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Variation of the canonical height in a family of rational maps

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Abstract. Let $d \geq 2$ be an integer, let $\mathbf{c} \in \bar{\mathbb{Q}}(t)$ be a rational map, and let $\mathbf{f}_t(z) := \frac{z^d+t}{z}$ be a family of rational maps indexed by t. For each $t = \lambda \in \bar{\mathbb{Q}}$, we let $\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ be the canonical height of $\mathbf{c}(\lambda)$ with respect to the rational map $\mathbf{f}_\lambda$; also we let $\widehat{h}_{\mathbf{f}}(\mathbf{c})$ be the canonical height of $\mathbf{c}$ on the generic fiber of the above family of rational maps. We prove that there exists a constant C depending only on $\mathbf{c}$ such that for each $\lambda \in \bar{\mathbb{Q}}$,

$$\left|\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) - \widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot h(\lambda)\right| \leq C.$$

In particular, we show that $\lambda \mapsto \widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ is a Weil height on $\mathbb{P}^1$. This improves a result of Call and Silverman, 1993, for this family of rational maps.

Contents

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1. Introduction

Let X be a curve defined over $\bar{\mathbb{Q}}$, let $\mathcal{V} \longrightarrow X$ be an algebraic family of varieties $\{\mathcal{V}_\lambda\}_{\lambda\in X}$, let $\Phi : \mathcal{V} \longrightarrow \mathcal{V}$ be an endomorphism with the property that there exists $d > 1$, and there exists a divisor $\mathcal{D}$ of $\mathcal{V}$ such that $\Phi^*(\mathcal{D}) = d \cdot \mathcal{D}$ (the equality takes place in $\mathrm{Pic}(\mathcal{V})$). Then for all but finitely many $\lambda \in X$, there is a well-defined canonical height $\widehat{h}_{\mathcal{V}_\lambda,\mathcal{D}_\lambda,\Phi_\lambda}$ on the fiber above λ. Let $P : X \longrightarrow \mathcal{V}$ be an arbitrary section; then for each $\lambda \in X(\bar{\mathbb{Q}})$, we denote by P_λ the corresponding point on $\mathcal{V}_\lambda$. Also, P can be viewed as an element of $V(\bar{\mathbb{Q}}(X))$ and thus we denote by $\widehat{h}_{V,D,\Phi}(P)$ the canonical height of P with respect to the action of Φ on the generic fiber (V, D) of $(\mathcal{V}, \mathcal{D})$. Extending a result of Silverman [17] for the variation of the usual canonical height in algebraic families of abelian varieties, Call and Silverman [4, Theorem 4.1] proved that

$$\widehat{h}_{\mathcal{V}_\lambda,\mathcal{D}_\lambda,\Phi_\lambda}(P_\lambda) = \widehat{h}_{V,D,\Phi}(P) \cdot h(\lambda) + o(h(\lambda)), \tag{1.0.1}$$

where $h(\lambda)$ is a Weil height on X. In the special case $\mathcal{V} \longrightarrow \mathbb{P}^1$ is an elliptic surface, Tate [22] improved the error term of (1.0.1) to $O(1)$ (where the implied constant depends on P only, and it is independent of λ). Working with families of abelian varieties which admit good completions of their Néron models (for more details, see [3]), Call proved general theorems regarding local canonical heights which yield the result of Tate [22] as a corollary. Furthermore, Silverman [18, 19, 20] proved that the difference of the main terms from (1.0.1) when $\mathcal{V} \longrightarrow \mathbb{P}^1$ is an elliptic surface, in addition to being bounded, varies quite regularly as a function of λ, breaking up into a finite sum of well-behaved functions at finitely many places. It is natural to ask whether there are other instances when the error term of (1.0.1) can be improved to $O_P(1)$.

In [9], Ingram showed that when Φ_λ is an algebraic family of polynomials acting on the affine line, then again the error term in (1.0.1) is $O(1)$ (when the parameter space X is the projective line). More precisely, Ingram proved that for an arbitrary parameter curve X, there exists $D = D(\mathbf{f}, P) \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ of degree $\widehat{h}_{\mathbf{f}}(P)$ such that $\widehat{h}_{\mathbf{f}_\lambda}(P_\lambda) = h_D(\lambda) + O(1)$. This result is an analogue of Tate's theorem [22] in the setting of arithmetic dynamics. Using this result and applying an observation of Lang [10, Chap. 5, Prop. 5.4], the error term can be improved to $O(h(\lambda)^{1/2})$ and furthermore, in the special case where $X = \mathbb{P}^1$ the error term can be replaced by $O(1)$. In [8], Ghioca, Hsia and Tucker showed that the error term is also uniformly bounded independent of $\lambda \in X$ (an arbitrary projective curve) when Φ_λ is an algebraic family of rational maps satisfying the properties:

(a) Each Φ_λ is superattracting at infinity, i.e., if $\Phi_\lambda = \frac{P_\lambda}{Q_\lambda}$ for algebraic families of polynomials $P_\lambda, Q_\lambda \in \bar{\mathbb{Q}}[z]$, then $\deg(P_\lambda) \geq 2 + \deg(Q_\lambda)$.
(b) The resultant of P_λ and Q_λ is a nonzero constant.

The condition (a) is very mild for applications; on the other hand condition (b) is restrictive. Essentially condition (b) asks that Φ_λ is a well-defined rational map of same degree as on the generic fiber, i.e., all fibers of Φ are *good*.

Our main result is to improve the error term of (1.0.1) to $O(1)$ for the algebraic family of rational maps $\mathbf{f}_t(z) = \frac{z^d+t}{z}$ where the parameter t varies on the projective line. We denote by $\widehat{h}_{\mathbf{f}_\lambda}$ the canonical height associated to $\mathbf{f}_\lambda$ for each $t = \lambda \in \bar{\mathbb{Q}}$, and we denote by $\widehat{h}_{\mathbf{f}}$ the canonical height on the generic fiber (i.e., with respect to the map $\mathbf{f}_t(z) := \frac{z^d+t}{z} \in \bar{\mathbb{Q}}(t)(z)$).

Theorem 1.1. *Let $\mathbf{c} \in \bar{\mathbb{Q}}(t)$ be a rational map, let $d \geq 2$ be an integer, and let $\{\mathbf{f}_t\}$ be the algebraic family of rational maps given by $\mathbf{f}_t(z) := \frac{z^d+t}{z}$. Then as $t = \lambda$ varies in $\bar{\mathbb{Q}}$ we have*

$$\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) = \widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot h(\lambda) + O(1), \tag{1.0.2}$$

where the constant in $O(1)$ depends only on $\mathbf{c}$, and it is independent of λ.

Alternatively, Theorem 1.1 yields that the function $\lambda \mapsto \widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ is a Weil height on $\mathbb{P}^1$ associated to the divisor $\widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot \infty \in \mathrm{Pic}(\mathbb{P}^1) \otimes \mathbb{Q}$.

We note that on the fiber $\lambda = 0$, the corresponding rational map Φ_0 has degree $d-1$ rather than d (which is the generic degree in the family Φ_λ). So, our result is the *first* example of an algebraic family of rational maps (which are neither totally ramified at infinity, nor Lattés maps, and also admit *bad* fibers) for which the error term in (1.0.1) is $O(1)$. In addition, we mention that the family $\mathbf{f}_t(z) = \frac{z^d+t}{z}$ for $t \in \mathbb{C}$ is interesting also from the complex dynamics point of view. Devaney and Morabito [14] proved that the Julia sets $\{J_t\}_{t\in\mathbb{C}}$ of the above maps converge to the unit disk as t converges to 0 along the rays $\mathrm{Arg}(t) = \frac{(2k+1)\pi}{d-1}$ for $k = 0, \ldots, d-1$, providing thus an example of a family of rational maps whose Julia sets have empty interior, but in the limit, these sets converge to a set with nonempty interior.

A special case of our Theorem 1.1 is when the starting point $\mathbf{c}$ is constant; in this case we can give a precise formula for the $O(1)$-constant appearing in (1.0.2).

Theorem 1.2. *Let $d \geq 2$ be an integer, let α be an algebraic number, let $K = \mathbb{Q}(\alpha)$ and let ℓ be the number of nonarchimedean places $|\cdot|_v$ of K satisfying $|\alpha|_v \notin \{0, 1\}$. If $\{\mathbf{f}_t\}$ is the algebraic family of rational maps given by $\mathbf{f}_t(z) := \frac{z^d+t}{z}$, then*

$$\left|\widehat{h}_{f_\lambda}(\alpha) - \widehat{h}_{\mathbf{f}}(\alpha) \cdot h(\lambda)\right| < 3d \cdot (1 + \ell + 2h(\alpha)),$$

as $t = \lambda$ varies in $\bar{\mathbb{Q}}$.

In particular, Theorem 1.2 yields an effective way for determining for any given $\alpha \in \bar{\mathbb{Q}}$ the set of parameters λ contained in a number field of bounded degree such that α is preperiodic for $\mathbf{f}_\lambda$. Indeed, if $\alpha \in \bar{\mathbb{Q}}$ then either $\alpha = 0$

and then it is preperiodic for all $\mathbf{f}_\lambda$, or $\alpha \neq 0$ in which case generically α is not preperiodic and $\widehat{h}_{\mathbf{f}}(\alpha) = \frac{1}{d}$ (see Proposition 3.1). So, if $\alpha \in \bar{\mathbb{Q}}^*$ is preperiodic for $\mathbf{f}_\lambda$ then $\widehat{h}_{\mathbf{f}_\lambda}(\alpha) = 0$ and thus, Theorem 1.2 yields that

$$h(\lambda) < 3d^2 \cdot (1 + \ell + 2h(\alpha)). \tag{1.0.3}$$

For example, if α is a root of unity, then $h(\lambda) < 3d^2$ for all parameters $\lambda \in \overline{\mathbb{Q}}$ such that α is preperiodic for $\mathbf{f}_\lambda$.

Besides the intrinsic interest in studying the above problem, recently it was discovered a very interesting connection between the variation of the canonical height in algebraic families and the problem of unlikely intersections in algebraic dynamics (for a beautiful introduction to this area, please see the book of Zannier [25]). Masser and Zannier [11, 12] proved that for the family of Lattés maps $\mathbf{f}_\lambda(z) = \frac{(z^2-\lambda)^2}{4z(z-1)(z-\lambda)}$ there exist at most finitely many $\lambda \in \bar{\mathbb{Q}}$ such that both 2 and 3 are preperiodic for $\mathbf{f}_\lambda$. Geometrically, their result says the following: given the Legendre family of elliptic curves E_λ given by the equation $y^2 = x(x-1)(x-\lambda)$, there exist at most finitely many $\lambda \in \bar{\mathbb{Q}}$ such that $P_\lambda := \left(2, \sqrt{2(2-\lambda)}\right)$ and $Q_\lambda := \left(3, \sqrt{6(3-\lambda)}\right)$ are simultaneously torsion points for E_λ. Later Masser and Zannier [13] extended their result by proving that for any two sections P_λ and Q_λ on any elliptic surface E_λ, if there exist infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion for E_λ then the two sections are linearly dependent over $\mathbb{Z}$. Their proof uses the recent breakthrough results of Pila and Zannier [15]. Moreover, Masser and Zannier exploit in a crucial way the existence of the analytic uniformization map for elliptic curves. Motivated by a question of Zannier, Baker and DeMarco [1] showed that for any $a, b \in \mathbb{C}$, if there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for $\mathbf{f}_\lambda(z) = z^d + \lambda$ (where $d \geq 2$), then $a^d = b^d$. Later their result was generalized by Ghioca, Hsia and Tucker [7] to arbitrary families of polynomials. The method of proof employed in both [1] and [7] uses an equidistribution statement (see [2, Theorem 7.52] and [5, 6]) for points of small canonical height on Berkovich spaces. Later, using the powerful results of Yuan and Zhang [23, 24] on metrized line bundles, Ghioca, Hsia and Tucker [8] proved the first results on unlikely intersections for families of rational maps and also for families of endomorphisms of higher dimensional projective spaces. The difference between the results of [1, 7, 8] and the results of [11, 12, 13] is that for arbitrary families of polynomials there is no analytic uniformization map as in the case of the elliptic curves. Instead one needs to employ a more careful analysis of the local canonical heights associated to the family of rational maps. This led the authors of [8] to prove the error term in (1.0.1) is $O(1)$ for the rational maps satisfying conditions (a)–(b) listed above. Essentially, in order to use the equidistribution results of Baker–Rumely, Favre–Rivera–Letelier, and Yuan–Zhang, one needs to show that certain metrics converge uniformly and in turn this relies on showing that the local canonical heights associated to the corresponding family of rational maps vary uniformly across the various fibers of the family; this leads to improving to $O(1)$ the error term in (1.0.1). It is of great interest to see whether the results on unlikely intersections can be extended to more general families of rational maps beyond families of Lattés maps [11, 12, 13], or of polynomials [1, 7], or of rational maps with good fibers for all points in the parameter space [8]. On the other hand, a preliminary step to ensure the strategy from [1, 8, 7] can be employed to proving new results on unlikely intersections in arithmetic dynamics is to improve to $O(1)$ the error term from (1.0.1). For example, using the exact strategy employed in [8], the results of our paper yield that if $\mathbf{c}_1(t), \mathbf{c}_2(t) \in \bar{\mathbb{Q}}(t)$ have the property that there exist infinitely many $\lambda \in \bar{\mathbb{Q}}$ such that both $\mathbf{c}_1(\lambda)$ and $\mathbf{c}_2(\lambda)$ are preperiodic under the action of $\mathbf{f}_\lambda(z) := \frac{z^d+\lambda}{z}$, then for *each* $\lambda \in \bar{\mathbb{Q}}$ we have that $\mathbf{c}_1(\lambda)$ is preperiodic for $\mathbf{f}_\lambda$ if and only if $\mathbf{c}_2(\lambda)$ is preperiodic for $\mathbf{f}_\lambda$. Furthermore, if in addition $\mathbf{c}_1, \mathbf{c}_2$ are constant, then the same argument as in [8] yields that for *each* $\lambda \in \bar{\mathbb{Q}}$, we have $\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_1) = \widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_2)$. Finally, this condition should yield that $\mathbf{c}_1 = \mathbf{c}_2$; however finding the exact relation between $\mathbf{c}_1$ and $\mathbf{c}_2$ is usually difficult (see the discussion from [7, 8]).

In our proofs we use in an essential way the decomposition of the (canonical) height in a sum of local (canonical) heights. So, in order to prove Theorems 1.1 and 1.2 we show first (see Proposition 4.5) that for all but finitely many places v, the contribution of the corresponding local height to $d^2 \cdot \widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ matches the v-adic contribution to the height for the second iterate $\mathbf{f}_\lambda^2(\mathbf{c}(\lambda))$. This allows us to conclude that $\left|\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) - \frac{h(\mathbf{f}_\lambda^2(\mathbf{c}(\lambda)))}{d^2}\right|$ is uniformly bounded as λ varies. Then, using that $\deg_\lambda(\mathbf{f}_\lambda^2(\mathbf{c}(\lambda))) = \widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot d^2$, an application of the height machine finishes our proof. The main difficulty lies in proving that for each place v the corresponding local contribution to $d^2 \cdot \widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ varies from the v-adic contribution to $h(\mathbf{f}_\lambda^2(\mathbf{c}(\lambda)))$ by an amount bounded solely in terms of v and of $\mathbf{c}$. In order to derive our conclusion we first prove the statement for the special case when $\mathbf{c}$ is constant. Actually, in this latter case we can prove (see Propositions 5.8 and 5.11) that $\left|\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) - \frac{h(\mathbf{f}_\lambda(\mathbf{c}(\lambda)))}{d}\right|$ is uniformly bounded as λ varies. Then for the general case of Proposition 4.5, we apply Propositions 5.8 and 5.11 to the first iterate of $\mathbf{c}(\lambda)$ under $\mathbf{f}_\lambda$. For our analysis, we split the proof into 3 cases:

(i) $|\lambda|_v$ is much larger than the absolute values of the coefficients of the polynomials $A(t)$ and $B(t)$ defining $\mathbf{c}(t) := \frac{A(t)}{B(t)}$.
(ii) $|\lambda|_v$ is bounded above and below by constants depending only on the absolute values of the coefficients of $A(t)$ and of $B(t)$.
(iii) $|\lambda|_v$ is very small.

The cases (i)–(ii) are not very difficult and the same proof is likely to work for more general families of rational maps (especially if ∞ is a superattracting point for the rational maps $\mathbf{f}_\lambda$; note that the case $d = 2$ for Theorems 1.1 and 1.2 requires a different approach). However, case (iii) is much harder, and here we use in an essential way the general form of our family of maps. It is not surprising that this is the hard case since $\lambda = 0$ is the only bad fiber of the family $\mathbf{f}_\lambda$. We do not know whether the error term of $O(1)$ can be obtained for the variation of the canonical height in more general families of rational maps. It seems that each time λ is close to a singularity of the family (i.e., λ is close v-adically to some λ_0 for which $\deg(\Phi_{\lambda_0})$ is less than the generic degree in the family) would require a different approach.

The plan of our paper is as follows. In the next section we setup the notation for our paper. Then in Section 3 we compute the height $\widehat{h}_{\mathbf{f}}(\mathbf{c})$ on the generic fiber of our dynamical system. We continue in Section 4 with a series of reductions of our main results; we reduce Theorem 1.1 to proving Proposition 4.5. We conclude by proving Theorem 1.2 in Section 5, and then finishing the proof of Proposition 4.5 in Section 6.

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2. Notation

2.1. Generalities. For a rational function $f(z)$, we denote by $f^n(z)$ its n-th iterate (for any $n \geq 0$, where f^0 is the identity map). We call a point P preperiodic if its orbit under f is finite.

For each real number x, we denote $\log^+ x := \log \max\{1, x\}$.

2.2. Good reduction for rational maps. Let K be a number field, let v be a nonarchimedean valuation on K, let $\mathfrak{o}_v$ be the ring of v-adic integers of K, and let k_v be the residue field at v. If $A, B \in K[z]$ are coprime polynomials, then $\varphi(z) := A(z)/B(z)$ has good reduction (see [21]) at all places v satisfying the properties:

(1) The coefficients of A and of B are in $\mathfrak{o}_v$.
(2) The leading coefficients of A and of B are units in $\mathfrak{o}_v$.
(3) The resultant of the polynomials A and B is a unit in $\mathfrak{o}_v$.

Clearly, all but finitely many places v of K satisfy the above conditions (1)–(3). In particular this yields that if we reduce modulo v the coefficients of both A and B, then the induced rational map $\overline{\varphi}(z) := \overline{A}(z)/\overline{B}(z)$ is a well-defined rational map defined over k_v of same degree as φ.

2.3. Absolute values. We denote by $\Omega_{\mathbb{Q}}$ the set of all (inequivalent) absolute values of $\mathbb{Q}$ with the usual normalization so that the product formula holds: $\prod_{v \in \Omega_{\mathbb{Q}}} |x|_v = 1$ for each nonzero $x \in \mathbb{Q}$. For each $v \in \Omega_{\mathbb{Q}}$, we fix an extension of $|\cdot|_v$ to $\bar{\mathbb{Q}}$.

2.4. Heights.

2.4.1. Number fields. Let K be a number field. For each $n \geq 1$, if $P := [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ then the Weil height of P is

$$h(P) := \frac{1}{[K:\mathbb{Q}]} \cdot \sum_{\sigma: K \longrightarrow \bar{\mathbb{Q}}} \sum_{v \in \Omega_{\mathbb{Q}}} \log \max\{|\sigma(x_0)|_v, \cdots, |\sigma(x_n)|_v\},$$

where the first summation runs over all embeddings $\sigma : K \longrightarrow \bar{\mathbb{Q}}$. The definition is independent of the choice of coordinates x_i representing P (by an application of the product formula) and it is also independent of the particular choice of number field K containing the coordinates x_i (by the fact that each place $v \in \Omega_{\mathbb{Q}}$ is defectless, as defined by [16]). In this paper we will be concerned mainly with the height of points in $\mathbb{P}^1$; furthermore, if $x \in \bar{\mathbb{Q}}$, then we identify x with $[x : 1] \in \mathbb{P}^1$ and define its height accordingly. The basic properties for heights which we will use are: for all $x, y \in \bar{\mathbb{Q}}$ we have:

(1) $h(x + y) \leq h(x) + h(y) + \log(2)$.
(2) $h(xy) \leq h(x) + h(y)$.
(3) $h(1/x) = h(x)$.

2.4.2. Function fields. We will also work with the height of rational functions (over $\bar{\mathbb{Q}}$). So, if L is any field, then the Weil height of a rational function $g \in L(t)$ is defined to be its degree.

2.5. Canonical heights.

2.5.1. Number fields. Let K be a number field, and let $f \in K(z)$ be a rational map of degree $d \geq 2$. Following [4] we define the canonical height of a point $x \in \mathbb{P}^1(\bar{\mathbb{Q}})$ as

$$\widehat{h}_f(x) = \lim_{n \to \infty} \frac{h(f^n(x))}{d^n}. \tag{2.0.4}$$

As proved in [4], the difference $|h(x) - \widehat{h}_f(x)|$ is uniformly bounded for all $x \in \mathbb{P}^1(\bar{\mathbb{Q}})$, the difference depending on f only. Also, $\widehat{h}_f(x) = 0$ if and only if x is a preperiodic point for f. If $x \in \bar{\mathbb{Q}}$ then we view it embedded in $\mathbb{P}^1$ as $[x : 1]$ and denote by $\widehat{h}_f(x)$ its canonical height under f constructed as above.

2.5.2. Function fields. Let L be an arbitrary field, let $f \in L(t)(z)$ be a rational function of degree $d \geq 2$, and let $x \in L(t)$. Then the canonical height $\widehat{h}_f(x) := \widehat{h}_f([x : 1])$ is defined the same as in (2.0.4).

2.6. Canonical heights for points and rational maps as they vary in algebraic families.

2.6.1. Number fields. If $\lambda \in \bar{\mathbb{Q}}^*$, $x = [A : B] \in \mathbb{P}^1(\bar{\mathbb{Q}})$ and $f_\lambda(z) := \frac{z^d+\lambda}{z}$, then we can define $\widehat{h}_{f_\lambda}(x)$ alternatively as follows. We let $A_{\lambda,[A:B],0} := A$ and $B_{\lambda,[A:B],0} := B$, and for each $n \geq 0$ we let

$$A_{\lambda,[A:B],n+1} := A^d_{\lambda,[A:B],n} + \lambda \cdot B^d_{\lambda,[A:B],n},$$
$$B_{\lambda,[A:B],n+1} := A_{\lambda,[A:B],n} \cdot B^{d-1}_{\lambda,[A:B],n}.$$

Then $f^n_\lambda([A : B]) = [A_{\lambda,[A:B],n} : B_{\lambda,[A:B],n}]$ and so,

$$\widehat{h}_{f_\lambda}(x) = \lim_{n\to\infty} \frac{h([A_{\lambda,[A:B],n} : B_{\lambda,[A:B],n}])}{d^n}.$$

Also, for each place v, we define the local canonical height of $x = [A : B]$ with respect to f_λ as

$$\widehat{h}_{f_\lambda,v}(x) = \lim_{n\to\infty} \frac{\log \max\left\{\left|A_{\lambda,[A:B],n}\right|_v, \left|B_{\lambda,[A:B],n}\right|_v\right\}}{d^n}. \tag{2.0.5}$$

If $x \in \bar{\mathbb{Q}}$ we view it embedded in $\mathbb{P}^1(\bar{\mathbb{Q}})$ as $[x : 1]$ and compute its canonical heights (both global and local) under f_λ as above starting with $A_{\lambda,x,0} := x$ and $B_{\lambda,x,0} := 1$.

For $x = A/B$ with $B \neq 0$, we get that

$$\begin{aligned} A_{\lambda,[A:B],n} &= A_{\lambda,x,n} \cdot B^{d^n}, \\ B_{\lambda,[A:B],n} &= B_{\lambda,x,n} \cdot B^{d^n}, \end{aligned} \tag{2.0.6}$$

for all $n \geq 0$. If in addition $A \neq 0$, then $B_{\lambda,[A:B],1} = A \cdot B^{d-1} \neq 0$ and then for all $n \geq 0$ we have

$$\begin{aligned} A_{\lambda,[A:B],n+1} &= A_{\lambda,f_\lambda(x),n} \cdot B^{d^n}_{\lambda,[A:B],1}, \\ B_{\lambda,[A:B],n+1} &= B_{\lambda,f_\lambda(x),n} \cdot B^{d^n}_{\lambda,[A:B],1}, \end{aligned} \tag{2.0.7}$$

and in general, if $B_{\lambda,[A:B],k_0} \neq 0$, then

$$\begin{aligned} A_{\lambda,[A:B],n+k_0} &= A_{\lambda,f^{k_0}_\lambda(x),n} \cdot B^{d^n}_{\lambda,[A:B],k_0}, \\ B_{\lambda,[A:B],n+k_0} &= B_{\lambda,f^{k_0}_\lambda(x),n} \cdot B^{d^n}_{\lambda,[A:B],k_0}. \end{aligned} \tag{2.0.8}$$

We will be interested also in studying the variation of the canonical height of a family of starting points parametrized by a rational map (in t) under the family $\{\mathbf{f}_t(z)\}$ of rational maps. As before, $\mathbf{f}_t(z) := \frac{z^d+t}{z}$, and for each $t = \lambda \in \bar{\mathbb{Q}}$ we get a map in the above family of rational maps. When we want to emphasize the fact that each $\mathbf{f}_\lambda$ (for $\lambda \in \bar{\mathbb{Q}}^*$) belongs to this family of rational maps (rather than being a single rational map), we will use the boldface letter $\mathbf{f}$ instead of f. Also we let $\mathbf{c}(t) := \frac{\mathbf{A}(t)}{\mathbf{B}(t)}$ where $\mathbf{A}, \mathbf{B} \in K[t]$

are coprime polynomials defined over a number field K. Again, for each $t = \lambda \in \bar{\mathbb{Q}}$ we get a point $\mathbf{c}(\lambda) \in \mathbb{P}^1(\bar{\mathbb{Q}})$.

We define $\mathbf{A}_{\mathbf{c},n}(t) \in K[t]$ and $\mathbf{B}_{\mathbf{c},n}(t) \in K[t]$ so that for each $n \geq 0$ we have $\mathbf{f}_t^n(\mathbf{c}(t)) = [\mathbf{A}_{\mathbf{c},n}(t) : \mathbf{B}_{\mathbf{c},n}(t)]$. In particular, for each $t = \lambda \in \bar{\mathbb{Q}}$ we have $\mathbf{f}_\lambda^n(\mathbf{c}(\lambda)) = [\mathbf{A}_{\mathbf{c},n}(\lambda) : \mathbf{B}_{\mathbf{c},n}(\lambda)]$.

We let $\mathbf{A}_{\mathbf{c},0}(t) := \mathbf{A}(t)$ and $\mathbf{B}_{\mathbf{c},0}(t) := \mathbf{B}(t)$. Our definition for $\mathbf{A}_{\mathbf{c},n}$ and $\mathbf{B}_{\mathbf{c},n}$ for $n = 1$ will depend on whether $\mathbf{A}(0)$ (or equivalently $\mathbf{c}(0)$) equals 0 or not. If $\mathbf{A}(0) \neq 0$, then we define

$$\begin{aligned} \mathbf{A}_{\mathbf{c},1}(t) &:= \mathbf{A}(t)^d + t\mathbf{B}(t)^d, \\ \mathbf{B}_{\mathbf{c},1}(t) &:= \mathbf{A}(t)\mathbf{B}(t)^{d-1}, \end{aligned} \tag{2.0.9}$$

while if $\mathbf{c}(0) = 0$, then

$$\begin{aligned} \mathbf{A}_{\mathbf{c},1}(t) &:= \frac{\mathbf{A}(t)^d + t\mathbf{B}(t)^d}{t}, \\ \mathbf{B}_{\mathbf{c},1}(t) &:= \frac{\mathbf{A}(t)\mathbf{B}(t)^{d-1}}{t}. \end{aligned} \tag{2.0.10}$$

Then for each positive integer n we let

$$\begin{aligned} \mathbf{A}_{\mathbf{c},n+1}(t) &:= \mathbf{A}_{\mathbf{c},n}(t)^d + t \cdot \mathbf{B}_{\mathbf{c},n}(t)^d, \\ \mathbf{B}_{\mathbf{c},n+1}(t) &:= \mathbf{A}_{\mathbf{c},n}(t) \cdot \mathbf{B}_{\mathbf{c},n}(t)^{d-1}. \end{aligned} \tag{2.0.11}$$

Whenever it is clear from the context, we will use $\mathbf{A}_n$ and $\mathbf{B}_n$ instead of $\mathbf{A}_{\mathbf{c},n}$ and $\mathbf{B}_{\mathbf{c},n}$ respectively. For each $t = \lambda \in \bar{\mathbb{Q}}$, the canonical height of $\mathbf{c}(\lambda)$ under the action of $\mathbf{f}_\lambda$ may be computed as follows:

$$\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) = \lim_{n\to\infty} \frac{h\left([\mathbf{A}_{\mathbf{c},n}(\lambda) : \mathbf{B}_{\mathbf{c},n}(\lambda)]\right)}{d^n}.$$

Also, for each place v, we define the local canonical height of $\mathbf{c}(\lambda)$ at v as follows:

$$\widehat{h}_{\mathbf{f}_\lambda,v}(\mathbf{c}(\lambda)) = \lim_{n\to\infty} \frac{\log\max\{|\mathbf{A}_{\mathbf{c},n}(\lambda)|_v, |\mathbf{B}_{\mathbf{c},n}(\lambda)|_v\}}{d^n}. \tag{2.0.12}$$

The limit in (2.0.12) exists, as proven in Corollary 5.3 (note that our definition of local canonical heights differs from the corresponding definition from [21, Chapter 5]).

The following is a simple observation based on (2.0.8): if $\lambda \in \bar{\mathbb{Q}}$ such that $\mathbf{B}_{\mathbf{c},k_0}(\lambda) \neq 0$, then for each $k_0, n \geq 0$ we have

$$\begin{aligned} \mathbf{A}_{\mathbf{c},n+k_0}(\lambda) &= \mathbf{B}_{\mathbf{c},k_0}(\lambda)^{d^n} \cdot A_{\lambda, f_\lambda^{k_0}(\mathbf{c}(\lambda)), n} \\ \mathbf{B}_{\mathbf{c},n+k_0}(\lambda) &= \mathbf{B}_{\mathbf{c},k_0}(\lambda)^{d^n} \cdot B_{\lambda, f_\lambda^{k_0}(\mathbf{c}(\lambda)), n}. \end{aligned} \tag{2.0.13}$$

2.6.2. Function fields. We also compute the canonical height of $\mathbf{c}(t)$ on the generic fiber of the family of rational maps $\mathbf{f}$ with respect to the action of $\mathbf{f}_t(z) = \frac{z^d+t}{z} \in \mathbb{Q}(t)(z)$ as follows

$$\widehat{h}_{\mathbf{f}}(\mathbf{c}) := \widehat{h}_{\mathbf{f}_t}(\mathbf{c}(t)) := \lim_{n\to\infty} \frac{h(\mathbf{f}_t^n(\mathbf{c}(t)))}{d^n} = \lim_{n\to\infty} \frac{\deg_t(\mathbf{f}_t^n(\mathbf{c}(t)))}{d^n}.$$

3. Canonical height on the generic fiber

For each $n \geq 0$, the map $t \longrightarrow \mathbf{f}_t^n(\mathbf{c}(t))$ is a rational map; so, $\deg(\mathbf{f}_t^n(\mathbf{c}(t)))$ will always denote its degree. Similarly, letting $\mathbf{f}(z) := \frac{z^d+t}{z} \in \overline{\mathbb{Q}}(t)(z)$ and $\mathbf{c}(t) := \frac{\mathbf{A}(t)}{\mathbf{B}(t)}$ for coprime polynomials $\mathbf{A}, \mathbf{B} \in \bar{\mathbb{Q}}[t]$, then $\mathbf{f}^n(\mathbf{c}(t))$ is a rational function for each $n \geq 0$. In this section we compute $\widehat{h}_{\mathbf{f}}(\mathbf{c})$. It is easier to split the proof into two cases depending on whether $\mathbf{c}(0) = 0$ (or equivalently $\mathbf{A}(0) = 0$) or not.

Proposition 3.1. *If* $\mathbf{c}(0) \neq 0$, *then*

$$\widehat{h}_{\mathbf{f}}(\mathbf{c}) = \frac{\deg(\mathbf{f}_t(\mathbf{c}(t)))}{d} = \frac{\deg(\mathbf{f}_t^2(\mathbf{c}(t)))}{d^2}.$$

Proof. According to (2.0.9) and (2.0.11) we have defined $\mathbf{A}_{\mathbf{c},n}(t)$ and $\mathbf{B}_{\mathbf{c},n}(t)$ in this case. It is easy to prove that $\deg(\mathbf{A}_n) > \deg(\mathbf{B}_n)$ for all positive integers n. Indeed, if $\deg(\mathbf{A}) > \deg(\mathbf{B})$, then an easy induction yields that $\deg(\mathbf{A}_n) > \deg(\mathbf{B}_n)$ for all $n \geq 0$. If $\deg(\mathbf{A}) \leq \deg(\mathbf{B})$, then $\deg(\mathbf{A}_1) = 1 + d \cdot \deg(\mathbf{B}) > d \cdot \deg(\mathbf{B}) \geq \deg(\mathbf{B}_1)$. Again an easy induction finishes the proof that $\deg(\mathbf{A}_n) > \deg(\mathbf{B}_n)$ for all $n \geq 1$.

In particular, we get that $\deg(\mathbf{A}_n) = d^{n-1} \cdot \deg(\mathbf{A}_1)$ for all $n \geq 1$. The following claim will finish our proof.

Claim 3.2. For each $n \geq 0$, $\mathbf{A}_n$ and $\mathbf{B}_n$ are coprime.

Proof of Claim 3.2. The statement is true for $n = 0$ by definition. Assume now that it holds for all $n \leq N$ and we'll show that $\mathbf{A}_{N+1}$ and $\mathbf{B}_{N+1}$ are coprime.

Assume there exists $\alpha \in \bar{\mathbb{Q}}$ such that the polynomial $t - \alpha$ divides both $\mathbf{A}_{N+1}(t)$ and $\mathbf{B}_{N+1}(t)$. First we claim that $\alpha \neq 0$. Indeed, if t would divide $\mathbf{A}_{N+1}$, then it would also divide $\mathbf{A}_N$ and inductively we would get that $t \mid \mathbf{A}_0(t) = \mathbf{A}(t)$, which is a contradiction since $A(0) \neq 0$. So, indeed $\alpha \neq 0$. But then from the fact that both $\mathbf{A}_{N+1}(\alpha) = 0 = \mathbf{B}_{N+1}(\alpha)$ (and $\alpha \neq 0$) we obtain from the recursive formula defining $\{\mathbf{A}_n\}_n$ and $\{\mathbf{B}_n\}_n$ that also $\mathbf{A}_N(\alpha) = 0$ and $\mathbf{B}_N(\alpha) = 0$. However this contradicts the assumption that $\mathbf{A}_N$ and $\mathbf{B}_N$ are coprime. Thus $\mathbf{A}_n$ and $\mathbf{B}_n$ are coprime for all $n \geq 0$. $\square$

Using the definition of $\widehat{h}_{\mathbf{f}}(\mathbf{c})$ we conclude the proof of Proposition 3.1. $\square$

If $\mathbf{c}(0) = 0$ (or equivalently $\mathbf{A}(0) = 0$) the proof is very similar, only that this time we use (2.0.10) to define $\mathbf{A}_1$ and $\mathbf{B}_1$.

Proposition 3.3. *If* $\mathbf{c}(0) = 0$*, then*

$$\widehat{h}_{\mathbf{f}}(\mathbf{c}) = \frac{\deg(\mathbf{f}_t(\mathbf{c}(t)))}{d} = \frac{\deg(\mathbf{f}_t^2(\mathbf{c}(t)))}{d^2}.$$

Proof. Since $t \mid \mathbf{A}(t)$ we obtain that $\mathbf{A}_1, \mathbf{B}_1 \in \bar{\mathbb{Q}}[t]$; moreover, they are coprime because $\mathbf{A}$ and $\mathbf{B}$ are coprime. Indeed, t does not divide $\mathbf{B}(t)$ and so, because t divides $\mathbf{A}(t)$ and $d \geq 2$, we conclude that t does not divide $\mathbf{A}_1(t)$. Now, if there exists some $\alpha \in \bar{\mathbb{Q}}^*$ such that both $\mathbf{A}_1(\alpha) = \mathbf{B}_1(\alpha) = 0$, then we obtain that also both $\mathbf{A}(\alpha) = \mathbf{B}(\alpha) = 0$, which is a contradiction.

Using that $\mathbf{A}_1$ and $\mathbf{B}_1$ are coprime, and also that $t \nmid \mathbf{A}_1$, the same reasoning as in the proof of Claim 3.2 yields that $\mathbf{A}_n$ and $\mathbf{B}_n$ are coprime for each $n \geq 1$.

Also, arguing as in the proof of Proposition 3.1, we obtain that $\deg(\mathbf{A}_n) > \deg(\mathbf{B}_n)$ for all $n \geq 1$. Hence,

$$\deg(\mathbf{f}_t^n(\mathbf{c}(t))) = \deg_t(\mathbf{A}_n(t)) = d^{n-2} \cdot \deg(\mathbf{f}_t^2(\mathbf{c}(t))) = d^{n-1} \cdot \deg(\mathbf{f}_t(\mathbf{c}(t))),$$

as desired. □

4. Reductions

With the above notation, Theorem 1.1 is equivalent with showing that

$$\lim_{n\to\infty} \frac{h([\mathbf{A}_{\mathbf{c},n}(\lambda) : \mathbf{B}_{\mathbf{c},n}(\lambda)])}{d^n} = \widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot h(\lambda) + O_{\mathbf{c}}(1). \tag{4.0.1}$$

In all of our arguments we assume $\lambda \neq 0$, and also that $\mathbf{A}$ and $\mathbf{B}$ are not identically equal to 0 (where $\mathbf{c} = \mathbf{A}/\mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \bar{\mathbb{Q}}[t]$ coprime). Obviously excluding the case $\lambda = 0$ does not affect the validity of Theorem 1.1 (the quantity $\widehat{h}_{\mathbf{f}_0}(\mathbf{c}(0))$ can be absorbed into the $O(1)$-constant). In particular, if $\lambda \neq 0$ then the definition of $\mathbf{A}_{\mathbf{c},1}$ and $\mathbf{B}_{\mathbf{c},1}$ (when $\mathbf{c}(0) = 0$) makes sense (i.e., we are allowed to divide by λ). Also, if $\mathbf{A}$ or $\mathbf{B}$ equal 0 identically, then $\mathbf{c}(\lambda)$ is preperiodic for $\mathbf{f}_\lambda$ for all λ and then again Theorem 1.1 holds trivially.

Proposition 4.1. *Let* $\lambda \in \bar{\mathbb{Q}}^*$*. Then for all but finitely many* $v \in \Omega_{\mathbb{Q}}$*, we have* $\log\max\{|\mathbf{A}_{\mathbf{c},n}(\lambda)|_v, |\mathbf{B}_{\mathbf{c},n}(\lambda)|_v\} = 0$ *for all* $n \in \mathbb{N}$*.*

Proof. First of all, for the sake of simplifying our notation (and noting that $\mathbf{c}$ and λ are fixed in this Proposition), we let $A_n := \mathbf{A}_{\mathbf{c},n}(\lambda)$ and $B_n := \mathbf{B}_{\mathbf{c},n}(\lambda)$.

From the definition of A_1 and B_1 we see that not both are equal to 0 (here we use also the fact that $\lambda \neq 0$ which yields that if both A_1 and B_1 are equal to 0 then $\mathbf{A}(\lambda) = \mathbf{B}(\lambda) = 0$, and this contradicts the fact that $\mathbf{A}$ and $\mathbf{B}$ are coprime). Let S be the set of all nonarchimedean places $v \in \Omega_{\mathbb{Q}}$ such that $|\lambda|_v = 1$ and also $\max\{|A_1|_v, |B_1|_v\} = 1$. Since not both A_1 and B_1 equal 0 (and also $\lambda \neq 0$), then all but finitely many nonarchimedean places v satisfy the above conditions.

Claim 4.2. If $v \in S$, then $\max\{|A_n|_v, |B_n|_v\} = 1$ for all $n \in \mathbb{N}$.

Proof of Claim 4.2. This claim follows easily by induction on n; the case $n=1$ follows by the definition of S. Since

$$\max\{|A_n|_v, |B_n|_v\} = 1$$

and $|\lambda|_v = 1$ then $\max\{|A_{n+1}|_v, |B_{n+1}|_v\} \le 1$. Now, if $|A_n|_v = |B_n|_v = 1$ then $|B_{n+1}|_v = 1$. On the other hand, if $\max\{|A_n|_v, |B_n|_v\} = 1 > \min\{|A_n|_v, |B_n|_v\}$, then $|A_{n+1}|_v = 1$ (because $|\lambda|_v = 1$). $\square$

Claim 4.2 finishes the proof of Proposition 4.1. $\square$

We let K be the finite extension of $\mathbb{Q}$ obtained by adjoining the coefficients of both $\mathbf{A}$ and $\mathbf{B}$ (we recall that $\mathbf{c}(t) = \mathbf{A}(t)/\mathbf{B}(t)$). Then $\mathbf{A}_n(\lambda) := \mathbf{A}_{\mathbf{c},n}(\lambda)$, $\mathbf{B}_n(\lambda) := \mathbf{B}_{\mathbf{c},n}(\lambda) \in K(\lambda)$ for each n and for each λ. Proposition 4.1 allows us to invert the limit from the left-hand side of (4.0.1) with the following sum

$$h([\mathbf{A}_n(\lambda) : \mathbf{B}_n(\lambda)]) = \frac{1}{[K(\lambda):\mathbb{Q}]} \cdot \sum_{\sigma: K(\lambda) \longrightarrow \bar{\mathbb{Q}}} \sum_{v \in \Omega_{\mathbb{Q}}} \log\max\{|\sigma(\mathbf{A}_n(\lambda))|_v, |\sigma(\mathbf{B}_n(\lambda))|_v\},$$

because for all but finitely many places v, we have

$$\log\max\{|\sigma(\mathbf{A}_n(\lambda))|_v, |\sigma(\mathbf{B}_n(\lambda))|_v\} = 0.$$

Also we note that $\sigma(\mathbf{A}_{\mathbf{c},n}(\lambda)) = \mathbf{A}_{\mathbf{c}^\sigma,n}(\sigma(\lambda))$ and $\sigma(\mathbf{B}_{\mathbf{c},n}(\lambda)) = \mathbf{B}_{\mathbf{c}^\sigma,n}(\sigma(\lambda))$, where $\mathbf{c}^\sigma(t) := \frac{\mathbf{A}^\sigma(t)}{\mathbf{B}^\sigma(t)}$ is the rational map obtained by applying the homomorphism $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ to each coefficient of $\mathbf{A}$ and of $\mathbf{B}$. Using the definition of the local canonical height from (2.0.12), we observe that (4.0.1) is equivalent with showing that

$$\frac{1}{[K(\lambda):\mathbb{Q}]} \sum_{v \in \Omega_{\mathbb{Q}}} \sum_{\sigma: K(\lambda) \longrightarrow \bar{\mathbb{Q}}} \widehat{h}_{\mathbf{f}_{\sigma(\lambda)},v}(\mathbf{c}^\sigma(\sigma(\lambda))) = \widehat{h}_{\mathbf{f}}(\mathbf{c})h(\lambda) + O_{\mathbf{c}}(1). \tag{4.2.1}$$

For each $v \in \Omega_{\mathbb{Q}}$, and each $n \ge 0$ we let

$$M_{\mathbf{c},n,v}(\lambda) := \max\{|\mathbf{A}_{\mathbf{c},n}(\lambda)|_v, |\mathbf{B}_{\mathbf{c},n}(\lambda)|_v\}.$$

When $\mathbf{c}$ is fixed, we will use the notation $M_{n,v}(\lambda) := M_{\mathbf{c},n,v}(\lambda)$; if λ is fixed then we will use the notation $M_{n,v} := M_{n,v}(\lambda)$. If v is also fixed, we will use the notation $M_n := M_{n,v}$.

Proposition 4.3. *Let $v \in \Omega_{\mathbb{Q}}$ be a nonarchimedean place such that:*

(i) *Each coefficient of $\mathbf{A}$ and of $\mathbf{B}$ is a v-adic integer.*
(ii) *The resultant of the polynomials $\mathbf{A}$ and $\mathbf{B}$, and the leading coefficients of both $\mathbf{A}$ and of $\mathbf{B}$ are v-adic units.*
(iii) *If the constant coefficient a_0 of $\mathbf{A}$ is nonzero, then a_0 is a v-adic unit.*

Then for each $\lambda \in \bar{\mathbb{Q}}^*$ we have $\frac{\log M_{\mathbf{c},n,v}(\lambda)}{d^n} = \frac{\log M_{\mathbf{c},1,v}(\lambda)}{d}$, for all $n \ge 1$.

Remarks 4.4.

(1) Since we assumed $\mathbf{A}$ and $\mathbf{B}$ are nonzero, then conditions (i)–(iii) are satisfied by all but finitely many places $v \in \Omega_{\mathbb{Q}}$.

(2) Conditions (i)–(ii) of Proposition 4.3 yield that $\mathbf{c}(t) = \mathbf{A}(t)/\mathbf{B}(t)$ has good reduction at v. On the other hand, if $\mathbf{A}(t)/t\mathbf{B}(t)$ has good reduction at v, then condition (iii) must hold.

Proof. Let $\lambda \in \bar{\mathbb{Q}}^*$, let $|\cdot| := |\cdot|_v$, let $A_n := \mathbf{A}_{\mathbf{c},n}(\lambda)$, $B_n := \mathbf{B}_{\mathbf{c},n}(\lambda)$, and $M_n := \max\{|A_n|, |B_n|\}$.

Assume first that $|\lambda| > 1$. Using conditions (i)-(ii), then $M_0 = |\lambda|^{\deg(\mathbf{c})}$. If $\mathbf{c}$ is nonconstant, then $M_0 > 1$; furthermore, for each $n \geq 1$ we have $|A_n| > |B_n|$ (because $\deg(\mathbf{A}_{\mathbf{c},n}(t)) > \deg(\mathbf{B}_{\mathbf{c},n}(t))$ for $n \geq 1$), and so, $M_n = M_1^{d^{n-1}}$ for all $n \geq 1$. On the other hand, if $\mathbf{c}$ is constant, then $|\mathbf{A}_1| = |\lambda| > |\mathbf{B}_1| = 1$, and then again for each $n \geq 1$ we have $M_n = M_1^{d^{n-1}}$. Hence Proposition 4.3 holds when $|\lambda| > 1$.

Assume $|\lambda| \leq 1$. Then it is immediate that $M_n \leq 1$ for all $n \geq 0$. On the other hand, because v is a place of good reduction for $\mathbf{c}$, we get that $M_0 = 1$. Then, assuming that $|\lambda| = 1$ we obtain

$$|\mathbf{A}_1(\lambda)| = |\mathbf{A}(\lambda)^d + \lambda\mathbf{B}(\lambda)^d| \text{ and } |\mathbf{B}_1(\lambda)| = |\mathbf{A}(\lambda)\mathbf{B}(\lambda)^{d-1}|.$$

Then Claim 4.2 yields that $M_n = 1$ for all $n \geq 1$, and so Proposition 4.3 holds when $|\lambda| = 1$.

Assume now that $|\lambda| < 1$, then either $|\mathbf{A}(\lambda)| = 1$ or $|\mathbf{A}(\lambda)| < 1$. If the former holds, then first of all we note that $\mathbf{A}(0) \neq 0$ since otherwise $|\mathbf{A}(\lambda)| \leq |\lambda| < 1$. An easy induction yields that $|A_n| = 1$ for all $n \geq 0$ (since $|B_n| \leq 1$ and $|\lambda| < 1$). Therefore, $M_n = 1$ for all $n \geq 0$. Now if $|\mathbf{A}(\lambda)| < 1$, using that $|\lambda| < 1$, we obtain that $a_0 = 0$. Indeed, if a_0 were nonzero, then $|a_0| = 1$ by our hypothesis (iii), and thus $|\mathbf{A}(\lambda)| = |a_0| = 1$. So, indeed $\mathbf{A}(0) = 0$, which yields that

$$A_1 = \frac{\mathbf{A}(\lambda)^d}{\lambda} + \mathbf{B}(\lambda)^d. \tag{4.4.1}$$

On the other hand, since v is a place of good reduction for $\mathbf{c}$, and $|\mathbf{A}(\lambda)| < 1$ we conclude that $|\mathbf{B}(\lambda)| = 1$. Thus (4.4.1) yields that $|A_1| = 1$ because $d \geq 2$ and $|\mathbf{A}(\lambda)| \leq |\lambda| < 1$. Because for each $n \geq 1$ we have $A_{n+1} = A_n^d + \lambda \cdot B_n^d$ and $|\lambda| < 1$, while $|B_n| \leq 1$, an easy induction yields that $|A_n| = 1$ for all $n \geq 1$.

This concludes the proof of Proposition 4.3. $\square$

The following result is the key for our proof of Theorem 1.1.

Proposition 4.5. *Let $v \in \Omega_{\mathbb{Q}}$. There exists a positive real number $C_{v,\mathbf{c}}$ depending only on v, and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ (but independent of λ) such that*

$$\left|\lim_{n\to\infty} \frac{\log\max\{|\mathbf{A}_{\mathbf{c},n}(\lambda)|_v, |\mathbf{B}_{\mathbf{c},n}(\lambda)|_v\}}{d^n} - \frac{\log\max\{|\mathbf{A}_{\mathbf{c},2}(\lambda)|_v, |\mathbf{B}_{\mathbf{c},2}(\lambda)|_v\}}{d^2}\right|$$
$$\leq C_{v,\mathbf{c}},$$

for all $\lambda \in \bar{\mathbb{Q}}^*$ *such that* $\mathbf{c}(\lambda) \neq 0, \infty$.

Propositions 4.3 and 4.5 yield Theorem 1.1.

Proof of Theorem 1.1. First of all we deal with the case that either $\mathbf{A}$ or $\mathbf{B}$ is the zero polynomial, i.e., $\mathbf{c} = 0$ or $\mathbf{c} = \infty$ identically. In both cases, we obtain that $\mathbf{B}_{\mathbf{c},n} = 0$ for all $n \geq 1$, i.e., $\mathbf{c}$ is preperiodic for $\mathbf{f}$ being always mapped to ∞. Then the conclusion of Theorem 1.1 holds trivially since $\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) = 0 = \widehat{h}_{\mathbf{f}}(\mathbf{c})$.

Secondly, assuming that both $\mathbf{A}$ and $\mathbf{B}$ are nonzero polynomials, we deal with the values of λ excluded from the conclusion of Proposition 4.5. Since there are finitely many $\lambda \in \bar{\mathbb{Q}}$ such that either $\lambda = 0$ or $\mathbf{A}(\lambda) = 0$ or $\mathbf{B}(\lambda) = 0$ we see that the conclusion of Theorem 1.1 is not affected by these finitely many values of the parameter λ; the difference between $\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ and $\widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot h(\lambda)$ can be absorbed in $O(1)$ for those finitely many values of λ. So, from now on we assume that $\lambda \in \bar{\mathbb{Q}}^*$ such that $\mathbf{c}(\lambda) \neq 0, \infty$.

For each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ let $S_{\mathbf{c}^\sigma}$ be the finite set of places $v \in \Omega_{\mathbb{Q}}$ such that either v is archimedean, or v does not satisfy the hypothesis of Proposition 4.3 with respect to $\mathbf{c}^\sigma$. Let $S = \bigcup S_{\mathbf{c}^\sigma}$, and let C be the maximum of all constants $C_{v,\mathbf{c}^\sigma}$ (from Proposition 4.5) over all $v \in S$ and all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Thus from Propositions 4.3 and 4.5 we obtain for each $\lambda \in \bar{\mathbb{Q}}^*$ such that $\mathbf{A}(\lambda), \mathbf{B}(\lambda) \neq 0$ we have

$$\begin{aligned}
&\left| \frac{h([\mathbf{A}_{\mathbf{c},2}(\lambda) : \mathbf{B}_{\mathbf{c},2}(\lambda)])}{d^2} - \widehat{h}_{f_\lambda}(\mathbf{c}(\lambda)) \right| \\
&= \left| \frac{1}{[K(\lambda) : \mathbb{Q}]} \sum_{\sigma} \sum_{v \in \Omega_{\mathbb{Q}}} \frac{\log \max\{|\mathbf{A}_{\mathbf{c}^\sigma,2}(\sigma(\lambda))|_v, |\mathbf{B}_{\mathbf{c}^\sigma,2}(\sigma(\lambda))|_v\}}{d^2} \right. \\
&\qquad\qquad \left. - \widehat{h}_{\mathbf{f}_{\sigma(\lambda)},v}(\mathbf{c}^\sigma(\sigma(\lambda))) \right| \\
&\leq \frac{1}{[K(\lambda) : \mathbb{Q}]} \sum_{\sigma} \sum_{v \in S} \left| \frac{\log \max\{|A_{\mathbf{c}^\sigma,2}(\sigma(\lambda))|_v, |B_{\mathbf{c}^\sigma,2}(\sigma(\lambda))|_v\}}{d^2} \right. \\
&\qquad\qquad \left. - \widehat{h}_{\mathbf{f}_{\sigma(\lambda)},v}(\mathbf{c}^\sigma(\sigma(\lambda))) \right| \\
&\leq C \cdot |S|,
\end{aligned}$$

where the outer sum is over all embeddings $\sigma : K(\lambda) \longrightarrow \bar{\mathbb{Q}}$.

Finally, since the rational map $t \mapsto g_2(t) := \frac{\mathbf{A}_{\mathbf{c},2}(t)}{\mathbf{B}_{\mathbf{c},2}(t)}$ has degree $d^2 \cdot \widehat{h}_{\mathbf{f}}(\mathbf{c})$ (see Propositions 3.1 and 3.3), [10, Theorem 1.8] yields that there exists a constant C_1 depending only on g_2 (and hence only on the coefficients of $\mathbf{c}$) such that for each $\lambda \in \bar{\mathbb{Q}}$ we have:

$$\left| \frac{h([\mathbf{A}_{\mathbf{c},2}(\lambda) : \mathbf{B}_{\mathbf{c},2}(\lambda)])}{d^2} - \widehat{h}_{\mathbf{f}}(\mathbf{c}) \cdot h(\lambda) \right| \leq C_1. \tag{4.5.1}$$

Using inequality (4.5.1) together with the inequality

$$\left|\frac{h([\mathbf{A}_{\mathbf{c},2}(\lambda):\mathbf{B}_{\mathbf{c},2}(\lambda)])}{d^2}-\widehat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))\right|\le C\cdot|S|,$$

we conclude the proof of Theorem 1.1 (note that S depends only on $\mathbf{c}$). □

5. The case of constant starting point

In this Section we complete the proof of Proposition 4.5 in the case $\mathbf{c}$ is a nonzero constant, and then proceed to proving Theorem 1.2. We start with several useful general results (not only for the case $\mathbf{c}$ is constant).

Proposition 5.1. *Let m and M be positive real numbers, let $d\ge 2$ and $k_0\ge 0$ be integers, and let $\{N_k\}_{k\ge 0}$ be a sequence of positive real numbers. If*

$$m\le\frac{N_{k+1}}{N_k^d}\le M$$

for each $k\ge k_0$, then

$$\left|\lim_{n\to\infty}\frac{\log N_k}{d^k}-\frac{\log N_{k_0}}{d^{k_0}}\right|\le\frac{\max\{-\log(m),\log(M)\}}{d^{k_0}(d-1)}.$$

Proof. We obtain that for each $k\ge k_0$ we have

$$\left|\frac{\log N_{k+1}}{d^{k+1}}-\frac{\log N_k}{d^k}\right|\le\frac{\max\{-\log(m),\log(M)\}}{d^{k+1}}.$$

The conclusion follows by adding the above inequalities for all $k\ge k_0$. □

We let $|\cdot|_v$ be an absolute value on $\bar{\mathbb{Q}}$. As before, for each $\mathbf{c}(t)\in\bar{\mathbb{Q}}(t)$ and for each $t=\lambda\in\bar{\mathbb{Q}}$ we let $M_{\mathbf{c},n,v}(\lambda):=\max\{|\mathbf{A}_{\mathbf{c},n}(\lambda)|_v,|\mathbf{B}_{\mathbf{c},n}(\lambda)|_v\}$ for each $n\ge 0$.

Proposition 5.2. *Consider $\lambda\in\bar{\mathbb{Q}}^*$ and $|\cdot|_v$ an absolute value on $\bar{\mathbb{Q}}$. Let $m\le 1\le M$ be positive real numbers. If $m\le|\lambda|_v\le M$, then for each $1\le n_0\le n$ we have*

$$\left|\frac{\log M_{n,v}(\lambda)}{d^n}-\frac{\log M_{n_0,v}(\lambda)}{d^{n_0}}\right|\le\frac{\log(2M)-\log(m)}{d^{n_0}(d-1)}$$

and therefore the sequence $\left\{\frac{\log M_{n,v}}{d^n}\right\}$ is Cauchy.

Corollary 5.3. *Consider $\lambda\in\bar{\mathbb{Q}}^*$ and $|\cdot|_v$ an absolute value on $\bar{\mathbb{Q}}$. Then for each $n_0\ge 1$ we have*

$$\left|\lim_{n\to\infty}\frac{\log M_{n,v}(\lambda)}{d^n}-\frac{\log M_{n_0,v}(\lambda)}{d^{n_0}}\right|\le\frac{\log(2\max\{1,|\lambda|_v\})-\log(\min\{1,|\lambda|_v\})}{d^{n_0}(d-1)}.$$

Proof of Proposition 5.2. We let $A_n := \mathbf{A}_{\mathbf{c},n}(\lambda)$, $B_n := \mathbf{B}_{\mathbf{c},n}(\lambda)$ and $M_{n,v} := M_{n,v}(\lambda)$.

Lemma 5.4. *Let $\lambda \in \bar{\mathbb{Q}}^*$ and let $|\cdot|_v$ be an absolute value on $\bar{\mathbb{Q}}$. If $|\lambda|_v \le M$, then for each $n \ge 1$, we have $M_{n+1,v} \le (M+1) \cdot M_{n,v}^d$.*

Proof of Lemma 5.4. Since $|\lambda|_v \le M$, we have that for each $n \in \mathbb{N}$, $|A_{n+1}|_v \le (M+1) \cdot M_{n,v}^d$ and also $|B_{n+1}|_v \le M_{n,v}^d$; so

$$M_{n+1,v} \le (M+1) \cdot M_{n,v}^d, \tag{5.4.1}$$

for each $n \ge 1$. □

Because $M \ge 1$, Lemma 5.4 yields that

$$M_{n+1,v} \le 2M \cdot M_{n,v}^d. \tag{5.4.2}$$

The following result will finish our proof.

Lemma 5.5. *If $\lambda \in \bar{\mathbb{Q}}^*$ and $|\cdot|_v$ is an absolute value on $\bar{\mathbb{Q}}$, then for each $n \ge 1$ we have*

$$M_{n+1,v} \ge \frac{\min\{|\lambda|_v, 1\}}{2\max\{|\lambda|_v, 1\}} \cdot M_{n,v}^d.$$

Proof of Lemma 5.5. We let $\ell := \min\{|\lambda|_v, 1\}$ and $L := \max\{|\lambda|_v, 1\}$. Now, if

$$\left(\frac{2L}{\ell}\right)^{\frac{1}{d}} \cdot |B_n|_v \ge |A_n|_v \ge \left(\frac{\ell}{2L}\right)^{\frac{1}{d}} \cdot |B_n|_v,$$

then $M_{n+1,v} \ge |B_{n+1}|_v \ge (\ell/2L)^{(d-1)/d} \cdot M_{n,v}^d$ (note that $\ell < 2L$). On the other hand, if

$$\text{either } \left|\frac{A_n}{B_n}\right|_v > \left(\frac{2L}{\ell}\right)^{\frac{1}{d}} \text{ or } \left|\frac{A_n}{B_n}\right|_v < \left(\frac{\ell}{2L}\right)^{\frac{1}{d}}$$

then $M_{n+1,v} \ge |A_{n+1}|_v > (\ell/2L) \cdot M_{n,v}^d$. Indeed, if $|A_n/B_n|_v > (2L/\ell)^{1/d} > 1$ then

$$|A_{n+1}|_v > |A_n|_v^d \cdot \left(1 - |\lambda|_v \cdot \frac{\ell}{2L}\right) \ge M_{n,v}^d \cdot \left(1 - \frac{\ell}{2}\right) \ge \frac{\ell}{2} \cdot M_{n,v}^d \ge \frac{\ell}{2L} \cdot M_{n,v}^d.$$

Similarly, if $|A_n/B_n|_v < (\ell/2L)^{1/d} < 1$ then

$$|A_{n+1}|_v > |B_n|_v^d \cdot \left(|\lambda|_v - \frac{\ell}{2L}\right) \ge M_{n,v}^d \cdot \left(\frac{\ell}{L} - \frac{\ell}{2L}\right) = \frac{\ell}{2L} \cdot M_{n,v}^d.$$

In conclusion, we get $\frac{\ell}{2L} \cdot M_{n,v}^d \le M_{n+1,v}$ for all n. □

Lemmas 5.4 and 5.5, and Proposition 5.1 finish the proof of Proposition 5.2. □

The next result shows that Proposition 4.5 holds when $\mathbf{c}$ is a constant α, and moreover $|\alpha|_v$ is large compared to $|\lambda|_v$. In addition, this result holds for $d > 2$; the case $d = 2$ will be handled later in Lemma 5.12.

Proposition 5.6. *Assume $d \geq 3$. Let $M \geq 1$ be a real number, let $|\cdot|_v$ be an absolute value on $\bar{\mathbb{Q}}$, let $\lambda, \alpha \in \bar{\mathbb{Q}}$, let $A_n := A_{\lambda,\alpha,n}$, $B_n := B_{\lambda,\alpha,n}$ and $M_{n,v} := \max\{|A_n|_v, |B_n|_v\}$ for $n \geq 0$. Let n_0 be a nonnegative integer. If $|\alpha|_v \geq |\lambda|_v/M \geq 2M$ then for $0 \leq n_0 \leq n$ we have*

$$\left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq \frac{\log(2)}{d^{n_0}(d-1)}.$$

In particular, since we know that for each given λ, $\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n}$ exists, we conclude that

$$\left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq \frac{\log(2)}{d^{n_0}(d-1)}.$$

Proof of Proposition 5.6. We prove by induction on n the following key result.

Lemma 5.7. *For each $n \geq 0$, we have $|A_n|_v \geq \frac{|\lambda|_v}{M} \cdot |B_n|_v$.*

Proof of Lemma 5.7. Set $|\cdot| := |\cdot|_v$. The case $n = 0$ is obvious since $A_0 = \alpha$ and $B_0 = 1$. Now assume $|A_n| \geq \frac{|\lambda|}{M} \cdot |B_n|$ and we prove the statement for $n+1$. Indeed, using that $|\lambda| \geq 2M^2$ and $d \geq 3$ we obtain

$$\begin{aligned}
|A_{n+1}| - \frac{|\lambda|}{M} \cdot |B_{n+1}| &\geq |A_n|^d - |\lambda| \cdot |B_n|^d - \frac{|\lambda|}{M} \cdot |A_n| \cdot |B_n|^{d-1} \\
&= |A_n|^d \cdot \left(1 - |\lambda| \cdot \frac{|B_n|^d}{|A_n|^d} - |\lambda| \cdot \frac{|B_n|^{d-1}}{|A_n|^{d-1}}\right) \\
&\geq |A_n|^d \cdot \left(1 - M^d \cdot |\lambda|^{1-d} - M^{d-1}|\lambda|^{2-d}\right) \\
&\geq |A_n|^d \cdot \left(1 - M^{2-d} \cdot 2^{1-d} - M^{3-d} \cdot 2^{2-d}\right) \\
&\geq |A_n|^d \cdot \left(1 - 2^{-2} - 2^{-1}\right) \\
&\geq 0,
\end{aligned}$$

as desired. □

Lemma 5.7 yields that $M_{n,v} = |A_n|_v$ for each n (using that $|\lambda|_v/M \geq 2M > 1$). Furthermore, Lemma 5.7 yields

$$\left|M_{n+1,v} - M_{n,v}^d\right| \leq |\lambda \cdot B_n^d|_v \leq M_{n,v}^d \cdot M^d |\lambda|_v^{1-d} \leq M_{n,v}^d \cdot M^{2-d} \cdot 2^{1-d} \leq \frac{1}{4} \cdot M_{n,v}^d,$$

because $|\lambda|_v \geq 2M^2$, $M \geq 1$ and $d-1 \geq 2$. Thus for each $n \geq 1$ we have

$$\frac{3}{4} \leq \frac{M_{n+1,v}}{M_{n,v}^d} \leq \frac{5}{4}. \tag{5.7.1}$$

Then Proposition 5.1 yields the desired conclusion. □

The next result yields the conclusion of Proposition 4.5 for when the starting point $\mathbf{c}$ is constant equal to α, and d is larger than 2.

Proposition 5.8. *Assume $d > 2$. Let $\alpha, \lambda \in \bar{\mathbb{Q}}^*$, let $|\cdot|_v$ be an absolute value, and for each $n \geq 0$ let $A_n := A_{\lambda,\alpha,n}$, $B_n := B_{\lambda,\alpha,n}$ and $M_{n,v} := \max\{|A_n|_v, |B_n|_v\}$. Consider $L := \max\{|\alpha|_v, 1/|\alpha|_v\}$. Then for all $n_0 \geq 1$ we have*

$$\left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq (3d-2)\log(2L).$$

Proof. We split our proof into three cases: $|\lambda|_v$ is large compared to $|\alpha|_v$; $|\lambda|_v$ and $|\alpha|_v$ are comparable, and lastly, $|\lambda|_v$ is very small. We start with the case $|\lambda|_v \gg |\alpha|_v$. Firstly, we note $L = \max\left\{|\alpha|_v, |\alpha|_v^{-1}\right\} \geq 1$.

Lemma 5.9. *If $|\lambda|_v > 8L^d$ then for integers $1 \leq n_0 \leq n$ we have*

$$\left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq \frac{\log(2)}{d^{n_0}(d-1)}. \tag{5.9.1}$$

Proof of Lemma 5.9. Since $|\lambda|_v > 8L^d$, then $|\alpha|_v^{d-1} < \frac{|\lambda|_v}{2|\alpha|_v}$ and therefore

$$|f_\lambda(\alpha)|_v = \left|\alpha^{d-1} + \frac{\lambda}{\alpha}\right|_v > \frac{|\lambda|_v}{2|\alpha|_v} \geq \frac{|\lambda|_v}{2L} \geq 4L.$$

This allows us to apply Proposition 5.6 coupled with (2.0.13) (with $k_0 = 1$) and obtain that for all $1 \leq n_0 \leq n$ we have

$$\begin{aligned}
&\left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \\
&= \frac{1}{d} \cdot \left|\frac{\log\max\left\{|A_{\lambda,f_\lambda(\alpha),n-1}|_v, |B_{\lambda,f_\lambda(\alpha),n-1}|_v\right\}}{d^{n-1}}\right. \\
&\qquad\qquad \left. - \frac{\log\max\left\{|A_{\lambda,f_\lambda(\alpha),n_0-1}|_v, |B_{\lambda,f_\lambda(\alpha),n_0-1}|_v\right\}}{d^{n_0-1}}\right| \\
&\leq \frac{1}{d} \cdot \frac{\log(2)}{d^{n_0-1}(d-1)},
\end{aligned}$$

as desired. □

Let $R = \frac{1}{4^d L^d}$. If $R \leq |\lambda|_v \leq 8L^d$, then Proposition 5.2 yields that for all $1 \leq n_0 \leq n$ we have

$$\left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq \frac{2d\log(4L)}{d^{n_0}(d-1)} \leq \log(4L). \tag{5.9.2}$$

So we are left to analyze the range $|\lambda|_v < R$.

Lemma 5.10. *If $|\lambda|_v < R$, then $\left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq (3d-2)\log(2L)$ for all integers $0 \leq n_0 \leq n$.*

Proof of Lemma 5.10. First we note that since $|\lambda|_v < R < 1$, Lemma 5.4 yields that $M_{n+1,v} \le 2 \cdot M_{n,v}^d$ and arguing as in the proof of Proposition 5.1 we obtain that for all $0 \le n_0 \le n$ we have

$$\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}} \le \frac{\log(2)}{d^{n_0}(d-1)}. \tag{5.10.1}$$

Next, we will establish a lower bound for the main term from (5.10.1). Since

$$|f_\lambda^0(\alpha)|_v = |\alpha|_v \ge \frac{1}{L} > \sqrt[d]{2R} > \sqrt[d]{2|\lambda|_v},$$

we conclude that the smallest integer n_1 (if it exists) satisfying

$$|f_\lambda^{n_1}(\alpha)|_v < \sqrt[d]{2|\lambda|_v}$$

is positive. We will now derive a lower bound for n_1 (if n_1 exists) in terms of L.

We know that for all $n \in \{0, \dots, n_1 - 1\}$ we have $|f_\lambda^n(\alpha)|_v \ge \sqrt[d]{2|\lambda|_v}$. Hence, for each $0 \le n \le n_1 - 1$ we have

$$|A_{n+1}|_v \ge |A_n|_v^d \cdot \left(1 - \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v^d}\right) \ge \frac{|A_n|_v^d}{2}. \tag{5.10.2}$$

On the other hand,

$$\frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v} \le \frac{|f_\lambda^n(\alpha)|_v^{d-1}}{2}. \tag{5.10.3}$$

So, for each $0 \le n \le n_1 - 1$ we have

$$|f_\lambda^{n+1}(\alpha)|_v \ge |f_\lambda^n(\alpha)|_v^{d-1} - \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v} \ge \frac{|f_\lambda^n(\alpha)|_v^{d-1}}{2}. \tag{5.10.4}$$

Therefore, repeated applications of (5.10.4) yield that for $0 \le n \le n_1$ we have

$$|f_\lambda^n(\alpha)|_v \ge \frac{|\alpha|_v^{(d-1)^n}}{2^{\frac{(d-1)^n-1}{d-2}}} \ge \frac{1}{L^{(d-1)^n} \cdot 2^{\frac{(d-1)^n-1}{d-2}}} \ge \frac{1}{(2L)^{(d-1)^n}}, \tag{5.10.5}$$

because $|\lambda|_v \ge 1/L$ and $d - 2 \ge 1$. So, if $|f_\lambda^{n_1}(\alpha)|_v < \sqrt[d]{2|\lambda|_v}$, then

$$\frac{1}{(2L)^{(d-1)^{n_1}}} < \sqrt[d]{2|\lambda|_v}.$$

Using now the fact that $\log(2) < \log(2L) \cdot (d-1)^{n_1}$ and that $d \le (d-1)^2 - 1$ (since $d \ge 3$) we obtain

$$\log\left(\frac{1}{|\lambda|_v}\right) < \log(2L) \cdot (d-1)^{n_1+2}. \tag{5.10.6}$$

Moreover, inequality (5.10.5) yields that for each $0 \le n \le n_1 - 1$, we have

$$|B_{n+1}|_v = |B_n|_v^d \cdot |f_\lambda^n(\alpha)|_v \ge |B_n|_v^d \cdot \frac{1}{(2L)^{(d-1)^n}}. \tag{5.10.7}$$

Combining (5.10.2) and (5.10.7) we get $M_{n+1,v} \geq \frac{M_{n,v}^d}{(2L)^{(d-1)^n}}$, if $0 \leq n \leq n_1 - 1$. So,

$$\frac{\log M_{n+1,v}}{d^{n+1}} \geq \frac{\log M_{n,v}}{d^n} - \log(2L) \cdot \left(\frac{d-1}{d}\right)^n. \tag{5.10.8}$$

Summing up (5.10.8) starting from $n = n_0$ to $N-1$ for some $N \leq n_1$, and using inequality (5.10.1) we obtain that for $0 \leq n_0 \leq n \leq n_1$ we have

$$\left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \leq d\log(2L). \tag{5.10.9}$$

Now, for $n \geq n_1$, we use Lemma 5.5 and obtain

$$M_{n+1,v} \geq \frac{\min\{|\lambda|_v, 1\}}{2\max\{|\lambda|_v, 1\}} \cdot M_{n,v}^d = \frac{|\lambda|_v}{2} \cdot M_{n,v}^d, \tag{5.10.10}$$

because $|\lambda|_v < R < 1$. Inequalities (5.10.1) and (5.10.10) yield that for all $n \geq n_0 \geq n_1$, we have

$$\begin{aligned} \left|\frac{\log(M_{n,v})}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| &\leq \log\left(\frac{2}{|\lambda|_v}\right) \cdot \sum_{n=n_0}^{n-1} \frac{1}{d^{n+1}} \\ &< \frac{2\log\left(\frac{1}{|\lambda|_v}\right)}{(d-1)\cdot d^{n_1}}. \end{aligned} \tag{5.10.11}$$

In establishing inequality (5.10.11) we also used the fact that $|\lambda|_v < R < 1/2$ and so, $\log(2/|\lambda|_v) < 2\log(1/|\lambda|_v)$. Combining inequalities (5.10.6), (5.10.9) and (5.10.11) yields that for all $0 \leq n_0 \leq n$ we have

$$\begin{aligned} \left|\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| &< d\log(2L) + \frac{2\cdot(d-1)^{n_1+2}\log(2L)}{(d-1)\cdot d^{n_1}} \\ &< \log(2L)\cdot(d + 2\cdot(d-1)) \\ &\leq (3d-2)\log(2L), \end{aligned}$$

as desired.

If on the other hand, we had that $|f_\lambda^n(\alpha)|_v \geq \sqrt[d]{2|\lambda|_v}$ for all $n \in \mathbb{N}$, we get that equation (5.10.9) holds for all $n \in \mathbb{N}$. Hence, in this case too, the Lemma follows. □

Lemmas 5.9 and 5.10 and Inequality (5.9.2) finish the proof of Proposition 5.6. □

For $d = 2$ we need a separate argument for proving Proposition 4.5 when $\mathbf{c}$ is constant.

Proposition 5.11. *Let $d = 2$, $\alpha, \lambda \in \bar{\mathbb{Q}}^*$, let $|\cdot|_v$ be an absolute value, and for each $n \geq 0$ let $A_n := A_{\lambda,\alpha,n}$, $B_n := B_{\lambda,\alpha,n}$ and*

$$M_{n,v} := \max\{|A_n|_v, |B_n|_v\}.$$

Let $L := \max\{|\alpha|_v, 1/|\alpha|_v\}$. *Then for all* $1 \le n_0 \le n$ *we have*

$$\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \le 4\log(2L).$$

In particular, since we know (by Corollary 5.3) that $\lim_{n\to\infty} \frac{\log M_{n,v}}{2^n}$ exists, we conclude that

$$\left|\lim_{n\to\infty}\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \le 4\log(2L).$$

Proof of Proposition 5.11. We employ the same strategy as for the proof of Proposition 5.8, but there are several technical difficulties for this case. Essentially the problem lies in the fact that ∞ is not a superattracting (fixed) point for $\mathbf{f}_\lambda(z) = \frac{z^2+\lambda}{z}$. So the main change is dealing with the case when $|\lambda|_v$ is large, but there are changes also when dealing with the case $|\lambda|_v$ is close to 0.

Lemma 5.12. *Assume* $|\lambda|_v > Q := (2L)^4$. *Then for integers* $1 \le n_0 \le n$ *we have*

$$\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| < \frac{5}{2}.$$

Proof of Lemma 5.12. Let k_1 be the smallest positive integer (if it exists) such that $|f_\lambda^{k_1}(\alpha)|_v < \sqrt{2|\lambda|_v}$. So, we know that $|f_\lambda^n(\alpha)|_v \ge \sqrt{2|\lambda|_v}$ if $1 \le n \le k_1 - 1$. We will show that $k_1 > \log_4\left(\frac{|\lambda|_v}{4L^2}\right) \ge 1$ (note that $|\lambda|_v > Q = (2L)^4$).

Claim 5.13. For each positive integer $n \le \log_4\left(\frac{|\lambda|_v}{2L}\right)$ we have

$$|f_\lambda^n(\alpha)|_v \ge \frac{|\lambda|_v}{2^n L}.$$

Proof of Claim 5.13. The claim follows by induction on n; the case $n = 1$ holds since $|\lambda|_v > (2L)^4$ and so,

$$|f_\lambda(\alpha)|_v \ge \frac{|\lambda|_v}{|\alpha|_v} - |\alpha|_v \ge \frac{|\lambda|_v}{L} - L \ge \frac{|\lambda|_v}{2L}.$$

Now, assume for $1 \le n \le \log_4\left(\frac{|\lambda|_v}{2L}\right)$ we have $|f_\lambda^n(\alpha)|_v \ge \frac{|\lambda|_v}{2^n L}$. Then

$$|f_\lambda^{n+1}(\alpha)|_v \ge |f_\lambda^n(\alpha)|_v - \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v} \ge \frac{|\lambda|_v}{2^n L} - 2^n L \ge \frac{|\lambda|_v}{2^{n+1} L},$$

because $|\lambda|_v \ge 4^n \cdot 2L$ since $n \le \log_4\left(\frac{|\lambda|_v}{2L}\right)$. This concludes our proof. ☐

Claim 5.13 yields that for each $1 \le n \le \log_4\left(\frac{|\lambda|_v}{4L^2}\right) < \log_4\left(\frac{|\lambda|_v}{2L}\right)$ we have

$$|f_\lambda^n(\alpha)|_v \ge \frac{|\lambda|_v}{2^n L} \ge \frac{|\lambda|_v}{\sqrt{\frac{|\lambda|_v}{4L^2}} \cdot L} > \sqrt{2|\lambda|_v}.$$

Hence,

$$k_1 > \log_4\left(\frac{|\lambda|_v}{4L^2}\right). \tag{5.13.1}$$

Now for each $1 \le n \le k_1 - 1$ we have

$$\frac{|A_n|_v}{|B_n|_v} = |f_\lambda^n(\alpha)|_v \ge \sqrt{2|\lambda|_v} > 1, \tag{5.13.2}$$

because $|\lambda|_v > Q > 2$ and so, $M_{n,v} = |A_n|_v$. Furthermore,

$$|f_\lambda^{k_1}(\alpha)|_v \ge |f_\lambda^{k_1-1}(\alpha)|_v - \frac{|\lambda|_v}{|f_\lambda^{k_1-1}(\alpha)|_v} \ge \sqrt{2|\lambda|_v} - \sqrt{\frac{|\lambda|_v}{2}} = \sqrt{\frac{|\lambda|_v}{2}} > 1.$$

Hence $M_{k_1} = |A_{k_1}|_v$ and therefore, for each $1 \le n \le k_1 - 1$, using (5.13.2) we have

$$|M_{n+1,v} - M_{n,v}^2| \le |\lambda|_v \cdot |B_n|_v^2 \le \frac{|A_n|_v^2}{2} = \frac{M_{n,v}^2}{2}.$$

Hence $\frac{M_{n,v}^2}{2} \le M_{n+1,v} \le \frac{3M_{n,v}^2}{2}$, and so

$$\left|\frac{\log M_{n+1,v}}{2^{n+1}} - \frac{\log M_{n,v}}{2^n}\right| < \frac{\log(2)}{2^{n+1}}, \tag{5.13.3}$$

for $1 \le n \le k_1 - 1$. The next result establishes a similar inequality to (5.13.3) which is valid for all $n \in \mathbb{N}$.

Claim 5.14. For each $n \ge 1$ we have $\frac{1}{2|\lambda|_v} \le \frac{M_{n+1,v}}{M_{n,v}^2} \le 2|\lambda|_v$.

Proof of Claim 5.14. The lower bound is an immediate consequence of Lemma 5.5 (note that $|\lambda|_v > Q > 1$), while the upper bound follows from Lemma 5.4. $\square$

Using Claim 5.14 we obtain that for all $n \ge 1$ we have

$$\left|\frac{\log M_{n+1,v}}{2^{n+1}} - \frac{\log M_{n,v}}{2^n}\right| \le \frac{\log(2|\lambda|_v)}{2^{n+1}}. \tag{5.14.1}$$

Using inequalities (5.13.1), (5.13.3) and (5.14.1) we obtain that for all $1 \le n_0 \le n$ we have

$$\begin{aligned} &\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \\ &\le \sum_{n=1}^{k_1-1} \frac{\log(2)}{2^{n+1}} + \sum_{n=k_1}^{\infty} \frac{\log(2|\lambda|_v)}{2^{n+1}} \\ &\le \frac{\log(2)}{2} + \frac{\log(2|\lambda|_v)}{2^{k_1}} \\ &\le \frac{\log(2)}{2} + \frac{\log(2|\lambda|_v)}{\sqrt{\frac{|\lambda|_v}{4L^2}}} \end{aligned} \tag{5.14.2}$$

$$\leq \frac{1}{2} + \frac{\log(2|\lambda|_v)}{\sqrt[4]{|\lambda|_v}} \qquad \text{(because } |\lambda|_v > (2L)^4\text{)}$$
$$< \frac{5}{2} \text{ (because } |\lambda|_v > Q \geq 16\text{)},$$

as desired.

If on the other hand for all $n \in \mathbb{N}$, we have that $|f_\lambda^n(\alpha)|_v \geq \sqrt{2|\lambda|_v}$, we get that equation (5.13.3) holds for all $n \in \mathbb{N}$. Hence, in this case too the Lemma follows. □

Let $R = \frac{1}{4L^2}$. If $R \leq |\lambda|_v \leq Q$ then for each $n_0 \geq 1$, Proposition 5.2 yields

$$\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \leq \frac{\log(2Q) - \log(R)}{2} < \frac{7\log(2L)}{2} < 4\log(2L). \tag{5.14.3}$$

Next we deal with the case $|\lambda|_v$ is small.

Lemma 5.15. *If $|\lambda|_v < R$, then for all $1 \leq n_0 \leq n$, we have*

$$\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \leq 3\log(2L).$$

Proof of Lemma 5.15. The argument is similar to that in the proof of Lemma 5.12, only that this time we do not know that $|f_\lambda^n(\alpha)|_v > 1$ (and therefore we do not know that $|A_n|_v > |B_n|_v$) because $|\lambda|_v$ is small. Also, the proof is similar to the proof of Lemma 5.10, but there are several changes due to the fact that $d = 2$.

We note that since $|\lambda|_v < R < 1$ then Lemma 5.4 yields

$$M_{n+1} \leq 2M_n^2. \tag{5.15.1}$$

Now, let n_1 be the smallest integer (if it exists) such that

$$|f_\lambda^{n_1}(\alpha)|_v < \sqrt{2|\lambda|_v}.$$

Note that $|f_\lambda^0(\alpha)|_v = |\alpha|_v \geq \frac{1}{L} \geq \sqrt{2|\lambda|_v}$ because $|\lambda|_v < R = \frac{1}{4L^2}$. Hence (if n_1 exists), we get that $n_1 \geq 1$. In particular, for each $0 \leq n \leq n_1 - 1$ we have $|f_\lambda^n(\alpha)|_v \geq \sqrt{2|\lambda|_v}$ and this yields

$$|A_{n+1}| \geq |A_n|^2 \cdot \left(1 - \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v^2}\right) \geq \frac{|A_n|^2}{2}. \tag{5.15.2}$$

On the other hand,

$$\frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v} \leq \frac{|f_\lambda^n(\alpha)|_v}{2}. \tag{5.15.3}$$

So, for each $0 \leq n \leq n_1 - 1$ we have

$$|f_\lambda^{n+1}(\alpha)|_v \geq |f_\lambda^n(\alpha)|_v - \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v} \geq \frac{|f_\lambda^n(\alpha)|_v}{2}. \tag{5.15.4}$$

Therefore, repeated applications of (5.15.4) yield for $n \le n_1$ that

$$|f_\lambda^n(\alpha)|_v \ge \frac{|\alpha|_v}{2^n} \ge \frac{1}{2^n L}, \tag{5.15.5}$$

because $|\alpha|_v \ge 1/L$. So, for each $n \ge 0$ we have

$$|B_{n+1}|_v = |B_n|_v^2 \cdot |f_\lambda^n(\alpha)|_v \ge |B_n|_v^2 \cdot \frac{1}{2^n L}. \tag{5.15.6}$$

Combining (5.15.2) and (5.15.6) we get

$$M_{n+1} \ge \frac{M_n^2}{L \cdot 2^{\max\{1,n\}}} \tag{5.15.7}$$

for all $n \ge 0$. Using (5.15.1) and (5.15.7) we obtain for $0 \le n \le n_1 - 1$ that

$$\left| \frac{\log M_{n+1}}{2^{n+1}} - \frac{\log M_n}{2^n} \right| \le \frac{\max\{1,n\} \cdot \log(2) + \log(L)}{2^{n+1}}. \tag{5.15.8}$$

Summing up (5.15.8) starting from $n = n_0$ to $n = n_1 - 1$ we obtain that for $1 \le n_0 \le n \le n_1$ we have

$$\begin{aligned} \left| \frac{\log M_n}{2^n} - \frac{\log M_{n_0}}{2^{n_0}} \right| &\le \sum_{k=n_0}^{n-1} \frac{k \log(2) + \log(L)}{2^{k+1}} \\ &< \log(2) + \log(L) = \log(2L). \end{aligned} \tag{5.15.9}$$

Using inequality (5.15.5) for $n = n_1$ yields $\frac{1}{2^{n_1} L} \le |f_\lambda^{n_1}(\alpha)|_v < \sqrt{2|\lambda|_v}$, and so,

$$\frac{1}{|\lambda|_v} < 4^{n_1} \cdot 2L^2. \tag{5.15.10}$$

Now, for $n \ge n_1$, we use Lemma 5.5 and obtain

$$M_{n+1} \ge \frac{\min\{|\lambda|_v, 1\}}{2 \max\{|\lambda|_v, 1\}} \cdot M_n^2 = \frac{|\lambda|_v}{2} \cdot M_n^2, \tag{5.15.11}$$

because $|\lambda| < R < 1$. Inequality (5.15.11) coupled with inequality (5.15.1) yields that for all $n \ge n_0 \ge n_1$, we have

$$\left| \frac{\log(M_n)}{2^n} - \frac{\log M_{n_0}}{2^{n_0}} \right| < \log\left(\frac{2}{|\lambda|_v} \right) \cdot \sum_{n=n_0}^{n-1} \frac{1}{2^{n+1}} < \frac{\log\left(\frac{2}{|\lambda|_v} \right)}{2^{n_1}}. \tag{5.15.12}$$

Combining inequalities (5.15.10), (5.15.8) and (5.15.12) yields that for all $1 \le n_0 \le n$ we have

$$\begin{aligned} \left| \frac{\log M_n}{2^n} - \frac{\log M_{n_0}}{2^{n_0}} \right| &< \log(2L) + \frac{(n_1+1)\log(4) + 2\log(L)}{2^{n_1}} \\ &< \log(2L) + \log(4) + \log(L) \\ &\le 3\log(2L), \end{aligned} \tag{5.15.13}$$

as desired. ∎

Lemmas 5.12 and 5.15, and inequality (5.14.3) complete the proof of Proposition 5.11. $\square$

Proof of Theorem 1.2. First we deal with the case $\alpha = 0$. In this case, $\alpha = 0$ is preperiodic under the action of the family $\mathbf{f}_\lambda$ and so, $\widehat{h}_{\mathbf{f}_\lambda}(\alpha) = 0 = h(\alpha)$. From now on, assume that $\alpha \neq 0$. Secondly, if $\lambda = 0$ (and $d \geq 3$) then $\widehat{h}_{\mathbf{f}_0}(\alpha) = h(\alpha)$ (since $\mathbf{f}_0(z) = z^{d-1}$) and thus

$$\widehat{h}_{\mathbf{f}_0}(\alpha) - \widehat{h}_{\mathbf{f}}(\alpha) \cdot h(\alpha) = \frac{d-1}{d} \cdot h(\alpha) \leq 6d \cdot h(\alpha),$$

and so the conclusion of Theorem 1.2 holds. So, from now on we assume both α and λ are nonzero.

Propositions 5.8 and 5.11 allow us to apply the same strategy as in the proof of Theorem 1.1 only that this time it suffices to compare $\widehat{h}_{f_\lambda}(\alpha)$ and $h([A_{\lambda,\alpha,1} : B_{\lambda,\alpha,1}])$. As before, we let S be the set of places of $\mathbb{Q}$ containing the archimedean place and all the nonarchimedean places v for which there exists some $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $|\sigma(\alpha)|_v \neq 1$. Since $\alpha \neq 0$, we have that S is finite; moreover $|S| \leq 1 + \ell$. So, applying Proposition 4.3 and Propositions 5.8 and 5.11 with $n_0 = 1$ (see also (2.0.12)) we obtain

$$\begin{aligned}
&\left|\frac{h([A_{\lambda,\alpha,1} : B_{\lambda,\alpha,1}])}{d} - \widehat{h}_{f_\lambda}(\alpha)\right| \\
&= \left|\frac{1}{[K(\lambda) : \mathbb{Q}]} \sum_{\sigma} \sum_{v \in \Omega_{\mathbb{Q}}} \frac{\log\max\{|A_{\sigma(\lambda),\sigma(\alpha),1}|_v, |B_{\sigma(\lambda),\sigma(\alpha),1}|_v\}}{d} \right.\\
&\qquad\qquad\left. - \widehat{h}_{\mathbf{f}_{\sigma(\lambda)},v}(\sigma(\alpha))\right| \\
&\leq \frac{1}{[K(\lambda) : \mathbb{Q}]} \sum_{\sigma} \sum_{v \in S} \left|\frac{\log\max\{|A_{\sigma(\lambda),\sigma(\alpha),1}|_v, |B_{\sigma(\lambda),\sigma(\alpha),1}|_v\}}{d} \right.\\
&\qquad\qquad\left. - \widehat{h}_{\mathbf{f}_{\sigma(\lambda)},v}(\sigma(\alpha))\right| \\
&\leq \frac{3d-2}{[K(\lambda) : \mathbb{Q}]} \sum_{\sigma : K(\lambda) \longrightarrow \bar{\mathbb{Q}}} \sum_{v \in S} \log\left(2\max\{|\sigma(\alpha)|_v, |\sigma(\alpha)|_v^{-1}\}\right) \\
&\leq \frac{3d-2}{[K(\lambda) : \mathbb{Q}]} \sum_{\sigma : K(\lambda) \longrightarrow \bar{\mathbb{Q}}} \sum_{v \in S} \left(\log(2) + \log^+|\sigma(\alpha)|_v + \log^+\left|\sigma\left(\frac{1}{\alpha}\right)\right|_v\right) \\
&\leq (3d-2) \cdot \left(|S| + h(\alpha) + h\left(\frac{1}{\alpha}\right)\right) \\
&\leq (3d-2) \cdot (1 + \ell + 2h(\alpha)).
\end{aligned}$$

On the other hand, using that $\widehat{h}_{\mathbf{f}}(\alpha) = 1/d$ (by Proposition 3.1) and also using the basic inequalities (1)–(3) for the Weil height from Subsection 2.4

we obtain

$$\begin{aligned}
&\left|\frac{h([A_{\lambda,\alpha,1} : B_{\lambda,\alpha,1}])}{d} - \widehat{h}_{\mathbf{f}}(\alpha)\cdot h(\lambda)\right| \\
&= \left|\frac{h\left(\alpha^{d-1}+\frac{\lambda}{\alpha}\right)}{d} - \frac{h(\lambda)}{d}\right| \\
&\le \frac{1}{d}\cdot\left(\left|h\left(\alpha^{d-1}+\frac{\lambda}{\alpha}\right) - h\left(\frac{\lambda}{\alpha}\right)\right| + \left|h\left(\frac{\lambda}{\alpha}\right) - h(\lambda)\right|\right) \\
&\le \frac{1}{d}\cdot((d-1)h(\alpha)+\log(2)+h(\alpha)) \\
&< h(\alpha)+1.
\end{aligned}$$

This finishes the proof of Theorem 1.2. □

Remark 5.16. Theorem 1.2 yields an effective method for finding all $\lambda\in\bar{\mathbb{Q}}$ such that a given point $\alpha\in\bar{\mathbb{Q}}$ is preperiodic under the action of $\mathbf{f}_\lambda$.

6. Proof of our main result

So we are left to proving Proposition 4.5 in full generality. We fix a place $v\in\Omega_{\mathbb{Q}}$. Before completing the proof of Proposition 4.5 we need one more result.

Proposition 6.1. *Assume $d>2$. Let $\alpha,\lambda\in\bar{\mathbb{Q}}$, and let $|\cdot|_v$ be an absolute value. We let $A_n := A_{\lambda,\alpha,n}$, $B_n := B_{\lambda,\alpha,n}$ and $M_{n,v} := \max\{|A_n|_v, |B_n|_v\}$. If $|\alpha|_v\ge 2$ and $|\lambda|_v\le\frac{1}{2}$ then for each $n_0\ge 0$ we have*

$$\left|\lim_{n\to\infty}\frac{\log M_{n,v}}{d^n} - \frac{\log M_{n_0,v}}{d^{n_0}}\right| \le \frac{\log(2)}{d^{n_0}(d-1)}.$$

Proof. First we claim that for each $n\ge 0$ we have $|f_\lambda^n(\alpha)|_v\ge 2$. Indeed, for $n=0$ we have $|\alpha|_v\ge 2$ as desired. We assume $|f_\lambda^n(\alpha)|_v\ge 2$ and since $|\lambda|_v\le 1/2$ we get that

$$|f_\lambda^{n+1}(\alpha)|_v \ge |f_\lambda^n(\alpha)|_v^{d-1} - \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v} \ge 4-\frac{1}{4} > 2.$$

Hence $M_{n,v} = |A_n|_v$ and we obtain that

$$\left|M_{n+1,v} - M_{n,v}^d\right| \le |\lambda|_v\cdot|B_n|_v^d \le \frac{M_{n,v}^d}{2\cdot\left|\frac{A_n}{B_n}\right|_v^d} = \frac{M_{n,v}^d}{2|f_\lambda^n(\alpha)|_v^d} \le \frac{M_{n,v}^d}{16}$$

because $d\ge 3$. Thus Proposition 5.1 yields the desired conclusion. □

The next result deals with the case $d=2$ in Proposition 6.1.

Proposition 6.2. *Assume $d=2$. Let $M\ge 1$ be a real number, let $\alpha,\lambda\in\bar{\mathbb{Q}}$, and let $|\cdot|_v$ be an absolute value. We let $A_n := A_{\lambda,\alpha,n}$, $B_n := B_{\lambda,\alpha,n}$ and*

$M_{n,v} := \max\{|A_n|_v, |B_n|_v\}$. *If* $|\alpha|_v \geq \frac{1}{M\cdot|\lambda|_v} \geq 2M$ *then for each* $0 \leq n_0 \leq n$ *we have*

$$\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \leq 4\log(2).$$

In particular, using Corollary 5.3 we obtain that for all $n_0 \geq 0$ we have

$$\left|\lim_{n\to\infty}\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \leq 4\log(2).$$

Proof of Proposition 6.2. Since $|\lambda|_v \leq \frac{1}{2M^2} \leq \frac{1}{2}$, using Lemmas 5.4 and 5.5 we obtain for all $n \geq 0$ that

$$\left|\frac{\log M_{n+1,v}}{2^{n+1}} - \frac{\log M_{n,v}}{2^n}\right| \leq \frac{\log\left(\frac{2}{|\lambda|_v}\right)}{2^{n+1}}. \tag{6.2.1}$$

We need to improve the above bound and in order to do this we prove a sharper inequality when n is small compared to $\frac{1}{|\lambda|_v}$. The strategy is similar to the one employed in the proof of Lemma 5.12.

First of all, since $|\lambda|_v < 1$, Lemma 5.4 yields that for all $n \geq 0$ we have

$$\frac{\log M_{n+1,v}}{2^{n+1}} - \frac{\log M_{n,v}}{2^n} \leq \frac{\log(2)}{2^{n+1}}. \tag{6.2.2}$$

We will prove a lower bound for the main term from (6.2.2) when n_0 and n are small compared to $\frac{1}{|\lambda|_v}$. First we prove that $|f_\lambda^n(\alpha)|_v$ is large when n is small.

Lemma 6.3. *For each integer* $n \leq \frac{3M^2}{4|\lambda|_v}$, *we have* $|f_\lambda^n(\alpha)|_v \geq \frac{3M}{2}$.

Proof of Lemma 6.3. We will prove the statement inductively. For $n = 0$, we know that $|f_\lambda^0(\alpha)|_v = |\alpha|_v \geq 2M$. If now for some $n \geq 0$ we have that $|f_\lambda^n(\alpha)|_v \geq \frac{3M}{2}$, then $|f_\lambda^{n+1}(\alpha)|_v \geq |f_\lambda^n(\alpha)|_v - \frac{2|\lambda|_v}{3M}$. Therefore, for all $n \leq \frac{3M^2}{4|\lambda|_v}$ we have

$$|f_\lambda^n(\alpha)|_v \geq |\alpha|_v - \frac{n\cdot 2|\lambda|_v}{3M^2} \geq \frac{3M}{2},$$

as desired. □

In conclusion, if we let n_1 be the smallest positive integer larger than $\frac{3M^2}{4|\lambda|}$ we know that for all $0 \leq n \leq n_1 - 1$ we have $|f_\lambda^n(\alpha)|_v \geq \frac{3}{2}$. In particular,

$$|f_\lambda^{n_1}(\alpha)|_v \geq |f_\lambda^{n_1-1}(\alpha)|_v - \frac{|\lambda|_v}{|f_\lambda^{n_1-1}(\alpha)|_v} \geq \frac{3}{2} - \frac{\frac{1}{2}}{\frac{3}{2}} > 1.$$

Therefore, $M_n = |A_n|$ for all $0 \leq n \leq n_1$, and moreover for $0 \leq n \leq n_1 - 1$ we have

(6.3.1)

$$M_{n+1} - M_n^2 = |A_{n+1}|_v - |A_n^2|_v \geq -|\lambda|_v \cdot |B_n|_v^2 = -M_n^2 \cdot \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v^2} \geq -\frac{4M_n^2}{9},$$

because $|\lambda|_v < 1$ and $|f_\lambda^n(\alpha)|_v \ge \frac{3}{2}$. Inequality (6.3.1) coupled with the argument from Proposition 5.1 yields that for all $0 \le n \le n_1 - 1$ we have

$$\frac{\log M_{n+1}}{2^{n+1}} - \frac{\log M_n}{2^n} > -\frac{\log(2)}{2^{n+1}}. \tag{6.3.2}$$

Using the definition of n_1 and inequalities (6.2.1), (6.2.2) and (6.3.2), we conclude that

$$\begin{aligned}\left|\frac{\log M_n}{2^n} - \frac{\log M_{n_0}}{2^{n_0}}\right| &\le \sum_{n=0}^{n_1-1}\frac{\log(2)}{2^{n+1}} + \sum_{n=n_1}^{\infty}\frac{\log\left(\frac{2}{|\lambda|}\right)}{2^{n+1}}\\ &\le \log(2) + \frac{\log\left(\frac{8n_1}{3M^2}\right)}{2^{n_1}}\\ &\le 4\log(2),\end{aligned}$$

for all $0 \le n_0 \le n$. ◻

Finally, we will establish the equivalent of Proposition 5.6 for $d = 2$.

Proposition 6.4. *Assume $d = 2$. Let $M \ge 1$ be a real number, let $\alpha, \lambda \in \bar{\mathbb{Q}}$, and let $|\cdot|_v$ be an absolute value. We let $A_n := A_{\lambda,\alpha,n}$, $B_n := B_{\lambda,\alpha,n}$ and $M_{n,v} := \max\{|A_n|_v, |B_n|_v\}$. If $|\alpha|_v \ge \frac{|\lambda|_v}{M} \ge 8M$, then for each $0 \le n_0 \le n$ we have*

$$\left|\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \le 1 + 8M.$$

In particular, using Corollary 5.3 we obtain that for all $n_0 \ge 0$ we have

$$\left|\lim_{n\to\infty}\frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}}\right| \le 1 + 8M.$$

Proof of Proposition 6.4. We know that $|\lambda|_v \ge 8M^2 > 1$. Thus, Lemmas 5.4 and 5.5 yield that for all $n \ge 0$ we have

$$\left|\frac{\log M_{n+1,v}}{2^{n+1}} - \frac{\log M_{n,v}}{2^n}\right| \le \frac{\log(2|\lambda|_v)}{2^{n+1}}. \tag{6.4.1}$$

As in the proof of Proposition 6.2, we will find a sharper inequality for small n. Arguing identically as in Claim 5.13, we obtain that for $0 \le n \le \log_4\left(\frac{|\lambda|_v}{2M}\right)$ we have

$$|f_\lambda^n(\alpha)|_v \ge \frac{|\lambda|_v}{2^n M} \ge 2^{n+1} \ge 2. \tag{6.4.2}$$

So, let n_1 be the smallest integer larger than $\log_4\left(\frac{|\lambda|_v}{2M^2}\right) - 1$. Since $|\lambda|_v \ge 8M^2$, we get that $n_1 \ge 1$. Also, by its definition, $n_1 \le \log_4\left(\frac{|\lambda|_v}{2M}\right)$; so, for each $0 \le n \le n_1$, inequality (6.4.2) holds, and thus $M_{n,v} = |A_n|_v$. Moreover, for $0 \le n \le n_1 - 1$ we get that

$$|M_{n+1,v} - M_{n,v}^2| = ||A_{n+1}|_v - |A_n^2|_v| \le |\lambda|_v \cdot |B_n|_v^2 = M_{n,v}^2 \cdot \frac{|\lambda|_v}{|f_\lambda^n(\alpha)|_v^2}$$
$$\le M_{n,v}^2 \cdot \frac{|\lambda|_v}{\frac{|\lambda|_v^2}{4^n M^2}} \le \frac{M_{n,v}^2}{2}.$$

So, using Proposition 5.1 we obtain that for all $0 \le n \le n_1 - 1$ we have

$$\left| \frac{\log M_{n+1,v}}{2^{n+1}} - \frac{\log M_{n,v}}{2^n} \right| \le \frac{\log(2)}{2^{n+1}}. \tag{6.4.3}$$

Using the definition of n_1 and inequalities (6.4.1) and (6.4.3) we conclude that

$$\begin{aligned} \left| \frac{\log M_{n,v}}{2^n} - \frac{\log M_{n_0,v}}{2^{n_0}} \right| &\le \sum_{n=0}^{n_1-1} \frac{\log(2)}{2^{n+1}} + \sum_{n=n_1}^{\infty} \frac{\log(2|\lambda|_v)}{2^{n+1}} \\ &\le \log(2) + \frac{\log(2|\lambda|_v)}{2^{n_1}} \\ &\le \log(2) + \frac{2\log(2|\lambda|_v)}{\sqrt{\frac{|\lambda|_v}{2M^2}}} \\ &\le \log(2) + 4M \cdot \frac{\log(2|\lambda|_v)}{\sqrt{|\lambda|_v}} \\ &< 1 + 8M \text{ (because } |\lambda|_v \ge 8), \end{aligned}$$

for all $0 \le n_0 \le n$. ☐

Our next result completes the proof of Proposition 4.5 by considering the case of nonconstant $\mathbf{c}(t) = \frac{\mathbf{A}(t)}{\mathbf{B}(t)}$, where $\mathbf{A}, \mathbf{B} \in \bar{\mathbb{Q}}[t]$ are nonzero coprime polynomials.

Proposition 6.5. *Assume $\mathbf{c}(\lambda) = \frac{\mathbf{A}(\lambda)}{\mathbf{B}(\lambda)} \in \bar{\mathbb{Q}}(\lambda)$ is nonconstant, and let $|\cdot|_v$ be any absolute value on $\bar{\mathbb{Q}}$. Consider $\lambda_0 \in \bar{\mathbb{Q}}^*$ such that $\mathbf{c}(\lambda_0) \ne 0, \infty$. For each $n \ge 0$ we let $\mathbf{A}_n := \mathbf{A}_{\mathbf{c},n}(\lambda_0)$, $\mathbf{B}_n := \mathbf{B}_{\mathbf{c},n}(\lambda_0)$ and*

$$M_{n,v} := \max\{|\mathbf{A}_n|_v, |\mathbf{B}_n|_v\}.$$

Then there exists a constant C depending only on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ (but independent of λ_0) such that

$$\left| \lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2} \right| \le C. \tag{6.5.1}$$

Proof. We let $\alpha := f_{\lambda_0}(\mathbf{c}(\lambda_0))$. Since λ_0 is fixed in our proof, so is α. On the other hand, we will prove that the constant C appearing in (6.5.1) does not depend on α (nor on λ_0).

We split our proof in three cases depending on $|\lambda_0|_v$. We first deal with the case of large $|\lambda_0|_v$. As proven in Propositions 3.1 and 3.3,

$$\deg_t(\mathbf{A}_{\mathbf{c},1}(t)) - \deg_t(\mathbf{B}_{\mathbf{c},1}(t)) \ge 1.$$

We let c_1 and c_2 be the leading coefficients of $\mathbf{A}_{\mathbf{c},1}(t)$ and $\mathbf{B}_{\mathbf{c},1}(t)$ respectively. Then, there exists a positive real number Q depending on v and the coefficients of $\mathbf{A}$ and $\mathbf{B}$ only, such that if $|\lambda|_v > Q$ then

$$\frac{|\mathbf{A}_{\mathbf{c},1}(\lambda)|_v}{|\mathbf{B}_{\mathbf{c},1}(\lambda)|_v} \geq \frac{|\lambda|_v \cdot |c_1|_v}{2|c_2|_v} \geq 8M,$$

where $M := 2\max\{1, |c_2/c_1|_v\}$. Our first step is to prove the following result.

Lemma 6.6. *If $|\lambda_0|_v > Q$ then*

$$\left|\lim_{n\to\infty}\frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \leq \frac{1 + 16\max\left\{1, \left|\frac{c_2}{c_1}\right|\right\}}{d}. \tag{6.6.1}$$

Proof of Lemma 6.6. Recall that $M := 2\max\{1, |c_2/c_1|_v\}$. Since $|\lambda_0|_v > Q$, then

$$|\alpha|_v = \frac{|\mathbf{A}_{\mathbf{c},1}(\lambda_0)|_v}{|\mathbf{B}_{\mathbf{c},1}(\lambda_0)|_v} \geq |\lambda_0|_v/M \geq 8M.$$

We apply the conclusion of Propositions 5.6 and 6.4 with $n_0 = 1$ and we conclude that

$$\left|\lim_{n\to\infty}\frac{\log\max\{|A_{\lambda_0,\alpha,n}|_v, |B_{\lambda_0,\alpha,n}|_v\}}{d^n} - \frac{\log\max\{|A_{\lambda_0,\alpha,1}|_v, |B_{\lambda_0,\alpha,1}|_v\}}{d}\right| \leq 1 + 8M.$$

On the other hand, using (2.0.13) with $k_0 = 1$ (note that by our assumption, $\mathbf{B}_{\mathbf{c},1}(\lambda_0) = \mathbf{A}(\lambda_0)\mathbf{B}(\lambda_0)^{d-1} \neq 0$) we obtain

$$\begin{aligned}&\left|\lim_{n\to\infty}\frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \\ &= \frac{1}{d}\cdot\left|\lim_{n\to\infty}\frac{\log\max\{|A_{\lambda_0,\alpha,n}|_v, |B_{\lambda_0,\alpha,n}|_v\}}{d^n} - \frac{\log\max\{|A_{\lambda_0,\alpha,1}|_v, |B_{\lambda_0,\alpha,1}|_v\}}{d}\right| \\ &\leq \frac{1+8M}{d},\end{aligned}$$

as desired. $\square$

We will now deal with the case when $|\lambda_0|_v$ is small. We will define another quantity, R, which will depend only on v and on the coefficients of A and of B, and we will assume that $|\lambda_0|_v < R$. The definition of R is technical since it depends on whether $\mathbf{c}(0)$ equals 0, ∞ or neither. However, the quantity R will depend on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ only (and will not depend on λ_0 nor on $\alpha = f_{\lambda_0}(\mathbf{c}(\lambda_0))$).

Assume $\mathbf{c}(0) \neq 0, \infty$ (i.e., $\mathbf{A}(0) \neq 0$ and $\mathbf{B}(0) \neq 0$). Let $c_3 := \mathbf{A}(0) \neq 0$ and $c_4 := \mathbf{B}(0) \neq 0$ be the constant coefficients of $\mathbf{A}$ and respectively of

$\mathbf{B}$. Then there exists a positive real number R depending on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ only, such that if $|\lambda|_v < R$, then

$$\frac{|c_3|_v}{2} < |\mathbf{A}(\lambda)|_v < \frac{3|c_3|_v}{2} \text{ and } \frac{|c_4|_v}{2} < |\mathbf{B}(\lambda)|_v < \frac{3|c_4|_v}{2}.$$

Hence $\frac{|c_3|_v}{3|c_4|_v} < |\mathbf{c}(\lambda)|_v < \frac{3|c_3|_v}{|c_4|_v}$. Then we can apply Propositions 5.8 and 5.11 with $n_0 = 2$ (coupled with (2.0.13) for $k_0 = 0$); we obtain that if $|\lambda_0|_v < R$ then

$$\begin{aligned}
(6.6.2)\qquad & \left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \\
&= \left|\lim_{n\to\infty} \frac{\log\max\{|A_{\lambda_0,\mathbf{c}(\lambda_0),n}|_v, |B_{\lambda_0,\mathbf{c}(\lambda_0),n}|_v\}}{d^n}\right. \\
&\qquad\left. - \frac{\log\max\{|A_{\lambda_0,\mathbf{c}(\lambda_0),2}|_v, |B_{\lambda_0,\mathbf{c}(\lambda_0),2}|_v\}}{d^2}\right| \\
&\le (3d-2)\log\left(2\max\{|\mathbf{c}(\lambda_0)|_v, |\mathbf{c}(\lambda_0)|_v^{-1}\}\right) \\
&\le (3d-2)\left(2 + \log\left(\max\left\{\frac{|c_3|_v}{|c_4|_v}, \frac{|c_4|_v}{|c_3|_v}\right\}\right)\right).
\end{aligned}$$

Assume $c(0) = \infty$. Then $\mathbf{A}(0) \neq 0$ but $\mathbf{B}(0) = 0$; in particular $\deg(\mathbf{B}) \geq 1$ since $\mathbf{B}$ is not identically equal to 0. We recall that $c_3 = A(0)$ is the constant coefficient of $\mathbf{A}$ (we know $c_3 \neq 0$). Also, let c_5 be the first nonzero coefficient of $\mathbf{B}$. Then there exists a positive real number R depending on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ only, such that if $0 < |\lambda|_v < R$ then

$$\frac{|c_3|_v}{2} < |\mathbf{A}(\lambda)|_v \quad\text{and}\quad |\mathbf{B}(\lambda)|_v < 2|c_5|_v \cdot |\lambda|_v,$$

and moreover

$$|\mathbf{c}(\lambda)|_v = \left|\frac{\mathbf{A}(\lambda)}{\mathbf{B}(\lambda)}\right|_v > \frac{1}{M \cdot |\lambda|_v} \geq 2M,$$

where $M = 4\max\{1, |c_5/c_3|_v\}$. Then applying Propositions 6.1 and 6.2 with $n_0 = 2$ (coupled with (2.0.13) for $k_0 = 0$) we conclude that if $|\lambda_0|_v < R$ then

$$\begin{aligned}
(6.6.3)\qquad & \left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \\
&= \left|\lim_{n\to\infty} \frac{\log\max\{|A_{\lambda_0,\mathbf{c}(\lambda_0),n}|_v, |B_{\lambda_0,\mathbf{c}(\lambda_0),n}|_v\}}{d^n}\right. \\
&\qquad\left. - \frac{\log\max\{|A_{\lambda_0,\mathbf{c}(\lambda_0),2}|_v, |B_{\lambda_0,\mathbf{c}(\lambda_0),2}|_v\}}{d^2}\right| \\
&\le 4\log(2).
\end{aligned}$$

Assume $\mathbf{c}(0) = 0$. Then $\mathbf{A}(0) = 0$ but $\mathbf{B}(0) \neq 0$; in particular $\deg(\mathbf{A}) \geq 1$ since $\mathbf{A}$ is not identically equal to 0. So, the constant coefficient of $\mathbf{B}$ is nonzero, i.e., $c_4 = c_5 = \mathbf{B}(0) \neq 0$ in this case. There are two cases: $\mathbf{A}'(0) = 0$ or not. First, assume $c_6 := \mathbf{A}'(0) \neq 0$. Then there exists a positive real number R depending on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ only such that if $0 < |\lambda|_v < R$ then

$$|\mathbf{A}_{\mathbf{c},1}(\lambda)|_v = \left|\frac{\mathbf{A}(\lambda)^d}{\lambda} + \mathbf{B}(\lambda)^d\right|_v \in \left(\frac{|c_4|_v^d}{2}, \frac{3|c_4|_v^d}{2}\right),$$
$$|\mathbf{B}_{\mathbf{c},1}(\lambda)|_v = \left|\frac{\mathbf{A}(\lambda)\mathbf{B}(\lambda)^{d-1}}{\lambda}\right|_v \in \left(\frac{\left|c_6c_4^{d-1}\right|_v}{2}, \frac{3\left|c_6c_4^{d-1}\right|_v}{2}\right).$$

Hence $\frac{|c_4|_v}{3|c_6|_v} \le |\alpha|_v \le \frac{3|c_4|_v}{|c_6|_v}$ (also note that we are using the fact that $\lambda_0 \neq 0$ and so the above inequalities apply to our case). Hence using Propositions 5.8 and 5.11 with $n_0 = 1$ (combined also with (2.0.13) for $k_0 = 1$, which can be used since $\mathbf{B}_{\mathbf{c},1}(\lambda_0) = \mathbf{A}(\lambda_0)\mathbf{B}(\lambda_0)^{d-1} \neq 0$) we obtain

$$\begin{aligned}(6.6.4)\qquad &\left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \\ &= \frac{1}{d}\cdot\left|\lim_{n\to\infty}\frac{\log\max\{|A_{\lambda_0,\alpha,n}|_v, |B_{\lambda_0,\alpha,n}|_v\}}{d^n}\right. \\ &\qquad\qquad \left. - \frac{\log\max\{|A_{\lambda_0,\alpha,1}|_v, |B_{\lambda_0,\alpha,1}|_v\}}{d}\right| \\ &\le \frac{3d-2}{d}\cdot\log\left(2\max\left\{|\alpha|_v, |\alpha|_v^{-1}\right\}\right) \\ &\le 3\cdot\left(2+\log\left(\max\left\{\left|\frac{c_4}{c_6}\right|_v, \left|\frac{c_6}{c_4}\right|_v\right\}\right)\right).\end{aligned}$$

Next assume $\mathbf{A}(0) = \mathbf{A}'(0) = 0$. So, let c_7 be the first nonzero coefficient of $\mathbf{A}$. Also, we recall that $c_4 = c_5 = \mathbf{B}(0) \neq 0$ in this case. Then there exists a positive real number R depending on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ only such that if $0 < |\lambda|_v < R$ then

$$\frac{|c_4|_v^d}{2} < |\mathbf{A}_{\mathbf{c},1}(\lambda)|_v \text{ and } |\mathbf{B}_{\mathbf{c},1}(\lambda)|_v < 2\left|c_7c_4^{d-1}\right|_v\cdot|\lambda|_v,$$

and moreover

$$\left|\frac{\mathbf{A}_{\mathbf{c},1}(\lambda)}{\mathbf{B}_{\mathbf{c},1}(\lambda)}\right|_v > \frac{1}{M\cdot|\lambda|_v} \ge 8M,$$

where $M := 4\max\left\{1, \frac{|c_7|_v}{|c_4|_v}\right\}$. Hence, if $|\lambda_0|_v < R$ (using also that $\mathbf{c}(\lambda_0) \neq 0, \infty$), we obtain

$$|\alpha|_v \ge \frac{1}{M\cdot|\lambda_0|_v} \ge 8M. \tag{6.6.5}$$

Then Propositions 6.1 and 6.2 with $n_0 = 1$ (combined with (2.0.13) for $k_0 = 1$, which can be used since $\mathbf{B}_{\mathbf{c},1}(\lambda_0) \neq 0$) yield

$$
\begin{aligned}
(6.6.6) \qquad & \left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \\
&= \frac{1}{d}\cdot\left|\lim_{n\to\infty} \frac{\log\max\{|A_{\lambda_0,\alpha,n}|_v, |B_{\lambda_0,\alpha,n}|_v\}}{d^n}\right. \\
&\qquad\qquad \left. - \frac{\log\max\{|A_{\lambda_0,\alpha,1}|_v, |B_{\lambda_0,\alpha,1}|_v\}}{d}\right| \\
&\leq \frac{4\log(2)}{d}
\end{aligned}
$$

On the other hand, Proposition 5.2 yields that if $R \leq |\lambda_0|_v \leq Q$ then

$$
(6.6.7) \qquad \left|\lim_{n\to\infty} \frac{\log M_{n,v}}{d^n} - \frac{\log M_{2,v}}{d^2}\right| \leq \frac{\log(2Q) - \log(R)}{4}.
$$

Noting that R and Q depend on v and on the coefficients of $\mathbf{A}$ and of $\mathbf{B}$ only, inequalities (6.6.1), (6.6.2), (6.6.3), (6.6.4), (6.6.6) and (6.6.7) yield the conclusion of Proposition 4.5. □

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