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Conway–Gordon type theorem for the complete four-partite graph $K_{3,3,1,1}$

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Abstract. We give a Conway–Gordon type formula for invariants of knots and links in a spatial complete four-partite graph $K_{3,3,1,1}$ in terms of the square of the linking number and the second coefficient of the Conway polynomial. As an application, we show that every rectilinear spatial $K_{3,3,1,1}$ contains a nontrivial Hamiltonian knot.

Contents

1. Introduction

Throughout this paper we work in the piecewise linear category. Let G be a finite graph. An embedding f of G into the Euclidean 3-space $\mathbb{R}^3$ is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph*. We denote the set of all spatial embeddings of G by $\mathrm{SE}(G)$. We call a subgraph γ of G which is homeomorphic to the circle a *cycle* of G and denote the set of all cycles of G by $\Gamma(G)$. We also call a cycle of G a *k-cycle* if it contains exactly k edges and denote the set of all k-cycles of G by $\Gamma_k(G)$. In particular, a k-cycle is said to be *Hamiltonian* if k equals the number of all vertices of G. For a positive integer n, $\Gamma^{(n)}(G)$ denotes the set of all cycles of G $(= \Gamma(G))$ if $n = 1$ and the set of all unions of n mutually disjoint cycles of G if $n \geq 2$. For an element γ in $\Gamma^{(n)}(G)$ and an element f in $\mathrm{SE}(G)$, $f(\gamma)$ is none other than a knot in $f(G)$ if $n = 1$ and an n-component link in $f(G)$ if $n \geq 2$. In particular, we call $f(\gamma)$ a *Hamiltonian knot* in $f(G)$ if γ is a Hamiltonian cycle.

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For an edge e of a graph G, we denote the subgraph $G \setminus \mathrm{int}e$ by $G - e$. Let $e = \overline{uv}$ be an edge of G which is not a loop, where u and v are distinct end vertices of e. Then we call the graph which is obtained from $G - e$ by identifying u and v the *edge contraction of G along e* and denote it by G/e. A graph H is called a *minor* of a graph G if there exists a subgraph G' of G and the edges $e_1, e_2, \ldots, e_m$ of G' each of which is not a loop such that H is obtained from G' by a sequence of edge contractions along $e_1, e_2, \ldots, e_m$. A minor H of G is called a *proper minor* if H does not equal G. Let $\mathcal{P}$ be a property of graphs which is *closed* under minor reductions; that is, for any graph G which does not have $\mathcal{P}$, all minors of G also do not have $\mathcal{P}$. A graph G is said to be *minor-minimal* with respect to $\mathcal{P}$ if G has $\mathcal{P}$ but all proper minors of G do not have $\mathcal{P}$. Then it is known that there exist finitely many minor-minimal graphs with respect to $\mathcal{P}$ [RS].

Let K_m be the *complete graph* on m vertices, namely the simple graph consisting of m vertices in which every pair of distinct vertices is connected by exactly one edge. Then the following are very famous in spatial graph theory, which are called the Conway–Gordon theorems.

Theorem 1.1 (Conway–Gordon [CG]).

(1) *For any element f in* $\mathrm{SE}(K_6)$,

$$\sum_{\gamma \in \Gamma^{(2)}(K_6)} \mathrm{lk}(f(\gamma)) \equiv 1 \pmod 2, \tag{1.1}$$

where lk *denotes the linking number.*

(2) *For any element f in* $\mathrm{SE}(K_7)$,

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) \equiv 1 \pmod 2, \tag{1.2}$$

where a_2 denotes the second coefficient of the Conway polynomial.

A graph is said to be *intrinsically linked* if for any element f in $\mathrm{SE}(G)$, there exists an element γ in $\Gamma^{(2)}(G)$ such that $f(\gamma)$ is a nonsplittable 2-component link, and to be *intrinsically knotted* if for any element f in $\mathrm{SE}(G)$, there exists an element γ in $\Gamma(G)$ such that $f(\gamma)$ is a nontrivial knot. Theorem 1.1 implies that K_6 (resp. K_7) is intrinsically linked (resp. knotted). Moreover, the intrinsic linkedness (resp. knottedness) is closed under minor reductions [NeTh] (resp. [FL]), and K_6 (resp. K_7) is minor-minimal with respect to the intrinsically linkedness [S] (resp. knottedness [MRS]).

A $\triangle Y$*-exchange* is an operation to obtain a new graph G_Y from a graph $G_\triangle$ by removing all edges of a 3-cycle $\triangle$ of $G_\triangle$ with the edges $\overline{uv}, \overline{vw}$ and $\overline{wu}$, and adding a new vertex x and connecting it to each of the vertices u, v and w as illustrated in Figure 1.1 (we often denote $\overline{ux} \cup \overline{vx} \cup \overline{wx}$ by Y). A $Y\triangle$*-exchange* is the reverse of this operation. We call the set of all graphs obtained from a graph G by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges the *G-family* and denote it by $\mathcal{F}(G)$. In particular, we denote the set of all graphs obtained from G by a finite sequence of $\triangle Y$-exchanges by $\mathcal{F}_{\triangle}(G)$. For example, it is well known that the K_6-family consists of exactly seven graphs as illustrated in Figure 1.2, where an arrow between two graphs indicates the application of a single $\triangle Y$-exchange. Note that $\mathcal{F}_{\triangle}(K_6) = \mathcal{F}(K_6) \setminus \{P_7\}$. Since P_{10} is isomorphic to the *Petersen graph*, the K_6-family is also called the *Petersen family*. It is also well known that the K_7-family consists of exactly twenty graphs, and there exist exactly six graphs in the K_7-family each of which does not belong to $\mathcal{F}_{\triangle}(K_7)$. Then the intrinsic linkedness and the intrinsic knottedness behave well under $\triangle Y$-exchanges as follows.

Proposition 1.2 (Sachs [S])**.**

(1) *If $G_{\triangle}$ is intrinsically linked, then G_Y is also intrinsically linked.*
(2) *If $G_{\triangle}$ is intrinsically knotted, then G_Y is also intrinsically knotted.*

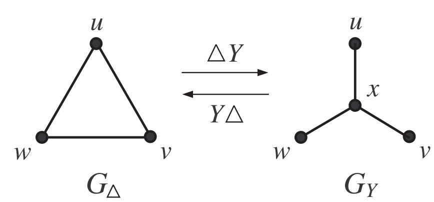


FIGURE 1.1.

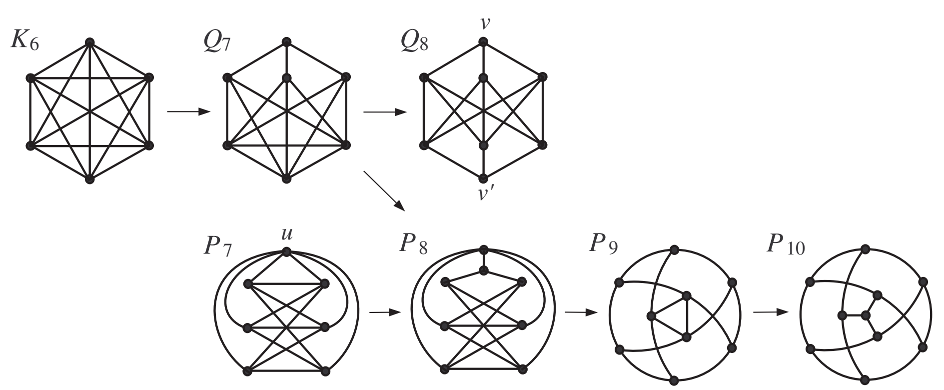


FIGURE 1.2.

Proposition 1.2 implies that any element in $\mathcal{F}_{\triangle}(K_6)$ (resp. $\mathcal{F}_{\triangle}(K_7)$) is intrinsically linked (resp. knotted). In particular, Robertson–Seymour–Thomas showed that the set of all minor-minimal intrinsically linked graphs equals the K_6-family, so the converse of Proposition 1.2(1) is also true [RST].

On the other hand, it is known that any element in $\mathcal{F}_\triangle(K_7)$ is minor-minimal with respect to the intrinsic knottedness [KS], but any element in $\mathcal{F}(K_7) \setminus \mathcal{F}_\triangle(K_7)$ is not intrinsically knotted [FN], [HNTY], [GMN], so the converse of Proposition 1.2(2) is not true. Moreover, there exists a minor-minimal intrinsically knotted graph which does not belong to $\mathcal{F}_\triangle(K_7)$ as follows. Let $K_{n_1,n_2,\ldots,n_m}$ be the *complete m-partite graph*, namely the simple graph whose vertex set can be decomposed into m mutually disjoint nonempty sets $V_1, V_2, \ldots, V_m$ where the number of elements in V_i equals n_i such that no two vertices in V_i are connected by an edge and every pair of vertices in the distinct sets V_i and V_j is connected by exactly one edge, see Figure 1.3 which illustrates $K_{3,3}$, $K_{3,3,1}$ and $K_{3,3,1,1}$. Note that $K_{3,3,1}$ is isomorphic to P_7 in the K_6-family, namely $K_{3,3,1}$ is a minor-minimal intrinsically linked graph. On the other hand, Motwani–Raghunathan–Saran claimed in [MRS] that it may be proven that $K_{3,3,1,1}$ is intrinsically knotted by using the same technique of Theorem 1.1, namely, by showing that for any element in $\mathrm{SE}(K_{3,3,1,1})$, the sum of a_2 over all of the Hamiltonian knots is always congruent to one modulo two. But Kohara–Suzuki showed in [KS] that the claim did not hold; that is, the sum of a_2 over all of the Hamiltonian knots is dependent to each element in $\mathrm{SE}(K_{3,3,1,1})$. Actually, they demonstrated the specific two elements f_1 and f_2 in $\mathrm{SE}(K_{3,3,1,1})$ as illustrated in Figure 1.4. Here $f_1(K_{3,3,1,1})$ contains exactly one nontrivial knot $f_1(\gamma_0)$ (= a trefoil knot, $a_2 = 1$) which is drawn by bold lines, where γ_0 is an element in $\Gamma_8(K_{3,3,1,1})$, and $f_2(K_{3,3,1,1})$ contains exactly two nontrivial knots $f_2(\gamma_1)$ and $f_2(\gamma_2)$ (= two trefoil knots) which are drawn by bold lines, where γ_1 and γ_2 are elements in $\Gamma_8(K_{3,3,1,1})$. Thus the situation of the case of $K_{3,3,1,1}$ is different from the case of K_7. By using another technique different from Conway–Gordon's, Foisy proved the following.

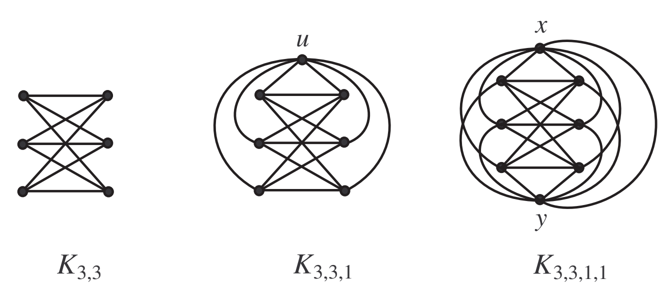


Figure 1.3.

Theorem 1.3 (Foisy [F02]). *For any element f in $\mathrm{SE}(K_{3,3,1,1})$, there exists an element γ in $\cup_{k=4}^{8}\Gamma_k(K_{3,3,1,1})$ such that $a_2(f(\gamma)) \equiv 1 \pmod 2$.*

Theorem 1.3 implies $K_{3,3,1,1}$ is intrinsically knotted. Moreover, Proposition 1.2(2) and Theorem 1.3 imply that any element G in $\mathcal{F}_\triangle(K_{3,3,1,1})$ is

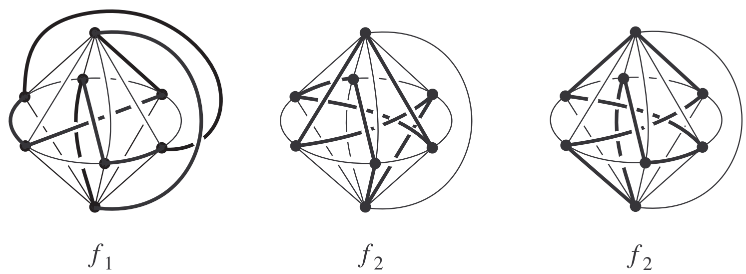


FIGURE 1.4.

also intrinsically knotted. It is known that there exist exactly twenty six elements in $\mathcal{F}_{\triangle}(K_{3,3,1,1})$. Since Kohara–Suzuki pointed out that each of the proper minors of G is not intrinsically knotted [KS], it follows that any element in $\mathcal{F}_{\triangle}(K_{3,3,1,1})$ is minor-minimal with respect to the intrinsic knottedness. Note that a $\triangle Y$-exchange does not change the number of edges of a graph. Since K_7 and $K_{3,3,1,1}$ have different numbers of edges, the families $\mathcal{F}_{\triangle}(K_7)$ and $\mathcal{F}_{\triangle}(K_{3,3,1,1})$ are disjoint.

Our first purpose in this article is to refine Theorem 1.3 by giving a kind of Conway–Gordon type formula for $K_{3,3,1,1}$ not over $\mathbb{Z}_2$, but integers $\mathbb{Z}$. In the following, $\Gamma^{(2)}_{k,l}(G)$ denotes the set of all unions of two disjoint cycles of a graph G consisting of a k-cycle and an l-cycle, and x and y denotes the two vertices of $K_{3,3,1,1}$ with valency seven. Then we have the following.

Theorem 1.4.

(1) *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$,

$$\begin{aligned}(1.3)\quad 4 \sum_{\gamma\in\Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\substack{\gamma\in\Gamma_7(K_{3,3,1,1})\\ \{x,y\}\not\subset\gamma}} a_2(f(\gamma)) \\ - 4\sum_{\gamma\in\Gamma'_6} a_2(f(\gamma)) - 4 \sum_{\substack{\gamma\in\Gamma_5(K_{3,3,1,1})\\ \{x,y\}\not\subset\gamma}} a_2(f(\gamma)) \\ = \sum_{\lambda\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 + 2 \sum_{\lambda\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 - 18,\end{aligned}$$

where Γ'_6 is a specific proper subset of $\Gamma_6(K_{3,3,1,1})$ which does not depend on f (see (2.31)).

(2) *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$,

$$(1.4)\qquad \sum_{\lambda\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 + 2 \sum_{\lambda\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 \geq 22.$$

We prove Theorem 1.4 in the next section. By combining the two parts of Theorem 1.4, we immediately obtain the following.

Corollary 1.5. *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$,

$$\sum_{\gamma\in\Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_7(K_{3,3,1,1})\\ \{x,y\}\not\subset\gamma}} a_2(f(\gamma)) - \sum_{\gamma\in\Gamma_6'} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_5(K_{3,3,1,1})\\ \{x,y\}\not\subset\gamma}} a_2(f(\gamma)) \geq 1. \tag{1.5}$$

Corollary 1.5 gives an alternative proof of the fact that $K_{3,3,1,1}$ is intrinsically knotted. Moreover, Corollary 1.5 refines Theorem 1.3 by identifying the cycles that might be nontrivial knots in $f(K_{3,3,1,1})$.

Remark 1.6. We see the left side of (1.5) is not always congruent to one modulo two by considering two elements f_1 and f_2 in $\mathrm{SE}(K_{3,3,1,1})$ as illustrated in Figure 1.4. Thus Corollary 1.5 shows that the argument over $\mathbb{Z}$ has a nice advantage. In particular, f_1 gives the best possibility for (1.5), and therefore for (1.4) by Theorem 1.4(1). Actually $f_1(K_{3,3,1,1})$ contains exactly fourteen nontrivial links all of which are Hopf links, where the six of them are the images of elements in $\Gamma_{3,5}^{(2)}(K_{3,3,1,1})$ by f_1 and the eight of them are the images of elements in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ by f_1.

As we said before, any element G in $\mathcal{F}_\triangle(K_7)\cup\mathcal{F}_\triangle(K_{3,3,1,1})$ is a minor-minimal intrinsically knotted graph. If G belongs to $\mathcal{F}_\triangle(K_7)$, then it is known that Conway–Gordon type formula over $\mathbb{Z}_2$ as in Theorem 1.1 also holds for G as follows.

Theorem 1.7 (Nikkuni–Taniyama [NT])**.** *Let G be an element in $\mathcal{F}_\triangle(K_7)$. Then, there exists a map ω from $\Gamma(G)$ to $\mathbb{Z}_2$ such that for any element f in* $\mathrm{SE}(G)$,

$$\sum_{\gamma\in\Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \equiv 1 \pmod 2.$$

Namely, for any element G in $\mathcal{F}_\triangle(K_7)$, there exists a subset Γ of $\Gamma(G)$ which depends on only G such that for any element f in $\mathrm{SE}(G)$, the sum of a_2 over all of the images of the elements in Γ by f is odd. On the other hand, if G belongs to $\mathcal{F}_\triangle(K_{3,3,1,1})$, we have a Conway–Gordon type formula over $\mathbb{Z}$ for G as in Corollary 1.5 as follows. We prove it in Section 3.

Theorem 1.8. *Let G be an element in $\mathcal{F}_\triangle(K_{3,3,1,1})$. Then, there exists a map ω from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element f in* $\mathrm{SE}(G)$,

$$\sum_{\gamma\in\Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \geq 1.$$

Our second purpose in this article is to give an application of Theorem 1.4 to the theory of rectilinear spatial graphs. A spatial embedding f of a graph G is said to be *rectilinear* if for any edge e of G, $f(e)$ is a straight line segment in $\mathbb{R}^3$. We denote the set of all rectilinear spatial embeddings of G by $\mathrm{RSE}(G)$. We can see that any simple graph has a rectilinear spatial embedding by taking all of the vertices on the spatial curve (t, t^2, t^3) in $\mathbb{R}^3$ and connecting every pair of two adjacent vertices by a straight line segment. Rectilinear spatial graphs appear in polymer chemistry as a mathematical model for chemical compounds, see [A] for example. Then by an application of Theorem 1.4, we have the following.

Theorem 1.9. *For any element f in $\mathrm{RSE}(K_{3,3,1,1})$,*

$$\sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) \geq 1.$$

We prove Theorem 1.9 in section 4. As a corollary of Theorem 1.9, we immediately have the following.

Corollary 1.10. *For any element f in $\mathrm{RSE}(K_{3,3,1,1})$, there exists a Hamiltonian cycle γ of $K_{3,3,1,1}$ such that $f(\gamma)$ is a nontrivial knot with $a_2(f(\gamma)) > 0$.*

Corollary 1.10 is an affirmative answer to the question of Foisy–Ludwig [FL, Question 5.8] which asks whether the image of every rectilinear spatial embedding of $K_{3,3,1,1}$ always contains a nontrivial Hamiltonian knot.

Remark 1.11.

(1) In [FL, Question 5.8], Foisy–Ludwig also asked that whether the image of every spatial embedding of $K_{3,3,1,1}$ (which may not be rectilinear) always contains a nontrivial Hamiltonian knot. As far as the authors know, it is still open.

(2) In addition to the elements in $\mathcal{F}_{\triangle}(K_7) \cup \mathcal{F}_{\triangle}(K_{3,3,1,1})$, many minor-minimal intrinsically knotted graph are known [F04], [GMN]. In particular, it has been announced by Goldberg–Mattman–Naimi that all of the thirty two elements in $\mathcal{F}(K_{3,3,1,1}) \setminus \mathcal{F}_{\triangle}(K_{3,3,1,1})$ are minor-minimal intrinsically knotted graphs [GMN]. Note that their method is based on Foisy's idea in the proof of Theorem 1.3 with the help of a computer.

2. Conway–Gordon type formula for $K_{3,3,1,1}$

To prove Theorem 1.4, we recall a Conway–Gordon type formula over $\mathbb{Z}$ for a graph in the K_6-family which is as below.

Theorem 2.1. *Let G be an element in $\mathcal{F}(K_6)$. Then there exist a map ω from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element f in $\mathrm{SE}(G)$,*

$$2 \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma))^2 - 1. \tag{2.1}$$

We remark here that Theorem 2.1 was shown by Nikkuni (for the case $G = K_6$) [N], O'Donnol ($G = P_7$) [O] and Nikkuni–Taniyama (for the others) [NT]. In particular, we use the following explicit formulae for Q_8 and P_7 in the proof of Theorem 1.4. For the other cases, see Hashimoto–Nikkuni [HN].

Theorem 2.2.

(1) (Hashimoto–Nikkuni [HN]). *For any element f in* $\mathrm{SE}(Q_8)$,

$$2\sum_{\gamma\in\Gamma_7(P_7)} a_2(f(\gamma)) + 2\sum_{\substack{\gamma\in\Gamma_6(Q_8)\\ v,v'\notin\gamma}} a_2(f(\gamma)) - 2\sum_{\substack{\gamma\in\Gamma_6(Q_8)\\ \gamma\cap\{v,v'\}\neq\emptyset}} a_2(f(\gamma)) = \sum_{\gamma\in\Gamma^{(2)}_{4,4}(Q_8)} \mathrm{lk}(f(\gamma))^2 - 1,$$

where v and v' are exactly two vertices of Q_8 with valency three.

(2) (O'Donnol [O]). *For any element f in* $\mathrm{SE}(P_7)$,

$$2\sum_{\gamma\in\Gamma_7(P_7)} a_2(f(\gamma)) - 4\sum_{\substack{\gamma\in\Gamma_6(P_7)\\ u\notin\gamma}} a_2(f(\gamma)) - 2\sum_{\gamma\in\Gamma_5(P_7)} a_2(f(\gamma)) = \sum_{\gamma\in\Gamma^{(2)}_{3,4}(P_7)} \mathrm{lk}(f(\gamma))^2 - 1,$$

where u is the vertex of P_7 with valency six.

By taking the modulo two reduction of (2.1), we immediately have the following fact containing Theorem 1.1(1).

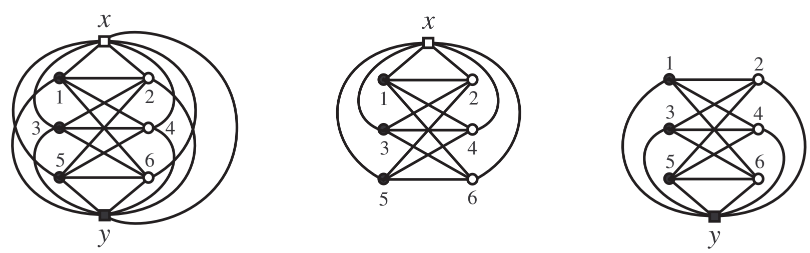
Corollary 2.3 (Sachs [S], Taniyama–Yasuhara [TY]). *Let G be an element in $\mathcal{F}(K_6)$. Then, for any element f in* $\mathrm{SE}(G)$,

$$\sum_{\gamma\in\Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma)) \equiv 1 \pmod 2.$$

Now we give labels for the vertices of $K_{3,3,1,1}$ as illustrated in the left figure in Figure 2.1. We also call the vertices $1, 3, 5$ and $2, 4, 6$ the *black vertices* and the *white vertices*, respectively. We regard $K_{3,3}$ as the subgraph of $K_{3,3,1,1}$ induced by all of the white and black vertices. Let G_x and G_y be two subgraphs of $K_{3,3,1,1}$ as illustrated in Figure 2.1(1) and (2), respectively. Since each of G_x and G_y is isomorphic to P_7, by applying Theorem 2.2(2) to $f|_{G_x}$ and $f|_{G_y}$ for an element f in $\mathrm{SE}(K_{3,3,1,1})$, it follows that

$$2\sum_{\gamma\in\Gamma_7(G_x)} a_2(f(\gamma)) - 4\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2\sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)) = \sum_{\gamma\in\Gamma^{(2)}_{3,4}(G_x)} \mathrm{lk}(f(\gamma))^2 - 1, \tag{2.2}$$

$$(2.3)\quad 2\sum_{\gamma\in\Gamma_7(G_y)} a_2(f(\gamma)) - 4\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2\sum_{\gamma\in\Gamma_5(G_y)} a_2(f(\gamma)) = \sum_{\gamma\in\Gamma^{(2)}_{3,4}(G_y)} \mathrm{lk}(f(\gamma))^2 - 1.$$



(1) (2)

FIGURE 2.1. (1) G_x, (2) G_y

Let γ be an element in $\Gamma(K_{3,3,1,1})$ which is a 8-cycle or a 6-cycle containing x and y. We will assign a type to γ as follows:

- γ is of *Type A* if the neighbor vertices of x in γ consist of both a black vertex and a white vertex (if and only if the neighbor vertices of y in γ consist of both a black vertex and a white vertex).
- γ is of *Type B* if the neighbor vertices of x in γ consist of only black (resp. white) vertices and the neighbor vertices of y in γ consist of only white (resp. black) vertices.
- γ is of *Type C* if γ contains the edge $\overline{xy}$.
- γ is of *Type D* if $\gamma \in \Gamma_6(K_{3,3,1,1})$ and the neighbor vertices of x and y in γ consist of only black or only white vertices.

Note that any element in $\Gamma_8(K_{3,3,1,1})$ is of Type A, B or C, and any element in $\Gamma_6(K_{3,3,1,1})$ containing x and y is of Type A, B, C or D.

On the other hand, let λ be an element in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$. We assign types to λ as follows:

- λ is of *Type A* if λ does not contain the edge $\overline{xy}$ and both x and y are contained in either connected component of λ.
- λ is of *Type B* if x and y are contained in different connected components of λ.
- λ is of *Type C* if λ contains the edge $\overline{xy}$.

Note that any element in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$ is of Type A, B or C.

Then the following three lemmas hold.

Lemma 2.4. *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$,

$$\sum_{\lambda\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 + 2 \sum_{\substack{\lambda\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})\\ \text{Type A}}} \mathrm{lk}(f(\lambda))^2 \tag{2.4}$$
$$= 4 \sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type A}}} a_2(f(\gamma)) - 4 \left\{ \sum_{\gamma\in\Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma\in\Gamma_7(G_y)} a_2(f(\gamma)) \right\}$$
$$+ 8 \sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma)) - 4 \sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma,\ \text{Type A}}} a_2(f(\gamma))$$
$$- 4 \left\{ \sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)) + \sum_{\gamma\in\Gamma_5(G_y)} a_2(f(\gamma)) \right\} + 10.$$

Proof. For $i = 1, 3, 5$ and $j = 2, 4, 6$, let us consider subgraphs $F^{(ij)}_x = (G_x - \overline{ij}) \cup \overline{iy} \cup \overline{jy}$ and $F^{(ij)}_y = (G_y - \overline{ij}) \cup \overline{ix} \cup \overline{jx}$ of $K_{3,3,1,1}$ as illustrated in Figure 2.2(1) and (2), respectively. Since each of $F^{(ij)}_x$ and $F^{(ij)}_y$ is homeomorphic to P_7, by applying Theorem 2.2(2) to $f|_{F^{(ij)}_x}$, it follows that

$$\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma^{(2)}_{3,5}(F^{(ij)}_x)\\ \gamma\in\Gamma_3(F^{(ij)}_x),\ \gamma'\in\Gamma_5(F^{(ij)}_x)\\ x\in\gamma,\ y\in\gamma'}} \mathrm{lk}(f(\lambda))^2 + \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma^{(2)}_{4,4}(F^{(ij)}_x)\\ x,y\in\gamma'}} \mathrm{lk}(f(\lambda))^2 \tag{2.5}$$
$$+ \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma^{(2)}_{3,4}(G_x)\\ \gamma\in\Gamma_3(G_x),\ \gamma'\in\Gamma_4(G_x)\\ \overline{ij}\not\subset\lambda,\ x\in\gamma}} \mathrm{lk}(f(\lambda))^2$$
$$= 2 \left\{ \sum_{\gamma\in\Gamma_8(F^{(ij)}_x)} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_7(G_x)\\ \overline{ij}\not\subset\gamma}} a_2(f(\gamma)) \right\}$$
$$- 4 \left\{ \sum_{\substack{\gamma\in\Gamma_7(F^{(ij)}_x)\\ x\notin\gamma,\ y\in\gamma}} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_6(K_{3,3})\\ \overline{ij}\not\subset\gamma}} a_2(f(\gamma)) \right\}$$
$$- 2 \left\{ \sum_{\substack{\gamma\in\Gamma_6(F^{(ij)}_x)\\ x,y\in\gamma}} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_5(G_x)\\ \overline{ij}\not\subset\gamma}} a_2(f(\gamma)) \right\} + 1.$$

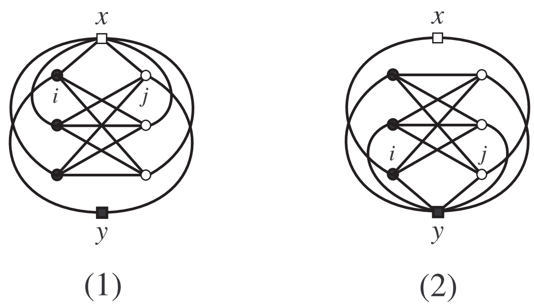


FIGURE 2.2. (1) $F_x^{(ij)}$, (2) $F_y^{(ij)}$ $(i = 1,3,5,\ j = 2,4,6)$

Let us take the sum of both sides of (2.5) over $i = 1,3,5$ and $j = 2,4,6$. For an element γ in $\Gamma_8(K_{3,3,1,1})$ of Type A, there uniquely exists $F_x^{(ij)}$ containing γ. This implies that

$$\sum_{i,j}\left(\sum_{\gamma\in\Gamma_8(F_x^{(ij)})} a_2(f(\gamma))\right) = \sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type A}}} a_2(f(\gamma)). \tag{2.6}$$

For an element γ of $\Gamma_7(G_x)$, there exist exactly four edges of $K_{3,3}$ which are not contained in γ. Thus γ is common for exactly four $F_x^{(ij)}$'s. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\gamma\in\Gamma_7(G_x)\\ \overline{ij}\not\subset\gamma}} a_2(f(\gamma))\right) = 4\sum_{\Gamma_7(G_x)} a_2(f(\gamma)). \tag{2.7}$$

For an element γ in $\Gamma_7(G_y)$, there uniquely exists $F_x^{(ij)}$ containing γ. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\gamma\in\Gamma_7(F_x^{(ij)})\\ x\notin\gamma,\ y\in\gamma}} a_2(f(\gamma))\right) = \sum_{\gamma\in\Gamma_7(G_y)} a_2(f(\gamma)). \tag{2.8}$$

For an element γ in $\Gamma_6(K_{3,3})$, there exist exactly three edges of $K_{3,3}$ which are not contained in γ. Thus γ is common for exactly three $F_x^{(ij)}$'s. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\gamma\in\Gamma_6(K_{3,3})\\ \overline{ij}\not\subset\gamma}} a_2(f(\gamma))\right) = 3\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma)). \tag{2.9}$$

For an element γ in $\Gamma_6(K_{3,3,1,1})$ containing x and y, if γ is of Type A, then there uniquely exists $F_x^{(ij)}$ containing γ. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\gamma\in\Gamma_6(F_x^{(ij)})\\ x,y\in\gamma}} a_2(f(\gamma))\right) = \sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma,\ \text{Type A}}} a_2(f(\gamma)). \tag{2.10}$$

For an element γ in $\Gamma_5(G_x)$, there exist exactly six edges of $K_{3,3}$ which are not contained in γ. Thus γ is common for exactly six $F_x^{(ij)}$'s. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\gamma\in\Gamma_5(G_x)\\ \overline{ij}\not\subset\gamma}} a_2(f(\gamma))\right) = 6\sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)). \tag{2.11}$$

For an element $\lambda=\gamma\cup\gamma'$ in $\Gamma_{3,5}^{(2)}(K_{3,3,1,1})$ where γ is a 3-cycle and γ' is a 5-cycle, if γ contains x and γ' contains y, then there uniquely exists $F_x^{(ij)}$ containing λ. This implies that

$$\begin{aligned}\sum_{i,j}\left(\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,5}^{(2)}(F_x^{(ij)})\\ \gamma\in\Gamma_3(F_x^{(ij)}),\ \gamma'\in\Gamma_5(F_x^{(ij)})\\ x\in\gamma,\ y\in\gamma'}} \mathrm{lk}(f(\lambda))^2\right)\\ = \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,5}^{(2)}(K_{3,3,1,1})\\ \gamma\in\Gamma_3(K_{3,3,1,1}),\ \gamma'\in\Gamma_5(K_{3,3,1,1})\\ x\in\gamma,\ y\in\gamma'}} \mathrm{lk}(f(\lambda))^2.\end{aligned} \tag{2.12}$$

For an element λ in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ of Type A, there uniquely exists $F_x^{(ij)}$ containing λ. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{4,4}^{(2)}(F_x^{(ij)})\\ x,y\in\gamma'}} \mathrm{lk}(f(\lambda))^2\right) = \sum_{\substack{\lambda\in\Gamma_{4,4}^{(2)}(K_{3,3,1,1})\\ \text{Type A}}} \mathrm{lk}(f(\lambda))^2. \tag{2.13}$$

For an element λ in $\Gamma_{3,4}^{(2)}(G_x)$, there exist exactly four edges of $K_{3,3}$ which are not contained in λ. Thus λ is common for exactly four $F_x^{(ij)}$'s. This implies that

$$\sum_{i,j}\left(\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,4}^{(2)}(G_x)\\ \gamma\in\Gamma_3(G_x),\ \gamma'\in\Gamma_4(G_x)\\ \overline{ij}\not\subset\lambda,\ x\in\gamma}} \mathrm{lk}(f(\lambda))^2\right) = 4\sum_{\lambda\in\Gamma_{3,4}^{(2)}(G_x)} \mathrm{lk}(f(\lambda))^2. \tag{2.14}$$

Thus by (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14), we have

$$\begin{aligned}
&\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{3,5}^{(2)}(K_{3,3,1,1})\\ \gamma\in\Gamma_3(K_{3,3,1,1}),\ \gamma'\in\Gamma_5(K_{3,3,1,1})\\ x\in\gamma,\ y\in\gamma'}} \mathrm{lk}(f(\lambda))^2 + \sum_{\substack{\lambda\in\Gamma_{4,4}^{(2)}(K_{3,3,1,1})\\ \text{Type A}}} \mathrm{lk}(f(\lambda))^2\\
&\quad + 4\sum_{\lambda\in\Gamma_{3,4}^{(2)}(G_x)} \mathrm{lk}(f(\lambda))^2\\
&= 2\sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type A}}} a_2(f(\gamma)) + 8\sum_{\gamma\in\Gamma_7(G_x)} a_2(f(\gamma))\\
&\quad - 4\sum_{\gamma\in\Gamma_7(G_y)} a_2(f(\gamma)) - 12\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma))\\
&\quad - 2\sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma,\ \text{Type A}}} a_2(f(\gamma)) 12\sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)) + 9.
\end{aligned} \tag{2.15}$$

Then by combining (2.15) and (2.2), we have

$$
\begin{aligned}
(2.16)\qquad & \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})\\ \gamma\in\Gamma_3(K_{3,3,1,1}),\ \gamma'\in\Gamma_5(K_{3,3,1,1})\\ x\in\gamma,\ y\in\gamma'}} \mathrm{lk}(f(\lambda))^2 + \sum_{\substack{\lambda\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})\\ \text{Type A}}} \mathrm{lk}(f(\lambda))^2\\
&= 2\sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type A}}} a_2(f(\gamma)) - 4\sum_{\gamma\in\Gamma_7(G_y)} a_2(f(\gamma))\\
&\quad + 4\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2\sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma,\ \text{Type A}}} a_2(f(\gamma))\\
&\quad - 4\sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)) + 5.
\end{aligned}
$$

By applying Theorem 2.2(2) to $f|_{F_y^{(ij)}}$ and combining the same argument as in the case of $F_x^{(ij)}$ with (2.3), we also have

$$
\begin{aligned}
(2.17)\qquad & \sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})\\ \gamma\in\Gamma_3(K_{3,3,1,1}),\ \gamma'\in\Gamma_5(K_{3,3,1,1})\\ y\in\gamma,\ x\in\gamma'}} \mathrm{lk}(f(\lambda))^2 + \sum_{\substack{\lambda\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})\\ \text{Type A}}} \mathrm{lk}(f(\lambda))^2\\
&= 2\sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type A}}} a_2(f(\gamma)) - 4\sum_{\gamma\in\Gamma_7(G_x)} a_2(f(\gamma)) + 4\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma))\\
&\quad - 2\sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ y,x\in\gamma,\ \text{Type A}}} a_2(f(\gamma)) - 4\sum_{\gamma\in\Gamma_5(G_y)} a_2(f(\gamma)) + 5.
\end{aligned}
$$

Then by adding (2.16) and (2.17), we have the result. □

Lemma 2.5. *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$,

$$
\begin{aligned}
(2.18)\qquad & \sum_{\substack{\lambda\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})\\ \text{Type B}}} \mathrm{lk}(f(\lambda))^2\\
&= 2\sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type B}}} a_2(f(\gamma)) + 4\sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma))\\
&\quad - 2\left\{\sum_{\substack{\gamma\in\Gamma_6(G_x)\\ x\in\gamma}} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_6(G_y)\\ y\in\gamma}} a_2(f(\gamma))\right\}\\
&\quad - 2\sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma,\ \text{TypeB}}} a_2(f(\gamma)) + 2.
\end{aligned}
$$

Proof. Let us consider subgraphs $Q_8^{(1)} = K_{3,3} \cup \overline{x1} \cup \overline{x3} \cup \overline{x5} \cup \overline{y2} \cup \overline{y4} \cup \overline{y6}$ and $Q_8^{(2)} = K_{3,3} \cup \overline{x2} \cup \overline{x4} \cup \overline{x6} \cup \overline{y1} \cup \overline{y3} \cup \overline{y5}$ of $K_{3,3,1,1}$ as illustrated in Figure 2.3(1) and (2), respectively. Since each of $Q_8^{(1)}$ and $Q_8^{(2)}$ is homeomorphic to Q_8, by applying Theorem 2.2(1) to $f|_{Q_8^{(1)}}$ and $f|_{Q_8^{(2)}}$, it follows that

$$
\begin{aligned}
\text{(2.19)} \quad \sum_{\lambda \in \Gamma_{4,4}^{(2)}(Q_8^{(i)})} \mathrm{lk}(f(\lambda))^2 = 2 & \sum_{\gamma \in \Gamma_8(Q_8^{(i)})} a_2(f(\gamma)) + 2 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \\
& - 2 \sum_{\substack{\gamma \in \Gamma_6(Q_8^{(i)}) \\ x \in \gamma,\ y \notin \gamma}} a_2(f(\gamma)) - 2 \sum_{\substack{\gamma \in \Gamma_6(Q_8^{(i)}) \\ x \notin \gamma,\ y \in \gamma}} a_2(f(\gamma)) \\
& - 2 \sum_{\substack{\gamma \in \Gamma_6(Q_8^{(i)}) \\ x, y \in \gamma}} a_2(f(\gamma)) + 1
\end{aligned}
$$

for $i = 1, 2$. By adding (2.19) for $i = 1, 2$, we have the result. $\square$

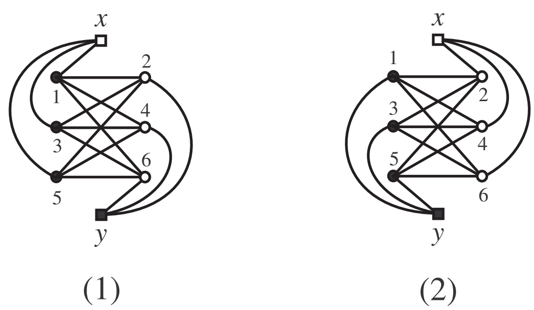


FIGURE 2.3. (1) $Q_8^{(1)}$, (2) $Q_8^{(2)}$

Lemma 2.6. *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$,

$$
\begin{aligned}
\text{(2.20)} \quad & \sum_{\substack{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1}) \\ \text{Type C}}} \mathrm{lk}(f(\lambda))^2 \\
& = 2 \sum_{\substack{\gamma \in \Gamma_8(K_{3,3,1,1}) \\ \text{Type C}}} a_2(f(\gamma)) - 8 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \\
& \quad - 2 \sum_{\substack{\gamma \in \Gamma_6(K_{3,3,1,1}) \\ x, y \in \gamma,\ \text{Type C}}} a_2(f(\gamma)) + 2.
\end{aligned}
$$

Proof. For $k = 1, 2, \ldots, 6$, let us consider subgraphs

$$F_x^{(k)} = (G_x - \overline{xk}) \cup \overline{xy} \cup \overline{ky},$$

$$F_y^{(k)} = (G_y - \overline{yk}) \cup \overline{kx} \cup \overline{yx},$$

of $K_{3,3,1,1}$ as illustrated in Figure 2.4(1) and (2), respectively. Since each of $F_x^{(k)}$ and $F_y^{(k)}$ is also homeomorphic to P_7, by applying Theorem 2.2(2) to $f|_{F_x^{(k)}}$, it follows that

$$\sum_{\substack{\lambda=\gamma\cup\gamma'\in\Gamma_{4,4}^{(2)}(F_x^{(ij)})\\ x,y\in\gamma,\ \text{Type C}}} \mathrm{lk}(f(\lambda))^2 + \sum_{\substack{\lambda\in\Gamma_{3,4}^{(2)}(G_x)\\ \overline{xk}\not\subset\lambda}} \mathrm{lk}(f(\lambda))^2 \tag{2.21}$$
$$= 2\left\{ \sum_{\gamma\in\Gamma_8(F_x^{(k)})} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_7(G_x)\\ \overline{xk}\not\subset\gamma}} a_2(f(\gamma)) \right\}$$
$$- 4 \sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma))$$
$$- 2\left\{ \sum_{\substack{\gamma\in\Gamma_6(F_x^{(k)})\\ x,y\in\gamma}} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_5(G_x)\\ \overline{xk}\not\subset\gamma}} a_2(f(\gamma)) \right\} + 1.$$

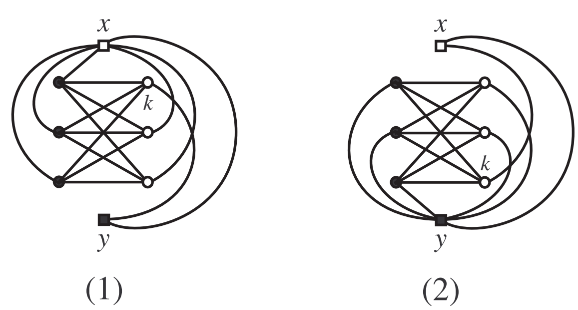


FIGURE 2.4. (1) $F_x^{(k)}$, (2) $F_y^{(k)}$ $(k = 1, 2, 3, 4, 5, 6)$

Let us take the sum of both sides of (2.21) over $k = 1, 2, \ldots, 6$. For an element γ in $\Gamma_8(K_{3,3,1,1})$, if γ is of Type C, then there uniquely exists $F_x^{(k)}$ containing γ. This implies that

$$\sum_k \left(\sum_{\gamma\in\Gamma_8(F_x^{(k)})} a_2(f(\gamma)) \right) = \sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type C}}} a_2(f(\gamma)). \tag{2.22}$$

For an element γ of $\Gamma_7(G_x)$, there exist exactly four edges which are incident to x such that they are not contained in γ. Thus γ is common for exactly four $F_x^{(k)}$'s. This implies that

$$\sum_k \left(\sum_{\substack{\gamma \in \Gamma_7(G_x) \\ \overline{xk} \not\subset \gamma}} a_2(f(\gamma)) \right) = 4 \sum_{\Gamma_7(G_x)} a_2(f(\gamma)). \tag{2.23}$$

It is clear that any element γ in $\Gamma_6(K_{3,3})$ is common for exactly six $F_x^{(k)}$'s. This implies that

$$\sum_k \left(\sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right) = 6 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)). \tag{2.24}$$

For an element γ in $\Gamma_6(K_{3,3,1,1})$ containing x and y, if γ is of Type C, then there uniquely exists $F_x^{(k)}$ containing γ. This implies that

$$\sum_k \left(\sum_{\substack{\gamma \in \Gamma_6(F_x^{(k)}) \\ x,y \in \gamma}} a_2(f(\gamma)) \right) = \sum_{\substack{\gamma \in \Gamma_6(K_{3,3,1,1}) \\ x,y \in \gamma,\ \text{Type C}}} a_2(f(\gamma)). \tag{2.25}$$

For an element γ of $\Gamma_5(G_x)$, there exist exactly four edges which are incident to x such that they are not contained in γ. Thus γ is common for exactly four $F_x^{(k)}$'s. This implies that

$$\sum_k \left(\sum_{\substack{\gamma \in \Gamma_5(G_x) \\ \overline{xk} \not\subset \gamma}} a_2(f(\gamma)) \right) = 4 \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)). \tag{2.26}$$

For an element $\lambda = \gamma \cup \gamma'$ in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$, if λ is of Type C, then there uniquely exists $F_x^{(k)}$ containing λ. This implies that

$$\sum_k \left(\sum_{\substack{\lambda = \gamma \cup \gamma' \in \Gamma_{4,4}^{(2)}(F_x^{(k)}) \\ x,y \in \gamma,\ \text{TypeC}}} \mathrm{lk}(f(\lambda))^2 \right) = \sum_{\substack{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1}) \\ \text{Type C}}} \mathrm{lk}(f(\lambda))^2. \tag{2.27}$$

For an element λ in $\Gamma_{3,4}^{(2)}(G_x)$, there exist exactly four edges which are incident to x such that they are not contained in λ. Thus λ is common for exactly four $F_x^{(k)}$'s. This implies that

$$\sum_k \left(\sum_{\substack{\lambda\in\Gamma_{3,4}^{(2)}(G_x)\\ \overline{xk}\not\subset\lambda}} \mathrm{lk}(f(\lambda))^2 \right) = 4 \sum_{\lambda\in\Gamma_{3,4}^{(2)}(G_x)} \mathrm{lk}(f(\lambda))^2. \tag{2.28}$$

Then by (2.21), (2.22), (2.23), (2.24), (2.25), (2.26), (2.27) and (2.28), we have

$$\begin{aligned}
&\sum_{\substack{\lambda\in\Gamma_{4,4}^{(2)}(K_{3,3,1,1})\\ \text{Type C}}} \mathrm{lk}(f(\lambda))^2 + 4 \sum_{\lambda\in\Gamma_{3,4}^{(2)}(G_x)} \mathrm{lk}(f(\lambda))^2 \\
&= 2 \sum_{\substack{\gamma\in\Gamma_8(K_{3,3,1,1})\\ \text{Type C}}} a_2(f(\gamma)) + 8 \sum_{\gamma\in\Gamma_7(G_x)} a_2(f(\gamma)) - 24 \sum_{\gamma\in\Gamma_6(K_{3,3})} a_2(f(\gamma)) \\
&\quad - 2 \sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma,\ \text{Type C}}} a_2(f(\gamma)) - 8 \sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)) + 6.
\end{aligned} \tag{2.29}$$

Then by combining (2.29) and (2.2), we have the reslut. We remark here that by by applying Theorem 2.2 (2) to $f|_{F_y^{(k)}}$ combining the same argument as in the case of $F_x^{(k)}$ with (2.3), we also have (2.20). $\square$

Proof of Theorem 1.4. (1) Let f be an element in $\mathrm{SE}(K_{3,3,1,1})$. Then by combining (2.4), (2.18) and (2.20), we have

(2.30)

$$\begin{aligned}
&\sum_{\lambda\in\Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 + 2 \sum_{\lambda\in\Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 \\
&= 4 \sum_{\gamma\in\Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \left\{ \sum_{\gamma\in\Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma\in\Gamma_7(G_y)} a_2(f(\gamma)) \right\} \\
&\quad - 4 \left\{ \sum_{\substack{\gamma\in\Gamma_6(G_x)\\ x\in\gamma}} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_6(G_y)\\ y\in\gamma}} a_2(f(\gamma)) + \sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ x,y\in\gamma\\ \text{Type A,B,C}}} a_2(f(\gamma)) \right\} \\
&\quad - 4 \left\{ \sum_{\gamma\in\Gamma_5(G_x)} a_2(f(\gamma)) + \sum_{\gamma\in\Gamma_5(G_y)} a_2(f(\gamma)) \right\} + 18.
\end{aligned}$$

Note that

$$\Gamma_k(G_x) \cup \Gamma_k(G_y) = \{\gamma \in \Gamma_k(K_{3,3,1,1}) \mid \{x, y\} \not\subset \gamma\}$$

for $k = 5, 7$. Moreover, we define a subset Γ'_6 of $\Gamma_6(K_{3,3,1,1})$ by

$$\text{(2.31)} \qquad \Gamma'_6 = \{\gamma \in \Gamma_6(G_x) \mid x \in \gamma\} \cup \{\gamma \in \Gamma_6(G_y) \mid y \in \gamma\} \\ \cup \{\gamma \in \Gamma_6(K_{3,3,1,1}) \mid x, y \in \gamma,\ \gamma \text{ is of Type A, B or C}\}.$$

Then we see that (2.30) implies (1.3).

(2) Let f be an element in $\mathrm{SE}(K_{3,3,1,1})$. Let us consider subgraphs $H_1 = Q_8^{(1)} \cup \overline{xy}$ and $H_2 = Q_8^{(2)} \cup \overline{xy}$ of $K_{3,3,1,1}$ as illustrated in Figure 2.5(1) and (2), respectively. For $i = 1, 2$, H_i has the proper minor $H'_i = H_i/\overline{xy}$ which is isomorphic to P_7. For a spatial embedding $f|_{H_i}$ of H_i, there exists a spatial embedding f' of H'_i such that $f'(H'_i)$ is obtained from $f(H_i)$ by contracting $f(\overline{xy})$ into one point. Note that this embedding is unique up to ambient isotopy in $\mathbb{R}^3$. Then by Corollary 2.3, there exists an element μ'_i in $\Gamma^{(2)}_{3,4}(H'_i)$ such that $\mathrm{lk}(f'(\mu'_i)) \equiv 1 \pmod 2$ $(i = 1, 2)$. Note that μ'_i is mapped onto an element μ_i in $\Gamma_{4,4}(H_i)$ by the natural injection from $\Gamma_{3,4}(H'_i)$ to $\Gamma_{4,4}(H_i)$. Since $f'(\mu'_i)$ is ambient isotopic to $f(\mu_i)$, we have $\mathrm{lk}(f(\mu_i)) \equiv 1 \pmod 2$ $(i = 1, 2)$. We also note that both μ_1 and μ_2 are of Type C in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$.

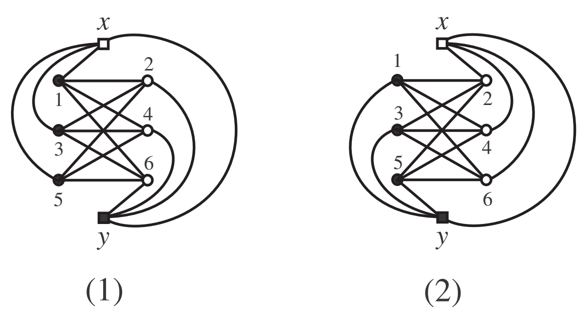


FIGURE 2.5. (1) H_1, (2) H_2

For $v = x, y$ and $i, j, k = 1, 2, \ldots, 6$ $(i \neq j)$, let $P_8^{(k)}(v; ij)$ be the subgraph of $K_{3,3,1,1}$ as illustrated in Figure 2.6 (1) if $v = y$, $k \in \{1, 3, 5\}$ and $i, j \in \{2, 4, 6\}$, (2) if $v = y$, $k \in \{2, 4, 6\}$ and $i, j \in \{1, 3, 5\}$, (3) if $v = x$, $k \in \{1, 3, 5\}$ and $i, j \in \{2, 4, 6\}$ and (4) if $v = x$, $k \in \{2, 4, 6\}$ and $i, j \in \{1, 3, 5\}$. Note that there exist exactly thirty six $P_8^{(k)}(v; ij)$'s and they are isomorphic to P_8 in the K_6-family. Thus by Corollary 2.3, there exists an element λ in $\Gamma^{(2)}(P_8^{(k)}(v; ij))$ such that $\mathrm{lk}(f(\lambda)) \equiv 1 \pmod 2$. All elements in $\Gamma^{(2)}(P_8^{(k)}(v; ij))$ consist of exactly four elements in $\Gamma^{(2)}_{3,5}(P_8^{(k)}(v; ij))$

and exactly four elements in $\Gamma_{4,4}^{(2)}(P_8^{(k)}(v;ij))$ of Type A or Type B because they do not contain the edge $\overline{xy}$. It is not hard to see that any element in $\Gamma_{3,5}^{(2)}(K_{3,3,1,1})$ is common for exactly two $P_8^{(k)}(v;ij)$'s, and any element in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ of Type A or Type B is common for exactly four $P_8^{(k)}(v;ij)$'s.

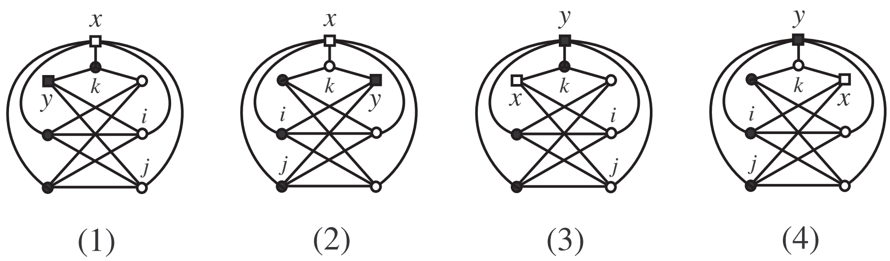


FIGURE 2.6. $P_8^{(k)}(v;ij)$

By (2.4), there exist a nonnegative integer m such that

$$\sum_{\lambda\in\Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 = 2m.$$

If $2m \geq 18$, since there exist at least two elements μ_1 and μ_2 in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ of Type C such that $\mathrm{lk}(f(\mu_i)) \equiv 1 \pmod 2$ $(i = 1, 2)$, we have

$$\sum_{\lambda\in\Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 + 2 \sum_{\lambda\in\Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 \geq 18 + 4 = 22.$$

If $2m \leq 16$, then there exist at least $(36 - 4m)/4 = 9 - m$ elements in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ of Type A or Type B such that each of the corresponding 2-component links with respect to f has an odd linking number. Then we have

$$\begin{aligned}\sum_{\lambda\in\Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 + 2 \sum_{\lambda\in\Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \mathrm{lk}(f(\lambda))^2 \\ \geq 2m + 2\{(9-m)+2\} = 22. \quad \square\end{aligned}$$

3. $\triangle Y$-exchange and Conway–Gordon type formulae

In this section, we give a proof of Theorem 1.8. Let $G_\triangle$ and G_Y be two graphs such that G_Y is obtained from $G_\triangle$ by a single $\triangle Y$-exchange. Let γ' be an element in $\Gamma(G_\triangle)$ which does not contain $\triangle$. Then there exists an element $\Phi(\gamma')$ in $\Gamma(G_Y)$ such that $\gamma' \setminus \triangle = \Phi(\gamma') \setminus Y$. It is easy to see that the correspondence from γ' to $\Phi(\gamma')$ defines a surjective map

$$\Phi : \Gamma(G_\triangle) \setminus \{\triangle\} \longrightarrow \Gamma(G_Y).$$

The inverse image of an element γ in $\Gamma(G_Y)$ by Φ contains at most two elements in $\Gamma(G_\triangle)\setminus\Gamma_\triangle(G_\triangle)$. Figure 3.1 illustrates the case that the inverse image of γ by Φ consists of exactly two elements. Let ω be a map from $\Gamma(G_\triangle)$ to $\mathbb{Z}$. Then we define the map $\tilde{\omega}$ from $\Gamma(G_Y)$ to $\mathbb{Z}$ by

$$\tilde{\omega}(\gamma) = \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma') \tag{3.1}$$

for an element γ in $\Gamma(G_Y)$.

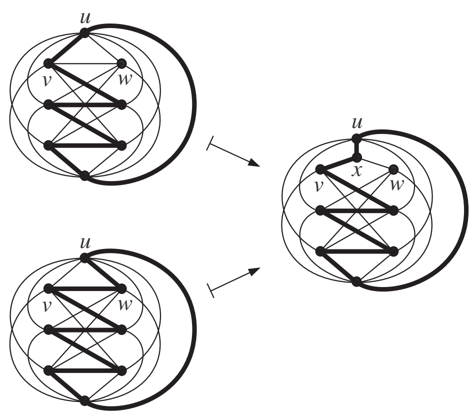


FIGURE 3.1.

Let f be an element in $\mathrm{SE}(G_Y)$ and D a 2-disk in $\mathbb{R}^3$ such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{f(u), f(v), f(w)\}$. Let $\varphi(f)$ be an element in $\mathrm{SE}(G_\triangle)$ such that $\varphi(f)(x) = f(x)$ for $x \in G_\triangle \setminus \triangle = G_Y \setminus Y$ and $\varphi(f)(G_\triangle) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$\varphi : \mathrm{SE}(G_Y) \longrightarrow \mathrm{SE}(G_\triangle).$$

Then we immediately have the following.

Proposition 3.1. *Let f be an element in $\mathrm{SE}(G_Y)$ and γ an element in $\Gamma(G_Y)$. Then, $f(\gamma)$ is ambient isotopic to $\varphi(f)(\gamma')$ for each element γ' in the inverse image of γ by Φ.*

Then we have the following lemma which plays a key role to prove Theorem 1.8. This lemma has already been shown in [NT, Lemma 2.2] in more general form, but we give a proof for the reader's convenience.

Lemma 3.2 (Nikkuni–Taniyama [NT]). *For an element f in* $\mathrm{SE}(G_Y)$,

$$\sum_{\gamma\in\Gamma(G_Y)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = \sum_{\gamma'\in\Gamma(G_\triangle)} \omega(\gamma')a_2(\varphi(f)(\gamma')).$$

Proof. Since $\varphi(f)(\triangle)$ is the trivial knot, we have

$$\sum_{\gamma'\in\Gamma(G_\triangle)} \omega(\gamma')a_2(\varphi(f)(\gamma')) = \sum_{\gamma'\in\Gamma(G_\triangle)\setminus\{\triangle\}} \omega(\gamma')a_2(\varphi(f)(\gamma')).$$

Note that

$$\Gamma(G_\triangle)\setminus\{\triangle\} = \bigcup_{\gamma\in\Gamma(G_Y)} \Phi^{-1}(\gamma).$$

Then, by Proposition 3.1, we see that

$$\begin{aligned}\sum_{\gamma'\in\Gamma(G_\triangle)\setminus\{\triangle\}} \omega(\gamma')a_2(\varphi(f)(\gamma')) &= \sum_{\gamma\in\Gamma(G_Y)}\left(\sum_{\gamma'\in\Phi^{-1}(\gamma)} \omega(\gamma')a_2(\varphi(f)(\gamma'))\right)\\ &= \sum_{\gamma\in\Gamma(G_Y)}\left(\sum_{\gamma'\in\Phi^{-1}(\gamma)} \omega(\gamma)a_2(f(\gamma))\right)\\ &= \sum_{\gamma\in\Gamma(G_Y)} \tilde{\omega}(\gamma)a_2(f(\gamma)).\end{aligned}$$ □

Proof of Theorem 1.8. By Corollary 1.5, there exists a map

$$\omega : \Gamma(K_{3,3,1,1}) \to \mathbb{Z}$$

such that for any element g in $\mathrm{SE}(K_{3,3,1,1})$,

$$\sum_{\gamma'\in\Gamma(K_{3,3,1,1})} \omega(\gamma')a_2(g(\gamma')) \geq 1. \tag{3.2}$$

Let G be a graph which is obtained from $K_{3,3,1,1}$ by a single $\triangle Y$-exchange and $\tilde{\omega}$ the map from $\Gamma(G)$ to $\mathbb{Z}$ as in (3.1). Let f be an element in $\mathrm{SE}(G)$. Then by Lemma 3.2 and (3.2), we see that

$$\sum_{\gamma\in\Gamma(G)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = \sum_{\gamma'\in\Gamma(K_{3,3,1,1})} \omega(\gamma')a_2(\varphi(f)(\gamma')) \geq 1.$$

By repeating this argument, we have the result. □

Remark 3.3. In Theorem 1.8, the proof of the existence of a map ω is constructive. It is also an interesting problem to give $\omega(\gamma)$ for each element γ in $\Gamma(G)$ concretely.

4. Rectilinear spatial embeddings of $K_{3,3,1,1}$

In this section, we give a proof of Theorem 1.9. For an element f in $\mathrm{RSE}(G)$ and an element γ in $\Gamma_k(G)$, the knot $f(\gamma)$ has *stick number* less than or equal to k, where the stick number $s(K)$ of a knot K is the minimum number of edges in a polygon which represents K. Then the following is well known.

Proposition 4.1 (Adams [A], Negami [Ne])**.** *For any nontrivial knot K, it follows that $s(K) \geq 6$. Moreover, $s(K) = 6$ if and only if K is a trefoil knot.*

We also show a lemma for a rectilinear spatial embedding of P_7 which is useful in proving Theorem 1.9.

Lemma 4.2. *For an element f in* $\mathrm{RSE}\,(P_7)$*,*

$$\sum_{\gamma \in \Gamma_7(P_7)} a_2(f(\gamma)) \geq 0.$$

Proof. Note that $a_2(\text{trivial knot}) = 0$ and $a_2(\text{trefoil knot}) = 1$. Thus by Proposition 4.1, $a_2(f(\gamma)) = 0$ for any element γ in $\Gamma_5(P_7)$ and $a_2(f(\gamma)) \geq 0$ for any element γ in $\Gamma_6(P_7)$. Moreover, by Corollary 2.3, we have

$$\sum_{\lambda \in \Gamma_{3,4}^{(2)}(P_7)} \mathrm{lk}(f(\lambda))^2 \geq 1. \tag{4.1}$$

Then Theorem 2.2(2) implies the result. □

Proof of Theorem 1.9. Let f be an element in $\mathrm{RSE}(\mathrm{K}_{3,3,1,1})$. Since G_x and G_y are isomorphic to P_7, by Lemma 4.2, we have

$$\sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \geq 0, \quad \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \geq 0. \tag{4.2}$$

Then by Corollary 1.5 and (4.2), we have

$$\begin{aligned}\sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) &\geq \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \\ &\quad + \sum_{\gamma \in \Gamma'_6} a_2(f(\gamma)) + \sum_{\substack{\gamma \in \Gamma_5(K_{3,3,1,1}) \\ \{x,y\} \not\subset \gamma}} a_2(f(\gamma)) + 1 \\ &\geq 0 + 0 + 0 + 0 + 1 \\ &= 1.\end{aligned}$$

□

Remark 4.3. All of knots with $s \leq 8$ and $a_2 > 0$ are 3_1, 5_1, 5_2, 6_3, a square knot, a granny knot, 8_{19} and 8_{20} (Calvo [C]). Therefore, Theorem 1.9 implies that at least one of them appears in the image of every rectilinear spatial embedding of $K_{3,3,1,1}$. On the other hand, it is known that the image of every rectilinear spatial embedding of K_7 contains a trefoil knot (Brown

[B], Ramírez Alfonsín [RA], Nikkuni [N]). It is still open whether the image of every rectilinear spatial embedding of $K_{3,3,1,1}$ contains a trefoil knot.

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