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Examples of parametrized families of elliptic functions with empty Fatou sets

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Abstract. In this paper, we investigate parametrized families of elliptic functions on real rectangular lattices. Although these functions have at most four critical values, we prove that they have at most one attracting or parabolic cycle of Fatou components. We find some families for which the Julia set is always the entire sphere.

Contents

1. Introduction

Hawkins showed in [10] that the Weierstrass elliptic $\wp$-function on any square lattice in the real rhombic position has Julia set equal to the entire sphere. The space of all real lattices can be parametrized by $\mathbb{C}\setminus\{0\}$, and the real rhombic lattices lie on the negative real axis. The main technique used in [10] to show that the Fatou set was empty involved extending a result of Singer's [21] on properties of real functions with negative Schwarzian derivatives to the context of elliptic functions.

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In this paper, we investigate elliptic functions of the form

$$f_{n,\Lambda,b}(z) = (\wp_\Lambda(z))^n + b,$$

where Λ is a real rectangular lattice, n is a positive integer, and b is a real number. The functions $f_{n,\Lambda,b}$ have order $2n$ and two, three, or four distinct critical values, depending on the shape of the lattice and the value of n. We show that $f_{n,\Lambda,b}$ restricted to the real line always has a negative Schwarzian derivative. Using the techniques developed in [10], we prove that the complex functions $f_{n,\Lambda,b}$ have at most one periodic cycle of Fatou components that is either attracting or parabolic.

We apply the results to families $f_{n,\Lambda,b}$ with $b = -e_1^n$, where e_1 is the real critical value of $\wp_\Lambda$. With this choice of b, these functions all have the property that the real critical value of $f_{n,\Lambda,b}$ is a pole. In this case, our main theorem implies that that the Julia set of $f_{n,\Lambda,b}$ is always the entire sphere on any real rectangular lattice, an open set in the parameter space of real lattices.

We also use our results to investigate the Weierstrass $\wp_\Lambda$-function on real rectangular lattices Λ. In [11], Hawkins and the author proved that the Julia set of $\wp_\Lambda$ on a real rectangular lattice is either the entire sphere or there exists at most three real periodic cycles that are superattracting or rationally neutral. The main theorem proved here implies that either the Julia set of $\wp_\Lambda$ is the entire sphere or there is exactly one attracting or rationally neutral real cycle. In [12], infinitely many real lattices for which the Julia set of $\wp_\Lambda$ is the entire sphere were found, but not one for every real rectangular equivalence class of lattices. We strengthen the result in [12] by finding infinitely many lattices in each real rectangular equivalence class for which the Julia set of $\wp_\Lambda$ is the entire sphere.

2. The iteration of elliptic functions

We begin with some preliminaries about elliptic functions, the Weierstrass $\wp$-function and period lattices. Let λ_1, λ_2 be nonzero complex numbers such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. A lattice $\Lambda \subset \mathbb{C}$ is defined by

$$\Lambda = [\lambda_1, \lambda_2] = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\};$$

we note that two different sets of vectors can generate the same lattice Λ.

If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. A lattice Λ is *real* if $\overline{\Lambda} = \Lambda$.

Definition 2.1.

(1) A lattice Λ is *real rectangular* if $\Lambda = [\lambda_1, \lambda_2]$ with $\lambda_1 \in \mathbb{R}$ and λ_2 purely imaginary.
(2) A lattice Λ is *real rhombic* if $\Lambda = [\lambda_1, \lambda_2]$ with $\lambda_2 = \overline{\lambda_1}$.

(3) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda\, i]$, for some $\lambda > 0$.)

In each of cases of Definition 2.1, the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to look rectangular, rhombic, or square respectively.

We begin with a meromorphic function $f\colon \mathbb{C} \to \mathbb{C}_\infty$, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ. For any $z \in \mathbb{C}$ and any lattice Λ, the *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

It is well-known that $\wp_\Lambda$ is meromorphic, is even, is periodic with respect to Λ, and has order 2. In the following, we will denote iteration of $\wp_\Lambda$ by $\wp_\Lambda^n$ or $\wp_\Lambda^n(z)$ and products of $\wp_\Lambda$ by $(\wp_\Lambda)^n$ or $(\wp_\Lambda(z))^n$.

The Weierstrass elliptic function and its derivative are related by the differential equation

$$(\wp_\Lambda'(z))^2 = 4(\wp_\Lambda(z))^3 - g_2 \wp_\Lambda(z) - g_3, \tag{1}$$

where

$$g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4},$$

and

$$g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}.$$

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore, given any g_2 and g_3 such that $g_2^3 - 27 g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [8], and we call such a lattice Λ a (g_2, g_3) lattice.

In this paper, we focus on real lattices. We say that $\wp_\Lambda$ is *real* if $z \in \mathbb{R}$ implies that $\wp_\Lambda(z) \in \mathbb{R} \cup \{\infty\}$.

Theorem 2.2 ([16]). *The following are equivalent:*

(1) $\wp_\Lambda$ *is real.*
(2) Λ *is a real lattice.*
(3) $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its invariants satisfy homogeneity properties:

Lemma 2.3 ([8]). *For lattices* Λ *and* Λ' *and for* $k \in \mathbb{C} \setminus \{0\}$:

(1) $\Lambda' = k\Lambda$ *if and only if* $g_2(\Lambda') = k^{-4} g_2(\Lambda)$ *and* $g_3(\Lambda') = k^{-6} g_3(\Lambda)$.
(2) *If* $\Lambda' = k\Lambda$ *then* $\wp_{\Lambda'}(ku) = k^{-2} \wp_\Lambda(u)$ *for all* $u \in \mathbb{C}$.

Verification of the homogeneity properties can be seen by substitution into the series definitions.

The following classical result characterizes all elliptic functions in terms of $\wp$ and $\wp'$.

Theorem 2.4 ([8]). *Every elliptic function f_Λ with period lattice Λ can be written as $f_\Lambda(z) = R(\wp_\Lambda(z)) + \wp_\Lambda{}'(z)Q(\wp_\Lambda(z))$, where R and Q are rational functions with complex coefficients. The converse is also true; namely, every f_Λ of this form is elliptic.*

We can determine the critical values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$. Define $\lambda_3 = \lambda_1 + \lambda_2$. For $j = 1, 2, 3$, notice that $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$. Substituting $z = \lambda_j/2$, we see that

$$\wp'_\Lambda(z) = 0 \text{ when } z = \frac{\lambda_j}{2} + \Lambda, \tag{2}$$

for $j = 1, 2, 3$. We use the notation

$$e_1 = \wp_\Lambda\left(\frac{\lambda_1}{2}\right), \quad e_2 = \wp_\Lambda\left(\frac{\lambda_2}{2}\right), \quad e_3 = \wp_\Lambda\left(\frac{\lambda_3}{2}\right) \tag{3}$$

to denote the critical values of $\wp_\Lambda$. Since e_1, e_2, e_3 are the distinct zeros of Equation (1), we also write

$$(\wp'_\Lambda(z))^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{4}$$

Equating like terms in Equations (1) and (4), we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{5}$$

It will be useful to have an expression for the second derivative of the Weierstrass elliptic function,

$$\wp''_\Lambda(z) = 6(\wp_\Lambda(z))^2 - \frac{g_2(\Lambda)}{2}. \tag{6}$$

The lattice shape relates to the properties and dynamics of the corresponding Weierstrass elliptic function to some extent, as discussed in [9, 10, 11, 12, 13, 14, 15]; however these papers also show that within a given shape equivalence class the dynamics vary widely.

2.1. Real rectangular lattices. In this section we recall some well-known results about the Weierstrass elliptic function on real rectangular lattices. By Theorem 2.2, Λ is real if and only if $g_2(\Lambda)$ and $g_3(\Lambda)$ are real, so we can identify a real lattice Λ with a point (g_2, g_3) in $\mathbb{R}^2$. We begin with a proposition that locates real rectangular lattices in the real (g_2, g_3) plane.

Denote $p(x) = 4x^3 - g_2x - g_3$, the polynomial associated with Λ.

Proposition 2.5 ([8]).

(1) *If Λ is real rectangular, then $g_2^3 - 27g_3^2 > 0$ and $g_2 > 0$; in this case the roots of p are three distinct real roots.*

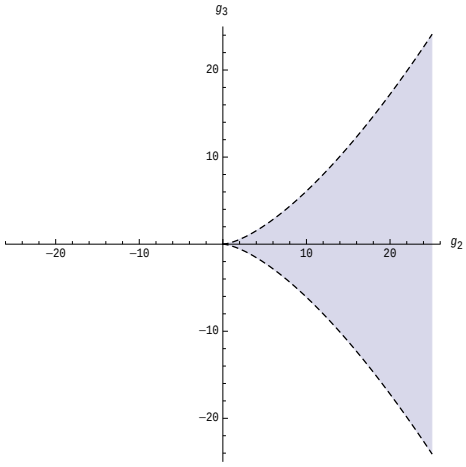

FIGURE 1. Points in (g_2, g_3) parameter space corresponding to real rectangular lattices.

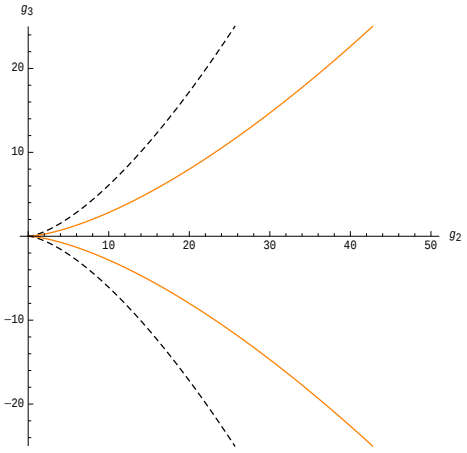

FIGURE 2. All points colored orange represent real lattices similar to the $(5, -1)$ lattice.

(2) *If Λ is real rectangular square, then $g_2 > 0$ and $g_3 = 0$; in this case the roots of p are $0, \pm\sqrt{g_2}/2$.*

The grey region in Figure 1 shows the locations of real rectangular lattices in (g_2, g_3) space. The curve $g_2^3 - 27g_3^2 = 0$ for which no lattice is defined is shown as a dotted black curve. We can use Lemma 2.3 to find all real lattices that are similar to a given real lattice. If Λ is the real lattice corresponding to the invariants (g_2, g_3), then parameters that lie on the planar curve

$$y^2 = g_3^2 x^3 / g_2^3$$

represent real lattices similar to Λ. In Figure 2, the orange curve represents the invariants of real lattices that are similar to the lattice Λ with invariants $(g_2, g_3) = (5, -1)$. In this case, the portion of the curve lying in the lower half plane represents lattices $k\Lambda$ where k is real. The portion of the curve lying in the upper half plane represents lattices $k\Lambda$ where k is purely imaginary; note that $k\Lambda$ is still real rectangular in this case.

Next, we state some properties of $\wp_\Lambda$ on any real rectangular lattice. The following proposition, which can be obtained using Equations (1) and (5), provides useful information for our study in subsequent sections.

Proposition 2.6 ([11]). *Let $\Lambda = [\lambda_1, \lambda_2]$ with $\lambda_1 > 0$ be a real rectangular lattice. Then:*

(1) *$\wp_\Lambda|_{\mathbb{R}} : \mathbb{R} \to [e_1, \infty]$ is piecewise monotonic and onto. Specifically, $\wp_\Lambda$ is strictly decreasing on $[0, \lambda_1/2]$ and strictly increasing on $[\lambda_1/2, \lambda_1]$, where $\lambda_1 > 0$ denotes the real period of Λ.*

(2) *The critical values* e_1, e_2, e_3 *of* $\wp_\Lambda$ *are all real.*
   (a) *If* $g_3 > 0$ *then* $e_2 < e_3 < 0 < e_1$, *and* $\wp_\Lambda$ *has a zero on the vertical line segment connecting* $\lambda_1/2$ *to* $(\lambda_1+\lambda_2)/2$.
   (b) *If* $g_3 < 0$ *then* $e_2 < 0 < e_3 < e_1$, *and* $\wp_\Lambda$ *has a zero on the horizontal line segment connecting* $\lambda_2/2$ *to* $(\lambda_1+\lambda_2)/2$.
   (c) *If* $g_3 = 0$ *(*Λ *is rectangular square) then*

$$e_1 = \sqrt{g_2}/2 > 0, \quad e_2 = -e_1, \quad \text{and} \quad e_3 = 0.$$

2.2. The family of functions $f_{n,\Lambda,b}$. The investigation in this paper focuses on elliptic functions of the form $f_{n,\Lambda,b}(z) = (\wp_\Lambda(z))^n + b$ with Λ a real rectangular lattice, n a positive integer, and $b \in \mathbb{R}$. Since $\wp_\Lambda$ is even and periodic with respect to Λ, so is $f_{n,\Lambda,b}$. Since $\wp_\Lambda$ has order two, $f_{n,\Lambda,b}$ has order $2n$. Many properties about the critical points, critical values, as well as the shape of $f_{n,\Lambda,b}$ restricted to $\mathbb{R}$, follow from properties of $\wp_\Lambda$; we collect these into one lemma.

Lemma 2.7. *Let* $\Lambda = [\lambda_1, \lambda_2]$ *with* $\lambda_1 > 0$ *be a real rectangular lattice.*

(1) *If* $n = 1$ *then the critical points of* $f_{1,\Lambda,b}$ *are*

$$\{\lambda_1/2, \lambda_2/2, \lambda_1/2 + \lambda_2/2\} + \Lambda,$$

and the critical values are $e_1 + b, e_2 + b$, *and* $e_3 + b$.

(2) *If* $n > 1$ *then the critical points of* $f_{n,\Lambda,b}$ *are*

$$\{\lambda_1/2, \lambda_2/2, \lambda_1/2 + \lambda_2/2, \wp_\Lambda^{-1}(0)\} + \Lambda,$$

and the critical values are $e_1^n + b, e_2^n + b, e_3^n + b$ *and* b.

(3) *The only real critical points of* $f_{n,\Lambda,b}$ *are* $\lambda_1/2 + k\lambda_1, k \in \mathbb{Z}$.

(4) $f_{n,\Lambda,b}\colon \mathbb{R} \to [e_1^n + b, \infty]$ *and* $f(\lambda_1/2) = e_1^n + b$.

(5) *The postcritical set of* $f_{n,\Lambda,b}$ *is real.*

(6) $f_{n,\Lambda,b}$ *is strictly decreasing on* $[0, \lambda_1/2]$ *and strictly increasing on* $[\lambda_1/2, \lambda]$.

Proof. If $n = 1$ then the critical points of $f_{1,\Lambda,b}$ are the roots of $\wp'_\Lambda$, which occur at $\{\lambda_1/2, \lambda_2/2, \lambda_1/2 + \lambda_2/2\} + \Lambda$ by Equation (2). The critical values of $\wp_\Lambda$ are defined in Equation (3), and thus the critical values of $f_{1,\Lambda,b}$ are $e_1 + b, e_2 + b$, and $e_3 + b$. If $n > 1$ then $f'_{n,\Lambda,b}(z) = n(\wp_\Lambda(z))^{n-1}\wp'_\Lambda(z)$, and so the critical points of $f_{n,\Lambda,b}$ occur at the roots of $\wp_\Lambda$ and the roots of $\wp'_\Lambda$, or $\{\lambda_1/2, \lambda_2/2, \lambda_1/2 + \lambda_2/2, \wp_\Lambda^{-1}(0)\} + \Lambda$. Thus for $n > 1$, $f_{n,\Lambda,b}$ has four critical values: $e_1^n + b, e_2^n + b, e_3^n + b$ and b (which may not be distinct).

By definition, if the real rectangular lattice $\Lambda = [\lambda_1, \lambda_2]$ has $\lambda_1 > 0$, then $\{\lambda_2/2, \lambda_1/2 + \lambda_2/2\} + \Lambda$ do not lie on the real line. By Proposition 2.6(2), the zeros of $\wp_\Lambda$ lie on the vertical line connecting $\lambda_1/2$ to $(\lambda_1 + \lambda_2)/2$ (and its translates) or the horizontal line connecting $\lambda_2/2$ to $(\lambda_1 + \lambda_2)/2$ (and its translates), and therefore the only real critical points are $\lambda_1/2 + k\lambda_1, k \in \mathbb{Z}$.

Parts (4) and (6) follow immediately from Proposition 2.6(2) and the fact that $f'_{n,\Lambda,b}(z) = n(\wp_\Lambda(z))^{n-1}\wp'_\Lambda(z)$. Proposition 2.6(2) implies that the

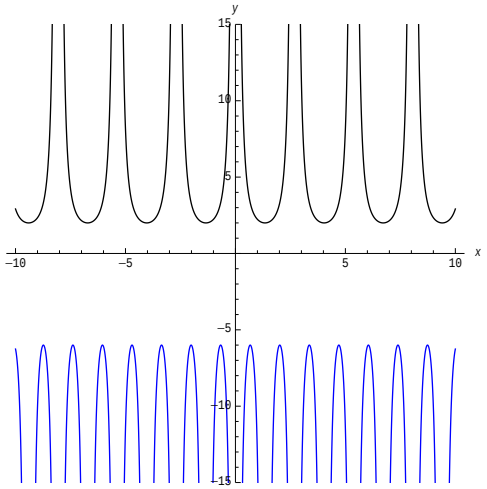

FIGURE 3. An example of a graph $f_{2,\Lambda,1}(x) = (\wp_\Lambda(x))^2 + 1$ on $\mathbb{R}$, where $(g_2, g_3) = (5, -1)$, in black, with its Schwarzian $S_{f_{2,\Lambda,1}}(x)$ (see Theorem 3.5) in blue.

critical values of $f_{n,\Lambda,b}$ are real, and thus so is the entire postcritical set using part (4). □

Since e_1, e_2, and e_3 are distinct, the critical values in Lemma 2.7(1) are distinct. However, the critical points and critical values discussed in Lemma 2.7(2) may not be distinct. If Λ is square then $\wp_\Lambda^{-1}(0) = (\lambda_1 + \lambda_2)/2 + \Lambda$, so there are only three equivalence classes of critical points. A simple example with fewer critical values occurs when Λ is square and $n = 2$; in this case, $e_2 = -e_1$ by Proposition 2.6(2c), so $(e_1)^2 + b = (e_2)^2 + b$. The black curve in Figure 3 shows part of a typical function in this family restricted to $\mathbb{R}$, $f_{2,\Lambda,1}(x) = (\wp_\Lambda(x))^2 + 1$, where the lattice Λ is defined by the parameters $g_2(\Lambda) = 5$ and $g_3(\Lambda) = -1$. The function shown in blue will be described in Section 3.

2.3. Julia and Fatou sets of elliptic functions. We review the basic dynamical definitions and properties for meromorphic functions which appear, for example, in [1, 2, 3, 7]. Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function, and let $f^k(z)$ denote the composition of f with itself k times. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^k\colon k \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{k\geq 0} f^{-k}(\infty)}$ is the largest open set where all iterates are defined. Since

$f(\mathbb{C}_\infty \backslash \overline{\bigcup_{k\geq 0} f^{-k}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{k\geq 0} f^{-k}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{k\geq 0} f^{-k}(\infty)}.$$

Let $\mathrm{Crit}(f)$ denote the set of critical points of f, *i.e.*,

$$\mathrm{Crit}(f) = \{z\colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. The *singular set* $\mathrm{Sing}(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S [8]. If f is Class S then f does not have wandering domains [2] or Baker domains [20]. The *postcritical set* of f is:

$$P(f) = \overline{\bigcup_{k\geq 1} f^k(\mathrm{Crit}(f))}.$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set

$$\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$$

a *p-cycle*. The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point.

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

Let $C = \{U_0, U_1, \ldots U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of immediate attractive basins or Leau domains, then

$$U_j \cap \mathrm{Sing}(f) \neq \emptyset$$

for some $0 \leq j \leq p-1$. If C is a cycle of Siegel Disks or Herman rings, then

$$\partial U_j \subset \overline{\bigcup_{k\geq 0} f^k(\mathrm{Sing}(f))}$$

for all $0 \leq j \leq p-1$. In particular, singular points are required for any type of preperiodic Fatou component.

In this paper, we focus exclusively on elliptic functions whose postcritical set is real and which map the real line to the real line. In this case, we can eliminate the possibility of Siegel disks or Herman rings. A version of the following proposition was proved for $\wp_\Lambda$ in [11], and we extend the result to the family $f_{n,\Lambda,b}$ on real rectangular lattices.

Proposition 2.8. *If $f_{n,\Lambda,b}(z) = (\wp_\Lambda(z))^n + b$, with $b \in \mathbb{R}$ and Λ a real rectangular lattice, then $f_{n,\Lambda,b}$ has no Siegel disks or Herman rings.*

Proof. Since $f_{n,\Lambda,b}$ is periodic with respect to Λ, we have

$$J(f_{n,\Lambda,b}) + \Lambda = J(f_{n,\Lambda,b})$$

and $F(f_{n,\Lambda,b}) + \Lambda = F(f_{n,\Lambda,b})$. If $C = \{U_0, U_1, \dots U_{p-1}\}$ is a cycle of Siegel disks or Herman rings, then

$$\partial U_j \subset \overline{\bigcup_{k\geq 0} f^k_{n,\Lambda,b}(\mathrm{Sing}(f_{n,\Lambda,b}))}$$

for all $0 \leq j \leq p-1$. (cf. [3], Theorem 7). By Lemma 2.7(5), the post-critical set of $f_{n,\Lambda,b}$ is contained in the real axis, and thus the closure of the postcritical set is a subset of $\mathbb{R} \cup \{\infty\}$. However, our cycle C must satisfy $\partial U_j \subset \mathbb{R}$ for all $0 \leq j \leq p-1$, which contradicts the periodicity of the Julia set with respect to Λ. □

3. The Schwarzian derivative

In this section, we prove a sharp bound on the number of nonrepelling periodic orbits for $f_{n,\Lambda,b}$ on a real rectangular lattice. For a general interval map, there can be more nonrepelling cycles than critical points; an example of a real polynomial function with one critical point and two attracting cycles can be found in [21]. Even in the case of elliptic functions it is possible to have such behavior. For example, $\wp_\Lambda$ on the real rhombic lattice Λ with invariants $g_2(\Lambda) = 26.56$ and $g_3(\Lambda) = -26.26$ has a real attracting fixed point that does not attract any real critical point (see Example 3.7 in [13]), and so the results of this paper cannot be extended to elliptic functions on arbitrary real rhombic lattices.

Schwarzian derivatives were first used in the context of the Weierstrass elliptic function in [10] to show that real rhombic square lattices have no non-repelling periodic cycles, and our method in this section follows the technique given there.

We recall the definition of the Schwarzian derivative.

Definition 3.1. If z is not a critical point or pole of a meromorphic function g, then the *Schwarzian derivative* of g at z is

$$S_g(z) = \frac{g'''(z)}{g'(z)} - \frac{3}{2}\left(\frac{g''(z)}{g'(z)}\right)^2.$$

Using the chain rule, we have that

$$S_{g\circ h}(z) = S_g(h(z))(h'(z))^2 + S_h(z)$$

at every point z for which $h(z)$ is defined. The chain rule immediately implies the following lemma.

Lemma 3.2. *If $S_g < 0$ and $S_h < 0$ then $S_{g\circ h} < 0$.*

3.1. The Schwarzian derivative of $f_{n,\Lambda,b}$. We focus our attention on the function $f_{n,\Lambda,b}(z) = (\wp_\Lambda(z))^n + b$ for $n \geq 1$ and $b \in \mathbb{R}$ on real rectangular lattices Λ.

Lemma 3.3. *Let Λ be a real rectangular lattice and let $b \in \mathbb{R}$.*

(1) *$S_{f_{n,\Lambda,b}}$ is an even elliptic function with poles at lattice points and half lattice points.*
(2) *$S_{f_{n,\Lambda,b}}$ is a real valued meromorphic function when restricted to $\mathbb{R}$.*

Proof. Let Λ be a real rectangular lattice, and let $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ be its invariants. For $n \geq 1$ we have $f'_{n,\Lambda,b} = n(\wp_\Lambda)^{n-1}\wp'_\Lambda$. If $n = 1$ then $f''_{1,\Lambda,b} = \wp''_\Lambda = 6(\wp_\Lambda)^2 - g_2/2$ by Equation (6). For $n > 1$, we use the chain rule and Equations (1) and (6) to obtain

$$
\begin{aligned}
f''_{n,\Lambda,b} &= n(\wp_\Lambda)^{n-1}\wp''_\Lambda + n(n-1)(\wp_\Lambda)^{n-2}(\wp'_\Lambda)^2 \\
&= n(\wp_\Lambda)^{n-1}\left(6(\wp_\Lambda)^2 - \frac{g_2}{2}\right) \\
&\quad + n(n-1)(\wp_\Lambda^{n-2})(4(\wp_\Lambda)^3 - g_2\wp_\Lambda - g_3) \\
&= (g_3 n - g_3 n^2)(\wp_\Lambda)^{n-2} + \left(\frac{g_2 n}{2} - g_2 n^2\right)(\wp_\Lambda)^{n-1} \\
&\quad + (2n + 4n^2)(\wp_\Lambda)^{n+1}.
\end{aligned} \tag{7}
$$

For all $n \geq 1$, we simplify by writing $f''_{n,\Lambda,b} = P_n(\wp_\Lambda)$, where P_n is a polynomial of degree $n+1$ with real coefficients.

Using the chain rule, we have $f'''_{n,\Lambda,b} = P'_n(\wp_\Lambda)\wp'_\Lambda$, and thus

$$
\begin{aligned}
S_{f_{n,\Lambda,b}} &= \frac{P'_n(\wp_\Lambda)\wp'_\Lambda}{n(\wp_\Lambda)^{n-1}\wp'_\Lambda} - \frac{3}{2}\left(\frac{P_n(\wp_\Lambda)}{n(\wp_\Lambda)^{n-1}\wp'_\Lambda}\right)^2 \\
&= \frac{P'_n(\wp_\Lambda)}{n(\wp_\Lambda)^{n-1}} - \frac{3}{2}\left(\frac{P_n(\wp_\Lambda)}{n(\wp_\Lambda)^{n-1}}\right)^2 \frac{1}{4(\wp_\Lambda)^3 - g_2\wp_\Lambda - g_3},
\end{aligned}
$$

by Equation (1). For $n = 1$ we substitute $f''_{1,\Lambda,b} = \wp''_\Lambda = 6(\wp_\Lambda)^2 - g_2/2$ to obtain

$$
\begin{aligned}
S_{f_{1,\Lambda,b}} &= \frac{-3\left(g_2^2 + 32 g_3\wp_\Lambda + 8 g_2(\wp_\Lambda)^2 + 16(\wp_\Lambda)^4\right)}{8(\wp'_\Lambda)^2} \\
&= \frac{-3\left(g_2^2 + 32 g_3\wp_\Lambda + 8 g_2(\wp_\Lambda)^2 + 16(\wp_\Lambda)^4\right)}{8(4(\wp_\Lambda)^2 - g_2\wp_\Lambda - g_3)}.
\end{aligned} \tag{8}
$$

For $n > 1$, we substitute Equation (7) for P_n and simplify.

$$\begin{aligned} S_{f_{n,\Lambda,b}} = \Big[&(.5g_3^2 - .5g_3^2n^2) + (21g_3 - .5g_2g_3 - 21g_3n + 1.5g_2g_3n - g_2g_3n^2)\wp_\Lambda \\ &+ (-73.5 + 21g_2 - g_2^2 - 21g_2n + 1.5g_2^2n - .5g_2^2n^2)(\wp_\Lambda)^2 \\ &+ (-16g_3 + 4g_3n^2)(\wp_\Lambda)^3 + (42 - 10g_2 + 84n - 6g_2n + 4g_2n^2)(\wp_\Lambda)^4 \\ &+ (2 - 8n^2)(\wp_\Lambda)^6\Big] \\ &\quad / \left(4(\wp_\Lambda)^5 - g_2(\wp_\Lambda)^3 - g_3(\wp_\Lambda)^2\right). \end{aligned}$$

Since $S_{f_{n,\Lambda,b}}$ is a rational function of $\wp_\Lambda$, it is elliptic by Theorem 2.4. It is clearly even since $\wp_\Lambda$ is even. Simplifying the denominator of $S_{f_{n,\Lambda,b}}$ when $n > 1$, we obtain $4(\wp_\Lambda)^5 - g_2(\wp_\Lambda)^3 - g_3(\wp_\Lambda)^2 = (\wp_\Lambda)^2\wp'_\Lambda$. The denominator is only zero when $\wp'_\Lambda = 0$ since $\wp_\Lambda \neq 0$ on the real line by Proposition 2.6(2). Therefore, the poles of $S_{f_{n,\Lambda,b}}$ occur at the poles of $\wp$ and the zeros of $\wp'$, which are the lattice points and half lattice points respectively.

Since $g_2, g_3 \in \mathbb{R}$ then by Theorem 2.2 either $S_{f_{n,\Lambda,b}}(z) \in \mathbb{R}$ or z is a pole. □

Next, we prove that the Schwarzian of $f_{1,\Lambda,b}(z) = \wp_\Lambda(z) + b$ is negative on every real rectangular lattice.

Proposition 3.4. *If Λ is a real rectangular lattice and $b \in \mathbb{R}$, then $S_{f_{1,\Lambda,b}}$ is negative for all real z for which it is defined.*

Proof. Let Λ be a real rectangular lattice with invariants g_2 and g_3. Since Λ is understood, we write $\wp_\Lambda = \wp$. From Equation (8),

$$S_{f_{1,\Lambda,b}} = \frac{-3\left(g_2^2 + 32g_3\wp + 8g_2(\wp)^2 + 16(\wp)^4\right)}{8(\wp')^2}.$$

Recall that $g_2 > 0$ by Proposition 2.5, and $\wp(z) > e_1 > 0$ on $\mathbb{R}$ by Proposition 2.6(2). Thus, if $g_3 \geq 0$ then $S_{f_{1,\Lambda,b}} < 0$ at all points which are not poles.

Consider the situation when $g_3 < 0$. Using the function

$$h = g_2^2 + 32g_3\wp + 8g_2(\wp)^2 + 16(\wp)^4$$

that appears in the numerator of the formulation of $S_{f_{1,\Lambda,b}}$ in Equation (8), we claim that $h > 0$. To prove the claim, we begin by applying Theorem 2.4 to observe that h is an even elliptic function with period lattice Λ. Theorem 2.2 implies that h maps $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$.

Next, we show that h has a minimum at $\lambda_1/2$. Taking the derivative, we obtain

$$h' = 16\wp'(2g_3 + g_2\wp + 4(\wp)^3).$$

Since $\wp' < 0$ on $(0, \lambda_1/2)$ by Proposition 2.6(1), if we show that the function $k = 2g_3 + g_2\wp + 4(\wp)^3$ is always positive on $(0, \lambda_1/2)$ then h is decreasing on $(0, \lambda_1/2)$. Proposition 2.6(2) also implies that $\wp(z) \geq e_1 > 0$ on $\mathbb{R}$, so

$$k = 2g_3 + g_2\wp + 4(\wp)^3 \geq 2g_3 + g_2e_1 + 4e_1^3.$$

Using the relationship between g_2, g_3 and the critical values e_1, e_2, and e_3 in Equation (5), we have

$$\begin{aligned} k \geq 2g_3 + g_2e_1 + 4e_1^3 &= 2(4e_1e_2e_3) - 4(e_1e_2 + e_2e_3 + e_1e_3)e_1 + 4e_1^3 \\ &= 4e_1(e_2 - e_1)(e_3 - e_1). \end{aligned}$$

By Proposition 2.6(2b), since $g_3 < 0$ we have that $e_2 < 0 < e_3 < e_1$. Therefore $k > 0$, and $h' < 0$ on $(0, \lambda_1/2)$. Thus h is decreasing on $(0, \lambda_1/2)$. Since h is even and periodic with respect to Λ, h is increasing on $(\lambda_1/2, \lambda_1)$ and thus h has a minimum at $\lambda_1/2$. We apply all three equations shown in Equation (5) to obtain that for $z \neq \lambda_1/2$,

$$\begin{aligned} h(z) > h\left(\frac{\lambda_1}{2}\right) &= g_2^2 + 32g_3\wp\left(\frac{\lambda_1}{2}\right) + 8g_2\left(\wp\left(\frac{\lambda_1}{2}\right)\right)^2 + 16\left(\wp\left(\frac{\lambda_1}{2}\right)\right)^4 \\ &= g_2^2 + 32g_3e_1 + 8g_2e_1^2 + 16e_1^4 \\ &= g_2^2 + 32(4e_1e_2e_3)e_1 + 8g_2e_1^2 + 16e_1^4 \\ &= g_2^2 + 128e_1^2(e_2e_3) + 8g_2e_1^2 + 16e_1^4 \\ &= g_2^2 + 128e_1^2\left(-\frac{g_2}{4} - e_1e_2 - e_1e_3\right) + 8g_2e_1^2 + 16e_1^4 \\ &= g_2^2 + 128e_1^2\left(-\frac{g_2}{4} + e_1^2\right) + 8g_2e_1^2 + 16e_1^4 \\ &= (g_2 - 12e_1^2)^2 \geq 0. \end{aligned}$$

This completes the proof of our claim that $h > 0$. Since $\lambda_1/2$ is a pole of $S_{f_{1,\Lambda,b}}$, we have that $S_{f_{1,\Lambda,b}} < 0$ at all z that are not lattice points or half lattice points. □

Next, we show that all functions $f_{n,\Lambda,b}$ with $n > 1$ have negative Schwarzian.

Theorem 3.5. *If Λ is a real rectangular lattice, $n > 1$, $b \in \mathbb{R}$, and*

$$f_{n,\Lambda,b} = (\wp_\Lambda)^n + b,$$

then $S_{f_{n,\Lambda,b}} < 0$ for all z that are not lattice or half lattice points.

Proof. Using Proposition 3.4 with $b = 0$, we have that $S_{f_{1,\Lambda,0}} = S_{\wp_\Lambda} < 0$ for all real rectangular lattices Λ. If $g(x) = x^n$ with $n \geq 2$, then

$$S_g = -\frac{(n-1)(n+1)}{2x^2}.$$

So $S_g < 0$ for all $x \neq 0$. Using Lemma 3.2, $S_{(\wp_\Lambda)^n} = S_{g\circ\wp_\Lambda} < 0$ at all points which are not lattice points of half lattice points. Finally, since $S_{(\wp_\Lambda)^n} = S_{f_{n,\Lambda,b}}$ for all $n \geq 1$, we have that $S_{f_{n,\Lambda,b}} < 0$. □

Figure 3 shows the Schwarzian $S_{f_{2,\Lambda,1}}(x)$ in blue for a typical function in this family, $f_{2,\Lambda,1}(x) = (\wp_\Lambda(x))^2 + 1$, where the lattice Λ is defined by the parameters $g_2(\Lambda) = 5$ and $g_3(\Lambda) = -1$.

3.2. Consequences of a negative Schwarzian. In [10], Hawkins extended Singer's result to the elliptic function $\wp_\Lambda$ defined on real rhombic square lattices. Rhombic square lattices have $g_2 < 0$ and $g_3 = 0$ and appear on the negative horizontal axis in Figure 1. The proof given in [10] that $\wp_\Lambda$ on a square lattice satisfied a Minimum Principle depended on properties of the square lattice; the proof shown here for $f_{n,\Lambda,b}$ on any real rectangular lattice is similar to that found in [4].

Lemma 3.6 (Minimum Principle). *Assume that Λ is a real rectangular lattice. Suppose we have a closed interval $I \subset \mathbb{R}$ with endpoints $l < r$, not containing any poles or critical points of $f_{n,\Lambda,b}$. Then*

$$|f'_{n,\Lambda,b}(z)| > \min\{|f'_{n,\Lambda,b}(l)|, |f'_{n,\Lambda,b}(r)|\}, \forall z \in (l, r).$$

Proof. Let z_0 be a critical point of $|f'_{n,\Lambda,b}|$. Then $f''_{n,\Lambda,b}(z_0) = 0$. Since $S_{f_{n,\Lambda,b}} < 0$ by Theorem 3.5, $f'_{n,\Lambda,b}$ and $f'''_{n,\Lambda,b}$ have opposite signs. If $f'_{n,\Lambda,b}(z_0)$ is negative, then z_0 is a local minimum of $f'_{n,\Lambda,b}$ and thus a local maximum of $|f'_{n,\Lambda,b}|$. If $f'_{n,\Lambda,b}(z_0) > 0$ then z_0 is a local maximum of $|f'_{n,\Lambda,b}|$. Thus $|f'_{n,\Lambda,b}|$ cannot have a local minimum in the interior of I. □

In [10], the Minimum Principle was used to extend Singer's Theorem on interval maps to the setting of the Weierstrass elliptic function on a real square lattice. The extension of Singer's Theorem given in [10] relied only on the Minimum Principle and generic properties of elliptic functions on real lattices; the proof for our setting follows identically so we do not provide it.

Before we state the theorem, we need to provide some definitions, following [10]. Given a real rectangular lattice Λ, we focus on the restriction of $f_{n,\Lambda,b}$ to the real line. Using Lemma 2.7(1), we know that $f_{n,\Lambda,b}(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}$. For any p-cycle

$$S = \{z_0, f_{n,\Lambda,b}(z_0), \ldots, f^{p-1}_{n,\Lambda,b}(z_0)\} \subset \mathbb{R},$$

we associate to it a set

$$B(S) = \{x \in \mathbb{R} \colon f^k_{n,\Lambda,b}(x) \to S \text{ as } k \to \infty\}.$$

The set S is *topologically attracting* if $B(S)$ contains an open interval, and in this case we call $B(S)$ the *real attracting basin* of S. The *real immediate attracting basin* of S is the union of components of $B(S)$ in $\mathbb{R}$ that contain points from S, and we denote this set by $B_0(S)$. Using Lemma 2.7(1), if $|(f^p_{n,\Lambda,b})'(z_0)| < 1$, then $S \subset [e^n_1 + b, \infty)$ and $B(S) \neq \emptyset$, so S is topologically attracting.

Theorem 3.7. *If Λ is a real rectangular lattice, $n \geq 1$, $b \in \mathbb{R}$, and*

$$f_{n,\Lambda,b} = (\wp_\Lambda)^n + b,$$

then:

(1) *The real immediate basin of attraction of a topologically attracting periodic orbit of $f_{n,\Lambda,b}$ contains a real critical point.*

(2) *If $y \in \mathbb{R}$ is in a rationally neutral p-cycle for $f_{n,\Lambda,b}$ then it is topologically attracting; i.e., there exists an open interval I such that for every $x \in I$, $\lim_{k\to\infty} f_{n,\Lambda,b}^{kp}(x) = y$.*

Lemma 2.7(1), (2) indicates that $f_{n,\Lambda,b}$ has three or four postcritical orbits that may have no relation. However, Theorem 3.7 forces a restriction on the number of nonrepelling cycles.

Proposition 3.8. *For every real rectangular lattice Λ and for every $n \geq 1$ and $b \in \mathbb{R}$, one of the following must occur:*

(1) $J(f_{n,\Lambda,b}) = \mathbb{C}_\infty$.
(2) *There exists exactly one (super)attracting or rationally neutral Fatou cycle for $f_{n,\Lambda,b}$ that contains a real critical point. In this case, the nonrepelling cycle is real, and a real critical point is contained in the cycle of Fatou components.*

Proof. Proposition 2.8 implies that $f_{n,\Lambda,b}$ has no Siegel disks or Herman rings. By Lemma 2.7(4), (5), $f_{n,\Lambda,b}$ maps $\mathbb{R}$ to $[e_1^n + b, \infty]$ and the postcritical set of $f_{n,\Lambda,b}$ is real. If there is an attracting or parabolic cycle of Fatou components, then the cycle must lie on the real axis and contain a real critical point $\lambda_1/2 + k\lambda_1, k \in \mathbb{Z}$ by Theorem 3.7. But since

$$f_{n,\Lambda,b}(\lambda_1/2 + k\lambda_1) = e_1^n + b$$

by Lemma 2.7(4), all real critical points have the same forward orbit, so there can only be one nonrepelling cycle by Theorem 3.7. □

4. Applications of the theorems

In this section, we choose specific values of b_Λ and apply the theorems of the previous section to the resulting families $f_{n,\Lambda,b}$.

4.1. Julia set the entire sphere. In this section, we investigate the family

$$h_{n,\Lambda}(z) = f_{n,\Lambda,-(\wp_\Lambda(\lambda_1/2))^n}(z) = (\wp_\Lambda(z))^n - (\wp_\Lambda(\lambda_1/2))^n.$$

Using Lemma 2.7, these functions all share the property that their minimum on $\mathbb{R}$ is $h_{n,\Lambda}(\lambda_1/2) = 0$. Using the lattice Λ with generators $(g_2, g_3) = (5, -1)$, we show examples of the functions $h_{1,\Lambda}(x)$ and $h_{2,\Lambda}(x)$ on $\mathbb{R}$ for this family in Figures 4 and 5.

For each integer $n > 0$, and for every real rectangular lattice, the real critical points of $h_{n,\Lambda}$ all land on the pole at 0. Therefore, the results of Section 3 enable us to show that the Julia set of $h_{n,\Lambda}$ is the entire sphere in this case.

Proposition 4.1. *For every real rectangular lattice Λ and for every $n \geq 1$, the function $h_{n,\Lambda}(z) = (\wp_\Lambda(z))^n - (\wp_\Lambda(\lambda_1/2))^n$ has $J(h_{n,\Lambda}) = \mathbb{C}_\infty$.*

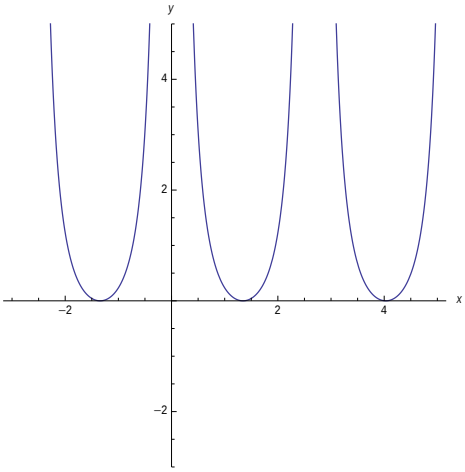

FIGURE 4. The graph of $h_{1,\Lambda}(x)$ on $\mathbb{R}$ when $(g_2, g_3) = (5, -1)$.

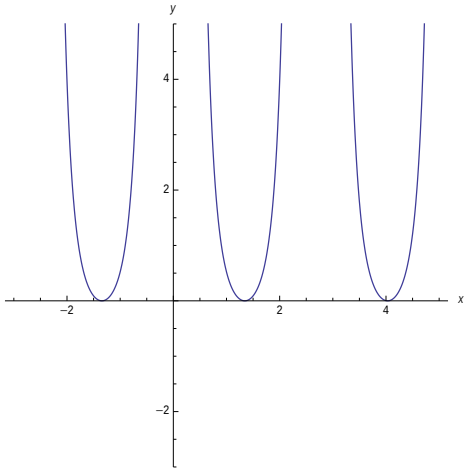

FIGURE 5. The graph of $h_{2,\Lambda}(x)$ on $\mathbb{R}$ when $(g_2, g_3) = (5, -1)$.

Proof. By Proposition 3.8, either $J(h_{n,\Lambda}) = \mathbb{C}_\infty$ or there exists exactly one (super)attracting or rationally neutral Fatou cycle for $h_{n,\Lambda}$ that contains a real critical point. If there is a nonrepelling cycle for $h_{n,\Lambda}$, the cycle must lie on the real axis and contain a real critical point. Since $h_{n,\Lambda}(\lambda_1/2) = 0$, we know that all real critical points are prepoles and hence belong to the Julia set. Therefore, the Fatou set is empty. □

4.2. Applications to $\wp_\Lambda$ in real lattice space. In the case where $n = 1$ and $b = 0$, we have that $f_{n,\Lambda,b} = \wp_\Lambda$ is the basic Weierstrass elliptic function on a real lattice. This function has been studied in [9]–[15], and these families of functions exhibit a wide variety of dynamical behaviors. If Λ is a rhombic square lattice then $J(\wp_\Lambda) = \mathbb{C}_\infty$ [10]. The rhombic square lattices lie on the line where $g_2 < 0$ and $g_3 = 0$ in Figure 1. In [12], examples of real lattices for which the Julia set of $\wp_\Lambda$ is the entire sphere were found, but not for every real rectangular equivalence class. In this section, we use Theorem 3.7 to find a countable number of real rectangular lattices in every similarity class for which the Julia set of $\wp_\Lambda$ is the entire sphere. We use the notation $\wp_\Lambda$ instead of $f_{n,\Lambda,b}$ throughout this section.

It will be helpful to identify a specified lattice within each shape equivalence class. We define the *standard lattice* within any real rectangular equivalence class as the lattice $\Gamma = [\gamma_1, \gamma_2]$ for which $\wp_\Gamma(\gamma_1/2) = 1$. Using the equations appearing in Equation (5) with $e_1 = 1$ we obtain

$$1 + e_2 + e_3 = 0, \quad e_2 e_3 = -\frac{g_2}{4}, \quad e_2 + e_3 + e_2 e_3 = \frac{g_3}{4},$$

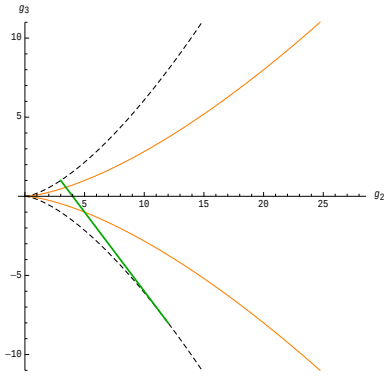

FIGURE 6. Real rectangular standard lattices are shown in green. All points colored orange represent real lattices similar to the $(5, -1)$ lattice.

and thus all real rectangular standard lattices lie on the line segment $g_3 = -g_2 + 4$ with $3 < g_2 < 12$ in real lattice space. (The ray when $g_2 > 12$ represents lattices for which $\wp(\lambda_1/2) > \wp(\lambda_1/2 + \lambda_2/2) = 1$.) Each curve in the parameter space representing a real rectangular lattice shape intersects this line segment exactly twice: once when the lattice is oriented horizontally, and once when the lattice is oriented vertically. We show the location of the standard lattices in green in Figure 6. All points colored orange represent real lattices similar to the $(5, -1)$ lattice.

Given any standard lattice, we can use the homogeneity property to find infinitely many similar lattices for which the real critical points land on a pole in one iteration. The following lemma follows from the homogeneity property in Lemma 2.3.

Lemma 4.2. [12] *Let $\Gamma = [\gamma_1, \gamma_2]$ be a standard real rectangular lattice, where γ_1 is chosen to be the smallest real positive lattice point. If m is any positive integer and $k = \sqrt[3]{1/(m\gamma_1)}$, then the lattice $\Lambda = k\Gamma$ has*

$$\wp_\Lambda(\lambda_1/2) = m\lambda_1$$

and thus $\wp_\Lambda(\lambda_1/2)$ is a pole.

Lemma 4.2 was used in [12] to find isolated examples for which the Julia set of $\wp_\Lambda$ was the entire sphere: namely, on real lattices for which we could show that the other two critical values were also poles. However, the results of Section 3 imply that even if the other two critical values are not poles, their orbits cannot be associated with Fatou components. As a consequence, given any real rectangular lattice, we can find infinitely many similar lattices Λ for which the Julia set of $\wp_\Lambda$ is the entire sphere.

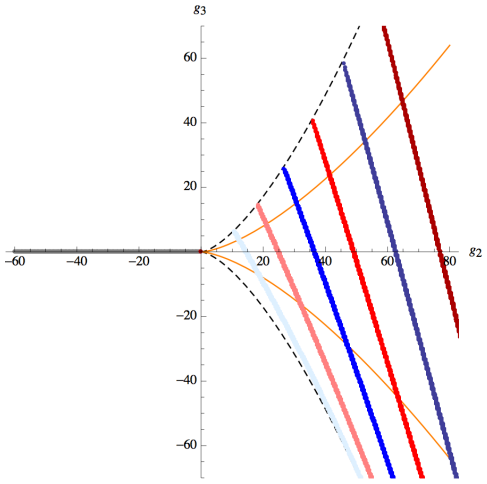

FIGURE 7. Lattices for which $J(\wp_\Lambda) = \mathbb{C}_\infty$ appear in increasing shades of blue for $m = 1$, $m = 2$, and $m = 3$ in Theorem 4.3. Real rhombic lattices (also having $J(\wp_\Lambda) = \mathbb{C}_\infty$) appear in grey. Lattices for which $\wp_\Lambda$ has a superattracting fixed point appear in increasing shades of red for $m = 1$, $m = 2$, and $m = 3$ in Lemma 4.4. All points colored orange represent real lattices similar to the $(5, -1)$ lattice.

Theorem 4.3. *Let $\Gamma = [\gamma_1, \gamma_2]$ be a standard real rectangular lattice, where γ_1 is chosen to be the smallest positive real lattice point. If m is any positive integer and $k = \sqrt[3]{1/(m\gamma_1)}$, then $\wp_\Lambda$ on the lattice $\Lambda = k\Gamma$ has $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

Proof. By Proposition 2.6(2) the postcritical set of $\wp_\Lambda$ is real. By Proposition 3.8, either $J(\wp_\Lambda) = \mathbb{C}_\infty$ or there exists exactly one (super)attracting or rationally neutral Fatou cycle for $\wp_\Lambda$ that contains a real critical point. By Lemma 4.2, all real critical points are prepoles, and therefore there are no nonrepelling cycles by Theorem 3.7. □

Recall that if Λ is a real rhombic square lattice then $J(\wp_\Lambda) = \mathbb{C}_\infty$ [10]; these lattices appear in grey in Figure 7 as the negative real axis. We show an approximation of the locus of parameters for the cases $m = 1, 2$, and 3 from Theorem 4.3 in increasingly darker shades of blue in Figure 7 (light blue corresponds to $m = 1$). We note that the locus of blue parameters in the real rectangular region do not form straight lines.

In [12], we found lattices for which the real critical point is superattracting.

Proposition 4.4. [12] *Let $\Gamma = [\gamma_1, \gamma_2]$ be a standard real rectangular lattice, where γ_1 is chosen to be the smallest positive real lattice point. If m is any odd positive integer and $k = \sqrt[3]{2/(m\gamma_1)}$, then the lattice $\Lambda = k\Gamma$ has $\wp_\Lambda(\lambda_1/2) = m\lambda_1/2$ and thus $\lambda_1/2$ is a superattracting fixed point.*

Using Proposition 3.8, the functions discussed in Proposition 4.4 have no other Fatou cycles. The locus of parameters for the cases $m = 1, 2$, and 3 from Proposition 4.4 appear in increasingly darker shades of red (pink corresponds to $m = 1$) in Figure 7.

References

[1] Baker, I. N.; Kotus, J.; Yinian, Lü. Iterates of meromorphic functions. I. *Ergodic Theory Dynam. Systems* **11** (1991), no. 2, 241–248. MR1116639 (92m:58113), Zbl 0711.30024.

[2] Baker, I. N.; Kotus, J.; Yinian, Lü. Iterates of meromorphic functions. IV. Critically finite functions. *Results Math.* **22** (1992), no. 3-4, 651–656. MR1189754 (94c:58166), Zbl 0774.30024.

[3] Bergweiler, Walter. Iteration of meromorphic functions. *Bull. Amer. Math. Soc.* (N.S.) **29** (1993), no. 2, 151–188. MR1216719 (94c:30033), Zbl 0791.30018, arXiv:math/9310226, doi: 10.1090/S0273-0979-1993-00432-4.

[4] Brin, Michael; Stuck, Garrett. Introduction to dynamical systems. *Cambridge University Press, Cambridge*, 2002. xii+240 pp. ISBN: 0-521-80841-3. MR1963683 (2003m:37001), Zbl 01849967, doi: 10.1017/CBO9780511755316.

[5] Devaney, Robert L.; Keen, Linda. Dynamics of tangent. *Dynamical systems (College Park, MD, 1986–87)*, 105–111, Lecture Notes in Math., 1342. *Springer, Berlin*, 1988. MR0970550 (90e:58093), Zbl 0662.30019, doi: 10.1007/BFb0082826.

[6] Devaney, Robert L.; Keen, Linda. Dynamics of maps with constant Schwarzian derivative. *Complex analysis, Joensuu 1987*, 93–100, Lecture Notes in Math., 1351. *Springer, Berlin*, 1988. MR0982075 (90b:58122), Zbl 0664.58023.

[7] Devaney, Robert L.; Keen, Linda. Dynamics of meromorphic maps: maps with polynomial Schwarzian derivative. *Ann. Sci. École Norm. Sup.* (4) **22** (1989), no. 1, 55–79. MR0985854 (90e:58071), Zbl 0666.30017.

[8] Du Val, Patrick. Elliptic functions and elliptic curves. London Mathematical Society Lecture Note Series, 9. *Cambridge Univ. Press, London-New York*, 1973. v+248 pp. MR0379512 (52 #417), Zbl 0261.33001.

[9] Hawkins, Jane. Smooth Julia sets of elliptic functions for square rhombic lattices. *Topology Proc.* **30** (2006), no. 1, 265–278. MR2280672 (2007j:37076), Zbl 1132.54021.

[10] Hawkins, Jane. A family of elliptic functions with Julia set the whole sphere. *J. Difference Equ. Appl.* **16** (2010), no. 5–6, 597–612. MR2642468 (2011f:37080), Zbl 1221.54045, doi: 10.1080/10236190903257859.

[11] Hawkins, Jane; Koss, Lorelei. Ergodic properties and Julia sets of Weierstrass elliptic functions. *Monatsh. Math.* **137** (2002), no. 4, 273–300. MR1947915 (2003j:37066), Zbl 1009.37034, doi: 10.1007/s00605-002-0504-1.

[12] Hawkins, Jane; Koss, Lorelei. Parametrized dynamics of the Weierstrass elliptic function. *Conform. Geom. Dyn.* **8** (2004), 1–35. MR2060376 (2005b:37089), Zbl 1076.30027, doi: 10.1090/S1088-4173-04-00103-1.

[13] Hawkins, Jane; Koss, Lorelei. Connectivity properties of Julia sets of Weierstrass elliptic functions. *Topology Appl.* **152** (2005), no. 1-2, 107–137. MR2160809 (2007b:37098), Zbl 1074.37029, doi: 10.1016/j.topol.2004.08.018.

[14] Hawkins, Jane; Koss, Lorelei; Kotus, Janina. Elliptic functions with critical orbits approaching infinity. *J. Difference Equ. Appl.* **16** (2010), no. 5–6, 613–630. MR2642469 (2011f:37081), Zbl 1230.37059, doi: 10.1080/10236190903203895.

[15] Hawkins, Jane M.; Look, Daniel M. Locally Sierpinski Julia sets of Weierstrass elliptic $\wp$ functions. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **16** (2006), no. 5, 1505–1520. MR2254870 (2007h:37068), Zbl 1185.37124, doi: 10.1142/S0218127406015453.

[16] Jones, Gareth A.; Singerman, David. Complex functions. An algebraic and geometric viewpoint. *Cambridge Univ. Press, Cambridge*, 1987. xiv+342 pp. ISBN: 0-521-30893-3; 0-521-31366-X. MR0890746 (89b:30001), Zbl 0608.30001, doi: 10.1002/zamm.19880680605.

[17] Kotus, Janina. Elliptic functions with critical points eventually mapped onto infinity. *Monatsh. Math.* **149** (2006), no. 2, 103–117. MR2264577 (2007i:37093), Zbl 1104.37033, doi: 10.1007/s00605-005-0373-5.

[18] Kotus, Janina; Urbański, Mariusz. Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. *Bull. London Math. Soc.* **35** (2003), no. 2, 269–275. MR1952406 (2003j:37067), Zbl 1023.37027, doi: 10.1112/S0024609302001686.

[19] Kotus, Janina; Urbański, Mariusz. Geometry and ergodic theory of non-recurrent elliptic functions. *J. Anal. Math.* **93** (2004), 35–102. MR2110325 (2005j:37065), Zbl 1092.37025, doi: 10.1007/BF02789304.

[20] Rippon, P. J.; Stallard, G. M. Iteration of a class of hyperbolic meromorphic functions. *Proc. Amer. Math. Soc.* **127** (1999), no. 11, 3251–3258. MR1610785 (2000b:30033), Zbl 0931.30017, doi: 10.1090/S0002-9939-99-04942-4.

[21] Singer, David. Stable orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.* **35** (1978), no. 2, 260–267. MR0494306 (58 #13206), Zbl 0391.58014, doi: 10.1137/0135020.

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