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Horospherical limit points of S-arithmetic groups

Dave Witte Morris and **Kevin Wortman**

Abstract. Suppose Γ is an S-arithmetic subgroup of a connected, semisimple algebraic group $\mathbf{G}$ over a global field Q (of any characteristic). It is well-known that Γ acts by isometries on a certain CAT(0) metric space $X_S = \prod_{v \in S} X_v$, where each X_v is either a Euclidean building or a Riemannian symmetric space. For a point ξ on the visual boundary of X_S, we show there exists a horoball based at ξ that is disjoint from some Γ-orbit in X_S if and only if ξ lies on the boundary of a certain type of flat in X_S that we call "Q-good." This generalizes a theorem of G. Avramidi and D. W. Morris that characterizes the horospherical limit points for the action of an arithmetic group on its associated symmetric space X.

Contents

1. Introduction

Definition 1.1 ([6, Defn. B]). Suppose the group Γ acts by isometries on the CAT(0) metric space X, and fix $x \in X$. A point ξ on the visual boundary of X is a *horospherical limit point* for Γ if every horoball based at ξ intersects the orbit $x \cdot \Gamma$. Notice that this definition is independent of the choice of x. Also note that if Λ is a finite-index subgroup of Γ, then ξ is a horospherical limit point for Λ if and only if it is a horospherical limit point for Γ.

In the situation where Γ is an arithmetic group, with its natural action on its associated symmetric space X, the horospherical limit points have a simple geometric characterization:

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Theorem 1.2 (Avramidi–Morris [1, Thm. 1.3])**.** *Let:*

- $\mathbf{G}$ *be a connected, semisimple algebraic group over* $\mathbb{Q}$*,*
- K *be a maximal compact subgroup of the Lie group* $\mathbf{G}(\mathbb{R})$*,*
- $X = K\backslash\mathbf{G}(\mathbb{R})$ *be the corresponding symmetric space of noncompact type (with the natural metric induced by the Killing form of* $\mathbf{G}(\mathbb{R})$*), and*
- Γ *be an arithmetic subgroup of* $\mathbf{G}$*.*

Then a point $\xi \in \partial X$ *is* ***not*** *a horospherical limit point for* Γ *if and only if* ξ *is on the boundary of some flat* F *in* X*, such that* F *is the orbit of a* $\mathbb{Q}$*-split torus in* $\mathbf{G}(\mathbb{R})$*.*

This note proves a natural generalization that allows Γ to be S-arithmetic (of any characteristic), rather than arithmetic. The precise statement assumes familiarity with the theory of Bruhat–Tits buildings [12], and requires some additional notation.

Notation 1.3.

(1) Let:
  - Q be a global field (of any characteristic),
  - $\mathbf{G}$ be a connected, semisimple algebraic group over Q,
  - S be a finite set of places of Q (containing all the archimedean places if the characteristic of Q is 0),
  - $G_v = \mathbf{G}(Q_v)$ for each $v \in S$, where Q_v is the completion of Q at v,
  - K_v be a maximal compact subgroup of G_v, for each $v \in S$, and
  - Z_S be the ring of S-integers in Q.

(2) Adding the subscript S to any symbol other than Z denotes the Cartesian product over all elements of S. Thus, for example, we have $G_S = \prod_{v\in S} G_v = \prod_{v \in S} \mathbf{G}(Q_v)$.

(3) For each $v \in S$, let

$$X_v = \begin{cases} \text{the symmetric space } K_v\backslash\mathbf{G}(Q_v) & \text{if } v \text{ is archimedean,} \\ \text{the Bruhat–Tits building of } \mathbf{G}(Q_v) & \text{if } v \text{ is nonarchimedean.} \end{cases}$$

Thus, $G_v = \mathbf{G}(Q_v)$ acts properly and cocompactly by isometries on the CAT(0) metric space X_v. So G_S acts properly and cocompactly by isometries on the CAT(0) metric space $X_S = \prod_{v\in S} X_v$.

Definition 1.4. We say a family $\{Y_t\}_{t\in\mathbb{R}}$ of subsets of X_S is *uniformly coarsely dense* in $X_S/\mathbf{G}(Z_S)$ if there exists $C > 0$, such that, for every $t \in \mathbb{R}$, each $\mathbf{G}(Z_S)$-orbit in X_S has a point that is at distance $< C$ from some point in Y_t.

See Definition 3.2 for the definition of a Q-good flat in X_S.

Theorem 1.5 (cf. [1, Cor. 4.5])**.** *For a point* ξ *on the visual boundary of* $X_S = \prod_{v\in S} X_v$*, the following are equivalent:*

(1) *ξ is a horospherical limit point for $\mathbf{G}(Z_S)$.*
(2) *ξ is not on the boundary of any Q-good flat.*
(3) *There does not exist a parabolic Q-subgroup $\mathbf{P}$ of $\mathbf{G}$, such that P_S fixes ξ, and $\mathbf{P}(Z_S)$ fixes some (or, equivalently, every) horosphere based at ξ.*
(4) *The horospheres based at ξ are uniformly coarsely dense in*

$$X_S/\mathbf{G}(Z_S).$$

(5) *The horoballs based at ξ are uniformly coarsely dense in $X_S/\mathbf{G}(Z_S)$.*
(6) *$\pi(\mathcal{B}) = X_S/\mathbf{G}(Z_S)$ for every horoball $\mathcal{B}$ based at ξ, where*

$$\pi\colon X_S \to X_S/\mathbf{G}(Z_S)$$

is the natural covering map.

Remark 1.6. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are proved in the following sections, by fairly straightforward adaptations of arguments in [1]. This suffices to establish the theorem, since:

- (1) ⇔ (6) is a restatement of Definition 1.1.
- (4) ⇒ (5) is obvious, because horoballs are bigger than horospheres.
- (5) ⇒ (1) is well-known (see, for example, [1, Lem. 2.3(⇐)]).

The minimal parabolic Q-subgroups of $\mathbf{G}$ are all conjugate under $\mathbf{G}(Q)$ [4, Thm. 4.13(b)], and the proof of Proposition 3.4 shows that the nonhorospherical limit points fixed by a given parabolic Q-subgroup are all contained in the boundary of a single Q-good flat, so Theorem 1.5 implies the following alternative characterization of the horospherical limit points:

Corollary 1.7 (cf. [1, Cor. 1.4])**.** *If B is the boundary of any maximal Q-good flat in X_S, then the set of horospherical limit points for $\mathbf{G}(Z_S)$ is the complement of $\bigcup_{g\in\mathbf{G}(Q)} Bg$.*

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2. Proof of (3) ⇒ (4)

(3) ⇒ (4) of Theorem 1.5 is the contrapositive of the following result.

Proposition 2.1 (cf. [1, Thm. 4.3])**.** *If the horospheres based at ξ are not uniformly coarsely dense in $X_S/\mathbf{G}(Z_S)$, then there is a parabolic Q-subgroup $\mathbf{P}$ of $\mathbf{G}$, such that:*

(1) *P_S fixes ξ.*
(2) *$\mathbf{P}(Z_S)$ fixes some (or, equivalently, every) horosphere based at ξ.*

Proof. We modify the proof of [1, Thm. 4.3] to deal with minor issues, such as the fact that G_S is not (quite) transitive on X_S. To avoid technical complications, assume $\mathbf{G}$ is simply connected. We begin by introducing yet more notation:

(Γ) Let $\Gamma = \mathbf{G}(Z_S)$.

(x) Let $x \in X_S$. If $v \in S$ is a nonarchimedean place, then we choose x so that its projection to X_v is a vertex.

(γ) Let $\gamma\colon \mathbb{R} \to X_S$ be a geodesic with $\gamma(0) = x$ and $\gamma(+\infty) = \xi$. Let $\gamma^+\colon [0,\infty) \to X$ be the forward geodesic ray of γ. For each $v \in S$, let γ_v be the projection of γ to X_v, so γ_v is a geodesic in X_v.

(F_S) For each $v \in S$, choose a maximal flat (or "apartment") F_v in X_v that contains γ_v. Then F_S is a maximal flat in X_S that contains γ.

(A_S) For each $v \in S$, there is a maximal Q_v-split torus A_v of $\mathbf{G}(Q_v)$, such that A_v acts properly and cocompactly on the Euclidean space F_v by translations. Then A_S acts properly and cocompactly on F_S (by translations).

(C_S) For each $v \in S$, choose a compact subset C_v of F_v, such that $C_v A_v = F_v$. Then $C_S A_S = F_S$.

(A_γ) Let $A_\gamma = \{\, a \in A_S \mid C_S\, a \cap \gamma \neq \emptyset \,\}$ and $A_\gamma^+ = \{\, a \in A_S \mid C_S\, a \cap \gamma^+ \neq \emptyset \,\}$.

($F_\perp, A_\perp$) Let $F_\perp$ be the (codimension-one) hyperplane in F_S that is orthogonal to the geodesic γ and contains x. Let

$$A_\perp = \{\, a \in A_S \mid C_S\, a \cap F_\perp \neq \emptyset \,\}.$$

(P_v^ξ, N_v) For each $v \in S$, let

$$P_v^\xi = \bigl\{\, g \in \mathbf{G}(Q_v) \bigm| \{\, aga^{-1} \mid a \in A_\gamma^+ \,\} \text{ is bounded} \,\bigr\},$$

so P_v^ξ is a parabolic Q_v-subgroup of $\mathbf{G}(Q_v)$ that fixes ξ. The Iwasawa decomposition [12, §3.3.2] allows us to choose a maximal horospherical subgroup N_v of $\mathbf{G}(Q_v)$ that is contained in P_v^ξ and is normalized by A_v, such that $F_v\, N_v = X_v$.

$\begin{pmatrix} P_v, M_v, \\ T_v, M_v^* \end{pmatrix}$ By applying the S-arithmetic generalization of Ratner's Theorem that was proved independently by Margulis-Tomanov [7] and Ratner [11] (or, if $\operatorname{char} Q \neq 0$, by applying a theorem of Mohammadi [8, Cor. 4.2]), we obtain an S-arithmetic analogue of [1, Cor. 2.13]. Namely, for some parabolic Q-subgroup $\mathbf{P}$ of $\mathbf{G}$, if we let $P_v = \mathbf{P}(Q_v)$ for each $v \in S$, and let $P_v = M_v T_v U_v$ be the Langlands decomposition over Q_v (so T_v is the maximal Q_v-split torus in the center of the reductive group $M_v T_v$, and U_v is the unipotent radical), then we have

$$N_S \subseteq M_S^* U_S \quad \text{ and } \quad M_S^* U_S \Gamma \subseteq \overline{N_S \Gamma},$$

where M_v^* is the product of all the isotropic almost-simple factors of M_v.

Since $N_v \subseteq P_v$ for every v (and P_S is parabolic), we have $U_S \subseteq N_S$ and $A_S \subset P_S$ (cf. proof of [1, Lem. 2.10]). Therefore, since all maximal Q_v-split tori of P_v are conjugate [2, Thm. 20.9(ii), p. 228], and $M_v^* T_v$ contains a maximal Q_v-split torus, there is no harm in assuming $A_S \subseteq M_S^* T_S$, by replacing $M_S^* T_S$ with a conjugate. Let $A_S^M = A_S \cap M_S = A_S \cap M_S^*$.

Note that N_v is in the kernel of every continuous homomorphism from P_v^ξ to $\mathbb{R}$. Since P_v^ξ acts continuously on the set of horospheres based at ξ, and these horospheres are parametrized by $\mathbb{R}$, this implies that N_v fixes every horosphere based at ξ. Then, since $F_S\, N_S = X_S$, we see that, for each $a \in A_\gamma$, the set $F_\perp\, a\, N_S$ is the horosphere based at ξ through the point xa. By the definition of $A_\perp$, this implies that the horosphere is at bounded Hausdorff distance from

$$\mathcal{H}_a = xaA_\perp\, N_S.$$

(Also note that every horosphere is at bounded Hausdorff distance from some $\mathcal{H}_a$, since A_S acts cocompactly on F_S.) We have

$$\overline{aA_\perp\, N_S\, \Gamma} \supseteq aA_\perp \cdot \overline{N_S\, \Gamma} \supseteq aA_\perp \cdot M_S^*\, U_S\, \Gamma. \tag{2.2}$$

We claim that $F_\perp A_S^M$ is not coarsely dense in F_S. Indeed, suppose, for the sake of a contradiction, that the set is coarsely dense. Then $A_\perp A_S^M$ is coarsely dense in A_S, which means there is a compact subset K_1 of A_S, such that $A_S = K_1 A_\perp A_S^M$. Also, the Iwasawa decomposition [12, §3.3.2] of each $\mathbf{G}(Q_v)$ implies there is a compact subset K_S of G_S, such that $K_S A_S N_S = G_S$. Then, for every $a \in A_\gamma$, we have

$$\begin{aligned} K_S K_1 \cdot aA_\perp M_S^* U_S = K_S a(K_1 A_\perp M_S^*) U_S &\supseteq K_S a A_S M_S^* U_S \\ &\supseteq K_S A_S N_S = G_S. \end{aligned}$$

Since the compact set $K_S K_1$ is independent of a, this (together with (2.2)) implies that the sets $\mathcal{H}_a$ are uniformly coarsely dense in X/Γ. This contradicts the fact that the horospheres based at ξ are not uniformly coarsely dense.

Since $F_\perp$ is a hyperplane of codimension one in F_S (and A_S^M is a group that acts by translations), the claim proved in the preceding paragraph implies $F_\perp = F_\perp A_S^M \supseteq xA_S^M$. This means that γ is orthogonal to the convex hull of xA_S^M.

On the other hand, we know that M_S centralizes T_S. Therefore, M_S fixes the endpoint ξ_T of any geodesic ray γ_T in the convex hull of xT_S. So M_S acts (continuously) on the set of horospheres based at ξ_T. However, M_S is the almost-direct product of compact groups and semisimple groups over local fields, so it has no has no nontrivial homomorphism to $\mathbb{R}$. (For the semisimple groups, this follows from the truth of the Kneser–Tits Conjecture [10, Thm. 7.6].) Since the horospheres are parametrized by $\mathbb{R}$, we conclude that M_S fixes every horosphere based at ξ_T. Hence A_S^M also fixes these horospheres. So xA_S^M is contained in the horosphere through x, which means the convex hull of xA_S^M must be perpendicular to the convex hull of xT_S. Since $A_S^M T_S$ has finite index in A_S, the conclusion of the preceding paragraph now implies that γ is contained in the convex hull of xT_S, so $C_{G_S}(T_S)$ fixes ξ.

We also have

$$P_S = M_S T_S U_S = C_{G_S}\big(T_S\big)\, U_S \subseteq C_{G_S}\big(T_S\big)\, N_S.$$

Since $C_{G_S}\big(T_S\big)$ and N_S each fix the point ξ, we conclude that P_S fixes ξ. This completes the proof of (1).

From here, the proof of (2) is almost identical to the proof of Thm. 4.3(2) in [1]. □

3. Proof of (2) ⇒ (3)

(2) ⇒ (3) of Theorem 1.5 is the contrapositive of Proposition 3.4 below.

Notation 3.1. Suppose $\mathbf{T}$ is a torus that is defined over Q. Let:

(1) $\mathcal{X}^*_Q(\mathbf{T})$ be the set of Q-characters of $\mathbf{T}$;

(2) $T_S^{(1)} = \left\{\, g \in T_S \;\middle|\; \prod_{v\in S}\big\|\chi(g_v)\big\|_v = 1,\ \forall \chi \in \mathcal{X}_Q(\mathbf{T}) \,\right\}$.

Definition 3.2. Suppose $\mathcal{F}$ is a flat in X_S (not necessarily maximal). We say $\mathcal{F}$ is *Q-good* if there exists a Q-torus $\mathbf{T}$, such that:

- $\mathbf{T}$ contains a maximal Q-split torus of $\mathbf{G}$.
- $\mathbf{T}$ contains a maximal Q_v-split torus A_v of G_v for every $v \in S$.
- $\mathcal{F}$ is contained in the maximal flat F_S that is fixed by A_S.
- $\mathcal{F}$ is orthogonal to the convex hull of an orbit of $T_S^{(1)}$ in F_S.

Remark 3.3. Q-good flats are a natural generalization of $\mathbb{Q}$-split flats. Indeed, the two notions coincide in the setting of arithmetic groups. Namely, suppose:

- Q is an algebraic number field.
- S is the set of all archimedean places of Q.
- $\mathbf{T}$ is a maximal Q-split torus in $\mathbf{G}$.
- $\mathbf{H} = \operatorname{Res}_{Q/\mathbb{Q}}\mathbf{G}$ is the $\mathbb{Q}$-group obtained from $\mathbf{G}$ by restriction of scalars.

Then T_S can be viewed as the real points of a $\mathbb{Q}$-torus in $\mathbf{H}(\mathbb{R})$, and $T_S^{(1)}$ is the group of real points of the $\mathbb{Q}$-anisotropic part of T_S. Thus, in this setting, the Q-good flats in the symmetric space of G_S are naturally identified with the $\mathbb{Q}$-split flats in the symmetric space of $\mathbf{H}(\mathbb{R})$.

Proposition 3.4 (cf. [1, Prop. 4.4])**.** *If there is a parabolic Q-subgroup $\mathbf{P}$ of $\mathbf{G}$, such that P_S fixes ξ, and $\mathbf{P}(Z_S)$ fixes every horosphere based at ξ, then ξ is on the boundary of a Q-good flat in X_S.*

Proof. Choose a maximal Q-split torus $\mathbf{R}$ of $\mathbf{P}$. The centralizer of $\mathbf{R}$ in $\mathbf{G}$ is an almost direct product $\mathbf{RM}$ for some reductive Q-subgroup $\mathbf{M}$ of $\mathbf{P}$.

Choose a Q-torus $\mathbf{L}$ of $\mathbf{M}$, such that $\mathbf{L}(Q_v)$ contains a maximal Q_v-split torus B_v of $\mathbf{M}(Q_v)$ for each $v \in S$. (This is possible when $\operatorname{char} Q = 0$ by [10, Cor. 3 of §7.1, p. 405], and the same proof works in positive characteristic, because a theorem of A. Grothendieck tells us that the variety of maximal tori is rational [5, Exp. XIV, Thm. 6.1, p. 334], [3, Thm. 7.9].) Let $\mathbf{T} = \mathbf{RL}$

and $A_v = \mathbf{R}(Q_v)B_v$, so that $\mathbf{T}$ is a Q-torus that contains the maximal Q-split torus $\mathbf{R}$ as well as the maximal Q_v-split tori A_v for all $v \in S$.

Let F_S be the maximal flat corresponding to A_S, and choose some $x \in F_S$. Since P_S fixes ξ, there is a geodesic $\gamma = \{\gamma_t\}$ in F_S, such that $\lim_{t\to\infty} \gamma_t = \xi$ (and $\gamma_0 = x$).

Now $\mathbf{T}(Z_S)$ is a cocompact lattice in $T_S^{(1)}$ (because the "Tamagawa number" of $\mathbf{T}$ is finite: see [10, Thm. 5.6, p. 264] if $\operatorname{char} Q = 0$; or see [9, Thm. IV.1.3] for the general case), and, by assumption, $\mathbf{T}(Z_S)$ fixes the horosphere through x. This implies that all of $T_S^{(1)}$ fixes this horosphere, so $x\,T_S^{(1)}$ is contained in the horosphere. Therefore, the convex hull of $x\,T_S^{(1)}$ is perpendicular to the geodesic γ, so γ is a Q-good flat. $\square$

4. Proof of (1) ⇒ (2)

(1) ⇒ (2) of Theorem 1.5 is the contrapositive of the following result.

Proposition 4.1 (cf. [1, Prop. 3.1] or [6, Thm. A])**.** *If ξ is on the boundary of a Q-good flat, then ξ is not a horospherical limit point for $\mathbf{G}(Z_S)$.*

Proof. Let:

- $\mathcal{F}$ be a Q-good flat, such that ξ is on the boundary of $\mathcal{F}$.
- γ be a geodesic in $\mathcal{F}$, such that $\lim_{t\to\infty} \gamma(t) = \xi$.
- $\mathbf{T}$, A_S, and F_S be as in Definition 3.2.
- $x = \gamma(0) \in F_S$.
- F_S be considered as a real vector space with Euclidean inner product, by specifying that the point x is the zero vector.
- C_x be a compact set, such that $C_x A_S = F_S$ (and $x \in C_x$).
- $\gamma^\perp$ be the orthogonal complement of the 1-dimensional subspace γ in the vector space F_S.
- $\gamma_A^\perp = \{\, a \in A_S \mid C_x a \cap \gamma^\perp \neq \emptyset \,\}$.
- $\gamma_A(t) \in A_S$, such that $\gamma(t) \in C_x \gamma_A(t)$, for each $t \in \mathbb{R}$.
- $\mathbf{R}$ be a maximal Q-split torus of $\mathbf{G}$ that is contained in $\mathbf{T}$.
- Φ be the system of roots of $\mathbf{G}$ with respect to $\mathbf{R}$.
- $\alpha^S \colon T_S \to \mathbb{R}^+$ be defined by $\alpha^S(g) = \prod_{v\in S} \big\|\alpha(g_v)\big\|_v$ for $\alpha \in \Phi$ (where $\|\cdot\|_v \circ \alpha$ is extended to be defined on all of $\mathbf{T}(Q_v)$ by making it trivial on the Q-anisotropic part).
- $\hat{\alpha}^S \colon F_S \to \mathbb{R}$ be the linear map satisfying $\hat{\alpha}^S(xa) = \log \alpha^S(a)$ for all $a \in A_S$.
- $\alpha^F \in F_S$, such that $\langle \alpha^F \mid y \rangle = \hat{\alpha}^S(y)$ for all $y \in F_S$.
- $\Phi^{++} = \{\, \alpha \in \Phi \mid \hat{\alpha}^S\big(\gamma(t)\big) > 0 \text{ for } t > 0 \,\}$.
- Δ be a base of Φ, such that Φ^+ contains Φ^{++}.
- $\Delta^{++} = \Delta \cap \Phi^{++}$.
- $\mathbf{P}_\alpha = \mathbf{R}_\alpha \mathbf{M}_\alpha \mathbf{N}_\alpha$ be the parabolic Q-subgroup corresponding to α, for $\alpha \in \Delta$, where:

- $\mathbf{R}_\alpha$ is the one-dimensional subtorus of $\mathbf{R}$ on which all roots in $\Delta \smallsetminus \{\alpha\}$ are trivial.
- $\mathbf{M}_\alpha$ is reductive with Q-anisotropic center.
- The unipotent radical $\mathbf{N}_\alpha$ is generated by the roots in Φ^+ that are *not* trivial on $\mathbf{R}_\alpha$.

Given any large $t \in \mathbb{R}^+$, we know $\hat{\alpha}^S\big(\gamma(t)\big)$ is large for all $\alpha \in \Delta^{++}$. By definition, we have $T_S^{(1)} = \bigcap_{\alpha\in\Delta} \ker \alpha^S$. Since γ is perpendicular to the convex hull of $x \cdot T_S^{(1)}$, this implies that $\gamma(t)$ is in the span of $\{\alpha^F\}_{\alpha\in\Delta}$. Also, for $\alpha \in \Delta$, we have

$$\langle \alpha^F \mid \gamma(t) \rangle = \hat{\alpha}^S\big(\gamma(t)\big) \geq 0.$$

There is no harm in renormalizing the metric on X_S by a positive scalar on each irreducible factor (cf. [1, Rem. 5.4]). This allows us to assume $\langle \alpha^F \mid \beta^F \rangle \leq 0$ whenever $\alpha \neq \beta$ (see Lemma 4.2 below). Therefore, for any $b \in \gamma_A^\perp$, there is some $\alpha \in \Delta$, such that $\hat{\alpha}^S\big(x\gamma_A(t)b\big)$ is large (see Lemma 4.3 below). This means $\alpha^S\big(\gamma_A(t)\, b\big)$ is large.

Since conjugation by the inverse of $\gamma_A(t)\, b$ contracts the Haar measure on $(N_\alpha)_S$ by a factor of $\alpha^S\big(\gamma_A(t)\, b\big)^k$ for some $k \in \mathbb{Z}^+$, and the action of N_S on $(N_\alpha)_S$ is volume-preserving, this implies that, for any $g \in \gamma_A(t)\, b\, N_S$, conjugation by the inverse of g contracts the Haar measure on $(N_\alpha)_S$ by a large factor. Since $\mathbf{N}_\alpha(Z_S)$ is a cocompact lattice in $(N_\alpha)_S$ (because the "Tamagawa number" of $\mathbf{N}_\alpha$ is finite: see [10, Thm. 5.6, p. 264] if $\operatorname{char} Q = 0$; or see [9, Thm. IV.1.3] for the general case), this implies there is some nontrivial $h \in \mathbf{N}_\alpha(Z_S)$, such that $\|ghg^{-1} - e\|$ is small. We conclude that ξ is not a horospherical limit point for $\mathbf{G}(Z_S)$ (cf. [1, Lem. 2.5(2)]). $\square$

Lemma 4.2. *Assume the notation of the proof of Proposition 4.1. The metric on X_S can be renormalized so that we have $\langle \alpha^F \mid \beta^F \rangle \leq 0$ for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$.*

Proof. When v is archimedean, the Killing form provides a metric on X_v. We now construct an analogous metric when v is nonarchimedean. To do this, let Φ_v be the root system of $\mathbf{G}$ with respect to the maximal Q_v-split torus A_v, let $\mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_v} \mathfrak{g}_\alpha$ be the corresponding weight-space decomposition of the Lie algebra of G_v, choose a uniformizer π_v of Q_v, let $\mathcal{X}_*(A_v)$ be the group of co-characters of A_v, and define a $\mathbb{Z}$-bilinear form

$$\langle \mid \rangle_v \colon \mathcal{X}_*(A_v) \times \mathcal{X}_*(A_v) \to \mathbb{R}$$

by

$$\langle \varphi_1 \mid \varphi_2 \rangle_v = \sum_{\alpha\in\Phi_v} v\Big(\alpha\big(\varphi_1(\pi_v)\big)\Big)\, v\Big(\alpha\big(\varphi_2(\pi_v)\big)\Big) \big(\dim \mathfrak{g}_\alpha\big).$$

This extends to a positive-definite inner product on $\mathcal{X}_*(A_v) \otimes \mathbb{R}$ (and the extension is also denoted by $\langle \mid \rangle_v$). It is clear that this inner product is invariant under the Weyl group, so it determines a metric on X_v [12, §2.3].

By renormalizing, we may assume that the given metric on X_v coincides with this one.

Let $\mathbf{E}$ be the Q-anisotropic part of $\mathbf{T}$. Then it is not difficult to see that $\mathcal{X}_*\big(\mathbf{R}\big) \otimes \mathbb{R}$ is the orthogonal complement of $\mathcal{X}_*\big(\mathbf{E}(Q_v)\big) \otimes \mathbb{R}$, with respect to the inner product $\langle \mid \rangle_v$ (cf. [1, Lem. 2.8]). Since every Q-root annihilates $\mathbf{E}(Q_v)$, this implies that the F_v-component α_v^F of α^F belongs to the convex hull of $x\,\mathbf{R}(Q_v)$, for every $\alpha \in \Phi$.

From [4, Cor. 5.5], we know that the Weyl group over Q is the restriction to $\mathbf{R}$ of a subgroup of the Weyl group over Q_v. So the restriction of $\langle \mid \rangle_v$ to $\mathcal{X}_*\big(\mathbf{R}\big) \otimes \mathbb{R}$ is invariant under the Q-Weyl group. Assume, for simplicity, that $\mathbf{G}$ is Q-simple, so the invariant inner product on $\mathcal{X}_*\big(\mathbf{R}\big) \otimes \mathbb{R}$ is unique (up to a positive scalar). (The general case is obtained by considering the simple factors individually.) This means that, after passing to the dual space $\mathcal{X}^*\big(\mathbf{R}\big) \otimes \mathbb{R}$, the inner product $\langle \mid \rangle_v$ must be a positive scalar multiple c_v of the usual inner product (for which the reflections of the root system Φ are isometries), so $\langle \alpha_v^F \mid \beta_v^F \rangle_v = c_v \langle \alpha \mid \beta \rangle$ for all $\alpha, \beta \in \Delta$. Since it is a basic property of bases in a root system that $\langle \alpha \mid \beta \rangle \leq 0$ whenever $\alpha \neq \beta$, we therefore have

$$\langle \alpha^F \mid \beta^F \rangle = \sum_{v \in S} \langle \alpha_v^F \mid \beta_v^F \rangle_v = \sum_{v \in S} c_v \langle \alpha \mid \beta \rangle = \sum_{v \in S} \big(> 0\big)\big(\leq 0\big) \leq 0. \qquad \square$$

Lemma 4.3 ([1, Lem. 2.6]). *Suppose:*

(1) $v, v_1, \ldots, v_n \in \mathbb{R}^k$, *with* $v \neq 0$.
(2) v *is in the span of* $\{v_1, \ldots, v_n\}$.
(3) $\langle v \mid v_i \rangle \geq 0$ *for all* i.
(4) $\langle v_i \mid v_j \rangle \leq 0$ *for* $i \neq j$.
(5) $T \in \mathbb{R}^+$.

Then, for all sufficiently large $t \in \mathbb{R}^+$ *and all* $w \perp v$, *there is some* i, *such that* $\langle tv + w | v_i \rangle > T$.

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(Dave Witte Morris) Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, T1K 6R4, Canada
Dave.Morris@uleth.ca
http://people.uleth.ca/~dave.morris/

(Kevin Wortman) Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090
wortman@math.utah.edu
http://www.math.utah.edu/~wortman/

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