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Asymptotic average shadowing property on nonuniformly expanding maps

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ABSTRACT. In this paper, we investigate the relationships between asymptotic average shadowing property for nonuniformly expanding maps with some notions in dynamical systems. We prove that if nonuniformly expanding (NUE) map f has asymptotic average shadowing property (AASP), then f is transitive and weakly mixing. Finally as a remark we show that if the C^2 diffeomorphism f is NUE and AASP then it has a unique SRB measure.

CONTENTS

1. Introduction

Topologically transitive systems have been studied extensively by many authors, see, e.g., [7, 20, 5, 4, 8, 21, 1, 14, 13, 11, 22]. Gu [9] introduced a new shadowing property, the asymptotic average shadowing property (abbreviated AASP), which is similar to the asymptotic pseudo orbit tracing property in shadowing way, and studied the relation between the AASP and transitivity. In fact he proved that a L-hyperbolic homeomorphism with the average shadowing property is topologically transitive.

AASP and its relations with other dynamical properties in dynamical systems have been studied extensively by many researchers, e.g., [16, 10, 12, 15, 17, 18]. In this paper we study nonuniformly expanding maps and show that such maps with asymptotic average shadowing property are transitive. We show in Remark 2 below that if the C^2 local diffeomorphism f is NUE with AASP, then f has a unique SRB measure.

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2. Some basic terminology

Let (X, d) be a compact metric space and let f be a self-homeomorphism of X. A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is called an orbit of f if for each $n \in \mathbb{Z}$ we have $x_{n+1} = f(x_n)$ and we call it a δ-pseudo-orbit of f if for each $n \in \mathbb{Z}$, we have

$$d(f(x_n), x_{n+1}) \leq \delta.$$

The homeomorphism f is said to have the shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ is ϵ-shadowed by the orbit $\{f^n(y) : n \in \mathbb{Z}\}$, for some y in X, i.e., for all $n \in \mathbb{Z}$ we have

$$d(f^n(y), x_n) < \epsilon.$$

For $\delta > 0$ a sequence $\{x_i\}_{i=0}^{\infty}$ in X is called a δ-average-pseudo-orbit of f if there exists a positive integer $N = N(\delta)$ such that for all $n \geq N$ and $k \in \mathbb{N}$ we have

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

A map f is said to have average shadowing property if for every $\epsilon > 0$ there is, $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-shadowed in average by the orbit of some point $y \in X$, that is

$$\limsup_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon.$$

Denote by $N_\epsilon(x)$ the open ball with center x and radius ϵ. A sequence $\{x_i\}_{i=0}^{\infty}$ in X is called an asymptotic-average pseudo orbit of f if

$$\lim_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

A sequence $\{x_i\}_{i=0}^{\infty}$ is said to be asymptotically shadowed in average by the point z in X if

$$\lim_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

We say that f has AASP if any asymptotic-average pseudo orbit of f, asymptotically shadowed in average by some point z in X.

The homeomorphism f is said to be topologically transitive if for any two nonempty open sets U, V, there is an integer l such that $f^l(U) \cap V \neq \emptyset$. It is said to be weakly mixing if $f \times f$ is topologically transitive. A map f is said to have the specification property if for any $\epsilon > 0$ there exists $L > 0$ such that for every $n \in \mathbb{N}$ and every finite sequence $y_1, y_2, \ldots, y_n \in M$, any natural numbers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_i - b_{i-1} \geq L$ and $2 \leq i \leq n$, there is a point $z \in M$ such that $d(f^k(z), f^k(y_i)) < \epsilon$, for every $1 \leq i \leq n$ and $a_i \leq k \leq b_i$.

3. Nonuniformly expanding

Let M be a C^∞ compact manifold with a Riemannain metric d and let $f : M \longrightarrow M$ be a homeomorphism which is a C^1 locally diffeomorphism. If there is a Riemannian metric $\|.\|$ on TM and $\lambda > 1$ such that

$$\|Df^n(x)v\| \geq \lambda^n \|v\|$$

for every $x \in M$ and $v \in T_x M$, then f is called expanding.

We say that f is nonuniformly expanding on a set $H \subset M$ if there is $\lambda > 0$ such that for every $x \in H$,

$$\liminf_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -\lambda.$$

A C^1 local diffeomorphism f is said to be nonuniformly expanding (NUE) if it is nonuniformly expanding on a set of full Lebesque measure.

Definition 1. For $\sigma < 1$, we say that n is a σ-hyperbolic time for a point $x \in M$ if for all $1 \leq k \leq n$,

$$\Pi_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k.$$

The following Propositions A and B can be obtained from [2, Lemma 5.2 and 5.4] (see Remark 1).

Proposition A. *For $0 < \sigma < 1$, there exists $\delta > 0$ such that if n is a σ-hyperbolic time for x, then there exists a neighborhood V_n of x such that:*

    (1) *f^n maps V_n diffeomorphically on to the ball of radius δ around $f^n(x)$.*
    (2) *For all $1 \leq k < n$ and $y, z \in V_n$, we have*

$$d(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{\frac{k}{2}} d(f^n(y), f^n(z)).$$

A σ-hyperbolic times for $x \in M$ is said to have positive frequency, if there is some $\theta > 0$ such that for large $n \in \mathbb{N}$ there are $l \geq \theta n$ and integers $1 \leq n_1 < n_2 < \cdots < n_l \leq n$ which are σ-hyperbolic times for x, in fact we have

$$\frac{1}{n} \#\{0 < k < n : k \text{ is a hyperbolic time for } x\} > \theta.$$

Proposition B. *Assume that f is NUE. Then there are $0 < \sigma < 1$ and $\theta > 0$ (depending only on λ in the definition NUE and the map f) such that the frequency of σ-hyperbolic times for a set H of full Lebesgue measure is greater than θ.*

Remark 1. By Lemma 5.4 in [2], there is positive frequency for any $x \in H$, where H is a set for which NUE is defined. In [2], Alves et al defined NUE for positive measure and so H has positive measure. But here we defined NUE for almost every point of M and so H has full Lebesgue measure. Therefore Proposition B can be obtained from Lemma 5.4 in [2].

Theorem 1. *If C^1 local diffeomorphism f is NUE and AASP, then f is transitive.*

Proof. Let U and V be arbitrary nonempty open subsets of M. Let H be the set of full measure on which f is nonuniformly expanding. Since every nonempty open subset has positive Lebesgue measure so $H \cap U \neq \phi$ and $H \cap V \neq \phi$. Consider $x \in H \cap U$ and $y \in H \cap V$. Let $\epsilon > 0$ be such that $N_\epsilon(x) \subset U$ and $N_\epsilon(y) \subset V$. There exists $\xi > 0$ such that if $d(e, z) < \xi$ then $d(f^{-1}(e), f^{-1}(z)) < \epsilon$.

By Proposition B, there are $0 < \sigma < \frac{\xi}{D}$ (where D is the diameter of M) and $\theta > 0$ such that the frequency of σ-hyperbolic times for the set H is greater than θ, and by Proposition A, there exists $\delta > 0$ such that if n_x and n_y are σ-hyperbolic time for x and y respectively, then there exists neighborhoods V_{n_x} of x and V_{n_y} of y such that f^{n_u} maps V_{n_u} diffeomorphically on to the ball of radius δ around $f^n(u)$ for $u \in \{x, y\}$. We use the method of Gu in [9] to construct an asymptotic average pseudo orbit, as follows. Let

$$\{w_i\} = \big\{ x, y, x, y, x, f(x), y, f(y), \ldots, x, f(x), \ldots, f^{2^{k-1}-1}(x), y,$$
$$f(y), \ldots, f^{2^{k-1}-2}(y), f^{2^{k-1}-1}(y), \ldots \big\}.$$

It is easy to see that for $2^k \leq n < 2^{k+1}$,

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=-n}^{i=n} d(f(w_i), w_{i+1}) < \frac{2(k+1) \cdot D}{n},$$

where D is the diameter of X. Hence

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=-n}^{i=n} d(f(w_i), w_{i+1}) = 0.$$

Thus, the sequence $(\{w_i\}_{0 < i < \infty})$ is an asymptotic average-pseudo-orbit of f. Hence it can be asymptotically shadowed in average by the orbit of f through some point z in X, that is,

$$\lim_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), w_i) = 0.$$

Claim. *There exist infinitely many σ-hyperbolic times n_x such that corresponding to every n_x there is a positive integer m_x such that*

$$d(f^{m_x}(z), f^{n_x}(u)) < \delta$$

$(u \in \{x, y\})$.

Using the claim there are σ-hyperbolic times n_x and n_y for x and y respectively such that

$$d(f^{m_x}(z), f^{n_x}(x)) < \delta$$

and

$$d(f^{m_y}(z), f^{n_y}(y)) < \delta,$$

for some positive integers m_x, m_y.

Notice that f^{n_u} maps V_{n_u} diffeomorphically on to the ball of radius δ around $f^n(u)$ for $u \in \{x, y\}$. By Proposition A(2), for all $c, e \in V_{n_u}$, $(u = x, y)$

$$d(f(c), f(e)) \leq \sigma^{\frac{n-1}{2}} d(f^n(c), f^n(e)) < \sigma^{\frac{n-1}{2}} D < \xi.$$

So $d(c, e) < \epsilon$. This show that $V_{n_x} \subset U$ and $V_{n_y} \subset V$, hence we have

$$f^{m_u}(z) \in f^{n_u}(V_u),$$

for $u = x, y$.

This shows that for some integer l, $f^l(U) \cap V \neq \phi$. Since U, V are arbitrary hence f is transitive. $\square$

Proof of Claim. Suppose on the contrary that there is a positive integer N such that for all σ-hyperbolic time $k > N$,

$$d(f^i(z), f^k(x)) > \delta$$

for any $i > 0$.

Then it would be obtained that for large n,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), w_i) \geq \frac{\delta}{n} \#\{N < k < n : k \text{ is a hyperbolic time for } x\} > \delta\theta.$$

So

$$\liminf_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), w_i) \geq \delta\theta,$$

which contradicts with AASP. $\square$

Theorem 2. *If the C^1 local diffeomorphism f is NUE with AASP, then f is weakly mixing.*

Proof. This is easy to see that if f is NUE and has AASP, then $f \times f$ is also NUE and has AASP. So Theorem 1 implies that $f \times f$ is topologically transitive. This means f is weakly mixing. $\square$

It is very difficult to study whether or not a concrete example has the asymptotic average shadowing. It is well-known that for the maps with the shadowing property the specification property and AASP are equivalent [15]. So if a map has the shadowing property, but does not have the specification property, then it does not have the average shadowing property. But it is very difficult to study whether a map has the shadowing property and does not have the specification property, even for the maps on an interval in $\mathbb{R}$. As an application of the main result of our paper (Theorem 1) we will now give some examples of nonuniformly expanding maps that are not transitive. By our main result such maps do not have the asymptotic average shadowing. We emphasize that in these examples it is not difficult to determine that the maps are not transitive and are NUE, although it is not easy to see by

definition that they do not have the asymptotic average shadowing. So our
main result can be useful in studying the maps with the AASP.

Example A. Let $f : [0,1] \longrightarrow [0,1]$ be given as follows: $f(0) = 0$, $f(1/6) = 1/2$, $f(1/3) = 0$, $f(2/3) = 1$, $f(5/6) = 1/2$, and $f(1) = 1$.
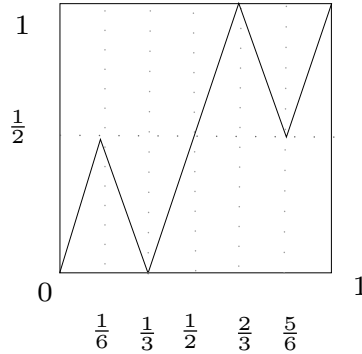
Let $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. We can easily see from above figure
that $f(I_0) = I_0$ and $f(I_1) = I_1$. So f is not transitive. Moreover for any
point x in $[0,1] \backslash \{1/6, 1/3, 2/3, 5/6\}$, by the above figure, $|(f^n)'(x)| = 3^n$
therefore f is NUE . By Theorem 1, f is not AASP.

Example B. Let $f : [0,1] \longrightarrow [0,1]$ be given as follows: $f(0) = 1/2$, $f(1/4) = 1$, $f(3/4) = 0$, and $f(1) = 1/2$.
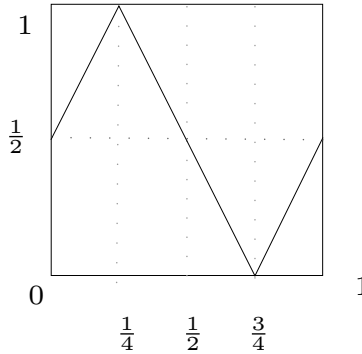
Let $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. We see that $f^2(I_0) = I_0$ and $f^2(I_1) = I_1$. So f^2 is not transitive. Moreover by above figure, similar the above
example f^2 is NUE. Therefore by Theorem 1, f^2 is not AASP so f is not
AASP. Note that if f has AASP, then for any psitive integer f^n has AASP.

4. SRB measure

Let μ be a Borel probability measure on M, invariant for f. We say that
μ is a SRB measure if for a positive Lebesgue measure set H and any point

$x \in H$ we have

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu,$$

for any continuous map $\varphi : M \longrightarrow \mathbb{R}$. We denote by $B(\mu)$, the basin of μ, as the set of those points $x \in M$ for which the above formula holds.

By the Birkhoff ergodic theorem every ergodic probability measure which is absolutely continuous with respect to the Lebesgue measure is a SRB measure.

The following lemma and theorem are proved in [2].

Lemma. *Let $G \subset M$ be with positive Lebesgue measure such that f is NUE on G. Then there exists some disk Δ with radius δ such that $m(\Delta \setminus G) = 0$.*

Theorem A. *Assume that f is NUE. Then there are ergodic absolutely continuous probability measures $\mu_1 \ldots \mu_p$ whose basins cover a full Lebesgue measure subset of M. Moreover, if μ is an invariant probability measure, then there are $\alpha_1 \geq 0, \ldots, \alpha_e \geq 0$ such that $\alpha_1 + \cdots + \alpha_e = 1$ and $\alpha_1 \mu_1 + \cdots + \alpha_e \mu_e = \mu$.*

Remark 2. If the C^2 local diffeomorphism f is NUE and AASP, then f has a unique SRB measure.

Indeed since f is NUE Theorem A implies that there are ergodic absolutely continuous probability measure $\mu_1 \ldots \mu_p$ whose basins cover a full Lebesgue measure subset of M. Assume that there are two distinct ergodic measure μ_1 and μ_2. Since $B(\mu_1)$ and $B(\mu_2)$ are positively invariant sets, then by the above lemma there are disks Δ_1 and Δ_2 such that $m(\Delta_i \setminus B(\mu_i)) = 0$, $i = 1, 2$. The transitivity of f and the invariance of $B(\mu_1)$ and $B(\mu_2)$ imply that $m(B(\mu_1) \cap B(\mu_2)) > 0$. Since distinct ergodic measures have disjoint basins we have a contradiction. This shows that f has a unique SRB measure.

A continuous map f from a compact metric space M to itself is said to be P-chaotic if f has the shadowing property and periodic points of f are dense in M.

Example C. Let $f : [0, 1] \longrightarrow [0, 1]$ be the tent map which is defined by

$$f(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Example 2.12 in [3] shows that f is P-chaotic and so is topologically mixing [3, Corollary 3.4]. By [6], f has the specification property. Hence by [15], f has AASP. On the other hand for any $x \in (0, 1) \setminus \frac{1}{2}$,

$$\liminf_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| = \liminf_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log(2) = -\log(2).$$

So f is NUE. Therefore by Remark 2, f has a unique SRB measure.

Example D. Let $g : [0, 1] \longrightarrow [0, 1]$ be quadratic map which is defined by

$$g(x) = 4x(1 - x).$$

We can see easily that the tent map f and g are topologically conjugate by $h(y) = \sin^2(\frac{\pi y}{2})$. Since conjugacy persevere AASP so g has AASP. Example 6.3 in [19] shows that g is NUE. Therefore g has a unique SRB measure.

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