD_P$. Therefore, in order to prove $T_0 = T_0^*U_0$, we need to show the following three steps:

(b_1) $D_{P^*}F_* = S^*D_{P^*} - D_PFP^*$.
(b_2) $PFD_P = F_*^*D_{P^*}P$.
(b_3) $PFP^* + F_*^*D_{P^*}^2 = F_*^*$.

For proving (b_1) let us set $G = D_{P^*}F_* + D_PFP^* - S^*D_{P^*}$. Obviously G maps $\mathcal{D}_{P^*}$ into $\mathcal{H}$ and

$$\begin{aligned}
GD_{P^*} &= D_{P^*}F_*D_{P^*} + D_PFP^*D_{P^*} - S^*D_{P^*}^2 \\
&= (S^* - SP^*) + (S - S^*P)P^* - S^*(I - PP^*), \text{ by } P^*D_{P^*} = D_PP^* \\
&= 0,
\end{aligned}$$

which proves (b_1). The other two parts, (b_2) and (b_3), follow from the facts that $PD_P = D_{P^*}P$ and $PF = F_*^*P|_{\mathcal{D}_P}$.

In the matrix of T_0, A_1 on $l^2(\mathcal{D}_P)$ is same as the multiplication operator M_{F+F^*z} on $l^2(\mathcal{D}_P)$. For $z = e^{i\theta} \in \mathbb{T}$ we have that

$$\begin{aligned}
\|F + F^*z\| &= \|F + e^{i\theta}F^*\| \\
&= \|e^{-i\theta/2}F + e^{i\theta/2}F^*\| \\
&= \sup_{\|x\|\leq 1} |\langle (e^{-i\theta/2}F + e^{i\theta/2}F^*)x, x\rangle| \\
&\qquad\qquad \text{(since } e^{-i\theta/2}F + e^{i\theta/2}F^* \text{ is self-adjoint)} \\
&\leq \omega(F) + \omega(F^*) \\
&\leq 2, \qquad \text{(since } \omega(F) \leq 1).
\end{aligned}$$

So by the Maximum Modulus Principle, $\|F + F^*z\| \leq 2$ for all $z \in \overline{\mathbb{D}}$. Thus, $\|A_1\| = \|M_{F+F^*z}\| = \|F + F^*z\| \leq 2$. Similarly we can show that $\|A_5\| \leq 2$. Also $\|S\| \leq 2$, because (S, P) is a Γ-contraction. Again by Lemma 1 of [15] we have that $\sigma(T_0) \subseteq \sigma(A_1) \cup \sigma(S) \cup \sigma(A_5)$. Therefore, $r(T_0) \leq 2$. Hence (T_0, U_0) is a Γ-unitary.

It is evident from the matrices of T_0 and U_0 that $P_{\mathcal{H}}(T_0^m U_0^n)|_{\mathcal{H}} = S^m P^n$ for all nonnegative integers m, n which proves that (T_0, U_0) dilates (S, P). The minimality of the Γ-unitary dilation (T_0, U_0) follows from the fact that $\mathcal{K}_0$ and U_0 are respectively the minimal unitary dilation space and minimal unitary dilation of P. Hence the proof is complete. □

An explicit Γ-isometric dilation of a Γ-contraction was provided in [11] (see Theorem 4.3 in [11]). Here we show that the Γ-isometric dilation can easily be obtained as the restriction of the Γ-unitary dilation described in the previous theorem.

Corollary 4.4. *Let $\mathcal{N}_0 \subseteq \mathcal{K}_0$ be defined as $\mathcal{N}_0 = \mathcal{H} \oplus l^2(\mathcal{D}_P)$. Then $\mathcal{N}_0$ is a common invariant subspace of T_0, U_0 and $(T^\flat, V^\flat) = (T_0|_{\mathcal{N}_0}, U_0|_{\mathcal{N}_0})$ is a minimal Γ-isometric dilation of (S, P).*

Proof. It is evident from the matrix form of T_0 and U_0 (from the previous theorem) that $\mathcal{N}_0 = \mathcal{H} \oplus l^2(\mathcal{D}_P) = H \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \cdots$ is a common invariant subspace of T_0 and U_0. Therefore by the definition of Γ-isometry, the restriction of (T_0, U_0) to the common invariant subspace $\mathcal{N}_0$, i.e, $(T^\flat, V^\flat)$ is a Γ-isometry. The matrices of $T^\flat$ and $V^\flat$ with respect to the decomposition $\mathcal{H} \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \cdots$ of $\mathcal{N}_0$ are the following:

$$T^\flat = \begin{bmatrix} S & 0 & 0 & 0 & \cdots \\ F^*D_P & F & 0 & 0 & \cdots \\ 0 & F^* & F & 0 & \cdots \\ 0 & 0 & F^* & F & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad V^\flat = \begin{bmatrix} P & 0 & 0 & 0 & \cdots \\ D_P & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is obvious from the matrices of $T^\flat$ and $V^\flat$ that the adjoint of $(T^\flat, V^\flat)$ is a Γ-co-isometric extension of (S^*, P^*). Therefore by Proposition 4.2, $(T^\flat, V^\flat)$ is a Γ-isometric dilation of (S, P). The minimality of this Γ-isometric dilation follows from the fact that $\mathcal{N}_0$ and $V^\flat$ are respectively the minimal isometric dilation space and minimal isometric dilation of P. Hence the proof is complete. $\square$

Remark 4.5. The minimal Γ-unitary dilation (T_0, U_0) described in Theorem 4.3 is the minimal Γ-unitary extension of minimal Γ-isometric dilation $(T^\flat, V^\flat)$ given in Corollary 4.4. The reason is that if there is any Γ-unitary extension (T, U) of $(T^\flat, V^\flat)$ then U is a unitary extension of $V^\flat$ and U_0 is the minimal unitary extension of $V^\flat$.

5. Functional models

Wold-decomposition breaks an isometry into two parts namely a unitary and a pure isometry. A pure isometry V is unitarily equivalent to the Toeplitz operator T_z on $H^2(\mathcal{D}_{V^*})$. We have an analogous Wold-decomposition for Γ-isometries in terms of a Γ-unitary and a pure Γ-isometry (Theorem 2.9(2)). Again Theorem 2.8 tells us that every Γ-unitary is nothing but the symmetrization of a pair of commuting unitaries. Therefore a standard model for pure Γ-isometries gives a complete picture of a Γ-isometry. In [12], a functional model for pure Γ-contractions has been described. When in particular we are concerned about pure Γ-isometries, it requires a much simpler effort to establish the model.

Theorem 5.1. *Let $(\hat{S}, \hat{P})$ be a commuting pair of operators on a Hilbert space $\mathcal{H}$. If $(\hat{S}, \hat{P})$ is a pure Γ-isometry then there is a unitary operator $U : \mathcal{H} \to H^2(\mathcal{D}_{\hat{P}^*})$ such that*

$$\hat{S} = U^* T_\varphi U, \text{ and } \hat{P} = U^* T_z U, \text{ where } \varphi(z) = \hat{F}_*^* + \hat{F}_* z.$$

Here $\hat{F}_*$ is the fundamental operator of $(\hat{S}^*, \hat{P}^*)$. Conversely, every such pair (T_{A+A^*z}, T_z) on $H^2(E)$ for some Hilbert space E with $\omega(A) \leq 1$ is a pure Γ-isometry.

Proof. First let us suppose that $(\hat{S}, \hat{P})$ is a pure Γ-isometry. Then $\hat{P}$ is a pure isometry and can be identified with T_z on $H^2(\mathcal{D}_{\hat{P}^*})$. Therefore, there is a unitary U from $\mathcal{H}$ onto $H^2(\mathcal{D}_{\hat{P}^*})$ such that $\hat{P} = U^* T_z U$. Since $\hat{S}$ is a commutant of $\hat{P}$, there exists $\varphi \in H^\infty(\mathcal{L}(D_{\hat{P}^*}))$ such that $T = U^* T_\varphi U$. As (T_φ, T_z) is a Γ-isometry, by the relation $T_\varphi = T_\varphi^* T_z$ (see Theorem 2.9), we have that

$$\varphi(z) = A + A^* z, \quad \text{for some } A \in \mathcal{L}(\mathcal{D}_{V^*}).$$

Also $\|T_\varphi\| = \|\varphi\|_\infty \leq 2$. Therefore, for any real θ,

$$\|A + A^* e^{i\theta}\| = \|Ae^{-i\theta/2} + A^* e^{i\theta/2}\| = \|2\mathrm{Re}(e^{-i\theta/2} A)\| \leq 2.$$

Therefore, $\omega(A) \leq 1$ by Lemma 3.1. It is evident from the proof of Theorem 3.3 that if (T_{A+A^*z}, T_z) is a Γ-isometry then A^* is the fundamental operator of the Γ-co-isometry $(T^*_{A+A^*z}, T^*_z)$. Denoting by $\hat{F}_*$, the fundamental operator of $(\hat{S}^*, \hat{P}^*)$, we have that $\hat{S} = U^* T_{\hat{F}_*^* + \hat{F}_* z} U$.

The proof to the converse is simple. The fact that (T_{A+A^*z}, T_z) on $H^2(E)$ is a Γ-isometry, when $\omega(A) \leq 1$, follows from Theorem 2.9(3). Moreover, since T_z is pure isometry, (T_{A+A^*z}, T_z) is a pure Γ-isometry. $\square$

The following result of one variable dilation theory is necessary for the proof of the model theorem for a Γ-contraction. We present a proof of it due to lack of a good reference.

Proposition 5.2. *If T is a contraction and V is its minimal isometric dilation then T^* and V^* have defect spaces of same dimension.*

Proof. Let T and V be defined on $\mathcal{H}$ and $\mathcal{K}$. Since V is the minimal isometric dilation of T we have

$$\mathcal{K} = \overline{\text{span}}\{p(V)h : \ h \in \mathcal{H} \text{ and } p \text{ is any polynomial in one variable}\}.$$

The defect spaces of T^* and V^* are respectively $\mathcal{D}_{T^*} = \overline{\text{Ran}}\ (I - TT^*)^{\frac{1}{2}}$ and $D_{V^*} = \overline{\text{Ran}}\ (I - VV^*)^{\frac{1}{2}}$. Let $\mathcal{N} = \overline{\text{Ran}}\ (I - VV^*)^{\frac{1}{2}}|_{\mathcal{H}}$. For $h \in \mathcal{H}$ and $n \geq 1$, we have

$$(I - VV^*)V^n h = V^n h - VV^*V^n h = 0,$$

as V is an isometry. Therefore, $(I - VV^*)p(V)h = p(0)(I - VV^*)h$ for any polynomial p in one variable. So $(I - VV^*)k \in \mathcal{N}$ for any $k \in \mathcal{K}$. This shows that $\overline{\text{Ran}}(I - VV^*) \subseteq \mathcal{N}$ and hence $\overline{\text{Ran}}(I - VV^*) = \mathcal{D}_{V^*} = \mathcal{N}$.

We now define for $h \in \mathcal{H}$,

$$\begin{aligned} L\text{Ran}(I - TT^*)^{\frac{1}{2}} &\to \text{Ran}(I - VV^*)^{\frac{1}{2}} \\ (I - TT^*)^{\frac{1}{2}}h &\mapsto (I - VV^*)^{\frac{1}{2}}h. \end{aligned}$$

We prove that L is an isometry. Since V^* is co-isometric extension of T^*, $TT^* = P_{\mathcal{H}}VV^*|_{\mathcal{H}}$ and thus we have $(I_{\mathcal{H}} - TT^*) = P_{\mathcal{H}}(I_{\mathcal{K}} - VV^*)|_{\mathcal{H}}$, that is, $D^2_{P^*} = P_{\mathcal{H}} D^2_{V^*}|_{\mathcal{H}}$. Therefore, for $h \in \mathcal{H}$,

$$\|D_{T^*}h\|^2 = \langle D^2_{P^*}h, h\rangle = \langle P_{\mathcal{H}} D^2_{V^*}h, h\rangle = \langle D^2_{V^*}h, h\rangle = \|D_{V^*}h\|^2,$$

and L is an isometry and this can clearly be extended to a unitary from $\mathcal{D}_{T^*}$ to $\mathcal{D}_{V^*}$. $\square$

The following is the model theorem of a Γ-contraction and is another main result of this section. This can be treated as a concrete form of the model given by Agler and Young (Theorem 1.3) in the sense that we have specified the model space and model operators.

Theorem 5.3. *Let (S,P) be a Γ-contraction on a Hilbert space $\mathcal{H}$. Let (T,V) on $\mathcal{K}_* = \mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots$ be defined as*

$$T = \begin{bmatrix} S & D_{P^*}F_* & 0 & 0 & \cdots \\ 0 & F_*^* & F_* & 0 & \cdots \\ 0 & 0 & F_*^* & F_* & \cdots \\ 0 & 0 & 0 & F_*^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad V = \begin{bmatrix} P & D_{P^*} & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ 0 & 0 & 0 & I & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where F_* is the fundamental operator of (S^*,P^*). Then:

(1) *(T,V) is a Γ-co-isometry, $\mathcal{H}$ is a common invariant subspace of T,V and $T|_{\mathcal{H}} = S$ and $V|_{\mathcal{H}} = P$.*
(2) *There is an orthogonal decomposition $\mathcal{K}_* = \mathcal{K}_1 \oplus \mathcal{K}_2$ into reducing subspaces of T and V such that $(T|_{\mathcal{K}_1}, V|_{\mathcal{K}_1})$ is a Γ-unitary and $(T|_{\mathcal{K}_2}, V|_{\mathcal{K}_2})$ is a pure Γ-co-isometry.*
(3) *$\mathcal{K}_2$ can be identified with $H^2(\mathcal{D}_V)$, where D_V has same dimension as that of $\mathcal{D}_P$. The operators $T|_{\mathcal{K}_2}$ and $V|_{\mathcal{K}_2}$ are respectively unitarily equivalent to $T_{B+B^*\bar{z}}$ and $T_{\bar{z}}$ defined on $H^2(\mathcal{D}_V)$, B being the fundamental operator of (T,V).*

Proof. It is evident from Corollary 4.4 that (T^*,V^*) is minimal Γ-isometric dilation of (S^*,P^*), where V^* is the minimal isometric dilation of P^*. Therefore by Proposition 4.2, (T,V) is Γ-co-isometric extension of (S,P). So we have that $\mathcal{H}$ is a common invariant subspace of T and V and $T|_{\mathcal{H}} = S$, $V|_{\mathcal{H}} = P$. Again since (T^*,V^*) is a Γ-isometry, by Theorem 2.9(2), there is an orthogonal decomposition $\mathcal{K}_* = \mathcal{K}_1 \oplus \mathcal{K}_2$ into reducing subspaces of T and V such that $(T|_{\mathcal{K}_1}, V|_{\mathcal{K}_1})$ is a Γ-unitary and $(T|_{\mathcal{K}_2}, V|_{\mathcal{K}_2})$ is a pure Γ-co-isometry. If we denote $(T|_{\mathcal{K}_1}, V|_{\mathcal{K}_1})$ by (T_1,V_1) and $(T|_{\mathcal{K}_2}, V|_{\mathcal{K}_2})$ by (T_2,V_2) then with respect to the orthogonal decomposition $\mathcal{K}_* = \mathcal{K}_1 \oplus \mathcal{K}_2$ we have

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}.$$

The fundamental equation $T - T^*V = D_V X D_V$ clearly becomes

$$\begin{bmatrix} T_1 - T_1^*V_1 & 0 \\ 0 & T_2 - T_2^*V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D_{V_2}X_2D_{V_2} \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Since $\mathcal{D}_V = \mathcal{D}_{V_2}$, the above form of the fundamental equation shows that (T,V) and (T_2,V_2) have the same fundamental operator. Now we apply Theorem 5.1 to the pure Γ-isometry $(T_2^*,V_2^*) = (T^*|_{\mathcal{K}_2}, V^*|_{\mathcal{K}_2})$ and get the following:

(i) $\mathcal{K}_2$ can be identified with $H^2(\mathcal{D}_{V_2}) = H^2(\mathcal{D}_V)$.
(ii) T_2^* and V_2^* can be identified with the Toeplitz operators T_{B^*+Bz} and T_z respectively defined on $H^2(\mathcal{D}_V)$, B being the fundamental operator of (T,V).

Therefore, $T|_{\mathcal{K}_2}$ and $V|_{\mathcal{K}_2}$ are respectively unitarily equivalent to $T_{B+B^*\bar{z}}$ and $T_{\bar{z}}$ defined on $H^2(\mathcal{D}_V)$. Also since V^* is the minimal isometric dilation of P^* by Proposition 5.2, $\mathcal{D}_V$ and $\mathcal{D}_P$ have same dimension. $\square$

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