

The signature of rational links

Khaled Qazaqzeh, Isam Al-Darabsah
and Aisheh Quraan

ABSTRACT. We give an explicit formula for the signature of any rational link in terms of the denominators of the canonical continued fraction of its slope.

CONTENTS

1. Introduction	183
2. Rational links and continued fraction	184
3. The Gordon–Litherland formula of the link signature	186
4. Main results	187
4.1. The signature of the Goeritz matrix $\sigma(G)$	188
4.2. The correction term $\mu(D)$	188
5. Examples and computer talk	190
References	193

1. Introduction

Goeritz defined some algebraic invariants of a knot type from a quadratic form that is obtained from a diagram of the given knot in [G]. The knot signature, however, is defined later by Trotter in [T] using a different notion of quadratic form. Murasugi in [M1] generalizes the work of Trotter for the case of links. The authors of [GL] defined a quadratic form that simultaneously generalizes the forms of Goeritz and Murasugi and relate the signature of this form to the signature of links. In particular, they describe a combinatorial way to calculate the signature of a given link from any link diagram.

Many people have computed this invariant for families of links. In particular, Shinohara in [Sh] has computed the signature of any rational link with slope $\frac{\alpha}{\beta}$ in terms of α and β . Later, the authors of [GJ] have computed the signature of the Goeritz matrix of any rational link. The signature of the Goeritz matrix is one of the two terms to compute the signature of any link.

Received December 8, 2013.

2010 *Mathematics Subject Classification.* 57M27.

Key words and phrases. Rational links, link signature.

The other term (the correction term) has been computed in [GJ] in many cases for rational links. The work in [GJ] is in terms of the denominators of any continued fraction of the slope of the given rational link.

In this paper, we give a formula to compute the signature of any rational link in terms of the denominators of the canonical continued fraction of the its slope that is given in the following theorem:

Theorem 1.1. *For the rational link L with canonical continued fraction $[b_1, b_2, \dots, b_n]$ of its slope $\frac{\alpha}{\beta}$, we have*

$$\begin{aligned} \sigma(L) &= \sigma(G) - \mu(D) \\ &= \sum_{i=1}^{\frac{n-1}{2}} b_{2i} + 1 - \sum_{i=1}^n \delta_i b_i, \end{aligned}$$

where δ_i is defined recursively in Lemma 4.4 and Proposition 4.5.

The above theorem generalizes the work of the authors in [GJ] and it gives another formula other than the one in [Sh] to compute the signature of any rational link in terms of the denominators of the canonical continued fraction of its slope.

The idea of computing the correction term in this paper can be applied to compute the correction term of links of braid index 3 that is one of the two terms of computing the signature.

Finally, we give a code for the above formula using Mathematica and we provide a table of knots of Rolfsen’s table in [BM] with their signature. We use the formula in the above theorem to confirm the signature of these knots.

2. Rational links and continued fraction

A continued fraction of the rational number $\frac{\alpha}{\beta}$ is a sequence of integers b_1, b_2, \dots, b_n such that

$$\frac{\alpha}{\beta} = b_1 + \frac{1}{b_2 + \frac{1}{\dots + \frac{1}{b_n}}}$$

The integers b_i are called the denominators of the continued fraction of the rational number $\frac{\alpha}{\beta}$. This continued fraction of $\frac{\alpha}{\beta}$ will be abbreviated by $[b_1, b_2, \dots, b_n]$.

A diagram of a rational link can be constructed from the denominators of any continued fraction of its slope that is a rational number of a pair of relatively prime integers α, β with $|\frac{\alpha}{\beta}| > 1$ and $\beta > 0$ by closing the 4-braid $\sigma_1^{b_1} \sigma_2^{-b_2} \sigma_1^{b_3} \dots$ in the manner shown in Figure 1, where σ_1, σ_2 are shown in Figure 2 and the multiplication is defined by concatenating from left to right. It is well known that for odd numerator α this diagram represents a

knot and for even numerator α it represents a two component link. A link diagram obtained from this construction is a diagram of the rational link $L_{\alpha/\beta}$ and it is characterized by the following theorem due to Schubert [S].

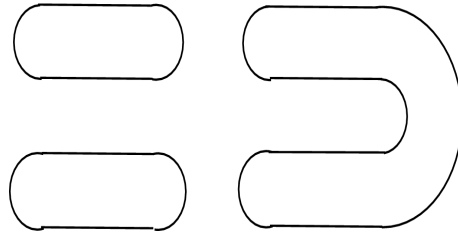


FIGURE 1. The closure of the 4-braid based on n being odd or even respectively

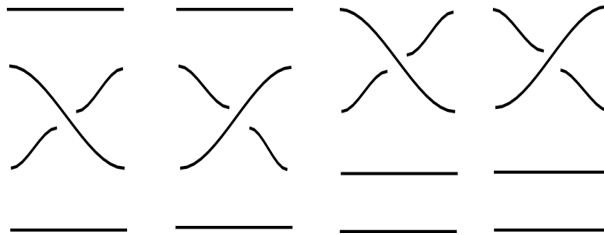


FIGURE 2. The 4-braids $\sigma_1, \sigma_1^{-1}, \sigma_2,$ and σ_2^{-1} respectively

Theorem 2.1. *Two rational links $L_{\alpha/\beta}$ and $L_{\alpha'/\beta'}$ are equivalent if and only if*

$$\alpha = \alpha',$$

$$\text{and } \beta^{\pm 1} \equiv \pm \beta' \pmod{\alpha}.$$

It is sufficient to consider the case when the number of denominators of the continued fraction n is odd and $b_i \geq 1$ for $i = 1, 2, \dots, n$ as a result of the following lemma.

Lemma 2.2. *There exists a canonical choice of continued fraction of $\frac{\alpha}{\beta} > 1$ of positive integers with n odd and $b_i \geq 1$ for $i = 1, 2, \dots, n$.*

Proof. We start with the rational number $\frac{\alpha}{\beta} > 1$ such that $\gcd(\alpha, \beta) = 1$ and $\alpha > \beta > 0$. Now we can apply the Euclidean algorithm to get

$$\begin{aligned} \alpha &= \beta b_1 + q_1, & 0 < q_1 < \beta \\ \beta &= q_1 b_2 + q_2, & 0 < q_2 < q_1 \\ q_1 &= q_2 b_3 + q_3, & 0 < q_3 < q_2 \\ &\vdots \\ q_{n-3} &= q_{n-2} b_{n-1} + q_{n-1}, & 0 < q_{n-1} < q_{n-2} \\ q_{n-2} &= q_{n-1} b_n. \end{aligned}$$

Now we have

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{\beta b_1 + q_1}{\beta} = b_1 + \frac{q_1}{\beta} = b_1 + \frac{1}{\frac{\beta}{q_1}} \\ &= b_1 + \frac{1}{b_2 + \frac{q_1}{q_2}} = \dots = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{\ddots + \frac{1}{b_n}}}}. \end{aligned}$$

In this way we get a unique continued fraction $[b_1, b_2, \dots, b_n]$ with $b_n \geq 2$ since $q_{n-1} < q_{n-2}$. Finally, if n is even then $[b_1, b_2, \dots, b_n - 1, 1]$ is the continued fraction with odd number of denominators. This continued fraction is unique as a result of the applying the Euclidean algorithm at each step. \square

Definition 2.3. The unique continued fraction obtained using the above lemma will be called the canonical continued fraction of $\frac{\alpha}{\beta}$ and the diagram obtained from the canonical continued fraction will be called the canonical diagram of the rational link whose slope is $\frac{\alpha}{\beta}$. It is easy to see that the canonical diagram is alternating.

Remark 2.4. The motivation behind the above definition and lemma is the work of the authors in [KL, Section. 2] for rational tangles.

3. The Gordon–Litherland formula of the link signature

We recall the Gordon–Litherland formula for link signature that was first introduced in [GL]. We color the regions of the complement of the diagram D of the oriented link L in \mathbb{R}^2 in a checkerboard fashion and denote the white regions by R_0, R_1, \dots, R_m . To each crossing c that is incident to two distinct white regions, we assign an incidence number $\mu(c)$ and type as shown in the Figure 3.

Let G be the associated symmetric integral Goeritz matrix of the diagram D that was first defined in [G] as follows:

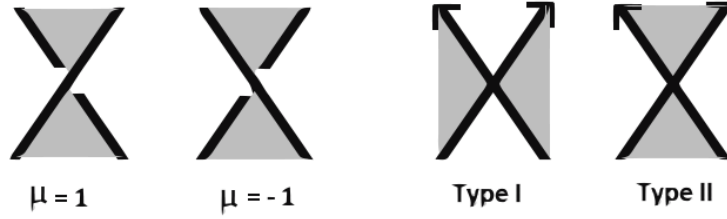


FIGURE 3. Incidence numbers and types of crossings

For any $0 \leq i \neq j \leq m$, let

$$g_{ij} = - \sum_{c \in R_i \cap R_j} \mu(c), \quad \text{and} \quad g_{ii} = - \sum_{i \neq j} g_{ij}.$$

Then G is the symmetric $m \times m$ matrix with entries g_{ij} for $1 \leq i, j \leq m$. We set

$$\mu(D) = \sum_{c \text{ of type II}} \mu(c).$$

Finally, the Gordon–Litherland formula for the link signature is

$$(1) \quad \sigma(L) = \sigma(G) - \mu(D),$$

where $\sigma(G), \mu(D)$ is the signature of G and correction term of the diagram D of L respectively.

4. Main results

For this section, we let L to be the rational link with slope $\frac{\alpha}{\beta}$ of canonical continued fraction $[b_1, b_2, \dots, b_n]$. We only consider the case with $\frac{\alpha}{\beta} > 1$ since the signature of the mirror image of any link is the opposite of the signature of the original link. Also, we let D the canonical diagram of the rational link L . Now, we color the regions of the diagram D such that the outside region is black and the most bottom region is white and numbered R_0 and the other white regions are numbered from left to right. Now we state the main theorem of this paper whose proof covers the rest of this section.

Theorem 4.1. *For the rational link L , we have*

$$\begin{aligned} \sigma(L) &= \sigma(G) - \mu(D) \\ &= \sum_{i=1}^{\frac{n-1}{2}} b_{2i} + 1 - \sum_{i=1}^n \delta_i b_i, \end{aligned}$$

where δ_i is defined recursively in Lemma 4.4 and Proposition 4.5.

We divide this section to two subsections the first to compute the signature of the Goeritz matrix, $\sigma(G)$, and the second is to compute the correction term, $\mu(D)$.

4.1. The signature of the Goeritz matrix $\sigma(G)$. In this subsection we want to compute the signature of the Goeritz matrix using the above coloring. It is easy to see that in this case we have the number of white regions $m + 1 = \sum_{i=1}^{\frac{n-1}{2}} b_{2i} + 2$. The following lemma gives us the general form of the matrix G for the canonical rational link diagram.

Lemma 4.2. *The associated Goeritz matrix of the canonical diagram D is an $m \times m$ square symmetric matrix with upper and lower diagonal entries are -1 , the entries of the main diagonal are*

$$b_1 + 1, \underbrace{2, 2, \dots, 2}_{(b_2-1)\text{-times}}, b_3 + 2, \underbrace{2, 2, \dots, 2}_{(b_4-1)\text{-times}}, \dots, b_n + 1$$

and the other entries are 0.

Proposition 4.3. *If A is an $l \times l$ square symmetric matrix with upper and lower diagonal entries are -1 , $a_{11} > 1$, $a_{ii} \geq 2$ for $2 \leq i \leq l$, and the other entries are 0, then A has signature equal to l . In particular, the above Goeritz matrix has signature equal to m .*

Proof. We use induction on l . We form the unimodular matrix B that consists of two blocks the first block is the 2×2 -matrix $\begin{pmatrix} 1 & 0 \\ \frac{1}{a_{11}} & 1 \end{pmatrix}$ and the second block is the $(l - 2) \times (l - 2)$ identity matrix. Now the product BAB^T yields a matrix with the same signature by using Sylvester law in [SJ] that consists of two blocks the first is the matrix of only one entry a_{11} , and the second matrix is an $(l - 1) \times (l - 1)$ -matrix that satisfies the same conditions stated in the proposition. Now the result follows by applying the induction hypothesis on the second block since $a_{22} - \frac{1}{a_{11}} > 1$. \square

4.2. The correction term $\mu(D)$. The set of all crossings in the canonical diagram D forms a partition of n elements such that i -th element of this partition contains all the crossings that form $\sigma_1^{b_i}$ if i is odd and $\sigma_2^{-b_i}$ if i is even in the braid form. It is easy to see that all crossings that belong to the same element of the partition will have the same type according to Figure 3. Moreover, the type of all crossings in the i -th element of the partition depends on the types of the crossings in the $(i - 2)$ -th and $(i - 1)$ -th elements of the partition. To simplify things, we use the notation $\delta_i = 1$ if the crossings of the i -th element of the partition is of type two and in the other case $\delta_i = 0$. Now using this notation, we can give a recursive relation that defines the type of the crossings in the i -th element of the partition in the following lemma:

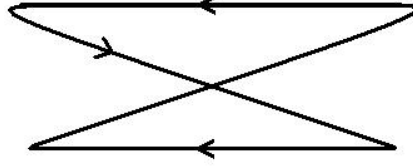


FIGURE 4. The closure of the diagram D when $\sigma_L = (23)$

Lemma 4.4. *For $i \geq 3$, we have*

$$\delta_i = \begin{cases} \delta_{i-2} + 1 \pmod{2}, & \text{if } b_{i-1} \text{ is odd, } i - 1 \text{ is even and } \delta_{i-1} \text{ is odd,} \\ \delta_{i-2} + 1 \pmod{2}, & \text{if } b_{i-1} \text{ is odd, } i - 1 \text{ is odd and } \delta_{i-1} \text{ is even,} \\ \delta_{i-2}, & \text{otherwise.} \end{cases}$$

Proof. Let D' be the associated link diagram obtained by the canonical continued fraction of n denominators with the i -th denominator equal to 1 if b_i is odd and 2 if b_i is even. It is easy to see that δ_i can be computed from the diagram D' . The crossings in the $i - 2, i - 1, i$ -th elements of the above partition of the crossings for the diagram D' formed by only three arcs. Now the above recursive relation follows by considering all possible orientations on these three arcs and all possible values of the $i - 2, i - 1$ and the i -th denominators of the diagram D' . \square

We want to compute the value of the correction term $\mu(D)$ for the canonical diagram D in terms of the denominators of the canonical continued fraction of $\frac{\alpha}{\beta}$. It is worth mentioning that any crossing of the i -th element of the partition in the canonical diagram D of type two will add one to the correction term $\mu(D)$ according to the above coloring and Figure 3 in both cases of i being odd or even.

The value of $\mu(D)$ depends on an associated permutation $\sigma_L \in S_3$ on the set $\{1, 2, 3\}$. This permutation is defined in terms of the denominators of the canonical continued fraction by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{b_1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{b_2} \dots \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{b_{2k+1}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \sigma_L(1) \\ \sigma_L(2) \\ \sigma_L(3) \end{pmatrix}.$$

Another way of defining this permutation is given by

$$\sigma_L = (23)^{b_{2k+1}}(12)^{b_{2k}}(23)^{b_{2k-1}} \dots (12)^{b_2}(23)^{b_1},$$

with multiplication in S_3 defined from left to right.

Proposition 4.5. *The correction term of the canonical diagram is*

$$\mu(D) = \sum_{i=1}^n \delta_i b_i$$

with:

- (1) $\delta_1 = \delta_2 = 1$ if $\sigma_L = (1), (123), (23)$ or (12) .
- (2) $\delta_1 = 0, \delta_2 = 1$ if $\sigma_L = (13)$, or (132) and b_1 is even.
- (3) $\delta_1 = 0, \delta_2 = 0$ if $\sigma_L = (13)$, or (132) with b_1 is odd.

In all of the above cases, δ_i is as defined in Lemma 4.4 for $i \geq 3$.

Proof. We prove the case where $\sigma_L = (23)$. In this case the canonical diagram of the rational link will be connected as in Figure 4. We choose the orientation in all cases in a way where the top arc always goes from right to left and if the diagram has two components then we can assume the orientation on the bottom arc goes from right to left since these two arcs will belong to different components.

It is clear to see that the crossings of the first two elements of the above partition are of type two. Now we use Lemma 4.4 to compute the correction term $\mu(D)$. \square

5. Examples and computer talk

In this section, we work out some examples and at the end we give the code that computes the signature for some of the rational knots in Rolfsen's table (see [BM] for that table).

Example 5.1. We compute the signature of the knot 8_4 in Rolfsen's table. It is known that the slope of this rational knot is $\frac{19}{5}$ with canonical continued fraction is given by $b_1 = 3, b_2 = 1$, and $b_3 = 4$. Hence we have $\sigma_{8_4} = (123)$ with $\delta_1 = \delta_2 = 1$, and $\delta_3 = 0$. Now we have $\mu(D) = 4$ and $\sigma(G) = 2$. Therefore, we obtain $\sigma(8_4) = -2$.

Example 5.2. We compute the signature of the rational link L whose slope is $\frac{75}{29}$ with canonical continued fraction is given by $b_1 = 2, b_2 = 1, b_3 = 1, b_4 = 2, b_5 = 2, b_6 = 1$, and $b_7 = 1$. Hence we have $\sigma_L = (123)$ with $\delta_1 = \delta_2 = 1$ and $\delta_i = 0$ for $i = 3, 4, 5, 6, 7$. Now we have $\mu(D) = 3$ and $\sigma(G) = 5$. Therefore, we obtain $\sigma(L) = \sigma(G) - \mu(D) = 5 - 3 = 2$.

Now we want to compute the signature of the above link by the formula given in [Sh, Theorem. 2]. The link L is a rational link of type (α, β) with $\alpha = 75$ and $\beta = 29$. Now consider the sequence $\beta, 2\beta, \dots, (\alpha - 1)\beta$ and take the remainder of dividing each element of this sequence by 2α to obtain a remainder r such that $-\alpha < r < \alpha$. We obtain a new sequence $\{r_1, r_2, \dots, r_{\alpha-1}\}$ where $r_i \neq 0$ for $i = 1, 2, \dots, \alpha - 1$. In particular for this


```

In[81]= ClearAll["Global`*"]
Clear[r]
r =;
s = Abs[r]; Print["s = ", s]
If[s >= 1, i = 1,
{ Print[" Enter r ≥ 1 or r ≤ -1"], Break[]}];
While[s ≠ Floor[s],
{b[i] = Floor[s], s = s - Floor[s], If[s ≠ 0, s = 1/s];
i++
}];
b[i] = Floor[s];
Which[i == 1,
{σ = 1 - b[i]; Print["b(1) = ", b[i]]; Print["σ(L) = ", σ]; Break[]},
Mod[i, 2] == 1, σ = Sum[b[2*j], {j, 1, ((i - 1) / 2)}] + 1,
Mod[i, 2] == 0,
{b[i] = b[i] - 1, b[i + 1] = 1, i = 1 + i, σ = Sum[b[2*j], {j, 1, ((i - 1) / 2)}] + 1
}];
A = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
For[j = 1, j ≤ i, j++,
{If[Mod[j, 2] == 0,
A = A.MatrixPower[{{0, 1, 0}, {1, 0, 0}, {0, 0, 1}}, b[j]],
A = A.MatrixPower[{{1, 0, 0}, {0, 0, 1}, {0, 1, 0}}, b[j]]
}];];
X = A.{{1}, {2}, {3}};
Print["X = ", X]
If[X == {{3}, {2}, {1}} || X == {{3}, {1}, {2}} && Mod[b[1], 2] == 0,
{μ = b[2]; δ[1] = 0; δ[2] = 1; For[j = 3, j ≤ i, j++, {δ[j] = δ[j - 2];
If[Mod[(j - 1), 2] == 0 && Mod[δ[j - 1], 2] == 1 && Mod[b[j - 1], 2] == 1,
δ[j] = Mod[δ[j] + 1, 2], If[Mod[(j - 1), 2] == 1 && Mod[δ[j - 1], 2] == 0 &&
Mod[b[j - 1], 2] == 1, δ[j] = Mod[δ[j] + 1, 2], δ[j] = δ[j - 2]]}}];];
If[X == {{3}, {2}, {1}} || X == {{3}, {1}, {2}} && Mod[b[1], 2] == 1,
{μ = 0; δ[1] = 0; δ[2] = 0; For[j = 3, j ≤ i, j++, {δ[j] = δ[j - 2];
If[Mod[(j - 1), 2] == 0 && Mod[δ[j - 1], 2] == 1 && Mod[b[j - 1], 2] == 1,
δ[j] = Mod[δ[j] + 1, 2], If[Mod[(j - 1), 2] == 1 && Mod[δ[j - 1], 2] == 0 &&
Mod[b[j - 1], 2] == 1, δ[j] = Mod[δ[j] + 1, 2], δ[j] = δ[j - 2]]}}];];
If[X == {{1}, {2}, {3}} || X == {{2}, {3}, {1}} ||
X == {{1}, {3}, {2}} || X == {{2}, {1}, {3}},
{μ = b[1] + b[2]; δ[1] = 1; δ[2] = 1; For[j = 3, j ≤ i, j++, {δ[j] = δ[j - 2];
If[Mod[(j - 1), 2] == 0 && Mod[δ[j - 1], 2] == 1 && Mod[b[j - 1], 2] == 1,
δ[j] = Mod[δ[j] + 1, 2], If[Mod[(j - 1), 2] == 1 && Mod[δ[j - 1], 2] == 0 &&
Mod[b[j - 1], 2] == 1, δ[j] = Mod[δ[j] + 1, 2], δ[j] = δ[j - 2]]}}];];
Do[Print["δ(", k), "] = ", δ[k], " b(", k), "] = ", b[k]], {k, 1, j - 1}
μ = μ + ∑n=3j-1 (Mod[δ[n], 2] * b[n]);
Print["σ(G) = ", σ]
Print["μ(D) = ", μ]
Print["σ(L) = ", σ - μ]

```

FIGURE 5. The Mathematica program

Slope	Continued fraction	σ_K	$\sigma(G)$	$\mu(D)$	$\sigma(L)$	Crossings type	knot
3	$b_1 = 3$	(23)	1	3	-2	$\delta_1 = 1$	3_1
5	$b_1 = 5$	(23)	1	5	-4	$\delta_1 = 1$	5_1
$\frac{5}{2}$	$b_1 = 2, b_2 = 1, b_3 = 1$	(132)	2	2	0	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 1$	4_1
7	$b_1 = 7$	(23)	1	7	-6	$\delta_1 = 1$	7_1
$\frac{7}{2}$	$b_1 = 3, b_2 = 1, b_3 = 1$	(13)	2	0	2	$\delta_1 = 0, \delta_2 = 0, \delta_3 = 0$	5_2
9	$b_1 = 9$	(23)	1	9	-8	$\delta_1 = 1$	9_1
$\frac{9}{2}$	$b_1 = 4, b_2 = 1, b_3 = 1$	(132)	2	2	0	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 1$	6_1
11	$b_1 = 11$	(23)	1	11	-10	$\delta_1 = 1$	11_1
$\frac{11}{2}$	$b_1 = 5, b_2 = 1, b_3 = 1$	(13)	2	0	2	$\delta_1 = 0, \delta_2 = 0, \delta_3 = 0$	7_2
$\frac{11}{3}$	$b_1 = 3, b_2 = 1, b_3 = 2$	(123)	2	4	-2	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 0$	6_2
$\frac{13}{2}$	$b_1 = 6, b_2 = 1, b_3 = 2$	(132)	2	2	0	$\delta_1 = 0, \delta_2 = 1, \delta_3 = 1$	8_1
$\frac{13}{3}$	$b_1 = 4, b_2 = 2, b_3 = 1$	(23)	3	7	-4	$\delta_1 = 1, \delta_2 = 1, \delta_3 = 1$	7_3
$\frac{13}{5}$	$b_1 = 2, b_2 = 1, b_3 = 1$ $b_4 = 1, b_5 = 1$	(123)	3	3	0	$\delta_1 = 1, \delta_2 = 1, \delta_3 = 0$ $\delta_4 = 0, \delta_5 = 0$	6_3
$\frac{15}{2}$	$b_1 = 7, b_2 = 1, b_3 = 1$	(13)	2	0	2	$\delta_1 = 0, \delta_2 = 0, \delta_3 = 0$	9_2
$\frac{15}{4}$	$b_1 = 3, b_2 = 1, b_3 = 3$	(13)	2	0	2	$\delta_1 = 0, \delta_2 = 0, \delta_3 = 0$	7_4

TABLE 1. Knot table

case, we have

$$\{i\beta\} = \{29, 58, 87, 116, 145, 174, 203, 232, 261, 290, 319, 348, 377, 406, 435, \\ 464, 493, 522, 551, 580, 609, 683, 667, 696, 725, 754, 783, 812, 841, 870, \\ 899, 928, 957, 986, 1015, 1044, 1073, 1102, 1131, 1160, 11189, 1218, \\ 1247, 1276, 1305, 1334, 1363, 1392, 1421, 1450, 1479, 1508, 1537, 1566, \\ 1595, 1624, 1653, 1682, 1711, 1740, 1769, 1798, 1827, 1856, 1885, 1914, \\ 1943, 1972, 2001, 2030, 2059, 2088, 2117, 2146\}$$

Therefore, we have the the sequence of remainders respectively as follows:

$$\{r_i\} = \{29, 58, -63, -34, -5, 24, 53, -68, -39, -10, 19, 48, -73, -44, -15, 14, \\ 43, 72, -49, -20, 9, 38, 67, -54, -25, 4, 33, 62, -59, -30, -1, 28, 57, \\ -64, -35, -6, 23, 52, -69, -40, -11, 18, 47, -74, -45, -16, 13, 42, \\ 71, -50, -21, 8, 37, 66, -55, -26, 3, 32, 61, -60, -31, -2, 27, 56, -65, \\ -36, -7, 22, 51, -70, -41, -12, 17, 46\}.$$

Finally using [Sh, Theorem. 2], the signature of L is equal to twice of the number of negative entries minus the total number of entries in the above set of remainders which is $76 - 74 = 2$.

Remark 5.3. We like to mention that the formula in [Sh, Theorem. 2] appears in [M2, Theorem. 9.3.6] but with the opposite sign.

We think that the fundamental time estimate in performing any algorithm that codes our formula is logarithmic time since the number of iterations to find the canonical continued fraction is less than or equal five times the number of digits of β according to Lamé's Theorem [M, Page. 21]. While the fundamental time estimate in performing any algorithm that codes the formula of Shinohara in [Sh] is polynomial time since we need to apply the Euclidean algorithm r -times.

Finally, we apply the above code for some of the knots in Rolfsen's table in [BM] and we show the details of the computations in the Table 1. The original code in Mathematica can be obtained from the site:

<https://drive.google.com/file/d/0ByaasUUX-ykfZEFGS21DTWnyZjQ/edit?usp=sharing>.

References

- [BM] BAR-NATAN, D.; MORRISON, S. The Mathematica package knot theory. The Knot Atlas. <http://katlas.math.toronto.edu/wiki/>.
- [G] GOERITZ, LEBRECHT. Knoten und quadratische Formen. *Math. Z.* **36** (1933), no.1, 647–654. [MR1545364](#), [Zbl 0006.42201](#), doi: [10.1007/BF01188642](#).
- [GJ] GALLASPY, M; JABUKA, S. The Goeritz matrix and the signature of a two bridge knot. 2009. [arXiv:arXiv:0910.2872v1](#).
- [GL] GORDON, C. MCA.; LITHERLAND, R. A. On the signature of a link. *Invent. Math.* **47** (1978), no. 1, 53–69. [MR0500905](#) (58 #18407), [Zbl 0391.57004](#).

- [KL] KAUFFMAN, LOUIS H.; LAMBROPOULOU, SOFIA. Unknots and molecular biology. *Milan J. Math.* **74** (2006), 227–263. [MR2278735](#) (2008d:57008), [Zbl 1214.57011](#), doi:[10.1007/s00032-006-0063-3](#).
- [M] MOLLIN, RICHARD A. Fundamental number theory with applications. Second edition. Discrete Mathematics and its Applications (Boca Raton). *Chapman and Hall/CRC, Boca Raton, FL*, 2008. x+369 pp. ISBN: 978-1-4200-6659-3; 1-4200-6659-5. [MR2404578](#) (2009c:11001), [Zbl 1175.11001](#),
- [M1] MURASUGI, K. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.* **117** (1965), 387–442. [MR0171275](#) (30 #1506), [Zbl 0137.17903](#), doi:[10.1090/S0002-9947-1965-0171275-5](#).
- [M2] MURASUGI, KUNIO Knot theory and its applications. Translated from the 1993 Japanese original by Bohdan Kurpita. *Birkhäuser, Boston, MA*, 1996. viii+341 pp. ISBN: 0-8176-3817-2. [MR1391727](#) (97g:57011), [Zbl 1138.57001](#), doi:[10.1007/978-0-8176-4719-3](#).
- [S] SCHUBERT, HORST. Knoten mit zwei Brücken. *Math. Z.* **65** (1956), 133–170. [MR0082104](#) (18,498e), [Zbl 0071.39002](#), doi:[10.1007/BF01473875](#).
- [Sh] SHINOHARA, YAICHI. On the signature of a link with two bridges. *Kwansei Gakuin Univ. Annual Stud.* **25** (1976), 111–119. [MR0563649](#) (81k:57005).
- [SJ] SYLVESTER, J. A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum positive and negative squares. *Philosophical Magazine* **4** (1852), no. 23, 138–142. <http://www.tandfonline.com/doi/abs/10.1080/14786445208647087#.Uvg640JdV-A>.
- [T] TROTTER, H. F. Homology of group systems with applications to knot theory. *Ann. of Math. (2)* **76** (1962), 464–498. [MR0143201](#) (26 #761), [Zbl 0108.18302](#), doi:[10.2307/1970369](#).

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KUWAIT UNIVERSITY, P. O. Box 5969, SAFAT-13060, KUWAIT, STATE OF KUWAIT
khaled@sci.kuniv.edu.kw

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN'S, NL A1C 5S7, P.O. Box 4200, CANADA
isam.matter@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, YARMOUK UNIVERSITY, IRBID, JORDAN, 21163
qaiush@yahoo.com

This paper is available via <http://nyjm.albany.edu/j/2014/20-10.html>.