A superclass of the posinormal operators

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ABSTRACT. The starting place is a brief proof of a well-known result, the hyponormality of C_k (the generalized Cesàro operator of order one) for $k \geq 1$. This leads to the definition of a superclass of the posinormal operators. It is shown that all the injective unilateral weighted shifts belong to this superclass.

Sufficient conditions are determined for an operator in this superclass to be posinormal and hyponormal. A connection is established between this superclass and some recently-published sufficient conditions for a lower triangular factorable matrix to be a hyponormal bounded linear operator on ℓ^2.

CONTENTS

1. Introduction

1.1. Preliminaries. If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $A \in B(H)$ is said to be *posinormal* if there exists a positive operator $P \in B(H)$ satisfying $AA^* = A^*PA$. The operator A is *coposinormal* if A^* is posinormal. From [7, Theorem 2.1] we know that A is posinormal if and only if

$$\gamma^2 A^*A \geq AA^*$$

for some $\gamma \geq 0$. A is *hyponormal* when $\gamma = 1$. The operator A is *dominant* if $\operatorname{Ran}(A - \lambda) \subset \operatorname{Ran}(A - \lambda)^*$ for all λ in the spectrum of A; A is dominant if and only if $A - \lambda$ is posinormal for all complex numbers λ [7, Proposition 3.5]. Hyponormal operators are necessarily dominant. If A is posinormal, then $\operatorname{Ker} A \subset \operatorname{Ker} A^*$; see [7, Corollary 2.3].

A lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its entries are

$$m_{ij} = \begin{cases} a_i c_j & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$$

where a_i depends only on i and c_j depends only on j. A factorable matrix is *terraced* (see [5], [6]) if $c_j = 1$ for all j.

1.2. The motivating example. For fixed $k > 0$, the generalized Cesàro matrices of order one are the terraced matrices C_k that occur when $a_i = \frac{1}{k+i}$ for all i.

Proposition 1. C_k *is posinormal for all* $k > 0$, *and* C_k *is hyponormal for all* $k \geq 1$.

Proof. If

$$Q :\equiv \operatorname{diag}\{k, 1, 1, 1, \dots\} \text{ and}$$
$$P :\equiv \operatorname{diag}\left\{ \frac{k+i}{k+i+1} : i = 0, 1, 2, 3, \dots \right\},$$

it can be verified that $C_k Q C_k^* = C_k^* P C_k$ for all $k > 0$. If $\delta :\equiv \max\left\{\frac{1}{k}, 1\right\}$, then for all f in ℓ^2 we have

$$\langle (\delta C_k^* C_k - C_k C_k^*) f, f \rangle = \langle (\delta C_k^* C_k - \delta C_k^* P C_k + \delta C_k Q C_k^* - C_k C_k^*) f, f \rangle$$
$$= \langle \delta (I - P) C_k f, C_k f \rangle + \langle (\delta Q - I) C_k^* f, C_k^* f \rangle \geq 0,$$

and this gives the result. $\qquad\qquad\square$

We note that the preceding proof first appeared in [9]; for different proofs of the hyponormality of C_k for $k \geq 1$, see [7], [12]. The key role played by the relationship $C_k Q C_k^* = C_k^* P C_k$ in the proof of Proposition 1 leads us to the definition of a very large class of operators.

2. Definition, properties, and examples of supraposinormal operators

Definition 1. If $A \in B(H)$, we say that A is *supraposinormal* if there exist positive operators P and Q on H such that $AQA^* = A^*PA$, where at least one of P, Q has dense range. It will sometimes be convenient to refer to the ordered pair (Q, P) as an *interrupter pair associated with A*.

It is straightforward to verify that supraposinormality is a unitary invariant. We note that a normal operator A is supraposinormal with interrupter pair (I, I).

As the name suggests, this superclass of operators contains all the posinormal operators (and hence all the hyponormal operators and all the invertible operators; see [7]), as well as all the coposinormal operators: If A is posinormal, then $AA^* = A^*PA$ for some positive operator P, so A is supraposinormal with interrupter pair (I, P). If A is coposinormal, then $A^*A = AQA^*$ for some positive operator Q, so A is supraposinormal with interrupter pair (Q, I).

Proposition 2. *The collection S of all supraposinormal operators on H forms a cone in $B(H)$, and S is closed under involution.*

Proof. It is easy to see that S is closed under scalar multiplication, so S contains all αA for $A \in S$ and $\alpha \geq 0$, and therefore S is a cone. Moreover, it is equally easy to see that A is supraposinormal if and only if A^* is supraposinormal, so S is closed under involution. $\qquad\square$

Theorem 1. *Suppose $A \in B(H)$ satisfies $AQA^* = A^*PA$ for positive operators $P, Q \in B(H)$.*

(a) *If Q has dense range, then A is supraposinormal and*

$$\operatorname{Ker} A \subset \operatorname{Ker} A^*.$$

(b) *If P has dense range, then A is supraposinormal and*

$$\operatorname{Ker} A^* \subset \operatorname{Ker} A.$$

(c) *If Q is invertible, then the supraposinormal operator A is posinormal.*
(d) *If P is invertible, then the supraposinormal operator A is coposinormal.*
(e) *If P and Q are both invertible, then A is both posinormal and coposinormal with $\operatorname{Ker} A = \operatorname{Ker} A^*$ and $\operatorname{Ran} A = \operatorname{Ran} A^*$.*

Proof. For (a) and (b), the proof is straightforward. For (c) and (d), the proof requires only a minor adjustment in the discussion of Douglas's Theorem at the beginning of Section 2 in [7]. The proof of (e) is also straightforward. $\qquad\square$

Corollary 1. *If $A \in B(H)$ is supraposinormal, then $\operatorname{Ker} A \subset \operatorname{Ker} A^*$ or $\operatorname{Ker} A^* \subset \operatorname{Ker} A$.*

Corollary 2. *If $A \in B(H)$ is posinormal with an invertible interrupter, then A is also coposinormal.*

Corollary 3. *If A is supraposinormal with interrupter pair (P, P) for some positive operator P, then $\operatorname{Ker} A = \operatorname{Ker} A^*$; also, $\sqrt{P}A\sqrt{P}$ is normal.*

Theorem 2. *Assume $A - \lambda$ is supraposinormal for distinct real values $\lambda = 0, r_1$, and r_2, and assume that the same interrupter pair (Q, P) serves $A - \lambda$*

in each of those three cases. Then $Q = P$ and $\mathrm{Ker}(A - \lambda) = \mathrm{Ker}(A - \lambda)^*$ when $\lambda = 0, r_1,$ and r_2.

Proof. Since $(A - \lambda)Q(A - \lambda)^* = (A - \lambda)^*P(A - \lambda)$ for $\lambda = 0, r_1,$ and r_2, we find that for $k = 1$ and 2,

$$(A - r_k)Q(A - r_k)^* = (A - r_k)^*P(A - r_k)$$

reduces to

$$PA + A^*P + r_kQ = QA^* + AQ + r_kP.$$

Thus $(r_1 - r_2)Q = (r_1 - r_2)P$, so $Q = P$. The assertion that $\mathrm{Ker}(A - \lambda) = \mathrm{Ker}(A - \lambda)^*$ for $\lambda = 0, r_1,$ and r_2 follows from Corollary 3. $\qquad\square$

Definition 2. For $A \in B(H)$, we say that A is *totally supraposinormal* if $A - \lambda$ is supraposinormal for all complex numbers λ.

Theorem 3. *If $A \in B(H)$ is totally supraposinormal and the same two positive operators $Q, P \in B(H)$ form an interrupter pair (Q, P) for $A - \lambda$ for all complex numbers λ, then $Q = P$; it also follows that*

$$\mathrm{Ker}(A - \lambda) = \mathrm{Ker}(A - \lambda)^*$$

for all λ.

Proof. This result is a consequence of Theorem 2 and Corollary 3. $\qquad\square$

We have already observed that the class of posinormal operators is included in the class of supraposinormal operators, which is included in the class of all operators. As we are about to see, the unilateral weighted shifts are enough to show that these inclusions are all proper.

Let $\{e_n\}$ denote the standard orthonormal basis for ℓ^2.

Proposition 3. *A unilateral weighted shift W with the weight sequence*

$$\{w_n : w_0 \neq 0\}_{n \geq 0}$$

is supraposinormal if and only if it is injective.

Proof. First assume that W is injective, so $w_n \neq 0$ for all n. If

$$Q :\equiv \mathrm{diag}\{|w_1|^2, |w_2|^2, |w_3|^2, \dots\}$$

and

$$P :\equiv \mathrm{diag}\{p_0, 0, |w_0|^2, |w_1|^2, |w_2|^2, \dots\}$$

for some $p_0 > 0$, then it is straightforward to verify that $WQW^* = W^*PW$. Since $\{|w_1|^2, |w_2|^2, |w_3|^2, \dots\}$ is a strictly positive sequence, Q is a one-to-one positive operator; thus Q has dense range, so W is supraposinormal.

Next we assume that W is not injective, so $w_n = 0$ for some $n > 0$; assume that n is the smallest integer for which this holds. We have $We_n = 0$ and $W^*e_n = \overline{w_{n-1}}e_{n-1} \neq 0$, so $\mathrm{Ker}\,W \not\subset \mathrm{Ker}\,W^*$. Also, $W^*e_0 = 0$ while $We_0 = w_0e_1 \neq 0$, so $\mathrm{Ker}\,W^* \not\subset \mathrm{Ker}\,W$. It follows from Corollary 1 that W is not supraposinormal. $\qquad\square$

Corollary 4. *Every injective unilateral weighted shift is supraposinormal.*

We note that a noninjective unilateral weighted shift can also be supra-posinormal, as the next example illustrates.

Example 1. Let W denote the unilateral weighted shift with $w_0 = 0$ and $w_n = 1$ for all $n \geq 1$. Glancing at the proof of Proposition 3, we take $Q = I$ and $P :\equiv \text{diag}\{1, 0, 0, 1, 1, 1, 1, \ldots\}$. One easily verifies that $WW^* = \text{diag}\{0, 0, 1, 1, 1, 1, \ldots\} = W^*PW$, so W is posinormal and hence supraposinormal.

In fact, it can be shown that if, for some nonnegative integer n, W satisfies

    (a) $w_k = 0$ for $0 \leq k \leq n$ and $w_k \neq 0$ for $k > n$ and
    (b) $\sup_{k>n} |\frac{w_k}{w_{k+1}}| < +\infty$,

then W is posinormal and hence also supraposinormal.

Proposition 4. *An injective unilateral weighted shift is posinormal if and only if* $\sup_n |\frac{w_n}{w_{n+1}}| < +\infty$.

Proof. See [4]. □

Proposition 3 gives us a collection of operators that are supraposinormal, as well as a collection of operators that are not supraposinormal. We emphasize the latter now with a specific example.

Example 2. Suppose W is the unilateral weighted shift with weights $w_{2n} = 1$ and $w_{2n+1} = 0$ for all n. It follows from Proposition 3 that W cannot be supraposinormal.

Next we present an example of a supraposinormal operator that is neither posinormal nor coposinormal.

Example 3. Let W denote the unilateral weighted shift with weights $w_{2n} = 1$ and $w_{2n+1} = 1/(n+1)$ for all n. By Proposition 3, W is supraposinormal. Since $\sup_n |\frac{w_n}{w_{n+1}}| = +\infty$, it follows from Proposition 4 that W is not posi-normal. Since $W^*e_0 = 0$ while $We_0 = e_1$, we see that $\text{Ker}\, W^* \not\subset \text{Ker}\, W$, so W is also not coposinormal.

We model the following proof on that for the motivating example.

Theorem 4. *Assume that the bounded linear operator A on H is supraposi-normal with $AQA^* = A^*PA$. If D is a positive operator satisfying*

$$(1) \qquad\qquad \delta_1 Q \geq D \geq \delta_2 P \geq 0$$

for some constants $\delta_1, \delta_2 > 0$, then $\sqrt{D}A\sqrt{D}$ is posinormal. If (1) holds for some pair of positive constants δ_1, δ_2 with $\delta_1 \leq \delta_2$, then $\sqrt{D}A\sqrt{D}$ is hyponormal.

Proof. We find that

$$\left\langle \left[\frac{\delta_1}{\delta_2}(\sqrt{D}A^*\sqrt{D})(\sqrt{D}A\sqrt{D}) - (\sqrt{D}A\sqrt{D})(\sqrt{D}A^*\sqrt{D}) \right] f, f \right\rangle$$

$$= \left\langle \left[\frac{\delta_1}{\delta_2}\sqrt{D}A^*DA\sqrt{D} - \delta_1\sqrt{D}A^*PA\sqrt{D} + \delta_1\sqrt{D}AQA^*\sqrt{D} \right.\right.$$

$$\left.\left. - \sqrt{D}ADA^*\sqrt{D} \right] f, f \right\rangle$$

$$= \delta_1 \left\langle \left(\frac{1}{\delta_2}D - P \right) A\sqrt{D}f, A\sqrt{D}f \right\rangle + \langle (\delta_1 Q - D)A^*\sqrt{D}f, A^*\sqrt{D}f \rangle \geq 0$$

for all f in ℓ^2, as needed. $\qquad\square$

Corollary 5. *If A is a supraposinormal operator on H with $AQA^* = A^*PA$ and*

$$(2) \qquad\qquad\qquad \delta_1 Q \geq I \geq \delta_2 P \geq 0$$

for some constants $\delta_1, \delta_2 > 0$, then A is posinormal. If (2) holds for some pair of positive constants δ_1, δ_2 with $\delta_1 \leq \delta_2$, then A is hyponormal.

We note that Theorem 4 and Corollary 5 above are restricted to the case where P is dominated by a multiple of Q.

In the next section we will apply this theorem and its corollary to factorable matrices. We note that a large collection of examples of supraposinormal factorable matrices M having interrupter pair (I, P) with P diagonal and $I \geq P$, in which case M is hyponormal, can be found in [10].

3. Application to factorable matrices

Throughout this section we will restrict our attention to those factorable matrices M that are lower triangular and give bounded linear operators on $H = \ell^2$.

3.1. Sufficient conditions for hyponormality.

Theorem 5. *If M is a supraposinormal factorable matrix with $MQM^* = M^*PM$, and if P, Q, and D are diagonal matrices satisfying*

$$Q \geq D \geq P \geq 0,$$

then $\sqrt{D}M\sqrt{D}$ is a hyponormal factorable matrix.

Proof. This is an immediate consequence of Theorem 4. $\qquad\square$

Example 4 (Generalized Cesàro matrices of order one for $k \geq 1$)**.** We have already seen that C_k is hyponormal for $k \geq 1$ and that C_k is posinormal for all $k > 0$. C_k is known to be nonhyponormal when $0 < k < 1$ (see [7]). If $k \geq 1$ and $D :\equiv \text{diag}\{d_i : i = 0, 1, 2, 3, \dots\}$ where $\frac{k}{k+1} \leq d_0 \leq k$ and $\frac{k+i}{k+i+1} \leq d_i \leq 1$ for $i = 1, 2, 3, \dots$, then Proposition 1 (proof) and Theorem 5 together guarantee that $\sqrt{D}C_k\sqrt{D}$ is another hyponormal factorable matrix.

We note that Theorem 1(d) can be used to prove that C_k is coposinormal for all $k > 0$.

The following proposition will be useful to us throughout the remainder of this section.

Proposition 5. *Assume that the factorable matrix $M = [a_i c_j]$ is a bounded operator on ℓ^2 with $a_i, c_j > 0$ for all i, j, and $\left\{\frac{a_k}{c_k}\right\}$ is strictly decreasing to 0. If*

$$P :\equiv \mathrm{diag}\left\{\frac{c_{k+1}a_k - c_k a_{k+1}}{c_k c_{k+1} a_k^2} : k = 0, 1, 2, \dots\right\} \in B(\ell^2)$$

and

$$Q :\equiv \mathrm{diag}\left\{\frac{1}{c_0 a_0}, \frac{c_{k+1}a_k - c_k a_{k+1}}{c_{k+1}^2 a_k a_{k+1}} : k = 0, 1, 2, \dots\right\} \in B(\ell^2),$$

then M is supraposinormal with interrupter pair (Q, P).

Proof. Once the hypothesis is assumed, it is straightforward to verify that

$$M^* P M = \begin{pmatrix} c_0 a_0 & c_0 a_1 & c_0 a_2 & c_0 a_3 & \cdots \\ c_0 a_1 & c_1 a_1 & c_1 a_2 & c_1 a_3 & \cdots \\ c_0 a_2 & c_1 a_2 & c_2 a_2 & c_2 a_3 & \cdots \\ c_0 a_3 & c_1 a_3 & c_2 a_3 & c_3 a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = M Q M^*.$$

Clearly the positive operators P and Q are one-to-one, so they both have dense range; thus M is supraposinormal with interrupter pair (Q, P). $\square$

Our goal now is to obtain restrictions on the sequences $\{a_i\}, \{c_j\}$ that are sufficient to guarantee that the factorable matrix $M = [a_i c_j] \in B(\ell^2)$ is hyponormal.

Theorem 6. *Assume that $M = [a_i c_j]$ is a bounded operator on ℓ^2 with $a_i, c_j > 0$ for all i, j, $\left\{\frac{a_k}{c_k}\right\}$ is strictly decreasing to 0, and $\left\{\frac{a_k c_k}{a_{k+1} c_{k+1}}\right\}$ is bounded. If there exists a $\delta > 0$ such that $0 < c_0 a_0 \leq \delta$ and*

$$\frac{1}{\delta}\frac{a_k}{c_k}(\delta - c_k a_k) \leq \frac{a_{k+1}}{c_{k+1}} \leq \frac{\delta \frac{a_k}{c_k}}{\delta + c_{k+1}^2 \frac{a_k}{c_k}}$$

for each nonnegative integer k, then M is hyponormal.

Proof. Assume the conditions of the hypothesis and also that

$$(3) \qquad \sup\left\{\frac{c_{k+1}a_k - c_k a_{k+1}}{c_k c_{k+1} a_k^2} : k = 0, 1, 2, \dots\right\} < \infty;$$

this is enough to guarantee that

$$\sup\left\{\frac{c_{k+1}a_k - c_k a_{k+1}}{c_{k+1}^2 a_k a_{k+1}} : k = 0, 1, 2, \dots\right\} < \infty$$

also. If P and Q are the operators defined in Proposition 5, then $M^*PM = MQM^*$. By Corollary 5, M will be hyponormal if there exists a $\delta > 0$ such that $\delta Q \geq I \geq \delta P > 0$, and this leads to inequalities stated in the theorem. We note that since $P \leq (1/\delta)I$, the assumption (3) was not needed in the hypothesis. $\square$

Example 5 (Toeplitz matrix)**.** Suppose that M is the factorable matrix with entries $m_{ij} = a_i c_j$ where $a_i = r^i$, $c_j = 1/r^j$ for all i, j where $0 < r < 1$. One easily verifies that Theorem 6 is satisfied with $\delta = 1/(1 - r^2)$, so M is hyponormal.

We note that the conditions presented in Theorem 6 are not necessary for the hyponormality of a factorable matrix. For consider the case when $c_j = 1$ for each j and $a_i = (i+3)/(i+2)^2$ for each i. This example is known to be hyponormal since it satisfies the hypothesis of [8, Theorem 2.2], but it does not satisfy the inequality in Theorem 6 since that would require $1 \geq \delta \geq 12/11$, an impossibility.

We point out that for $\delta = 1$, the inequality in Theorem 6 reduces to the result in [11], which was obtained using a somewhat different approach, without invoking posinormality; for several examples that hold for $\delta = 1$, see that paper.

3.2. Some nonhyponormal examples. We now investigate posinormality and coposinormality for some nonhyponormal, supraposinormal factorable matrices.

Example 6 (Fibonacci matrix; see [3])**.** We recall that the Fibonacci sequence $\{f_n\}$ is defined by the linear recurrence equations $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2.$$

Let M denote the factorable matrix with nonzero entries

$$m_{ij} = a_i c_j$$

where $a_i = 1/(f_{i+1}f_{i+2})$ and $c_j = f_{j+1}^2$ for $j \leq i$. When $n \geq 3$,

$$c_n a_1 \prod_{j=2}^{n-1} (c_0 a_0 - c_j a_j)/(c_0^n a_0^n) = f_{n+1}/f_n > 1,$$

so it follows from [11, Theorem 2], that M is not a dominant operator and is hence also not hyponormal. If

$$P :\equiv \text{diag}\{(f_{n+2}^2 f_{n+3} - f_{n+1}^3)/(f_{n+1}f_{n+2}f_{n+3}) : n = 0, 1, 2, 3, \dots\}$$

and

$$Q :\equiv \text{diag}\{1, (f_{n+2}^2 f_{n+3} - f_{n+1}^3)/f_{n+2}^3 : n = 0, 1, 2, 3, \dots\}$$
$$= \text{diag}\{(f_{n+1}^2 f_{n+2} - f_n^3)/f_{n+1}^3 : n = 0, 1, 2, 3, \dots\},$$

it follows from Proposition 5 that $M^*PM = MQM^*$. It can be verified that

$$I \leq Q \leq 2I \text{ and } (1/2)I \leq P \leq 4I;$$

thus

$$Q \geq I \geq (1/4)P,$$

so by Corollary 5, M is posinormal. Similarly, since

$$2P \geq I \geq (1/2)Q,$$

M is also coposinormal.

Example 7 (q-Cesàro matrix for $q > 1$; see [1], [13]). If M is the factorable matrix with nonzero entries

$$m_{ij} = a_i c_j$$

where $a_i = (q-1)/(q^{i+1} - 1)$ and $c_j = q^j$ for $0 \leq j \leq i$, and if

$$P := \text{diag}\{(q^{n+1} - 1)(q^{n+2} + q^{n+1} - 1)/[q^{n+1}(q^{n+2} - 1)] : n = 0, 1, 2, 3, \dots\}$$

and

$$\begin{aligned}
Q &:\equiv \text{diag}\{1, (q^{n+2} + q^{n+1} - 1)/q^{n+2} : n = 0, 1, 2, 3, \dots\} \\
&= \text{diag}\{1 + 1/q - 1/q^{n+1} : n = 0, 1, 2, 3, \dots\},
\end{aligned}$$

it follows from Proposition 5 that $MQM^* = M^*PM$. It is straightforward to check that

$$Q \geq I \geq (1/2)P \text{ and } (q+1)P \geq I \geq [1/(q+1)]Q,$$

so we know from Corollary 5 that M is both posinormal and coposinormal for $q > 1$. It is demonstrated in [11] that M is not dominant and hence also not hyponormal.

Example 8 (q-Cesàro matrix for $0 < q < 1$; see [2], [13]). If M is the factorable matrix with nonzero entries

$$m_{ij} = a_i c_j$$

where $a_i = (1-q)q^i/(1 - q^{i+1})$ and $c_j = 1/q^j$ for $0 \leq j \leq i$, and if

$$P := \text{diag}\{(1 - q^{n+1})(1 + q - q^{n+2})/(1 - q^{n+2}) : n = 0, 1, 2, 3, \dots\}$$

and

$$\begin{aligned}
Q &:\equiv \text{diag}\{1, 1 + q - q^{n+2} : n = 0, 1, 2, 3, \dots\} \\
&= \text{diag}\{1 + q - q^{n+1} : n = 0, 1, 2, 3, \dots\},
\end{aligned}$$

it follows from Proposition 5 that $MQM^* = M^*PM$. One can easily check that

$$Q \geq I \geq (1/2)P \quad \text{and} \quad [(q+1)/q]P \geq I \geq [q/(q+1)]Q,$$

so M is both posinormal and coposinormal for $0 < q < 1$. We know from [11] that M is not dominant and not hyponormal.

4. Conclusion

We close with two questions that seem natural but whose answers have not been found here:

(1) Does there exist an operator A that is totally supraposinormal but neither posinormal nor coposinormal?

(2) Does there exist an operator A that is totally supraposinormal but neither dominant nor codominant?

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