Some results on radial symmetry in partial differential equations

Amin Farjudian and Behrouz Emamizadeh

ABSTRACT. In this paper we will discuss three different problems which share the same conclusions. In the first one we revisit the well known Faber–Krahn inequality for the principal eigenvalue of the p-Laplace operator with zero homogeneous Dirichlet boundary conditions. Motivated by Chatelain, Choulli, and Henrot, 1996, we show in case the equality holds in the Faber–Krahn inequality, the domain of interest must be a ball. In the second problem we consider a generalization of the well known torsion problem and accordingly define a quantity that we name the p-torsional rigidity of the domain of interest. We maximize this quantity relative to a set of domains having the same volume, and prove that the optimal domain is a ball. The last problem is very similar in spirit to the second one. We consider a Hamilton–Jacobi boundary value problem, and define a quantity to be maximized relative to a set of domains having fixed volume. Again, we prove that the optimal domain is a ball. The main tools in our analysis are the method of domain derivatives, an appropriate generalized version of the Pohozaev identity, and the classical symmetrization techniques.

CONTENTS

1. Introduction

This paper is devoted to three different problems which share the same conclusions. The first one is of isoperimetric type, where we show that under

certain constraints the domain of interest must be a ball. The remaining two problems are of optimization types with volume constraint, and the conclusion of both of them is that the optimal shape is a ball. We use three main keys to reach our goals. One is the method of domain derivatives, the other is an appropriate generalization of the Pohozaev identity, and the last one is the symmetrization method. Let us describe each of our three problems:

Problem 1. Consider the eigenvalue problem:

$$
(1) \qquad \begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}
$$

in which $D \subseteq \mathbb{R}^N$ is a bounded domain with C^1 boundary, and

$$
\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) \quad (p > 1),
$$

stands for the usual p-Laplace operator. It is well known that (1) has a sequence of eigenvalues:

$$
0 < \lambda_p(D) := \lambda_1^{(p)} < \lambda_2^{(p)} \leq \lambda_3^{(p)} \leq \cdots \to \infty.
$$

We denote the first eigenvalue (commonly referred to as the principal eigenvalue) by $\lambda_p(D)$, which has a variational formulation:

$$
(2) \qquad \lambda_p(D) = \inf \left\{ \int_D |\nabla u|^p dx \, : \, u \in W_0^{1,p}(D), \, \int_D u^p dx = 1 \right\}.
$$

The infimum in (2) is attained for a unique u which (together with the infimum value λ) satisfies the following integral equation:

$$
\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda \int_D |u|^{p-2} uv \, dx, \quad (\forall v \in W_0^{1,p}(D)).
$$

Moreover, $u \in C^{1,\alpha}(\overline{D})$, and it is positive throughout D. Using a simple symmetrization argument one can derive the Faber–Krahn inequality:

$$
(3) \qquad\qquad\qquad \lambda_p(B) \leq \lambda_p(D),
$$

where B is any N-dimensional ball satisfying $|B| = |D|$. Henceforth, for a measurable set $E \subseteq \mathbb{R}^N$, we will write $|E|$ to denote the N-dimensional Lebesgue measure of E. Our first main result concerns the equality in (3).

Problem 2. The second problem is related to the following boundary value problem:

$$
\begin{cases} -\Delta_p u = 1 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}
$$

We will introduce the p-torsional rigidity of the domain D, and pose a maximization problem where the admissible set comprises domains with fixed volume. We will see that this issue can be settled using the method of domain derivatives. However, we will then look at a more general optimization

problem and observe that the method of domain derivatives fails this time, hence we apply the method of symmetrization to overcome the difficulty.

Problem 3. The last problem we will consider in this paper is related to the following Hamilton–Jacobi system:

$$\begin{cases} K(|\nabla u|) = 1 & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}$$

Similar to the second problem, we will define a quantity, which is merely the integral of a function composed with a solution of the Hamilton–Jacobi system (see (46) on page 252), and maximize it under the constraint that the volume of D is fixed. Again, for this problem the method of domain derivatives will not be applicable, hence we will have to once again use the symmetrization techniques. However, prior to applying these techniques we will need a Talenti type inequality which will be derived using the well known co-area formula. Finally, we will prove that the optimal domain has to be a ball.

2. Preliminaries and Faber–Krahn inequality

As mentioned in the introduction one of the main tools we use is the domain derivative. In this section we review some well known facts about the domain derivative in the context of the eigenvalue problem (1).

Let $V \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a smooth vector field. Let D^t denote the image of D under the mapping $\phi_t(x) = x + tV(x)$. Note that since ϕ_t is a diffeomorphism, for small t, we infer D^t is open and has C^1 boundary. Let $u^t \in W_0^{1,p}(D^t)$ denote the normalized eigenfunction (i.e., $\|u^t\|_p = 1$) corresponding to $\lambda_p(D^t)$. So, u^t satisfies:

$$(4) \qquad \begin{cases} -\Delta_p u^t = \lambda \, (u^t)^{p-1} & \text{in } D^t \\ u^t = 0 & \text{on } \partial D^t. \end{cases}$$

Next, we define

$$\forall x \in D: \; u'(x) = \lim_{t \to 0^+} \frac{u^t(x) - u(x)}{t}$$

and

$$\lambda_p'(D) = \lim_{t \to 0^+} \frac{\lambda_p(D^t) - \lambda_p(D)}{t}.$$

We note that both of these limits exist [Hen06]. We call u' the domain derivative of u in the direction of the vector field V. Similarly, λ_p' is called the domain derivative of λ_p with respect to V. In [Sim80] it is proved that $u' \in W^{1,p}(D)$ and satisfies:

$$(5) \qquad \begin{cases} -\nabla \cdot W = \lambda_p'(D) \, u^{p-1} + (p-1) \, \lambda_p(D) \, u^{p-2} u' & \text{in } D \\ u' = -\frac{\partial u}{\partial \nu} \, V \cdot \nu & \text{on } \partial D, \end{cases}$$

in which:

- ν, in the boundary condition, stands for the unit outward normal vector on ∂D.
- $W = (p-2)|\nabla u|^{p-1}\frac{\nabla u \cdot \nabla u'}{|\nabla u|^3}\,\nabla u + |\nabla u|^{p-2}\nabla u'$, (also see [EZR08]).

We first derive a formula for the domain derivative of λ_p.

Lemma 2.1. *The following formula holds:*

$$(6) \qquad \lambda_p'(D) = (1-p)\int_{\partial D}|\nabla u|^p(V\cdot\nu)\,d\mathcal{H}^{N-1}.$$

Proof. Again for simplicity we set

$$W = (p-2)|\nabla u|^{p-1}\frac{\nabla u \cdot \nabla u'}{|\nabla u|^3}\,\nabla u + |\nabla u|^{p-2}\nabla u'.$$

Multiplying the differential equation in (1) by u' and integrating the result over D, we obtain:

$$(7) \quad \int_D|\nabla u|^{p-2}\nabla u\cdot\nabla u'\,dx - \int_D\nabla\cdot(u'|\nabla u|^{p-2}\nabla u)\,dx = \lambda_p(D)\int_D u^{p-1}u'dx.$$

An application of the Divergence theorem to the second integral on the left-hand side of (7) yields:

$$(8) \quad \int_D|\nabla u|^{p-2}\nabla u\cdot\nabla u'dx - \int_{\partial D}u'|\nabla u|^{p-2}(\nabla u\cdot\nu)\,d\mathcal{H}^{N-1}$$
$$= \lambda_p(D)\int_D u^{p-1}u'dx.$$

Using the boundary condition $u' = -\frac{\partial u}{\partial\nu}V\cdot\nu$, and noting that $\frac{\partial u}{\partial\nu} = -|\nabla u|$, we infer from (8) that:

$$(9) \quad \int_D|\nabla u|^{p-2}\nabla u\cdot\nabla u'dx + \int_{\partial D}|\nabla u|^p(V\cdot\nu)d\mathcal{H}^{N-1} = \lambda_p(D)\int_D u^{p-1}u'dx.$$

Next, we multiply the differential equation in (5) by u and integrate the result over D:

$$(10) \quad \int_D W\cdot\nabla u\,dx - \int_D\nabla\cdot(uW)\,dx$$
$$= \lambda_p'(D)\int_D u^p dx + (p-1)\lambda_p(D)\int_D u^{p-1}u'dx.$$

The second integral on the left-hand side of (10) vanishes by the divergence theorem. Note that:

$$(11) \qquad W\cdot\nabla u = (p-1)|\nabla u|^{p-2}\nabla u\cdot\nabla u'.$$

From (10) and (11) we obtain:

$$(12) \quad (p-1)\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla u' dx$$

$$= \lambda_p'(D)\int_D u^p dx + (p-1)\lambda_p(D)\int_D u^{p-1}u' dx.$$

From (9) and (12) we readily deduce (6). $\qquad\square$

The next result is a generalized version of the Pohozaev identity [DMS03]. This version of the Pohozaev identity fits the differential structure of (1) and the regularity of its solutions.

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary. Assume $\mathcal{L} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is C^1, and the function $\xi \to \mathcal{L}(s, \xi)$ is strictly convex for each $s \in \mathbb{R}$. Suppose $u \in C^1(\overline{\Omega})$ is a weak solution of the following boundary value problem:*

$$\begin{cases} -div\,(\nabla_\xi \mathcal{L}(w, \nabla w)) + D_s\mathcal{L}(w, \nabla w) = 0 & in\ \Omega \\ w = 0 & on\ \partial\Omega. \end{cases}$$

Then

$$\int_{\partial\Omega} \left(\mathcal{L}(0, \nabla u) - \nabla_\xi \mathcal{L}(0, \nabla u) \cdot \nabla u\right)(h \cdot \nu)d\mathcal{H}^{N-1}$$

$$= \int_\Omega (div\ h)\mathcal{L}(u, \nabla u)dx - \sum_{i,j=1}^N \int_\Omega \left(D_j u D_i h_j + u D_i a\right) D_{\xi_i}\mathcal{L}(u, \nabla u)\,dx$$

$$- \int_\Omega a\left(\nabla_\xi \mathcal{L}(u, \nabla u) \cdot \nabla u + u D_s\mathcal{L}(u, \nabla u)\right)dx$$

for every $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega}; \mathbb{R}^N)$.

Corollary 2.3. *Let $\lambda_p(D)$ be the principal eigenvalue of (1), and u the unique corresponding eigenfunction. Then:*

$$(13) \qquad \frac{1}{p}\int_{\partial D} |\nabla u|^p (x \cdot \nu)d\mathcal{H}^{N-1} = \lambda_p(D)\int_D u^p dx.$$

Proof. The equation (13) easily follows from Lemma 2.2 by setting:

$$\mathcal{L}(s, \xi) = \frac{1}{p}(|\xi|^p + s^p), \quad h = x, \quad a = \frac{p(p-1)\lambda_p(D) + 2N - p}{2p}. \qquad \square$$

We are now ready to state the main result of this section.

Theorem 2.4. *If equality holds in the Faber–Krahn inequality (3), then D is a translation of B. In particular, D must be a ball.*

Remark 2.5. In [CCH96], the authors give an elementary proof of Theorem 2.4 for the case $p = 2$. Their proof uses two key tools: one is the domain derivative and the other the Pohozaev identity. We use the same approach

for the general case including $p \neq 2$. To this end, we derive a formula for the domain derivative of $\lambda_p(D)$, and recall a version of the generalized Pohozaev identity that fits the structure of the differential operator in (1). We should mention that one can find various versions of the generalized Pohozaev identity in the literature, which differ from each other depending on the structure of the differential operator and the regularity of the solution. In many versions they require C^2 regularity of the solution for the identity to hold, but this restriction can not be used for the p-Laplace eigenproblem (1) when $p \in (1,2)$ as in this case the best regularity is $C^{1,\alpha}$. Luckily, there is a version which is suitable for our case in which the regularity requirement is merely $C^1(\overline{D})$.

Proof of Theorem 2.4. We begin by assuming $\lambda_p(\hat{D}) = \lambda_p(B)$. Recalling the Faber–Krahn inequality (3), we infer $\hat{D}$ is a minimizer of $\lambda_p(D)$ relative to $\mathcal{A} := \{D \subseteq \mathbb{R}^n : |D| = |B|\}$. Whence an application of the Lagrange multiplier method yields:

$$(14) \qquad \lambda'_p(\hat{D}) = \gamma \, \mathrm{Vol}'(\hat{D}), \quad \text{(for any vector field } V),$$

where $\mathrm{Vol}(\cdot)$ denotes the volume operator. For the left hand side of (14), we can use the formula (6). The right-hand side of (14) is computed easily as follows. First of all:

$$\frac{\mathrm{Vol}(\hat{D}^t) - \mathrm{Vol}(\hat{D})}{t} = \frac{1}{t}\left(\int_{\hat{D}^t} dx - \int_{\hat{D}} dx\right)$$
$$= \frac{1}{t}\left(\int_{\hat{D}} |\det(J)|\, dx - \int_{\hat{D}} dx\right),$$

in which J stands for the Jacobian of the map ϕ_t. So,

$$\det(J) = \det\left(\delta_{ij} + t\frac{\partial V_i}{\partial x_j}\right),$$

in which δ is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

Since $\det(J) = 1 + t\,\nabla \cdot V + O(t^2)$, we deduce

$$\mathrm{Vol}'(\hat{D}) = \lim_{t \to 0^+} \frac{\mathrm{Vol}(\hat{D}^t) - \mathrm{Vol}(\hat{D})}{t} = \int_{\hat{D}} \nabla \cdot V\, dx = \int_{\partial\hat{D}} V \cdot \nu\, d\sigma.$$

From the above, we can see that (14) can be rewritten as follows:

$$(15) \qquad (1-p)\int_{\partial\hat{D}} |\nabla u|^p V \cdot \nu\, d\sigma = \gamma \int_{\partial\hat{D}} V \cdot \nu\, d\sigma \quad (\forall V).$$

Clearly, (15) implies $|\nabla u| = c$ on $\partial\hat{D}$, where c is a positive constant. Note that $\partial u/\partial\nu = -|\nabla u|$ on $\partial\hat{D}$, hence $|\nabla u|^{p-2}(\partial u/\partial\nu) = -c^{p-1}$ on $\partial\hat{D}$. So, by

integration over $\partial\hat{D}$ we obtain

$$(16) \qquad \int_{\partial\hat{D}} |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}\, d\sigma = -c^{p-1}\|\partial\hat{D}\|,$$

where $\|\partial\hat{D}\|$ denotes the surface measure of $\partial\hat{D}$. An application of the Divergence theorem to the left hand side of (16), and recalling the differential equation satisfied by u, yields:

$$(17) \qquad \lambda\int_{\hat{D}} u^{p-1}\, dx = c^{p-1}\|\partial\hat{D}\|,$$

where we have written λ instead of $\lambda_p(\hat{D})$, for simplicity. On the other hand, Corollary 2.3 coupled with the fact that $|\nabla u| = c$ on $\partial\hat{D}$ imply:

$$\frac{1}{p}c^p n|\hat{D}| = \lambda\int_{\hat{D}} u^p\, dx = \lambda,$$

since $\int_{\hat{D}} u^p\, dx = 1$. Whence, $p\lambda = c^p n|\hat{D}|$, which together with (17) would give:

$$(18) \qquad \lambda^{\frac{p}{p-1}}\left(\int_{\hat{D}} u^{p-1}\, dx\right)^{\frac{p}{p-1}} = c^p\|\partial\hat{D}\|^{\frac{p}{p-1}}.$$

From (18) and $p\lambda = c^p n|\hat{D}|$, we obtain

$$(19) \qquad \lambda^{\frac{1}{p-1}}\left(\int_{\hat{D}} u^{p-1}\, dx\right)^{\frac{p}{p-1}} = p\frac{\|\partial\hat{D}\|^{\frac{p}{p-1}}}{n|\hat{D}|}.$$

It is worth noting that (19) is a generic identity satisfied for any $u \in W_0^{1,p}(\hat{D})$ such that

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} & \text{in } \hat{D} \\ u = 0 \text{ and } |\nabla u| = c & \text{on } \partial\hat{D} \\ \int_{\hat{D}} u^p\, dx = 1. \end{cases}$$

From the Pólya–Szygo inequality (see, e.g., [BZ88]), we obtain

$$(20) \qquad \lambda_p(\hat{D}) = \lambda_p(B) \le \int_B |\nabla u^*|^p\, dx \le \int_{\hat{D}} |\nabla u|^p\, dx = \lambda_p(\hat{D}).$$

Here, u^* denotes the Schwarz symmetrization of u (see [Kaw85]). Observe that all inequalities in (20) must in fact be equalities. So, since $\int_B u^{*p}\, dx = 1$, we infer that u^* is the unique eigenfunction corresponding to $\lambda_p(B)$. Since u^* is radial, $|\nabla u^*|$ must be constant on ∂B. Therefore, the identity (19) is

applicable to u^* as well, which in turn implies:

$$
\begin{aligned}
(21) \qquad p\frac{\|\partial \hat{D}\|^{\frac{p}{p-1}}}{n|\hat{D}|} &= \lambda_p(\hat{D})^{\frac{1}{p-1}} \left(\int_{\hat{D}} u^{p-1} \, dx \right)^{\frac{p}{p-1}} \\
&= \lambda_p(B)^{\frac{1}{p-1}} \left(\int_B u^{*p-1} \, dx \right)^{\frac{p}{p-1}} \\
&= p\frac{\|\partial B\|^{\frac{p}{p-1}}}{n|B|}.
\end{aligned}
$$

Since $|\hat{D}| = |B|$, (21) implies $\|\partial \hat{D}\| = \|\partial B\|$. Whence, by the classical isoperimetric inequality (see, e.g., [HLP88]), we conclude that $\hat{D}$ is a translation of B. □

3. Torsional rigidity optimization

Consider:

$$
(22) \qquad \begin{cases} -\Delta_p u = 1 & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}
$$

where $D \subseteq \mathbb{R}^N$ is a smooth bounded domain. When $p = 2$, (22) becomes the well known Saint-Venant (torsion) problem [KM93]. By

$$
u_D \in W_0^{1,p}(D) \cap C^{1,\alpha}(\overline{D})
$$

we denote the unique solution of (22). Note that u_D is positive throughout D.

3.1. The simple case. A quantity which we call the p-torsional rigidity of D, is introduced as follows:

$$
\Phi_1(D) = \int_D u_D \, dx.
$$

We are interested in the maximization problem:

$$
(23) \qquad \sup_{D \in \mathcal{A}} \Phi_1(D),
$$

in which

$$
\mathcal{A} = \{D \subseteq \mathbb{R}^N : \ D \text{ is a bounded smooth domain with } |D| = \epsilon\},
$$

for some $\epsilon > 0$.

Theorem 3.1. *The optimization problem* (23) *has a unique solution B (modulo translations), which is the ball centred at the origin with radius* $\left(\frac{\epsilon}{\omega_N}\right)^{1/N}$. *Here ω_N denotes the volume of the unit N-ball.*

Proof. We begin by showing that B is a solution of (23). To this end, let us fix $D \in \mathcal{A}$. We have:

$$(24) \qquad \Phi_1(D) = \int_D u_D \, dx = \int_B u_D^* \, dx \le \int_B u_B dx = \Phi_1(B),$$

where we have used the following inequality attributed to Giorgio Talenti [Tal79]:

$$(25) \qquad u_D^*(x) \le u_B(x), \quad (\forall x \in B).$$

Notice that u_B is the solution of (22) with D replaced by B. From (24) we infer that B is a solution of (23), as desired.

We now address the uniqueness. Let us suppose $\hat{D} \in \mathcal{A}$ is a solution of (23). To simplify notation we set $\hat{u} = u_{\hat{D}}$. So $\hat{u}$ satisfies:

$$(26) \qquad \begin{cases} -\Delta_p \hat{u} = 1 & \text{in } \hat{D} \\ \hat{u} = 0 & \text{on } \partial\hat{D}. \end{cases}$$

We use the method of domain derivatives. Fix a vector field

$$V \in C^2(\mathbb{R}^N, \mathbb{R}^N).$$

As in Section 2, $\hat{u}'$ denotes the domain derivative of $\hat{u}$ in the direction of V. Similar to (5), we derive:

$$(27) \qquad \begin{cases} \nabla \cdot \left((p-2)|\nabla\hat{u}|^{p-1} \frac{\nabla\hat{u} \cdot \nabla\hat{u}'}{|\nabla\hat{u}|^3} \nabla\hat{u} + |\nabla\hat{u}|^{p-2}\nabla\hat{u}' \right) = 0 & \text{in } \hat{D} \\ \hat{u}' = -\frac{\partial\hat{u}}{\partial\nu} \, V \cdot \nu & \text{on } \partial\hat{D}. \end{cases}$$

Multiplying the differential equation in (26) by $\hat{u}'$, integrating the result over $\hat{D}$, and finally applying the Divergence theorem, yields:

$$(28) \qquad \int_{\hat{D}} \hat{u}' \, dx = -\int_{\hat{D}} \hat{u}' \Delta_p \hat{u} \, dx$$

$$= \int_{\hat{D}} |\nabla\hat{u}|^{p-2}\nabla\hat{u} \cdot \nabla\hat{u}' \, dx - \int_{\partial\hat{D}} \hat{u}' \, |\nabla\hat{u}|^{p-2}\frac{\partial\hat{u}}{\partial\nu} \, d\mathcal{H}^{N-1}$$

$$= \int_{\hat{D}} |\nabla\hat{u}|^{p-2}\nabla\hat{u} \cdot \nabla\hat{u}' \, dx + \int_{\partial\hat{D}} |\nabla\hat{u}|^p \, (V \cdot \nu) \, d\mathcal{H}^{N-1}.$$

Let us now set:

$$W := (p-2)|\nabla\hat{u}|^{p-1}\frac{\nabla\hat{u} \cdot \nabla\hat{u}'}{|\nabla\hat{u}|^3}\nabla\hat{u} + |\nabla\hat{u}|^{p-2}\nabla\hat{u}'.$$

Hence, the boundary value problem (27) reduces to:

$$(29) \qquad \begin{cases} \nabla \cdot W = 0 & \text{in } \hat{D} \\ \hat{u}' = -\frac{\partial\hat{u}}{\partial\nu} \, (V \cdot \nu) & \text{on } \partial\hat{D}. \end{cases}$$

Multiplying the differential equation in (29) by $\hat{u}$, integrating the result over $\hat{D}$, and applying the Divergence theorem yields:

$$(30) \qquad \int_{\hat{D}} |\nabla\hat{u}|^{p-2}\nabla\hat{u} \cdot \nabla\hat{u}' \, dx = 0.$$

The combination of (28) and (30) leads to:

$$(31) \qquad \int_{\hat{D}} \hat{u}' \, dx = \int_{\partial \hat{D}} |\nabla \hat{u}|^p \, (V \cdot \nu) \, d\mathcal{H}^{N-1}.$$

The left-hand side of (31) is precisely the domain derivative of Φ_1 at $\hat{D}$ in the direction of V which we denote by $\frac{\partial \Phi_1}{\partial V}(\hat{D})$. So, we have

$$(32) \qquad \frac{\partial \Phi_1}{\partial V}(\hat{D}) = \int_{\partial \hat{D}} |\nabla \hat{u}|^p \, (V \cdot \nu) \, d\mathcal{H}^{N-1}.$$

Recall that $\hat{D}$ is assumed to be a solution of (23), hence we can apply the Lagrange multiplier method to deduce existence of a constant C, independent of V, such that

$$(33) \qquad \frac{\partial \Phi_1}{\partial V}(\hat{D}) = C \, \frac{\partial \, \mathrm{Vol}}{\partial V}(\hat{D}),$$

where "Vol" stands for the volume operator. By using the Hadamard formula:

$$(34) \qquad \frac{\partial \, \mathrm{Vol}}{\partial V}(\hat{D}) = \int_{\partial \hat{D}} V \cdot \nu \, d\mathcal{H}^{N-1},$$

From (32), (33), and (34) we obtain:

$$(35) \qquad \int_{\partial \hat{D}} |\nabla \hat{u}|^{p-2} V \cdot \nu \, d\sigma = C \int_{\partial \hat{D}} V \cdot \nu \, d\mathcal{H}^{N-1}.$$

Since (35) holds for every vector field V, we deduce

$$(36) \qquad |\nabla \hat{u}| = C^{1/(p-2)} \qquad \text{on } \partial \hat{D}.$$

By joining (36) to (26), we derive the system

$$(37) \qquad \begin{cases} -\Delta_p \hat{u} = 1 & \text{in } \hat{D} \\ \hat{u} = 0 & \text{on } \partial \hat{D} \\ |\nabla \hat{u}| = C^{1/(p-2)} & \text{on } \partial \hat{D}. \end{cases}$$

This system is an overdetermined problem that has been investigated in [GL89], where the authors prove (37) is solvable if and only if D is a ball. This completes the proof of the theorem. $\qquad \square$

3.2. A generalization. The main result of this subsection is again related to the boundary value problem (22).

Let $F \in C[0, \infty)$ be C^1 over $(0, \infty)$ and assume that is satisfies $F' \geq a > 0$ on $(0, \infty)$. Consider the quantity:

$$\Phi_2(D) = \int_D F(u_D) \, dx,$$

where u_D is the unique solution of (22). We are interested in the following maximization problem:

$$(38) \qquad \sup_{D \in \mathcal{A}} \Phi_2(D),$$

where $\mathcal{A} = \{D \subseteq \mathbb{R}^N : D \text{ is a bounded smooth domain with } |D| = \epsilon\}$. Our main result in this subsection is as follows:

Theorem 3.2. *The conclusion of Theorem* 3.1 *still holds for the maximization* (38).

It turns out that the method of domain derivatives is not applicable in the present situation. In fact, this method could only be applicable if F were an affine function, a case which we have no interest in since it will essentially be a replica of the case considered in the previous subsection. To overcome this difficulty we use the method of rearrangements. Let us first recall the following result:

Lemma 3.3 ([BZ88]). *Let* $V \in W_0^{1,p}(\mathbb{R}^N)$ *be non-negative with compact support, and let* $M := \text{ess sup } V$ *(which may be infinity). Then:*

    (i) *The following inequality holds:*

$$(39) \qquad \int_{\mathbb{R}^N} |\nabla V|^p dx \geq \int_{\mathbb{R}^N} |\nabla V^*|^p dx.$$

    (ii) *If equality holds in* (39), *then for all* $0 \leq \alpha < M$, *the set* $V^{-1}(\alpha, \infty)$ *is a translation of the ball* $V^{*-1}(\alpha, \infty)$.

Proof of Theorem 3.2. Proving that B is a solution of (38) is easy. Indeed,

$$(40) \qquad \Phi_2(D) = \int_D F(u_D) \, dx$$

$$= \int_B F(u_D^*) \, dx \leq \int_B F(u_B) \, dx$$

$$= \Phi_2(B), \quad (\forall D \in \mathcal{A}),$$

where we have used the monotonicity of F and Talenti's inequality (25). Thus, B is a solution of (38), as desired.

Now, we address the question of uniqueness. To this end, suppose $D \in \mathcal{A}$ is a solution of (38). Whence, the inequality in (40) would in fact be equality for the optimal solution D. Thus,

$$(41) \qquad \int_B (F(u_B) - F(u_D^*)) \, dx = 0.$$

Since $\forall x \in B : F(u_B(x)) - F(u_D^*(x)) = F'(c(x)) \, (u_B(x) - u_D^*(x))$, from (41) we obtain:

$$(42) \qquad \int_B F'(c(x)) \, (u_B(x) - u_D^*(x)) \, dx = 0.$$

Recalling $F' \geq a > 0$, from (42), we get:

$$(43) \qquad a \int_B (u_B - u_D^*) \, dx \leq 0.$$

On the other hand, $u_D^* \le u_B$, so (43) implies $u_B = u_D^*$, throughout B. Note that:

$$(44) \qquad |D| = \int_D |\nabla u_D|^p dx \ge \int_B |\nabla u_D^*|^p dx = \int_B |\nabla u_B|^p dx = |B|,$$

where we have used (39). Since $|D| = |B|$, the inequality in (44) is in fact equality. So, in particular we obtain: $\int_D |\nabla u_D|^p dx = \int_B |\nabla u_D^*|^p dx$. Whence, by Lemma 3.3, we deduce $D = u_D^{-1}(0, \infty)$ is a translation of $u_D^{*-1}(0, \infty) = B$. Hence, D is a ball, as desired. $\qquad\square$

4. An optimization related to Hamilton–Jacobi problems

In this section we consider an optimization problem in the same spirit as the one considered in the previous section. This new optimization is related to the following Hamilton–Jacobi problem:

$$(45) \qquad \begin{cases} K(|\nabla u|) = 1 & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}$$

where $K : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. From [GN84] we know that (45) has a solution $u \in W^{1,\infty}(D) \cap C(\overline{D})$ but it may not be unique. Henceforth, by a solution of (45) we understand a non-negative solution, which always exists simply because of the identity $K(\,|\nabla|u|\,|) = K(|\nabla u|)$. Next, we define the quantity:

$$(46) \qquad \Phi_3(D) = \int_D F(u_D)\, dx,$$

where u_D is a solution of (45), and F is the same function as the one in the previous section. The optimization problem we are interested in is the following:

$$(47) \qquad \sup_{D \in \mathcal{A}} \Phi_3(D),$$

in which $\mathcal{A}$ is the same as the set in the previous section. Our main result in this section is:

Theorem 4.1. *The maximization problem* (47) *has the unique solution B (modulo translations), which is the ball centred at the origin with radius* $\left(\frac{\epsilon}{\omega_N}\right)^{\frac{1}{N}}$.

We are going to apply the rearrangement methods to prove Theorem 4.1. But first we need to derive a Talenti type inequality that fits our new circumstances.

Lemma 4.2. *Suppose u_D is a solution of* (45). *Then:*

$$(48) \qquad u_D^*(x) \le u_B(x), \quad x \in B.$$

Proof. Recall the distribution function of $u \equiv u_D$:

$$\lambda_u(t) = |\{x \in D : u(x) \geq t\}|.$$

Then:

(49) $\quad \lambda_u(t) = \dfrac{1}{K^{-1}(1)} \displaystyle\int_{u \geq t} |\nabla u| \, dx = \dfrac{1}{K^{-1}(1)} \int_t^M \left(\int_{u = \tau} d\mathcal{H}^{N-1} \right) d\tau,$

by the co-area formula (see, e.g., [Mor09]). From (49), we readily obtain:

(50) $\quad -\lambda_u'(t) = \dfrac{1}{K^{-1}(1)} \| \partial \{u \geq t\} \| \geq \dfrac{1}{K^{-1}(1)} N \omega_N^{\frac{1}{N}} \lambda_u^{1 - \frac{1}{N}}(t)$

where we have used the isoperimetric inequality. From (50), after integrating from 0 to t, we obtain:

(51)
$$t \leq \frac{K^{-1}(1)}{N \omega_N^{1/N}} \int_0^t \frac{-\lambda_u'(s)}{\lambda_u^{1 - 1/N}(s)} \, ds$$

$$= \frac{K^{-1}(1)}{N \omega_N^{1/N}} \int_{\lambda_u(t)}^{|D|} \frac{d\tau}{\tau^{1 - 1/N}}$$

$$= K^{-1}(1) \left(R - \left(\frac{\lambda_u(t)}{\omega_N} \right)^{\frac{1}{N}} \right).$$

Finally, by substituting $t = u^*(x)$ in (51), noting that $\lambda_u(u^*(x)) = \omega_N |x|^N$, we infer:

$$u^*(x) \leq K^{-1}(1) \left(R - |x| \right) \equiv u_B(x), \quad x \in B,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We have:

(52) $\quad \Phi_3(D) = \displaystyle\int_D F(u_D) dx = \int_B F(u_D^*) dx \leq \int_B F(u_B) dx$

$$= \Phi_3(B), \quad \forall D \in \mathcal{A}.$$

So, B is a solution of (47). Note that in (52), we have used Lemma 4.2, and the monotonicity of F.

To prove uniqueness, we assume $D \in \mathcal{A}$ is a solution of (47). Similar to the argument in the previous section, we deduce $u_D^* = u_B$. Therefore:

(53)
$$|D| = \frac{1}{(K^{-1}(1))^p} \int_D |\nabla u_D|^p \, dx$$

$$\geq \frac{1}{(K^{-1}(1))^p} \int_B |\nabla u_D^*|^p \, dx$$

$$= \frac{1}{(K^{-1}(1))^p} \int_B |\nabla u_B|^p dx$$

$$= |B|.$$

Since $|D| = |B|$, we see that the inequality in (53) is in fact equality. So, in particular, we obtain:

$$\int_D |\nabla u_D|^p dx = \int_B |\nabla u_D^*|^p dx.$$

Whence, from Lemma 3.3, we see that $D = u^{-1}(0, \infty)$ must be a translation of $B = u_B^{-1}(0, \infty)$. This completes the proof of the theorem. $\qquad\square$

References

[BZ88]   Brothers, John E.; Ziemer, William P. Minimal rearrangements of Sobolev functions. *J. Reine Angew. Math.* **384** (1988), 153–179. MR0929981 (89g:26013), Zbl 0633.46030, doi: 10.1515/crll.1988.384.153.

[CCH96] Chatelain, T.; Choulli, M.; Henrot, A. Some new ideas for a Schiffer's conjecture. *Modelling and optimization of distributed parameter systems*, 90–97. *Chapman & Hall, New York*, 1996. MR1388521 (97a:35038), Zbl 0881.35083, doi: 10.1007/978-0-387-34922-0_7.

[DMS03] Degiovanni, Marco; Musesti, Alessandro; Squassina, Marco. On the regularity of solutions in the Pucci–Serrin identity. *Calc. Var. Partial Differential Equations* **18** (2003), 317–334. MR2018671 (2005d:35067), Zbl 1046.35039, doi: 10.1007/s00526-003-0208-y.

[EZR08] Emamizadeh, B.; Zivari-Rezapour, M. Monotonicity of the principal eigenvalue of the p-Laplacian in an annulus. *Proc. Am. Math. Soc.* **136** (2008), no. 5, 1725–1731. MR2373602 (2009b:35105), Zbl 1148.35060, doi: 10.1090/S0002-9939-07-09153-8.

[GL89]   Garofalo, Nicola; Lewis, John L. A symmetry result related to some overdetermined boundary value problems. *Amer. J. Math.* **111** (1989), no. 1, 9–33. MR0980297 (90d:35212), Zbl 0681.35016, `http://www.jstor.org/stable/2374477`.

[GN84]   Giarrusso, E.; Nunziante, D. Symmetrization in a class of first-order Hamilton-Jacobi equations. *Nonlinear Anal.* **8** (1984), no. 4, 289–299. MR0739660 (85e:35026), Zbl 0543.35014, doi: 10.1016/0362-546X(84)90031-2.

[Hen06] Henrot, Antoine. Extremum problems for eigenvalues of elliptic operators. Frontiers in Mathematics. *Birkhäuser Verlag, Basel*, 2006. x+202 pp. ISBN: 978-3-7643-7705-2; 3-7643-7705-4. MR2251558 (2007h:35242), Zbl 1109.35081, doi: 10.1007/3-7643-7706-2.

[HLP88] Hardy, G. H.; Littlewood, J. E.; Pólya, G. Inequalities. Reprint of the 1952 edition. Cambridge Mathematical Library. *Cambridge University Press, Cambridge*, 1988. xii+324 pp. ISBN: 0-521-35880-9. MR0944909 (89d:26016), Zbl 0634.26008.

[Kaw85] Kawohl, Bernhard. Rearrangements and convexity of level sets in PDE. Lecture Notes in Mathematics, 1150. *Springer-Verlag, Berlin*, 1985. iv+136 pp. ISBN: 3-540-15693-3. MR0810619 (87a:35001), Zbl 0593.35002, doi: 10.1007/BFb0075060.

[KM93]   Keady, Grant; McNabb, Alex. The elastic torsion problem: solutions in convex domains. *New Zealand J. Math.* **22** (1993), no. 2, 43–64. MR1244022 (94k:73019), Zbl 0814.35133.

[Mor09] Morgan, Frank. Geometric measure theory. A beginner's guide. Fourth edition. *Elsevier/Academic, Amsterdam*, 2009. viii+249 pp. ISBN: 978-0-12-374444-9. MR2455580 (2009i:49001), Zbl 1179.49050.

[Sim80] Simon, J. Differentiation with respect to the domain in boundary value problems. *Numer. Funct. Anal. Optim.* **2** (1980), no. 7–8, 649–687. MR0619172 (83m:49032), Zbl 0471.35077, doi: 10.1080/01630563.1980.10120631.

[Tal79] Talenti, Giorgio. Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. *Ann. Mat. Pura Appl* (4) **120** (1979), 160–184. MR0551065 (81i:35068), Zbl 0419.35041, doi: 10.1007/BF02411942.

Faculty of Science and Engineering, University of Nottingham Ningbo China, 199 Taikang East Road, Ningbo, 315100, China
Amin.Farjudian@nottingham.edu.cn

Behrouz.Emamizadeh@nottingham.edu.cn