New York Journal of Mathematics
New York J. Math. **19** (2013) 729–792.

On groupoids with involutions and their cohomology

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Abstract. We extend the definitions and main properties of graded extensions to the category of locally compact groupoids endowed with involutions. We introduce Real Čech cohomology, which is an equivariant-like cohomology theory suitable for the context of groupoids with involutions. The Picard group of such a groupoid is discussed and is given a cohomological picture. Eventually, we generalize Crainic's result, about the differential cohomology of a proper Lie groupoid with coefficients in a given representation, to the topological case.

Contents

Received April 29, 2012; revised October 23, 2013.
2010 *Mathematics Subject Classification.* 22A22; 55N32; 53C08.
Key words and phrases. Real groupoids, groupoid cohomology, graded extensions.
Partially suported by the German Research Foundation (DFG) via the IRTG 1133.

ISSN 1076-9803/2013

0. Introduction

A Real[1] object in a category $\mathcal{C}$ is a pair (A,f) consisting of an object $A\in \mathrm{Ob}(\mathcal{C})$ together with an element $f\in \mathrm{Isom}_{\mathcal{C}}(A,A)$, called the *Real structure*, such that $f^2=\mathbf{1}_A$. For instance, an Atiyah Real space (X,τ) [2] is nothing but a Real object in the category of locally compact spaces. We are particularly interested in the category $\mathfrak{G}_s$ [25] of locally compact Hausdorff groupoids with *strict homomorphisms* [15, 16] as morphisms; we shall refer to Real objects in $\mathfrak{G}_s$ as *Real groupoids*. For example, let $\mathbb{WP}^n_{(a_1,\dots,a_n)}$ be the weighted projective orbifold [1] associated to the pairwise coprime integers $a_1,\dots,a_n$; then together with the coordinate-wise complex conjugation, $\mathbb{WP}^n_{(a_1,\dots,a_n)}$ is a Real groupoid.

A morphism of Real groupoids is a morphism in $\mathfrak{G}_s$ intertwining the Real structures. We may also speak of a Real strict homomorphism. Real groupoids form a category $\mathfrak{RG}_s$ in which morphisms are Real strict homomorphisms. Moreover, they are the objects of a 2-category $\mathfrak{RG}(2)$ defined as follows. Let $(\mathcal{G},\rho),(\Gamma,\varrho)\in\mathrm{Ob}(\mathfrak{RG}_s)$. A *generalized homomorphism* [7, 9, 16, 25] $\Gamma\xrightarrow{Z}\mathcal{G}$ is said to be *Real* if Z is given a Real structure τ such that the moment maps and the groupoid actions respect some coherent compatibility conditions with respect to the Real structures. A morphism of Real generalized homomorphisms $(Z,\tau)\longrightarrow(Z',\tau')$ is a morphism of generalized homomorphisms $Z\longrightarrow Z'$ intertwining the Real structures. Henceforth, 1-morphisms in $\mathfrak{RG}(2)$ are Real generalized homomorphisms and 2-morphisms are morphisms of Real generalized homomorphisms. All functorial properties we deal with in this paper are however discussed in the category $\mathfrak{RG}$ defined as $\mathfrak{RG}(2)$ "up to 2-isomorphisms".

In [21], a Čech cohomology theory for topological groupoids is defined as the Čech cohomology of simplicial topological spaces, and it is shown that the well-known isomorphism between $\mathbb{S}^1$-central extensions of a discrete groupoid $\mathcal{G}$ and the second cohomology group [19, 11] of $\mathcal{G}$ with coefficients in the sheaf of germs of $\mathbb{S}^1$-valued functions also holds in the general case; *i.e.*, $\mathrm{Ext}(\mathcal{G},\mathbb{S}^1)\cong\check{H}^2(\mathcal{G}_\bullet,\mathbb{S}^1)$. We define here an analogous theory $\check{H}R^*$

[1]Note the capitalization, used to avoid confusion with a module over $\mathbb{R}$ or a real manifold.

that fits well the context of Real groupoids. This theory was motivated by the classification of groupoid C^*-dynamical systems endowed with involutions [17]. These can be thought of as a generalization of continuous-trace C^*-algebras with involutions. Specifically, it is known [20] that given such a C^*-algebra A, its spectrum X admits a Real structure τ, and its Dixmier–Douady invariant $\delta(A) \in \check{H}^2(X, \mathbb{S}^1)$ is such that $\overline{\delta(A)} = \tau^*\delta(A)$, where the "bar" is the complex conjugation in $\mathbb{S}^1$. In fact, thinking of X as a Real groupoid, we will see that all 2-cocycles satisfying the latter relation are classified by $\check{H}R^2(X, \mathbb{S}^1)$, where $\mathbb{S}^1$ is endowed with the complex conjugation. $\check{H}R^*$ appears then to provide the right cohomological interpretation of C^*-dynamical systems with involutions.

We try, to the extent possible, to make the present paper self-contained. We start by collecting, in Section 1, a number of notions and results about Real groupoids most of which are adapted from many sources in the literature [15, 19, 25]; specifically, we define the group $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ of (equivalence classes of) Real graded S-central extensions over a Real groupoid $\mathcal{G}$, by a given Real abelian group S. In Section 2, we introduce Real Čech cohomology, following closely [21]. While $\check{H}R^*$ behaves almost like a $\mathbb{Z}_2$-equivariant cohomology theory, we will see that it is actually not. Geometric interpretations of the cohomology groups $\check{H}R^1(\mathcal{G}_\bullet, \mathrm{S})$ and $\check{H}R^2(\mathcal{G}_\bullet, \mathrm{S})$, for a Real Abelian group S, are given. Finally, we generalize a result by Crainic [4] (on the differential cohomology groups of a proper Lie groupoid) to topological proper (Real) groupoid.

1. Real groupoids and Real graded extensions

Recall [19, 16, 25] that a *strict homomorphism* between two groupoids $\mathcal{G} \rightrightarrows X$ and $\Gamma \rightrightarrows Y$ is a functor $\varphi : \Gamma \longrightarrow \mathcal{G}$ given by a map $Y \longrightarrow X$ on objects and a map $\Gamma^{(1)} \longrightarrow \mathcal{G}^{(1)}$ on arrows, both denoted again by φ, which preserve the groupoid structure maps, *i.e.*, $\varphi(s(\gamma)) = s(\varphi(\gamma))$, $\varphi(r(\gamma)) = r(\varphi(\gamma))$, $\varphi(\mathbf{1}_y) = \mathbf{1}_{\varphi(y)}$ and $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1)\varphi(\gamma_2)$ (hence $\varphi(\gamma^{-1}) = \varphi(\gamma)^{-1}$), for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and $y \in Y$. Unless otherwise specified, all our groupoids are topological groupoids which are supposed to be Hausdorff and locally compact.

1.1. Real groupoids.

Definition 1.1. A *Real* groupoid is a groupoid $\mathcal{G} \rightrightarrows X$ together with a strict 2-periodic homeomorphism $\rho : \mathcal{G} \longrightarrow \mathcal{G}$. The homeomorphism ρ is called a *Real structure on* $\mathcal{G}$. Such a groupoid will be denoted by a pair $(\mathcal{G}, \rho)$.

Example 1.2. Any topological Real space (X, ρ) in the sense of Atiyah [2] can be viwed as a Real groupoid whose the unit space and the space of morphisms are identified with X; i.e., the operations in this Real groupoid is defined by $s(x) = r(x) = x$, $x \cdot x = x$, $x^{-1} = x$.

Example 1.3. Any group with involution can be viewed as a Real groupoid with unit space identified with the unit element. Such a group will be called Real.

Lemma 1.4. *Let G be an abelian group equipped with an involution $\tau : G \longrightarrow G$ (*i.e.*, a Real structure). Set*

$$\Re(\tau) := \{g \in G \mid \tau(g) = g\} = {}^{\mathbb{R}}G, \quad \Im(\tau) := \{g \in G \mid \tau(g) = -g\}.$$

Then,

$$G \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong (\Re(\tau) \oplus \Im(\tau)) \otimes \mathbb{Z}\left[\frac{1}{2}\right]. \tag{1.1}$$

If τ is understood, we will write ${}^{\mathfrak{I}}G$ for $\Im(\tau)$. We call $\Re(\tau)$ and $\Im(\tau)$ the Real part and the imaginary part of G, respectively.

Proof. For all $g \in G$, one has $g + \tau(g) \in {}^{\mathbb{R}}G$, and $g - \tau(g) \in {}^{\mathfrak{I}}G$. Therefore, after tensoring G with $\mathbb{Z}[1/2]$, every $g \in G$ admits a unique decomposition

$$g = \frac{g + \tau(g)}{2} + \frac{g - \tau(g)}{2} \in \mathbb{Z}[1/2] \otimes \left({}^{\mathbb{R}}G \oplus {}^{\mathfrak{I}}G\right). \qquad \square$$

Example 1.5. Let $n \in \mathbb{N}^*$. Suppose ρ is a Real structure on the additive group $\mathbb{R}^n$. Then there exists a unique decomposition $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ such that ρ is determined by the formula

$$\rho(x, y) = (\mathbf{1}_p \oplus (-\mathbf{1}_q))(x, y) := (x, -y),$$

for all $(x, y) = (x_1, \cdots, x_p, y_1, \cdots, y_q) \in \mathbb{R}^p \oplus \mathbb{R}^q$.

For each pair $(p, q) \in \mathbb{N}$, we will write $\mathbb{R}^{p,q}$ for the additive group $\mathbb{R}^{p+q}$ equipped with the Real structure $(\mathbf{1}_p \oplus (-\mathbf{1}_q))$.

Define the Real space $\mathrm{S}^{p,q}$ as the invariant subset of $\mathbb{R}^{p,q}$ consisting of elements $u \in \mathbb{R}^{p+q}$ of norm 1. For $q = p$, $\mathrm{S}^{p,p}$ is clearly identified with the Real space $\mathbb{S}^p$ whose Real structure is given by the coordinate-wise complex conjugation. Notice that ${}^{r}\mathrm{S}^{p,q} = \mathrm{S}^{p,0}$.

Example 1.6. Let (X, ρ) be a topological Real space. Consider the fundamental groupoid $\pi_1(X)$ over X whose arrows from $x \in X$ to $y \in X$ are homotopy classes of paths (relative to end-points) from x to y and the partial multiplication given by the concatenation of paths. The involution ρ induces a Real structure on the groupoid as follows: if $[\gamma] \in \pi_1(X)$, we set $\rho([\gamma])$ the homotopy classes of the path $\rho(\gamma)$ defined by $\rho(\gamma)(t) := \rho(\gamma(t))$ for $t \in [0, 1]$.

Two Real structures ρ and ρ' on $\mathcal{G}$ are said to be *conjugate* if there exists a strict homeomorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}$ such that $\rho' = \phi \circ \rho \circ \phi^{-1}$. In this case we say that the Real groupoids $(\mathcal{G}, \rho)$ and $(\mathcal{G}, \rho')$ are equivalent.

Definition 1.7. We write ${}^{r}\mathcal{G} \rightrightarrows {}^{r}X$ (or ${}^{\rho}\mathcal{G}$ when there is a risk of confusion) for the the subgroupoid of $\mathcal{G} \rightrightarrows X$ by ρ.

Lemma 1.8. *Let $\mathcal{G}$ and Γ be Real groupoids, and let $\phi : \Gamma \longrightarrow \mathcal{G}$ be a Real groupoid homomorphism, then $\phi({}^r\Gamma)$ is a full subgroupoid of ${}^r\mathcal{G} \rightrightarrows {}^rX$. If in addition ϕ is an isomorphism, then ${}^r\Gamma \cong {}^r\mathcal{G} \rightrightarrows {}^rX$.*

In particular, if ρ_1 and ρ_2 are two conjugate Real structures on $\mathcal{G}$, then ${}^{\rho_1}\mathcal{G} \cong {}^{\rho_2}\mathcal{G}$.

Proof. This is obvious since $\phi(\bar{\gamma}) = \overline{\phi(\gamma)}$ for all $\gamma \in \Gamma$. □

Remark 1.9. Note that the converse of the second statement of the above lemma is false in general. For instance, consider the Real group $\mathbb{S}^1$ whose Real structure is given by the complex conjugation, and the Real group $\mathbb{Z}_2$ (with the trivial Real structure). We have ${}^r\mathbb{S}^1 = \{\pm 1\} \cong \mathbb{Z}_2 = {}^r\mathbb{Z}_2$.

The following is an example of groupoids with equivalent Real structures.

Example 1.10. Recall ([8, IV.3]) that a Riemannian manifold X is called *globally symmetric* if each point $x \in X$ is an isolated fixed point of an involutory isometry $s_x : X \longrightarrow X$; i.e., s_x is a diffeomorphism verifying $s_x^2 = \mathrm{Id}_X$ and $s_x(x) = x$. Moreover, for every two points $x, y \in X$, s_x and s_y are related through the formula $s_x \circ s_y \circ s_x = s_{s_x(y)}$. Given such a space, each point $x \in X$ defines a Real structure on X which leaves x fixed. However, let x and y be two different points in X and let $z \in X$ be such that $y = s_z(x)$. Then, we get $s_z \circ s_x \circ s_z = s_y$ which means that the diffeomorphism $s_z : X \longrightarrow X$ implements an equivalence $s_x \sim s_y$. But since x and y are arbitrary, it turns out that all of the Real structures s_x are equivalent. Thus, all of the Real spaces (X, s_x) are equivalent to each others.

Now, recall [8, IV. Theorem 3.3] that if G denotes the identity component of $I(X)$, where the latter is the group of isometries on X, then the map $\sigma_{x_0} : g \longmapsto s_{x_0} g s_{x_0}$ is an involutory automorphism in G, for any arbitrary $x_0 \in X$. It follows that all of the points of X give rise to equivalent Real groups (G, σ_x).

From now on, by a Real structure on a groupoid, we will mean a representative of a conjugation class of Real structures. Moreover, we will sometimes put $\bar{g} := \rho(g)$, and write $\mathcal{G}$ instead of $(\mathcal{G}, \rho)$ when ρ is understood.

Definition 1.11 (Real covers)**.** Let (X, ρ) be a Real space. We say that an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X is *Real* if $\mathcal{U}$ is invariant with respect to the Real structure ρ; *i.e.*, $\rho(U_i) \in \mathcal{U}, \forall i \in I$. Alternatively, $\mathcal{U}$ is Real if I is equipped with an involution $i \longmapsto \bar{i}$ such that $U_{\bar{i}} = \rho(U_i)$ for all $i \in I$.

Remark 1.12. Observe that Real open covers always exist for all locally compact Real space X. Indeed, let $\mathcal{V} = \{V_{i'}\}_{i' \in I'}$ be an open cover of the space X. Let $I := I' \times \{\pm 1\}$ be endowed with the involution $(i', \pm 1) \longmapsto (i', \mp 1)$. Next, put $U_{(i', \pm 1)} := \rho^{(\pm 1)}(V_{i'})$, where $\rho^{(+1)}(g) := g$, and $\rho^{(-1)}(g) := \rho(g)$ for $g \in \mathcal{G}$.

Definition 1.13 (Real action)**.** Let (Z, τ) be a locally compact Hausdorff Real space. A (continuous) right Real action of $(\mathcal{G}, \rho)$ on (Z, τ) is given by a continuous open map $\mathfrak{s} : Z \longrightarrow X$ (called the *generalized source map*) and a continuous map $Z \times_{\mathfrak{s},X,r} \mathcal{G} \longrightarrow Z$, denoted by $(z, g) \longmapsto zg$, such that:

(a) $\tau(zg) = \tau(z)\rho(g)$ for all $(z, g) \in Z \times_{\mathfrak{s},X,r} \mathcal{G}$.
(b) $\rho(\mathfrak{s}(z)) = \mathfrak{s}(\tau(z))$ for all $z \in Z$.
(c) $\mathfrak{s}(zg) = s(g)$.
(d) $z(gh) = (zg)h$ for $(z, g) \in Z \times_{\mathfrak{s},X,r} \mathcal{G}$ and $(g, h) \in \mathcal{G}^{(2)}$.
(e) $z\mathfrak{s}(z) = z$ for any $z \in Z$ where we identify $\mathfrak{s}(z)$ with its image in $\mathcal{G}$ by the inclusion $X \hookrightarrow \mathcal{G}$.

If such a Real action is given, we say that (Z, τ) is a (right) Real $\mathcal{G}$-space.

Likewise a (continuous) left Real action of $(\mathcal{G}, \rho)$ on (Z, τ) is determined by a continuous Real open surjection $\mathfrak{r} : Z \longrightarrow X$ (the *generalized range map* of the action) and a continuous Real map $\mathcal{G} \times_{s,X,\mathfrak{r}} Z \longrightarrow Z$ satisfying the appropriate analogues of conditions (a), (b), (c), (d) and (e) above.

Given a right Real action of $(\mathcal{G}, \rho)$ on (Z, τ) with respect to $\mathfrak{s}$, let

$$\Psi : Z \times_{\mathfrak{s},X,r} \mathcal{G} \longrightarrow Z \times Z$$

be defined by the formula $\Psi(z, g) = (z, zg)$. Then we say that the action is *free* if this map is one-to-one (or in other words if the equation $zg = z$ implies $g = \mathfrak{s}(z)$. The action is called *proper* if Ψ is proper.

Notations 1.14. *If we are given such a right (resp. left) Real action of $(\mathcal{G}, \rho)$ on (Z, τ), and if there is no risk of confusion, we will write $Z * \mathcal{G}$ (resp. $\mathcal{G} * Z$) for $Z \times_{\mathfrak{s},X,r} \mathcal{G}$ (resp. for $\mathcal{G} \times_{s,X,\mathfrak{r}} Z$).*

1.2. Real $\mathcal{G}$-bundles.

Definition 1.15. Let $(\mathcal{G}, \rho)$ be a Real groupoid. A Real (right) $\mathcal{G}$-bundle over a Real space (Y, ϱ) is a Real (right) $\mathcal{G}$-space (Z, τ) with respect to a map $\mathfrak{s} : Z \longrightarrow X$, together with a Real map $\pi : Z \longrightarrow Y$ satisfying the relation $\pi(zg) = \pi(z)$ for any $(z, g) \in Z \times_{\mathfrak{s},X,r} \mathcal{G}$, and such that for any $y \in Y$, the induced map

$$\tau_y : Z_y \longrightarrow Z_{\varrho(y)}$$

on the fibres is $\mathcal{G}$-antilinear in the sense that for $(z, g) \in Z_y \times_{\mathfrak{s},X,r} \mathcal{G}$ we have

$$\tau_y(zg) = \tau_y(z)\rho(g)$$

as an element in $Z_{\varrho(y)}$.

Such a bundle (Z, τ) is said to be *principal* if:

(i) $\pi : Z \longrightarrow Y$ is *locally split* (i.e., it is surjective and admits local sections).
(ii) The map $Z \times_{\mathfrak{s},X,r} \mathcal{G} \longrightarrow Z \times_Y Z$, $(z, g) \longmapsto (z, zg)$ is a Real homeomorphism.

Remarks 1.16.

(1) **The unit bundle.** Given a Real groupoid $(\mathcal{G},\rho)$, its space of arrows $\mathcal{G}^{(1)}$ is a $\mathcal{G}$-principal Real bundle over X. Indeed, the projection is the range map $r:\mathcal{G}^{(1)}\longrightarrow X$, the generalized source map is given by s and the action is just the partial multiplication on $\mathcal{G}$. This bundle is denoted by $U(\mathcal{G})$ and is called the *unit* bundle of $\mathcal{G}$ (see [16]).

(2) **Pull-back.** Let

$$\begin{array}{ccc} Z & \xrightarrow{\mathfrak{s}} & X \\ {\scriptstyle\pi}\downarrow & & \\ Y & & \end{array}$$

be a $\mathcal{G}$-principal Real bundle and $f:Y'\longrightarrow Y$ be a Real continuous map. Then the pull-back $f^*Z:=Y'\times_Y Z$ equipped with the involution (ϱ',τ) has the structure of a $\mathcal{G}$-principal Real bundle over Y'. Indeed, the right Real $\mathcal{G}$-action is given by the $\mathcal{G}$-action on Z and the generalized source map is $\mathfrak{s}'(y',z):=\mathfrak{s}(z)$.

(3) **Trivial bundles.** From the previous two remarks, we see that if (Z,τ) is any Real space together with a Real map $\varphi:Z\longrightarrow X$, then we get a $\mathcal{G}$-principal Real bundle $\varphi^*U(\mathcal{G})$ over Z; its total space being the space $Z\times_{\varphi,X,r}\mathcal{G}$. A Bundle of this form is called *trivial* while a $\mathcal{G}$-principal Real bundle which is locally of this form is called *locally trivial*.

1.3. Generalized morphisms of Real groupoids.

Definition 1.17. A generalized morphism from a Real groupoid (Γ,ϱ) to a Real groupoid $(\mathcal{G},\rho)$ consists of a Real space (Z,τ), two maps

$$Y\xleftarrow{\mathfrak{r}} Z\xrightarrow{\mathfrak{s}} X\ ,$$

a left (Real) action of Γ with respect to $\mathfrak{r}$, a right (Real) action of $\mathcal{G}$ with respect to $\mathfrak{s}$, such that:

(i) The actions commute, i.e., if $(z,g)\in Z\times_{\mathfrak{s},X,r}\mathcal{G}$ and $(\gamma,z)\in\Gamma\times_{s,Y,\mathfrak{r}}Z$ we must have $\mathfrak{s}(\gamma z)=\mathfrak{s}(z)$, $\mathfrak{r}(zg)=\mathfrak{r}(z)$ so that $\gamma(zg)=(\gamma z)g$.

(ii) The maps $\mathfrak{s}$ and $\mathfrak{r}$ are Real in the sense that $\mathfrak{s}(\tau(z))=\rho(\mathfrak{s}(z))$ and $\mathfrak{r}(\tau(z))=\varrho(\mathfrak{r}(z))$ for any $z\in Z$.

(iii) $\mathfrak{r}:Z\longrightarrow Y$ is a locally trivial $\mathcal{G}$-principal Real bundle.

Example 1.18. Let $f:\Gamma\longrightarrow\mathcal{G}$ be a Real strict morphism. Let us consider the fibre product $Z_f:=Y\times_{f,X,r}\mathcal{G}$ and the maps $\mathfrak{r}:Z_f\longrightarrow Y,\ (y,g)\longmapsto y$ and $\mathfrak{s}:Z_f\longrightarrow X,\ (y,g)\longmapsto s(g)$. For $(\gamma,(y,g))\in\Gamma\times_{s,Y,\mathfrak{r}}Z_f)$, we set $\gamma.(y,g):=(r(\gamma),f(\gamma)g)$ and for $((y,g),g')\in Z_f\times_{\mathfrak{s},X,r}\mathcal{G}$ we set $(y,g).g':=(y,gg')$. Using the definition of a strict morphism, it is easy to check that these maps are well-defined and make Z_f into a generalized morphism from Γ to $\mathcal{G}$. Furthermore, the map τ on Z_f defined by $\tau(y,g):=(\varrho(y),\rho(g))$ is a Real involution and then Z_f is a Real generalized morphism.

Definition 1.19. A morphism between two such morphisms (Z,τ) and (Z',τ') is a Γ-$\mathcal{G}$-equivariant Real map $\varphi : Z \longrightarrow Z'$ such that $\mathfrak{s} = \mathfrak{s}' \circ \varphi$ and $\mathfrak{r} = \mathfrak{r}' \circ \varphi$. We say that the Real generalized homomorphism (Z,τ) and (Z',τ') are `isomorphic` if there exists such a φ which is at the same time a homeomorphism.

Compositions of Real generalized morphisms are defined by the following proposition.

Proposition 1.20. *Let (Z',τ') and (Z'',τ'') be Real generalized homomorphisms from (Γ,ϱ) to $(\mathcal{G}',\rho')$ and from $(\mathcal{G}',\rho')$ to $(\mathcal{G},\rho)$ respectively. Then*

$$Z = Z' \times_{\mathcal{G}'} Z'' := (Z' \times_{\mathfrak{s}',\mathcal{G}'^{(0)},\mathfrak{r}''} Z'')/_{(z',z'')\sim(z'g',g'^{-1}z'')}$$

with the obvious Real involution, defines a Real generalized morphism from $\Gamma \rightrightarrows Y$ *to* $\mathcal{G} \rightrightarrows X$ *.*

Proof. Let us first describe the structure maps

$$Y \xleftarrow{\ \mathfrak{r}\ } Z \xrightarrow{\ \mathfrak{s}\ } X$$

and the actions.

For $(z',z'') \in Z$ we set $\mathfrak{r}(z',z'') := \mathfrak{r}'(z')$ and $\mathfrak{s}(z',z'') := \mathfrak{s}''(z'')$. These are well-defined and since

$$\mathfrak{s}(z'g',g'^{-1}z'') = \mathfrak{s}''(g'^{-1}z'') = \mathfrak{s}''(z''),$$
$$\mathfrak{r}(z'g',g'^{-1}z'') = \mathfrak{r}'(z'g') = \mathfrak{s}'(z'),$$

from Definition 1.17(i). The actions are defined by $\gamma.(z',z'') := (\gamma z',z'')$ and $(z',z'').g := (z',z''g)$ for $(\gamma,(z',z'')) \in \Gamma \times_{s,Y,\mathfrak{r}} Z$ and $((z',z''),g) \in Z \times_{\mathfrak{s},X,r} \mathcal{G}$ while the Real involution is the obvious one:

$$\tau(z',z'') := (\tau'(z'),\tau''(z'')).$$

Now to show the local triviality of Z, notice that from (3) of Remarks 1.16, Z' and Z'' are locally of the form $U \times_{\varphi',\mathcal{G}'^{(0)},r'} \mathcal{G}'$ and $V \times_{\varphi'',X,r} \mathcal{G}$ respectively, where $\varphi' : U \longrightarrow \mathcal{G}'^{(0)}$ and $\varphi'' : V \longrightarrow X$ are Real continuous maps, U and V subspaces of Y and $\mathcal{G}'^{(0)}$ respectively. It turns out that by construction, Z is locally of the form $W \times_{\varphi,\mathcal{G}'^{(0)},r} \mathcal{G}$ where $W = U \times_{\varphi',\mathcal{G}'^{(0)}} V$. ☐

Definition 1.21. Given two Real generalized morphisms $(\Gamma,\varrho) \xrightarrow{(Z,\tau)} (\mathcal{G}',\rho')$ and $(\mathcal{G}',\rho') \xrightarrow{(Z',\tau')} (\mathcal{G},\rho)$, we define their composition

$$(Z' \circ Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$$

to be $(Z \times_{\mathcal{G}'} Z', \tau \times \tau')$.

Remark 1.22. It is easy to check that the composition of Real generalized homomorphisms is associative. For instance, if

$$\Gamma \xrightarrow{(Z_1,\rho_1)} \mathcal{G}_1 \xrightarrow{(Z_2,\rho_2)} \mathcal{G}_2 \xrightarrow{(Z_3,\rho_3)} \mathcal{G}$$

are given Real generalized morphisms, we get two Real generalized morphisms $Z = Z_1 \times_{\mathcal{G}_1} (Z_2 \times_{\mathcal{G}_2} Z_3)$ and $Z' = (Z_1 \times_{\mathcal{G}_1} Z_2) \times_{\mathcal{G}_2} Z_3$ between (Γ, ϱ) and $(\mathcal{G}, \rho)$; notice that here Z and Z' carry the obvious Real involutions. Moreover, the map $Z \longrightarrow Z'$, $(z_1, (z_2, z_3)) \longmapsto ((z_1, z_2), z_3)$ is a Γ-$\mathcal{G}$-equivariant Real homeomorphism. Hence, there exists a category $\mathfrak{RG}$ whose objects are Real locally compact groupoids and morphisms are isomorphism classes of Real generalized homomorphisms.

Lemma 1.23. *Let $f_1, f_2 : \Gamma \to \mathcal{G}$ be two Real strict homomorphisms. Then f_1 and f_2 define isomorphic Real generalized homomorphisms if and only if there exists a Real continuous map $\varphi : Y \longrightarrow \mathcal{G}$ such that*

$$f_2(\gamma) = \varphi(r(\gamma)) f_1(\gamma) \varphi(s(\gamma))^{-1}.$$

Proof. Le $\Phi : Z_{f_1} \longrightarrow Z_{f_2}$ be a Real Γ-$\mathcal{G}$-equivariant homeomorphism, where $Z_{f_i} = Y \times_{f_i, X, r} \mathcal{G}$. Then from the commutative diagrams

$$\begin{tikzcd} Y & Z_{f_1} \arrow[l, "pr_1"'] \arrow[r, "s\circ pr_2"] \arrow[d, "\Phi"] & X \\ & Z_{f_2} \arrow[ul, "pr_1"] \arrow[ur, "s\circ pr_2"'] & \end{tikzcd}$$

we have $\Phi(x, g) = (x, h)$ with $s(g) = s(h)$; and then there exists a unique element $\varphi(x) \in \mathcal{G}$ such that $h = \varphi(x) g$. To see that this defines a continuous map $\varphi : Y \longrightarrow \mathcal{G}$, notice that for any $x \in Y$, the pair $(x, f_1(x))$ is an element in Z_{f_1}, then $\varphi(x)$ is the unique element in $\mathcal{G}$ such that

$$\Phi(x, f_1(x)) = (x, \varphi(x) f_1(x)).$$

Furthermore, since Φ is Real,

$$\Phi(\varrho(x), \rho(f_1(x))) = (\varrho(x), \rho(\varphi(x))\rho(f_1(x))),$$

which shows that $\varphi(\varrho(x)) = \rho(\varphi(x))$ for any $x \in Y$; *i.e.*, φ is Real.

Now for $\gamma \in \Gamma$, take $x = s(\gamma)$, then from the Γ-equivariance of Φ, we have

$$\Phi(\gamma \cdot (s(\gamma), f_1(s(\gamma)))) = \Phi(r(\gamma), f_1(\gamma)) = \gamma \cdot \Phi(s(\gamma), f_1(s(\gamma)));$$

so that

$$(r(\gamma), \varphi(r(\gamma)) f_1(\gamma)) = (r(\gamma), f_2(\gamma) \varphi(s(\gamma)))$$

and $f_2(\gamma) \cdot r(\varphi(s(\gamma))) = \varphi(r(\gamma)) f_1(\gamma) \varphi(s(\gamma))$; but $r(\varphi(s(\gamma))) = s(f_2(\gamma))$ by definition of φ and this gives the desired relation.

The converse is easy to check by working backwards. $\square$

1.4. Morita equivalence. Let (Γ, ϱ) and $(\mathcal{G}, \rho)$ be two Real groupoids. Suppose that $f : (\Gamma, \varrho) \longrightarrow (\mathcal{G}, \rho)$ is an isomorphism in the category $\mathfrak{RG}_s$. In this case, we say that (Γ, ϱ) and $(\mathcal{G}, \rho)$ are *strictly equivalent* and we write $(\Gamma, \varrho) \sim_{strict} (\mathcal{G}, \rho)$. Now, consider the induced Real generalized morphisms $(Z_f, \tau_f) : (\Gamma, \varrho) \longrightarrow (\mathcal{G}, \rho)$ and $(Z_{f^{-1}}, \tau_{f^{-1}}) : (\mathcal{G}, \rho) \longrightarrow (\Gamma, \varrho)$. Define `the inverse of` Z_f by $Z_f^{-1} := \mathcal{G} \times_{r, X, f} Y$ with the obvious Real structure also denoted by τ_f. The map $Z_{f^{-1}} \longrightarrow Z_f^{-1}$ defined by $(x,\gamma) \longmapsto (f(\gamma), f^{-1}(x))$ is clearly a $\mathcal{G}$-Γ-equivariant Real homeomorphism; hence, $(Z_{f^{-1}}, \tau_{f^{-1}})$ and (Z_f^{-1}, τ_f) are isomorphic Real generalized morphisms from $(\mathcal{G},\rho)$ to (Γ,ϱ). Notice that Z_f^{-1} is Z_f as space; thus, (Z_f, τ_f) is at the same time a Real generalized morphism from (Γ,ϱ) to $(\mathcal{G},\rho)$ and from $(\mathcal{G},\rho)$ to (Γ,ϱ). Furthermore, it is simple to check that $Z_f \circ Z_f^{-1}$ and $Z_{\mathrm{Id}_{\mathcal{G}}}$ define isomorphic Real generalized morphisms from $(\mathcal{G},\rho)$ into itself, and likewise, $Z_f^{-1} \circ Z_f$ and Z_{Id_Γ} are isomorphic Real generalized morphisms from (Γ,ϱ) into itself.

Definition 1.24. Two Real groupoids (Γ,ϱ) and $(\mathcal{G},\rho)$ are said to be *Morita equivalent* if there exists a Real space (Z,τ) that is at the same time a Real generalized morphism from Γ to $\mathcal{G}$ and from $\mathcal{G}$ to Γ; that is to say that $Y \xleftarrow{\mathfrak{r}} Z$ is a $\mathcal{G}$-principal *Real* bundle and $Z \xrightarrow{\mathfrak{s}} X$ is a Γ-principal Real bundle.

Remark 1.25. Given a Morita equivalence $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$, its inverse, denoted by (Z^{-1},τ), is (Z,τ) as Real space, and if $\flat : (Z,\tau) \longrightarrow (Z^{-1},\tau)$ is the identity map, the left Real $\mathcal{G}$-action on (Z^{-1},τ) is given by $g \cdot \flat(z) := \flat(z \cdot g^{-1})$, and the right Real Γ-action is given by $\flat(z) \cdot \gamma := \flat(\gamma^{-1} \cdot z)$; (Z^{-1},τ) is the corresponding Real generalized morphism from $(\mathcal{G},\rho)$ to (Γ,ϱ).

The discussion before Definition 1.24 shows that the Real generalized morphism induced by a Real strict morphism is actually a Morita equivalence. However, the converse is not true. Moreover, there is a functor

$$\mathfrak{RG}_s \longrightarrow \mathfrak{RG}, \tag{1.2}$$

where $\mathfrak{RG}_s$ is the category whose objects are Real locally compact groupoids and whose morphisms are Real strict morphisms, given by

$$f \longmapsto Z_f.$$

Definition 1.26 (Real cover groupoid)**.** Let $\mathcal{G} \rightrightarrows X$ be a Real groupoid. Let $\mathcal{U} = \{U_j\}$ be a Real open cover of X. Consider the disjoint union $\coprod_{j\in J} U_j = \{(j,x) \in J \times X \ : \ x \in U_j\}$ with the Real structure $\rho^{(0)}$ given by $\rho^{(0)}(j,x) := (\bar{\jmath}, \rho(x))$ and define a Real local homeomorphism given by the projection $\pi : \coprod_j U_j \longrightarrow X,\ (j,x) \longmapsto x$. Then the set

$$\mathcal{G}[\mathcal{U}] := \{(j_0,g,j_1) \in J \times \mathcal{G} \times J \ : \ r(g) \in U_{j_0}, s(g) \in U_{j_1}\},$$

endowed with the involution $\rho^{(1)}(j_0,g,j_1) := (\bar{j_0}, \rho(g), \bar{j_1})$ has a structure of a *Real* locally compact groupoid whose unit space is $\coprod_j U_j$. The range and source maps are defined by $\tilde{r}(j_0,g,j_1) := (j_0, r(g))$ and $\tilde{s}(j_0,g,j_1) := (j_1, s(g))$; two triples are composable if they are of the form (j_0,g,j_1) and (j_1,h,j_2), where $(g,h) \in \mathcal{G}^{(2)}$, and their product is given by $(j_0,g,j_1) \cdot (j_1,h,j_2) := (j_0, gh, j_2)$. The inverse of (j_0,g,j_1) is (j_1, g^{-1}, j_0).

It is a matter of simple verifications to check the following:

Lemma 1.27. *Let* $\mathcal{G} \rightrightarrows X$ *be a Real groupoid, and* $\mathcal{U}$ *a Real open cover of* X*. Then the Real generalized morphism* $Z_\iota : \mathcal{G}[\mathcal{U}] \longrightarrow \mathcal{G}$ *induced from the canonical Real morphism*

$$\iota : \mathcal{G}[\mathcal{U}] \longrightarrow \mathcal{G},\ (j_0, g, j_1) \longmapsto g,$$

is a Morita equivalence between $(\mathcal{G}[\mathcal{U}], \rho)$ *and* $(\mathcal{G}, \rho)$*.*

Definition 1.28. Let

$$\begin{array}{ccc} Z & \xrightarrow{\mathfrak{s}} & X \\ {\scriptstyle\pi}\big\downarrow & & \\ Y & & \end{array}$$

be a locally trivial $\mathcal{G}$-principal Real bundle. A section $\mathsf{s} : Y \longrightarrow Z$ is said to be Real if $\mathsf{s} \circ \varrho = \tau \circ \mathsf{s}$. Moreover, given a Real open cover $\{U_j\}_{j\in J}$ of Y, we say that a family of local sections $\mathsf{s}_j : U_j \longrightarrow Z$ is *globally Real* if for any $j \in J$, we have

$$\mathsf{s}_{\bar{j}} \circ \varrho = \tau \circ \mathsf{s}_j. \tag{1.3}$$

Lemma 1.29. *Any locally trivial* $\mathcal{G}$*-principal Real bundle* $\pi : Z \longrightarrow Y$ *admits a globally Real family of local sections* $\{\mathsf{s}_j\}_{j\in J}$ *over some Real open cover* $\{U_j\}$*.*

Proof. Choose a *local trivialization* $(U_i, \varphi_i)_{i\in I}$ of Z; *i.e.*, $\varphi_i : U_i \longrightarrow X$ are continuous maps such that $\pi^{-1}(U_i) =: Z_{U_i} \cong U_i \times_{\varphi_i, X, r} \mathcal{G}$ with $\tau_{Z_{U_i}} = (\varrho, \rho)$. It turns out that $Z_{U_{(i,\epsilon)}} \cong U_{(i,\epsilon)} \times_{\varphi_i^\epsilon, X, r} \mathcal{G}$, where

$$\varphi_i^\epsilon := \rho^\epsilon \circ \varphi_i \circ \varrho^\epsilon : U_{(i,\epsilon)} \longrightarrow X$$

is a well-defined continuous map and $U_{(i,\epsilon)} := \varrho^\epsilon(U_i)$ for $(i, \epsilon) \in I \times \mathbb{Z}_2$. However, for $(i, \epsilon) \in I \times \mathbb{Z}_2$, there is a homeomorphism

$$U_{(i,\epsilon)} \times_{\varphi_i^\epsilon, X, r} \mathcal{G} \xrightarrow{(\varrho,\rho)} U_{\overline{(i,\epsilon)}} \times_{\varphi_i^{\epsilon+1}, X, r} \mathcal{G}\ .$$

Now, putting $\mathsf{s}_{(i,\epsilon)} : U_{(i,\epsilon)} \longrightarrow Z,\ x \longmapsto (x, \varphi_i^\epsilon(x))$, we obtain the desired sections. $\square$

For the remainder of this subsection we will need the following construction.

Let (Z, τ) be a Real space and (Γ, ϱ) a Real groupoid together with a continuous Real map $\varphi : Z \longrightarrow Y$. Then we define an induced groupoid $\varphi^*\Gamma$ over Z in which the arrows from z_1 to z_2 are the arrows in Γ from $\varphi(z_1)$ to $\varphi(z_2)$; i.e.,

$$\varphi^*\Gamma := Z \times_{\varphi, Y, r} \Gamma \times_{s, Y, \varphi} Z\ ,$$

and the product is given by $(z_1, \gamma_1, z_2).(z_2, \gamma_2, z_3) = (z_1, \gamma_1\gamma_2, z_3)$ whenever γ_1 and γ_2 are composable, while the inverse is given by

$$(z, \gamma, z')^{-1} = (z', \gamma^{-1}, z).$$

Moreover, the triple (ρ, ϱ, ρ) defines a Real structure $\varphi^*\varrho$ on $\varphi^*\Gamma$ making it into a Real groupoid $(\varphi^*\Gamma, \varphi^*\varrho)$ that we will call *the pull-back* of Γ over Z via φ.

Lemma 1.30. *Given a continuous locally split Real open map $\varphi : Z \longrightarrow Y$, then the Real groupoids Γ and $\varphi^*\Gamma$ are Morita equivalent.*

Proof. Consider the Real strict homomorphism

$$\tilde{\varphi} : \varphi^*\Gamma \ni (z_1, \gamma, z_2) \longmapsto \gamma \in \Gamma.$$

Then by Example 1.18 we obtain a Real generalized homomorphism

$$Z \xleftarrow{\pi_1} Z_{\tilde{\varphi}} \xrightarrow{s\circ\pi_2} Y$$

with $Z_{\tilde{\varphi}} := Z \times_{\tilde{\varphi},Y,r} \Gamma$, π_1 and π_2 the obvious projections, and where $Z \hookrightarrow \varphi^*\Gamma$ by $z \longmapsto (z, \varphi(z), z)$. Now using the constructions of Example 1.18, it is very easy to check that $Z_{\tilde{\varphi}}$ is in fact a Morita equivalence. $\square$

Proposition 1.31. *Two Real groupoids (Γ, ϱ) and $(\mathcal{G}, \rho)$ are Morita equivalent if and only if there exist a Real space (Z, τ) and two continuous Real maps $\varphi : Z \longrightarrow Y$ and $\varphi' : Z \longrightarrow X$ such that $\varphi^*\Gamma \cong (\varphi')^*\mathcal{G}$ under a Real (strict) homeomorphism.*

Proof. Let $Y \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} X$ be a Morita equivalence. Let us define

$$\Gamma \ltimes Z * Z \rtimes \mathcal{G} := \{(\gamma, z_1, z_2, g) \in (\Gamma \times_{s,Y,\mathfrak{r}} Z) \times (Z \times_{\mathfrak{s},X,r} \mathcal{G}) \mid z_1 g = \gamma z_2\}.$$

This defines a Real groupoid over Z whose range and source maps are defined by the second and the third projection respectively, the product is given by

$$(\gamma, z_1, z_2, g) \cdot (\gamma', z_2, z_3, g') = (\gamma\gamma', z_1, z_3, gg'),$$

provided that $\gamma, \gamma' \in \Gamma^{(2)}$ and $g, g' \in \mathcal{G}^{(2)}$, and the inverse of (γ, z_1, z_2, g) is $(\gamma^{-1}, z_2, z_1, g^{-1})$. Now, for a given triple $(z_1, \gamma, z_2) \in \mathfrak{r}^*\Gamma$, the relations $\mathfrak{r}(z_1) = r(\gamma)$ and $\mathfrak{r}(z_2) = s(\gamma)$ give $\mathfrak{r}(\gamma z_2) = \mathfrak{r}(z_1)$; then since $\mathfrak{r} : Z \longrightarrow Y$ is a Real $\mathcal{G}$-principal bundle, there exists a unique $g \in \mathcal{G}$ such that $\gamma z_2 = z_1 g$. This gives an injective homomorphism

$$\Psi : \mathfrak{r}^*\Gamma \longrightarrow \Gamma \ltimes Z * Z \rtimes \mathcal{G},$$
$$(z_1, \gamma, z_2) \longmapsto (\gamma, z_1, z_2, g),$$

which respects the Real structures. In the other hand, the map

$$\Phi : \Gamma \ltimes Z * Z \rtimes \mathcal{G} \longrightarrow \mathfrak{r}^*\Gamma,$$
$$(\gamma, z_1, z_2, g) \longmapsto (z_1, \gamma, z_2),$$

is a well-defined Real homomorphism that is injective and Real. Moreover, these two maps are, by construction, inverse to each other so that we have a Real homeomorphism $\mathfrak{r}^*\Gamma \cong \Gamma \ltimes Z * Z \rtimes \mathcal{G}$. Furthermore, since $\mathfrak{s} : Z \longrightarrow X$ is a Real Γ-principal bundle, we can use the same arguments to show that $\mathfrak{s}^*\mathcal{G} \cong \Gamma \ltimes Z * Z \rtimes \mathcal{G}$ under a Real homeomorphism.

Conversely, if $\varphi : Z \longrightarrow Y$ and $\varphi' : Z \longrightarrow X$ are given continuous Real maps and $f : \varphi^*\Gamma \longrightarrow (\varphi')^*X$ is a Real homeomorphism of groupoids, then the induced Real generalized homomorphism

$$\varphi^*\Gamma \xrightarrow{Z_f} (\varphi')^*\mathcal{G}$$

is a Morita equivalence and Lemma 1.30 completes the proof. □

The following example provides a characterization of groupoids Morita equivalent to a given Real space.

Example 1.32. Let $(X,\rho),(Y,\varrho)$ be a locally compact Hausdorff Real spaces, and let $\pi : (Y,\varrho) \longrightarrow (X,\rho)$ be a continuous locally split Real open map. Form the Real groupoid $Y^{[2]} \rightrightarrows Y$, where $Y^{[2]}$ is the fibered-product $Y \times_{\pi,X,\pi} Y$ equipped with the obvious Real structure; the groupoid structure on $Y^{[2]}$ is:

$$\begin{aligned} s(y_1,y_2) &:= y_2; & r(y_1,y_2) &:= y_1; \\ (y_1,y_2)^{-1} &:= (y_2,y_1); & (y_1,y_2)\cdot(y_2,y_3) &:= (y_1,y_3). \end{aligned}$$

Then the Real groupoids $Y^{[2]} \rightrightarrows Y$ and $X \rightrightarrows X$ are Morita equivalent. Indeed, we have $\pi^*X \sim_{Morita} X$, thanks to Lemma 1.30; but π^*X clearly identifies with $Y^{[2]}$ as Real groupoids.

Conversely, suppose (Γ,ϱ) is a Real groupoids Morita equivalent to X. Then in view of Proposition 1.31, there is a Real space (Z,τ), two continuous locally split Real open maps $\mathfrak{s} : Z \longrightarrow X, \mathfrak{r} : Z \longrightarrow Y$ such that $\mathfrak{s}^*X \cong \mathfrak{r}^*\Gamma$ as Real groupoids over Z. In particular, $\mathfrak{r} : Z \longrightarrow Y$ is a principal Real X-bundle, so that the Real space Y is homeomorphic to the quotient Real space $Z/X = Z$. Thus, we have isomorphism of Real spaces

$$\mathfrak{r}^*\Gamma = Z \times_Y \Gamma \times_Y Z \cong Y \times_Y \Gamma \times_Y Y \cong \Gamma.$$

Moreover, we have $\mathfrak{s}^*X \cong Z^{[2]}$ as Real spaces. Therefore, the Real groupoids $\Gamma \rightrightarrows Y$ and $Z^{[2]} \rightrightarrows Z$ as isomorphic.

Proposition 1.33 (Cf. Proposition 2.3 [25])**.** *Any Real generalized morphism*

$$Y \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} X$$

is obtained by composition of the canonical Morita equivalence between (Γ,ϱ) *and* $(\Gamma[\mathcal{U}],\varrho)$*, where* $\mathcal{U}$ *is an open cover of* Y*, with a Real strict morphism* $f_{\mathcal{U}} : \Gamma[\mathcal{U}] \longrightarrow \mathcal{G}$ *(i.e., its induced morphism in the category* $\mathfrak{RG}$*).*

Proof. From Lemma 1.30, there is a Real Morita equivalence $Z_{\tilde{\mathfrak{r}}} : \mathfrak{r}^*\Gamma \longrightarrow \Gamma$ and the Real homeomorphism $\mathfrak{r}^*\Gamma \cong \Gamma \ltimes Z * Z \rtimes \mathcal{G}$ induces a Real strict homomorphism $f : \mathfrak{r}^*\Gamma \longrightarrow \mathcal{G}$ given by the fourth projection, and hence a Real generalized homomorphism $Z_f : \mathfrak{r}^*\Gamma \longrightarrow \mathcal{G}$. Furthermore, by using the construction of these generalized homomorphisms, it is easy to check that the composition $Z_{\tilde{\mathfrak{r}}} \times_\Gamma Z$ is $\mathfrak{r}^*\Gamma$-$\mathcal{G}$-equivariently homeomorphic to Z (under a Real homeomorphism); i.e., the diagram

$$\begin{tikzcd} \Gamma \arrow[dr, "Z"'] & \mathfrak{r}^*\Gamma \arrow[l, "Z_{\tilde{\mathfrak{r}}}"', "\cong"] \arrow[d, "Z_f"] \\ & \mathcal{G} \end{tikzcd}$$

is commutative in the category $\mathfrak{RG}$.

Consider a Real open cover $\mathcal{U} = \{U_j\}$ of Y together with a globally Real family of local sections $\mathsf{s}_j : U_j \longrightarrow Z$ of $\mathfrak{r} : Z \longrightarrow Y$. Then, setting $(j_0, \gamma, j_1) \longmapsto (\mathsf{s}_{j_0}(r(\gamma)), \gamma, \mathsf{s}_{j_1}(s(\gamma)))$ for $(j_0, \gamma, j_1) \in \Gamma[\mathcal{U}]$, we get a Real strict homomorphism $\tilde{\mathsf{s}} : \Gamma[\mathcal{U}] \longrightarrow \mathfrak{r}^*\Gamma$ such that the composition $\Gamma[\mathcal{U}] \longrightarrow \mathfrak{r}^*\Gamma \longrightarrow \Gamma$ is the canonical map ι described in Example 1.26. Then, $f \circ \tilde{\mathsf{s}} : \Gamma[\mathcal{U}] \longrightarrow \mathcal{G}$ is the desired Real strict homomorphism. $\square$

This proposition leads us to think of a Real generalized homomorphism from a Real groupoid (Γ, ϱ) to a Real groupoid $(\mathcal{G}, \rho)$ as a Real strict morphism $f_{\mathcal{U}} : (\Gamma[\mathcal{U}], \varrho) \longrightarrow (\mathcal{G}, \rho)$, where $\mathcal{U}$ is a Real open cover of Y.

To refine this point of view, given two Real groupoids (Γ, ϱ) and $(\mathcal{G}, \rho)$, let Ω denote the collection of such pairs $(\mathcal{U}, f_{\mathcal{U}})$. We say that two pairs $(\mathcal{U}, f_{\mathcal{U}})$ and $(\mathcal{U}', f_{\mathcal{U}'})$ are *isomorphic* provided that $Z_{f_{\mathcal{U}}} \circ Z^{-1}_{\iota_{\mathcal{U}}} \cong Z_{f_{\mathcal{U}'}} \circ Z^{-1}_{\iota_{\mathcal{U}'}}$, where $\iota_{\mathcal{U}} : (\Gamma[\mathcal{U}], \varrho) \longrightarrow (\Gamma, \varrho)$ and $\iota_{\mathcal{U}'} : (\Gamma[\mathcal{U}'], \varrho) \longrightarrow (\Gamma, \varrho)$ are the canonical morphisms; this clearly defines an equivalence relation. We denote by $\Omega((\Gamma, \varrho), (\mathcal{G}, \rho))$ the set of isomorphism classes of elements of Ω.

Let $(\mathcal{U}, f_{\mathcal{U}}) : (\Gamma, \varrho) \longrightarrow (\mathcal{G}'.\rho')$ be an equivalence class in $\Omega((\Gamma, \varrho), (\mathcal{G}', \rho'))$ and let $(\mathcal{V}, f_{\mathcal{V}}) : (\mathcal{G}', \rho') \longrightarrow (\mathcal{G}, \rho)$ be an element in $\Omega((\mathcal{G}', \rho'), (\mathcal{G}, \rho))$. Let $\iota_{\mathcal{G}'} : \mathcal{G}'[\mathcal{V}] \longrightarrow \mathcal{G}'$ be the canonical morphism, and let $Z^{-1}_{\iota_{\mathcal{G}'}} : (\mathcal{G}', \rho') \longrightarrow (\mathcal{G}'[\mathcal{V}], \rho')$ be the inverse of $Z_{\iota_{\mathcal{G}'}}$. Next, we apply Proposition 1.33 to the Real generalized morphism $Z^{-1}_{\iota_{\mathcal{G}'}} \circ Z_{f_{\mathcal{U}}} : \Gamma[\mathcal{U}] \longrightarrow \mathcal{G}'[\mathcal{V}]$ to get a Real open cover $\mathcal{U}'$ of Y containing $\mathcal{U}$ and a Real strict morphism $\varphi_{\mathcal{U}'} : (\Gamma[\mathcal{U}'], \varrho) \longrightarrow (\mathcal{G}'[\mathcal{V}], \rho')$. Then, we pose

$$(\mathcal{V}, f_{\mathcal{V}}) \circ (\mathcal{U}, f_{\mathcal{U}}) := (\mathcal{U}', f_{\mathcal{U}'}), \tag{1.4}$$

with $f_{\mathcal{U}'} = f_{\mathcal{V}} \circ \varphi_{\mathcal{U}'}$; thus we get an element of $\Omega((\Gamma, \varrho), (\mathcal{G}, \rho))$. It follows that there exists a category $\mathfrak{RG}_\Omega$ whose objects are Real groupoids, and in which a morphism from (Γ, ϱ) to $(\mathcal{G}, \rho)$ is a class $(\mathcal{U}, f_{\mathcal{U}})$ in $\Omega((\Gamma, \varrho), (\mathcal{G}, \rho))$.

Example 1.34. Any Real strict morphism $f : (\Gamma, \varrho) \longrightarrow (\mathcal{G}, \rho)$ can be identified with the pair (Y, f), by considering the trivial Real open cover Y consisting of one set, and by viewing the groupoid Γ as the cover groupoid $\Gamma[Y]$. In particular, $\mathfrak{RG}_s$ is a subcategory of $\mathfrak{RG}_\Omega$.

Example 1.35. Suppose that $(Z, \tau) : (\Gamma, \varrho) \longrightarrow (\mathcal{G}, \rho)$ is a Real generalized morphism. Then, Proposition 1.33 provides a unique class $(\mathcal{U}, f_{\mathcal{U}}) \in \Omega((\Gamma, \varrho), (\mathcal{G}, \rho))$.

Remark 1.36. Note that a class $(\mathcal{U}, f_{\mathcal{U}}) \in \Omega\left((\Gamma,\varrho),(\mathcal{G},\rho)\right)$ is an isomorphism in $\mathfrak{RG}_\Omega$ if there exists $(\mathcal{V}, f_{\mathcal{V}}) \in \Omega\left((\mathcal{G},\rho),(\Gamma,\varrho)\right)$ such that

$$Z_{f_{\mathcal{U}}} \circ Z_{\iota_{\mathcal{U}}}^{-1} \circ Z_{f_{\mathcal{V}}} \cong Z_{\iota_{\mathcal{V}}} \text{ and } Z_{f_{\mathcal{V}}} \circ Z_{\iota_{\mathcal{V}}}^{-1} \circ Z_{f_{\mathcal{U}}} \cong Z_{\iota_{\mathcal{U}}}, \tag{1.5}$$

where $\iota_{\mathcal{U}} : (\Gamma[\mathcal{U}],\varrho) \longrightarrow (\Gamma,\varrho)$ and $\iota_{\mathcal{V}} : (\mathcal{G}[\mathcal{U}],\rho) \longrightarrow (\mathcal{G},\rho)$ are the canonical morphisms.

Proposition 1.37. *Define* $\mathsf{F} : \mathfrak{RG} \longrightarrow \mathfrak{RG}_\Omega$ *by*

$$\mathsf{F}(Z,\tau) := (\mathcal{U}, f_{\mathcal{U}}), \tag{1.6}$$

where, if $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ *is a class of Real generalized morphisms,* $(\mathcal{U}, f_{\mathcal{U}})$ *is the class of pairs corresponding to* (Z,τ)*.*

Then F *is a functor; furthermore,* F *is an isomorphism of categories.*

Proof. Suppose that $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G}',\rho')$, $(Z',\tau') : (\mathcal{G}',\rho') \longrightarrow (\mathcal{G},\rho)$ are morphisms in $\mathfrak{RG}$. Let

$$\begin{aligned}\mathsf{F}(Z' \circ Z, \tau \times \tau') &= (\mathcal{U}, f_{\mathcal{U}}) \in \Omega\left((\Gamma,\varrho),(\mathcal{G},\rho)\right),\\ \mathsf{F}(Z,\tau) &= (\mathcal{U}', f_{\mathcal{U}'}) \in \Omega\left((\Gamma,\varrho),(\mathcal{G}',\rho')\right),\\ \mathsf{F}(Z',\tau') &= (\mathcal{V}, f_{\mathcal{V}}) \in \Omega\left((\mathcal{G}',\rho'),(\mathcal{G},\rho)\right).\end{aligned}$$

Consider a Real open cover $\tilde{\mathcal{U}}$ of Y containing $\mathcal{U}'$ and a Real morphism $\varphi_{\tilde{\mathcal{U}}} : (\Gamma[\tilde{\mathcal{U}}],\varrho) \longrightarrow (\mathcal{G}'[\mathcal{V}],\rho')$ such that $Z_{\varphi_{\tilde{\mathcal{U}}}} \circ Z_i^{-1} \cong Z_{\iota_{\mathcal{V}}}^{-1} \circ Z_{f_{\mathcal{U}'}}$ as Real generalized morphisms from $(\Gamma[\mathcal{U}'],\varrho)$ to $(\mathcal{G}'[\mathcal{V}],\rho')$, where

$$i : (\Gamma[\tilde{\mathcal{U}}],\varrho) \longrightarrow (\Gamma[\mathcal{U}'],\varrho) \quad \text{and} \quad \iota_{\mathcal{V}} : (\mathcal{G}'[\mathcal{V}],\rho') \longrightarrow (\mathcal{G}',\rho')$$

are the canonical morphisms. Note that if $\iota_{\tilde{\mathcal{U}}} : (\Gamma[\tilde{\mathcal{U}}],\varrho) \longrightarrow (\Gamma,\varrho)$ is the canonical morphism, then $\iota_{\tilde{\mathcal{U}}} = \iota_{\mathcal{U}'} \circ i$; hence, $Z_{\iota_{\tilde{\mathcal{U}}}}^{-1} \cong Z_i^{-1} \circ Z_{\iota_{\mathcal{U}'}}^{-1}$ by functoriality.

On the other hand, $\mathsf{F}(Z',\tau') \circ \mathsf{F}(Z,\tau) = (\mathcal{V}, f_{\mathcal{V}}) \circ (\mathcal{U}, f_{\mathcal{U}}) = (\tilde{\mathcal{U}}, f_{\tilde{\mathcal{U}}})$, where $f_{\tilde{\mathcal{U}}} = f_{\mathcal{V}} \circ \varphi_{\tilde{\mathcal{U}}}$. Henceforth,

$$Z_{f_{\tilde{\mathcal{U}}}} \circ Z_{\iota_{\tilde{\mathcal{U}}}}^{-1} \cong Z_{f_{\mathcal{V}}} \circ Z_{\varphi_{\tilde{\mathcal{U}}}} \circ Z_i^{-1} \circ Z_{\iota_{\mathcal{U}'}}^{-1} \cong Z_{f_{\mathcal{V}}} \circ Z_{\iota_{\mathcal{V}}}^{-1} \circ Z_{f_{\mathcal{U}'}} \circ Z_{\iota_{\mathcal{U}'}}^{-1} \cong Z' \circ Z,$$

which shows that $\mathsf{F}(Z' \circ Z, \tau \times \tau') \cong \mathsf{F}(Z',\tau') \circ \mathsf{F}(Z,\tau)$, and thus F is a functor.

Now, it is not hard to see that we get an inverse functor for F by defining

$$\mathsf{Z} : \mathfrak{RG}_\Omega \longrightarrow \mathfrak{RG}, (\mathcal{U}, f_{\mathcal{U}}) \longmapsto (Z_{f_{\mathcal{U}}} \circ Z_{\iota_{\mathcal{U}}}^{-1}, \tau), \tag{1.7}$$

where τ is defined in an obvious way. □

1.5. Real graded twists.

In this section we define *Real graded twists.*

Definition 1.38 (Cf. [11, §2])**.** Let $\Gamma \rightrightarrows Y$ be a Real groupoid and let $\mathbb{S}$ be a Real Abelian group. A *Real graded* $\mathbb{S}$*-twist* $(\widetilde{\Gamma},\delta)$ over Γ consists of the following data:

(i) a Real groupoid $\widetilde{\Gamma}$ whose unit space is Y, together with a Real strict homomorphism $\pi : \widetilde{\Gamma} \longrightarrow \Gamma$ that restricts to the identity in Y,

(ii) a (left) Real action of S on $\widetilde{\Gamma}$ compatible with the partial product in $\widetilde{\Gamma}$ making $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma$ a (left) Real S-principal bundle,

(iii) a strict homomorphism $\delta : \Gamma \longrightarrow \mathbb{Z}_2$, called *the grading*, such that $\delta(\bar{\gamma}) = \delta(\gamma)$ for any $\gamma \in \Gamma$.

In this case we refer to the triple $(\widetilde{\Gamma}, \Gamma, \delta)$ as a *Real graded* S-*twist*, and it is sometimes symbolized by the "extension"

$$\begin{tikzcd} \mathrm{S} \arrow[r] & \widetilde{\Gamma} \arrow[r, "\pi"] & \Gamma \arrow[d, "\delta"] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

Example 1.39 (The trivial twist)**.** Given Real groupoid Γ, we form the product groupoid $\Gamma \times \mathrm{S}$ and we endow it with the Real structure $\overline{(\gamma, \lambda)} := (\bar{\gamma}, \bar{\lambda})$ for. Let S act on $\Gamma \times \mathrm{S}$ by multiplication with the second factor. Then $\mathcal{T}_0 := (\Gamma \times \mathrm{S}, 0)$ is a Real graded twist of Γ, where $0 : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ is the zero map. This element is called *the trivial Real graded* S-*twist over* Γ.

Example 1.40. Let Y be a locally compact Real space and $\{U_i\}_{i \in I \times \{\pm 1\}}$ be a good Real open. Let us consider the Real groupoid $Y[\mathcal{U}] \rightrightarrows \coprod_i U_i$, and the space $Y \times \mathrm{S}$ together with the Real structure $(y, \lambda) \longmapsto (\bar{y}, \bar{\lambda})$ and the Real S-action given by the multiplication on the second factor. We write $x_{i_0 i_1}$ for $(i_1, x, i_1) \in Y[\mathcal{U}]$. There is a canonical Real morphism $\delta : Y[\mathcal{U}] \longrightarrow \mathbb{Z}_2$ given by $\delta(x_{i_0 i_1}) := \varepsilon_0 + \varepsilon_1$ for $i_0 = (i'_0, \varepsilon_0),\ i_1 = (i'_1, \varepsilon_1) \in I$. Then, a Real graded S-twist $(\widetilde{\Gamma}, Y[\mathcal{U}], \delta)$ consists of a family of principal Real S-bundles $\widetilde{\Gamma}_{ij} \cong U_{ij} \times \mathrm{S}$ subject to the multiplication

$$(x_{i_0 i_1}, \lambda_1) \cdot (x_{i_1 i_2}, \lambda_2) = (x_{i_0 i_2}, \lambda_1 \lambda_2 c_{i_0 i_1 i_2}(x)),$$

where $c = \{c_{i_0 i_1 i_2}\}$ is a family of continuous maps $c_{i_0 i_1 i_2} : U_{i_0 i_1 i_2} \longrightarrow \mathrm{S}$ which is a 2-cocycle such that $c_{\bar{i}_0 \bar{i}_1 \bar{i}_2}(\bar{x}) = \overline{c_{i_0 i_1 i_2}(x)}$ for all $x \in U_{i_0 i_1 i_2} = U_{i_0} \cap U_{i_1} \cap U_{i_2}$. The pair (δ, c) will be called *the Dixmier–Douady class of* $(\widetilde{\Gamma}, Y[\mathcal{U}], \delta)$ (see Section 2.12).

Example 1.41. Let $\Gamma \rightrightarrows Y$ be a Real groupoid, and let $J : \Lambda \longrightarrow Y$ be a Real S-principal bundle. Then the tensor product $r^*\Lambda \otimes \overline{s^*\Lambda}$, which is a Real S-principal bundle over Γ, naturally admits the structure of Real groupoid over Y, so that $(r^*\Lambda \otimes \overline{s^*\Lambda}, 0)$ is a Real graded S-twist over Γ.

There is an obvious notion of strict morphism of Real graded S-twists. For instance, two Real graded S-twists $(\widetilde{\Gamma}_1, \Gamma, \delta_1)$ and $(\widetilde{\Gamma}_2, \Gamma, \delta_2)$ are isomorphic if there exists a Real S-equivariant isomorphism of groupoids $f : \widetilde{\Gamma}_1 \longrightarrow \widetilde{\Gamma}_2$

such that the diagram

$$\begin{tikzcd} \widetilde{\Gamma}_1 \arrow[r, "\pi_1"] \arrow[d, "f"'] & \Gamma \\ \widetilde{\Gamma}_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

commutes in the category $\mathfrak{RG}_s$. In particular, we say that $(\widetilde{\Gamma}, \delta)$ is *strictly trivial* if it isomorphic to the trivial Real graded groupoid $(\Gamma \times \mathrm{S}, 0)$. By $\widehat{\mathrm{TwR}}(\Gamma, \mathrm{S})$ we denote the set of strict isomorphism classes of Real graded S-twists over Γ. The class of $(\widetilde{\Gamma}, \delta)$ in $\widehat{\mathrm{TwR}}(\Gamma, \mathrm{S})$ is denoted by $[\widetilde{\Gamma}, \delta]$.

Definition 1.42 (Cf. [11, 23, 6])**.** Given two Real graded S-twists $\mathcal{T}_1 = (\widetilde{\Gamma}_1, \delta_1)$ and $\mathcal{T}_2 = (\widetilde{\Gamma}_2, \delta_2)$ over $\mathcal{G}$, we define their tensor product

$$\mathcal{T}_1 \hat{\otimes} \mathcal{T}_2 = (\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2, \delta_1 + \delta_2)$$

by the *Baer sum* of $\mathcal{T}_1$ and $\mathcal{T}_2$ defined as follows. Define the groupoid $\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2$ as the quotient

$$\widetilde{\Gamma}_1 \times_\Gamma \widetilde{\Gamma}_2 / \mathrm{S} := \{(\tilde{\gamma_1}, \tilde{\gamma_2}) \in \widetilde{\Gamma}_1 \times_{\pi_1, \Gamma, \pi_2} \widetilde{\Gamma}_2\} /_{(\tilde{\gamma_1}, \tilde{\gamma_2}) \sim (\lambda \tilde{\gamma_1}, \lambda^{-1} \tilde{\gamma_2})}, \tag{1.8}$$

where $\lambda \in \mathrm{S}$, together with the obvious Real structure. The projection $\pi_1 \otimes \pi_2$ is just π_i and $\delta = \delta_1 + \delta_2$ is given by $\delta(\gamma) = \delta_1(\gamma) + \delta_2(\gamma)$.
The product in the Real groupoid $\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2$ is

$$(\tilde{\gamma_1}, \tilde{\gamma_2})(\tilde{\gamma'_1}, \tilde{\gamma'_2}) := (-1)^{\delta_2(\gamma_2)\delta_1(\gamma'_1)} (\tilde{\gamma_1}\tilde{\gamma'_1}, \tilde{\gamma_2}\tilde{\gamma'_2}), \tag{1.9}$$

whenever this does make sense and where $\gamma_i = \pi_2(\tilde{\gamma_i}),\ i = 1, 2$.

Lemma 1.43 ([23, p.4])**.** *Given* $[\widetilde{\Gamma}_i, \delta_i] \in \widehat{\mathrm{TwR}}(\Gamma, \mathrm{S}), i = 1, 2$*, set*

$$[\widetilde{\Gamma}_1, \delta_1] + [\widetilde{\Gamma}_2, \delta_2] := [\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2, \delta_1 + \delta_2].$$

Then, under this sum, $\widehat{\mathrm{TwR}}(\Gamma, \mathrm{S})$ *is an Abelian group whose zero element is given by the class of the trivial element* $\mathcal{T}_0 = (\mathcal{G} \times \mathrm{S}, 0)$.

Proof. The tensor product defined above is commutative in $\widehat{\mathrm{TwR}}(\Gamma, \mathrm{S})$. Indeed, the groupoid $\widetilde{\Gamma}_2 \hat{\otimes} \widetilde{\Gamma}_1 = \widetilde{\Gamma}_2 \times_\Gamma \widetilde{\Gamma}_1 / \mathrm{S}$ is endowed with the multiplication

$$(\tilde{\gamma_2}, \tilde{\gamma_1})(\tilde{\gamma'_2}, \tilde{\gamma'_1}) = (-1)^{\delta_1(\gamma_1)\delta_2(\gamma'_2)} (\tilde{\gamma_2}\tilde{\gamma'_2}, \tilde{\gamma_1}\tilde{\gamma'_1}).$$

Then the map

$$\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2 \longrightarrow \widetilde{\Gamma}_2 \hat{\otimes} \widetilde{\Gamma}_1\ , (\tilde{\gamma_1}, \tilde{\gamma_2}) \longmapsto (-1)^{\delta_1(\gamma_1)\delta_2(\gamma_2)} (\tilde{\gamma_2}, \tilde{\gamma_1})$$

is a Real S-equivariant isomorphism of groupoids.

Now define the inverse of $(\widetilde{\Gamma}, \delta)$ is $(\widetilde{\Gamma}^{\mathrm{op}}, \delta)$ where $\widetilde{\Gamma}^{op}$ is $\widetilde{\Gamma}$ as a set but, together with the same Real structure, but the S-principal bundle structure is replaced by the conjugate one, *i.e.*, $\lambda \tilde{\gamma}^{\mathrm{op}} = (\bar{\lambda}\tilde{\gamma})^{\mathrm{op}}$, and the product $*_{\mathrm{op}}$ in $\widetilde{\Gamma}^{op}$ is

$$\tilde{\gamma} *_{\mathrm{op}} \tilde{\gamma}' := (-1)^{\delta(\gamma)\delta(\gamma')} \tilde{\gamma}\tilde{\gamma}'.$$

Now it is easy to see that the map

$$\Gamma \times \mathbb{S} \longrightarrow \widetilde{\Gamma} \times_{\Gamma} \widetilde{\Gamma}^{\rm op}/\mathbb{S} \ , (\gamma, \lambda) \longmapsto (\lambda\tilde{\gamma}, \tilde{\gamma}) \ ,$$

where $\tilde{\gamma} \in \widetilde{\Gamma}$ is any lift of $\gamma \in \Gamma$, is an isomorphism. $\square$

We have the following criteria of strict triviality; the proof is the same as in [25, Proposition 2.8].

Proposition 1.44. *Let $(\widetilde{\Gamma}, \delta)$ be a Real graded $\mathbb{S}$-twist over the Real groupoid $\Gamma \rightrightarrows Y$. The following are equivalent:*

(i) *$(\widetilde{\Gamma}, \delta)$ is strictly trivial.*
(ii) *$\delta(\gamma) = 0, \forall \gamma \in \Gamma$, and there exists a Real strict homomorphism $\sigma : \Gamma \longrightarrow \widetilde{\Gamma}$ such that $\pi \circ \sigma = \mathrm{Id}$.*
(iii) *$\delta(\gamma) = 0, \forall \gamma \in \Gamma,$, and there exists a Real $\mathbb{S}$-equivariant groupoid homomorphism $\varphi : \widetilde{\Gamma} \longrightarrow \mathbb{S}$.*

Example 1.45. Let $J : \Lambda \longrightarrow Y$ be a Real $\mathbb{S}$-principal bundle with a Real (left) Γ-action that is compatible with the $\mathbb{S}$-action; in other words $Y \xleftarrow[J]{} \Lambda \longrightarrow \star$ is a Real generalized homomorphism from Γ to $\mathbb{S}$. Then, the Real Γ-action induces an $\mathbb{S}$-equivariant isomorphism $\Lambda_{s(\gamma)} \ni v \longmapsto \gamma \cdot v \in \Lambda_{r(\gamma)}$ for every $\gamma \in \Gamma$. Hence, there is a Real $\mathbb{S}$-equivariant groupoid isomorphism $\varphi : r^*\Lambda \otimes \overline{s^*\Lambda} \longrightarrow \Gamma \times \mathbb{S}$ defined as follows. If $(v, \flat(w)) \in \Lambda_{r(\gamma)} \otimes \overline{\Lambda}_{s(\gamma)}$, there exists a unique $\lambda \in \mathbb{S}$ such that $\gamma \cdot w = v \cdot \lambda$. We then set

$$\varphi([v, \flat(w)]) := (\gamma, \lambda).$$

The inverse of φ is $\varphi'(\gamma, \lambda) := [v_\gamma, \overline{\gamma^{-1} \cdot v_\gamma}]$, where for $\gamma \in \Gamma$, v_γ is any lift of $r(\gamma)$ through the projection J.

Observe that the set of Real graded $\mathbb{S}$-twists of the from $(r^*\Lambda \otimes \overline{s^*\Lambda}, 0)$ over Γ (see Example 1.41) is a subgroup of $\widehat{\mathrm{TwR}}(\Gamma, \mathbb{S})$. By $\widehat{\mathrm{extR}}(\Gamma, \mathbb{S})$ we denote the quotient of $\widehat{\mathrm{TwR}}(\Gamma, \mathbb{S})$ by this subgroup.

Let us show that $\widehat{\mathrm{extR}}(\cdot, \mathbb{S})$ is functorial in the category $\mathfrak{RG}_s$. Let Γ, Γ' be two Real groupoids, and let $f : \Gamma' \longrightarrow \Gamma$ be a morphism in $\mathfrak{RG}_s$. Suppose that $\mathcal{T} = (\widetilde{\Gamma}, \delta)$ is a Real graded $\mathbb{S}$-twist over Γ. Then, the pull-back

$$f^*\widetilde{\Gamma} := \widetilde{\Gamma} \times_{\pi, \Gamma, f} \Gamma'$$

of the Real $\mathbb{S}$-principal bundle $\pi : \widetilde{\Gamma} \longrightarrow \Gamma$, on which the Real groupoid structure is the one induced from the product Real groupoid $\widetilde{\Gamma} \times \Gamma'$, defines a Real graded twist

$$\begin{tikzcd} f^*\mathcal{T} := \ \mathbb{S} \arrow[r] & f^*\widetilde{\Gamma} \arrow[r, "f^*\pi"] & \Gamma' \arrow[d, "f^*\delta"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{1.10}$$

where $f^*\pi(\tilde{\gamma},\gamma') := \gamma'$, $f^*\delta(\gamma') := \delta(f(\gamma')) \in \mathbb{Z}_2$, and the Real left S-action on $f^*\widetilde{\Gamma}$ being given by $\lambda\cdot(\tilde{\gamma},\gamma') = (\lambda\tilde{\gamma},\gamma')$. Suppose now that $\mathcal{T}_i = (\widetilde{\Gamma}_i,\delta_i)$, $i=1,2$ are representatives in $\widehat{\mathrm{extR}}(\Gamma,\mathrm{S})$. Then,

$$f^*(\mathcal{T}_1\hat{\otimes}\mathcal{T}_2) = f^*\mathcal{T}_1\hat{\otimes}f^*\mathcal{T}_2;$$

indeed,

$$\begin{aligned} f^*(\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2) = \left(\widetilde{\Gamma}_1\times_\Gamma\widetilde{\Gamma}_2/\mathrm{S}\right)\times_\Gamma\Gamma' &\cong \left((\Gamma_1\times_\Gamma\Gamma')\times_\Gamma(\widetilde{\Gamma}_2\times_\Gamma\Gamma')\right)/\mathrm{S} \\ &= f^*\widetilde{\Gamma}_1\hat{\otimes}f^*\widetilde{\Gamma}_2. \end{aligned}$$

Moreover, it is easily seen that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent in $\widehat{\mathrm{extR}}(\Gamma,\mathrm{S})$, then so are $f^*\mathcal{T}_1$ and $f^*\mathcal{T}_2$. Thus, f induces a morphism of Abelian groups $f^*:\widehat{\mathrm{extR}}(\Gamma,\mathrm{S})\longrightarrow\widehat{\mathrm{extR}}(\Gamma',\mathrm{S})$. We then have proved this:

Lemma 1.46. *The correspondence*

$$\begin{aligned} \widehat{\mathrm{extR}}(\cdot,\mathrm{S}) &: \mathfrak{RG}_s\longrightarrow\mathfrak{Ab}, \\ \Gamma\longmapsto\widehat{\mathrm{extR}}(\Gamma,\mathrm{S}), &\quad f\longmapsto f^*, \end{aligned} \tag{1.11}$$

where $\mathfrak{Ab}$ is the category of Abelian groups, is a contravariant functor. In particular, $\widehat{\mathrm{extR}}(\mathcal{G},\mathrm{S})$ is invariant under Real strict isomorphisms.

1.6. Real graded central extensions. In this subsection we introduce Real graded central extensions of Real groupoids, by adapting [11, 12, 6, 23] to our context.

Definition 1.47. Let $(\widetilde{\Gamma}_i,\Gamma_i,\delta_i), i=1,2$, be Real graded S-twists. Then a Real generalized homomorphism $Z:\widetilde{\Gamma}_1\longrightarrow\widetilde{\Gamma}_2$ is said to be S-equivariant if there is a Real action of S on Z such that

$$(\lambda\tilde{\gamma}_1)\cdot z\cdot\tilde{\gamma}_2 = \tilde{\gamma}_1\cdot(\lambda z)\cdot\tilde{\gamma}_2 = \tilde{\gamma}_1\cdot z\cdot(\lambda\tilde{\gamma}_2),$$

for any $(\lambda,\tilde{\gamma}_1,z,\tilde{\gamma}_2)\in\mathrm{S}\times\widetilde{\Gamma}_1\times Z\times\widetilde{\Gamma}_2$ such that these products make sense. We refer to $Z:(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\longrightarrow(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$ as a generalized morphism of Real graded S-twists. In particular, if Z is an isomorphism, the two Real graded S-twists are said to be *Morita equivalent*; in this case we write $(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\sim(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$.

Lemma 1.48. *Let $Z:(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\longrightarrow(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$ be a generalized morphism. Then the* S*-action on Z is free and the Real space Z/S (with the obvious involution) is a Real generalized homomorphism from Γ_1 to Γ_2.*

Proof. Same as [25, Lemma 2.10]. □

Definition 1.49. Let $\mathcal{G}$ be a Real groupoid and S an abelian Real group. A *Real graded* S*-central extension* of $\mathcal{G}$ consists of a triple $(\widetilde{\Gamma},\Gamma,\delta,P)$, where $(\widetilde{\Gamma},\Gamma,\delta)$ is a Real graded S-twist, and P is a (Real) Morita equivalence $\Gamma\longrightarrow\mathcal{G}$.

Definition 1.50. We say that $(\widetilde{\Gamma}_1, \Gamma_1, \delta_1, P_1)$ and $(\widetilde{\Gamma}_2, \Gamma_2, \delta_2, P_2)$ are Morita equivalent if there exists a Morita equivalence $Z : (\widetilde{\Gamma}_1, \Gamma_1, \delta_1) \longrightarrow (\widetilde{\Gamma}_2, \Gamma_2, \delta_2)$ such that the diagrams

$$\begin{tikzcd} \Gamma_1 \arrow[r, "Z/\mathrm{S}"] \arrow[dr, "P_1"'] & \Gamma_2 \arrow[d, "P_2"] \\ & \mathcal{G} \end{tikzcd} \tag{1.12}$$

and

$$\begin{tikzcd} \Gamma_1 \arrow[r, "Z/\mathrm{S}"] \arrow[dr, "\delta_1"'] & \Gamma_2 \arrow[d, "\delta_2"] \\ & \mathbb{Z}_2 \end{tikzcd} \tag{1.13}$$

commute in the category $\mathfrak{RG}$. Such a Z is also called *an equivalence bimodule* of Real graded S-central extensions. The set of Morita equivalence classes of Real graded S-central extensions of $\mathcal{G}$ is denoted by $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$.

The set $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ admits a natural structure of abelian group described in the following way. Assume that $\mathbb{E}_i = (\widetilde{\Gamma}_i, \Gamma_i, \delta_i, P_i),\ i = 1, 2$, are two given Real graded S-central extensions of $\mathcal{G}$, then $Y_1 \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} Y_2$ is a Morita equivalence between Γ_1 and Γ_2, where $Z = P_1 \times_{\mathcal{G}} P_2$. But from Proposition 1.31 there exists a Real homeomorphism $f : \mathfrak{s}^*\Gamma_2 \longrightarrow \mathfrak{r}^*\Gamma_1$. Now one can see that the maps $\pi : \mathfrak{r}^*\widetilde{\Gamma}_1 \longrightarrow \mathfrak{r}^*\Gamma_1,\ (z, \tilde{\gamma}_1, z') \longmapsto (z, \pi_1(\tilde{\gamma}_1), z')$ and $\pi' : \mathfrak{s}^*\widetilde{\Gamma}_2 \longrightarrow \mathfrak{r}^*\Gamma_1(z, \tilde{\gamma}_2, z') \longmapsto \pi \circ f(z, \tilde{\gamma}_2, z')$ define two Real S-principal bundles and then $(\mathfrak{r}^*\widetilde{\Gamma}_1, \delta)$ and $(\mathfrak{s}^*\widetilde{\Gamma}_2, \delta)$, where $\delta := \delta_1 \circ pr_2$, define elements of $\widehat{\mathrm{extR}}(\mathfrak{r}^*\Gamma_1, \mathrm{S})$. Therefore, we can form the tensor product $(\mathfrak{r}^*\widetilde{\Gamma}_1 \hat{\otimes} \mathfrak{s}^*\widetilde{\Gamma}_2, \delta \otimes \delta)$ are Real graded S-groupoid over $\mathfrak{r}^*\Gamma_1$. Moreover, $\mathfrak{r}^*\Gamma_1 \sim_{Morita} \Gamma_1$; then, if $P : \mathfrak{r}^*\Gamma_1 \longrightarrow \mathcal{G}$ is a Real Morita equivalence, we obtain a Real graded S-central extension of $\mathcal{G}$ by setting

$$\mathbb{E}_1 \hat{\otimes} \mathbb{E}_2 := (\mathfrak{r}^*\widetilde{\Gamma}_1 \hat{\otimes} \mathfrak{s}^*\widetilde{\Gamma}_2, \mathfrak{r}^*\Gamma_1, \delta, P), \tag{1.14}$$

that we will call *the tensor product of* $\mathbb{E}_1$ *and* $\mathbb{E}_2$. Thus, we define the sum

$$[\mathbb{E}_1] + [\mathbb{E}_2] := [\mathbb{E}_1 \hat{\otimes} \mathbb{E}_2],$$

which is easily seen to be well-defined in $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$. The inverse $\mathbb{E}^{\mathrm{op}}$ of $\mathbb{E}$ is $(\widetilde{\Gamma}^{\mathrm{op}}, \Gamma, \delta, P)$. Notice that $\widehat{\mathrm{extR}}(\mathcal{G}, \mathrm{S})$ is naturally a subgroup of $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ by identifying a Real graded S-twist $(\widetilde{\Gamma}, \mathcal{G}, \delta)$ with the Real graded S-central extension $(\widetilde{\Gamma}, \mathcal{G}, \delta, \mathcal{G})$. We summarize this in the next lemma.

Lemma 1.51. *Under the sum defined above,* $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ *is an abelian group whose zero element is the class of the trivial Real graded* S*-central extension* $(\mathcal{G} \times \mathrm{S}, \mathcal{G}, 0, \mathcal{G})$.

When the Real structure is trivial, then we recover the usual definition of graded central extensions (see [6] for instance) of $\mathcal{G}$ by the group $\mathbb{Z}_2$.

Proposition 1.52. *Suppose that* $\mathcal{G} \rightrightarrows X$ *is equipped with a trivial Real structure. Then*

$$\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathbb{S}^1) \cong \widehat{\mathrm{Ext}}(\mathcal{G}, \mathbb{Z}_2).$$

Example 1.53. Suppose $\mathcal{G}$ reduces to a Real space X. Then following Example 1.32, a Real graded S-central extension of X is a triple $(\widetilde{\Gamma}, Y^{[2]}, \delta)$, where Y is a Real space together with a continuous locally split Real open map $\pi : Y \longrightarrow X$, and $\delta : Y^{[2]} \longrightarrow \mathbb{Z}_2$ is a Real morphism.

In particular, suppose ρ is trivial. Then, by Proposition 1.52, giving a Real graded $\mathbb{S}^1$-central extension of X amounts to giving a *real bundle gerbe*

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r] & \widetilde{\Gamma} \arrow[d] & \\
 & Y^{[2]} \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d, "\pi"] \\
 & & X
\end{tikzcd}$$

in the sense of Mathai, Murray, and Stevenson [14], together with an augmentation $\delta : Y^{[2]} \longrightarrow \mathbb{Z}_2$.

1.7. Functoriality of $\widehat{\mathbf{ExtR}}(\cdot, \mathrm{S})$. The aim of this subsection is to show that $\widehat{\mathrm{ExtR}}(\cdot, \mathrm{S})$ is functorial in the category $\mathfrak{RG}$, and hence that the group $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ invariant under Morita equivalence. To do this, we will need the following:

Proposition 1.54. *Let* $\mathcal{G} \rightrightarrows X$ *be a Real groupoid. Then, there is an isomorphism of abelian groups*

$$\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S}) \cong \varinjlim_{\mathcal{U}} \widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}], \mathrm{S}). \tag{1.15}$$

Before giving the proof of this proposition, we have to describe the sum in the inductive limit

$$\varinjlim_{\mathcal{U}} \widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}], \mathrm{S}).$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two Real open covers of X, and let $\mathcal{T}_i = (\tilde{\mathcal{G}}_i, \mathcal{G}[\mathcal{U}_i], \delta_i)$ be Real graded S-groupoids over $\mathcal{G}[\mathcal{U}_i]$, $i = 1, 2$. Let $(\mathcal{V}, f_{\mathcal{V}}) \in \Omega\,(\mathcal{G}[\mathcal{U}_1], \mathcal{G}[\mathcal{U}_2])$ be the unique class corresponding to the Real Morita equivalence $Z_{\iota_{\mathcal{U}_1}}^{-1} \circ Z_{\iota_{\mathcal{U}_2}}$ from $\mathcal{G}[\mathcal{U}_1]$ to $\mathcal{G}[\mathcal{U}_2]$. $\mathcal{V}$ is a Real open cover of X containing $\mathcal{U}_1$, and

$$f_{\mathcal{V}} : \mathcal{G}[\mathcal{V}] \longrightarrow \mathcal{G}[\mathcal{U}_2]$$

is a Real strict morphism. Denote by $\iota_{\mathcal{V},\mathcal{U}_1}$ the canonical Real morphism $\mathcal{G}[\mathcal{V}] \longrightarrow \mathcal{G}[\mathcal{U}_1]$. Then, the tensor product of $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_1 \hat{\otimes} \mathcal{T}_2 := \iota^*_{\mathcal{V},\mathcal{U}_1}\mathcal{T}_1 \hat{\otimes} f^*_{\mathcal{V}}\mathcal{T}_2, \tag{1.16}$$

which defines a Real graded S-groupoids over the Real groupoid $\mathcal{G}[\mathcal{V}]$.

Proof of Proposition 1.54. For a Real graded S-central extension $\mathbb{E} = (\widetilde{\Gamma}, \Gamma, \delta, P)$ of $\mathcal{G}$, let $(\mathcal{V}, f_{\mathcal{V}}) \in \Omega(\mathcal{G}, \Gamma)$ be the isomorphism in $\mathfrak{RG}_\Omega$ corresponding to the Morita equivalence $P^{-1} : \mathcal{G} \longrightarrow \Gamma$. Setting

$$\mathcal{T}_{\mathbb{E}} := \begin{array}{ccccc} \mathrm{S} & \longrightarrow & f^*_{\mathcal{V}}\widetilde{\Gamma} & \xrightarrow{f^*_{\mathcal{V}}\pi} & \mathcal{G}[\mathcal{V}] \\ & & & & \Big\downarrow{\scriptstyle \delta\circ f_{\mathcal{V}}} \\ & & & & \mathbb{Z}_2 \end{array} \tag{1.17}$$

we get a Real graded S-groupoid over $\mathcal{G}[\mathcal{V}]$. It is not hard to check that this provides us the desired isomorphism of abelian groups; the inverse is given by the formula

$$\mathbb{E}_{\mathcal{T}} := (\tilde{\mathcal{G}}, \mathcal{G}[\mathcal{U}], \delta, Z_{\iota_{\mathcal{U}}}), \tag{1.18}$$

for a Real graded S-twist $\mathcal{T} = (\tilde{\mathcal{G}}, \mathcal{G}[\mathcal{U}], \delta)$. □

From this proposition, it is now possible to define the *pull-back* of a Real graded S-central extension via a Real generalized morphism. More precisely, we have

Definition and Proposition 1.55. Let $\mathcal{G}$ and $\mathcal{G}'$ be Real groupoids, and let $Z : \mathcal{G}' \longrightarrow \mathcal{G}$ be a Real generalized morphism. Let $\mathbb{E} = (\widetilde{\Gamma}, \Gamma, \delta, P)$ is be a representative in $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$, and $\mathcal{T}_{\mathbb{E}} = (f^*_{\mathcal{V}}\widetilde{\Gamma}, \mathcal{G}[\mathcal{V}], \delta \circ f_{\mathcal{V}})$ its image in $\varinjlim_{\mathcal{U}} \widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}], \mathrm{S})$ (see the proof of Proposition 1.54). Let

$$(\mathcal{W}, f_{\mathcal{W}}) \in \Omega\left(\mathcal{G}', \mathcal{G}[\mathcal{V}]\right)$$

be the morphism in $\mathfrak{RG}_\Omega$ corresponding to the Real generalized morphism $Z^{-1}_{\iota_{\mathcal{V}}} \circ Z : \mathcal{G}' \longrightarrow \mathcal{G}[\mathcal{V}]$. Then

$$Z^*\mathbb{E} := \mathbb{E}_{f^*_{\mathcal{W}}\mathcal{T}_{\mathbb{E}}}. \tag{1.19}$$

is a Real graded S-central extension of the Real groupoid $\mathcal{G}'$; it is called the *pull-back of* $\mathbb{E}$ *along* Z

Now the following is straightforward.

Corollary 1.56. *There is a contravariant functor*

$$\widehat{\mathrm{ExtR}}(\cdot, \mathrm{S}) : \mathfrak{RG} \longrightarrow \mathfrak{Ab}, \tag{1.20}$$

which sends a Real groupoid $\mathcal{G}$ *to the abelian group* $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$. *In particular,* $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ *is invariant under Morita equivalences.*

2. Real Čech cohomology

2.1. *Real* simplicial spaces. We start by recalling some preliminary notions. For each zero integer $n \in \mathbb{N}$, we set $[n] = \{0, \ldots, n\}$. Recall [21] that the simplicial (resp. pre-simplicial) category Δ (resp. Δ') is the category whose objects are the sets $[n]$, and whose morphisms are the nondecreasing (resp. increasing) maps $f : [m] \longrightarrow [n]$. For $n \in \mathbb{N}$, we denote by $\Delta^{(N)}$ the N-truncated full subcategory of Δ whose objects are those $[k]$ with $k \leq N$.

Definition 2.1. A *Real simplicial (resp. pre-simplicial, N-simplicial) topological space* consists of a contravariant functor from Δ (resp. Δ', $\Delta^{(N)}$) to the category $\mathfrak{RTop}$ whose objects are topological Real spaces and morphisms are continuous Real maps. A morphism of Real simplicial (resp. pre-simplicial, ...) spaces is a morphism of such functors.

More concretely, a Real (pre-)simplicial space is given by a family

$$(X_\bullet, \rho_\bullet) = (X_n, \rho_n)_{n \in \mathbb{N}}$$

of topological Real spaces, and for every map $f : [m] \longrightarrow [n]$ we are given a continuous Real map (called *face* or *degeneracy map* depending which of m and n is larger) $\tilde{f} : (X_n, \rho_n) \longrightarrow (X_m, \rho_m)$, satisfying the relation $\widetilde{f \circ g} = \tilde{g} \circ \tilde{f}$ whenever f and g are composable.

Definition 2.2. Let $(X_\bullet, \rho_\bullet)$ be a Real simplicial space. For any $N \in \mathbb{N}$, the *N-skeleton* of $(X_\bullet, \rho_\bullet)$ is the Real simplicial space $(X_\bullet, \rho_\bullet)^N$ "of dimension N"; that is, $(X_n, \rho_n)^N = (X_n, \rho_n)$ for $n \leq N$, and $(X_n, \rho_n)^N = (X_N, \rho_N)$ for all $n \geq N + 1$.

Let $\varepsilon_i^n : [n-1] \longrightarrow [n]$ be the unique increasing injective map that avoids i, and let $\eta_i^n : [n+1] \longrightarrow [n]$ be the unique nondecreasing surjective map such that i is reached twice; that is,

$$\varepsilon_i^n(k) = \begin{cases} k, & \text{if } k \leq i-1, \\ k+1, & \text{if } k \geq i, \end{cases} \tag{2.1}$$

$$\eta_i^n(k) = \begin{cases} k, & \text{if } k \leq i; \\ k-1, & \text{if } k \geq i+1. \end{cases}$$

We will omit the superscript n if there is no ambiguity.

If $(X_\bullet, \rho_\bullet)$ is a Real simplicial space, it is straightforward to check that the face and degeneracy maps

$$\tilde{\varepsilon}_i^n : (X_n, \rho_n) \longrightarrow (X_{n-1}, \rho_{n-1}),$$
$$\tilde{\eta}_i^n : (X_n, \rho_n) \longrightarrow (X_{n+1}, \rho_{n+1}),$$

$i = 0, \ldots, n$ satisfy the following *simplicial identities*:

$$\tilde{\varepsilon}_i^{n-1} \tilde{\varepsilon}_j^n = \tilde{\varepsilon}_{j-1}^{n-1} \tilde{\varepsilon}_i^n \text{ if } i \leq j-1, \tag{2.2}$$
$$\tilde{\eta}_i^{n+1} \tilde{\eta}_j^n = \tilde{\eta}_{j+1}^{n+1} \tilde{\eta}_i^n \text{ if } i \leq j,$$

$$\tilde{\varepsilon}_i^{n+1}\tilde{\eta}_j^n = \tilde{\eta}_{j-1}^{n-1}\tilde{\varepsilon}_i^n \text{ if } i \leq j-1,$$
$$\tilde{\varepsilon}_i^{n+1}\tilde{\eta}_j^n = \tilde{\eta}_j^{n-1}\tilde{\varepsilon}_{i-1}^n \text{ if } i \geq j+2,$$
$$\tilde{\varepsilon}_j^{n+1}\tilde{\eta}_j^n = \tilde{\varepsilon}_{j+1}^{n+1}\tilde{\eta}_j^n = \mathrm{Id}_{X_n}.$$

Conversely, let $(X_n, \rho_n)_{n\in\mathbb{N}}$ be a sequence of topological Real spaces together with maps satisfying (2.2). Then thanks to [13, Theorem 5.2], there is a unique Real simplicial structure on $(X_\bullet, \rho_\bullet)$ such that $\tilde{\varepsilon}_i$ and $\tilde{\eta}_i$ are the face and degeneracy maps respectively.

Example 2.3 (Cf. [24, §2.3])**.** Consider the pair groupoid

$$[n]\times[n] \rightrightarrows [n];$$

that is, the product is $(i,j)(j,k)) := (i,k)$ and the inverse of (i,j) is (j,i).

If $(\mathcal{G}, \rho)$ is a topological Real groupoid, we define

$$\mathcal{G}_n := \mathrm{Hom}([n]\times[n], \mathcal{G})$$

as the space of strict morphisms from the groupoid $[n]\times[n] \rightrightarrows [n]$ to $\mathcal{G} \rightrightarrows X$. We obtain a Real structure on $\mathcal{G}_n$ by defining $\rho_n(\varphi) := \rho\circ\varphi$, for $\varphi \in \mathcal{G}_n$. Any $f \in \mathrm{Hom}_\Delta([m],[n])$ (or $f \in \mathrm{Hom}_{\Delta'}([m],[n])$) naturally gives rise to a strict morphism $f\times f : [m]\times[m] \longrightarrow [n]\times[n]$, which, in turn, induces a Real map $\tilde{f} : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_m, \rho_m)$ given by $\tilde{f}(\varphi) := \varphi\circ(f\times f)$ for $\varphi \in \mathcal{G}_n$. Hence, we obtain a Real simplicial space $(\mathcal{G}_\bullet, \rho_\bullet)$.

Notice that the groupoid

$$[n]\times[n] \rightrightarrows [n]$$

is generated by elements $(i-1,i), 1 \leq i \leq n$; indeed, given an element $(i,j) \in [n]\times[n]$, we can suppose that $i \leq j$ (otherwise, we take its inverse (j,i)), and then $(i,j) = (i,i+1)\ldots(j-1,j)$. It turns out that any strict morphism $\varphi : [n]\times[n] \longrightarrow \mathcal{G}$ is uniquely determined by its images $\varphi(i-1,i) \in \mathcal{G}$; hence, the well-defined Real map

$$\mathcal{G}_n \longrightarrow \mathcal{G}^{(n)},\ \varphi \longmapsto (g_1,\ldots,g_n),$$

where $g_i := \varphi(i-1,i),\ 1 \leq i \leq n$, and

$$\mathcal{G}^{(n)} := \{(h_1,\ldots,h_n) \mid s(h_i) = r(h_{i-1}),\ i = 1,\ldots,n\},$$

identifies $(\mathcal{G}_n, \rho_n)$ with $(\mathcal{G}^{(n)}, \rho^{(n)})$, where $\rho^{(n)}$ is the obvious Real structure on the fibred product $\mathcal{G}^{(n)}$. Therefore, using this identification, the face maps $\tilde{\varepsilon}_i^n : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_{n-1}, \rho_{n-1})$ of $(\mathcal{G}_\bullet, \rho_\bullet)$ are given by:

$$\begin{aligned}
\tilde{\varepsilon}_0^n(g_1,g_2,\ldots,g_n) &= (g_2,\ldots,g_n),\\
\tilde{\varepsilon}_i^n(g_1,g_2,\ldots,g_n) &= (g_1,\ldots,g_ig_{i+1},\ldots,g_n),\ 1 \leq i \leq n-1,\\
\tilde{\varepsilon}_n^n(g_1,g_2,\ldots,g_n) &= (g_1,\ldots,g_{n-1}),
\end{aligned} \tag{2.3}$$

and for $n = 1$, by $\tilde{\varepsilon}_0^1(g) = s(g)$, $\tilde{\varepsilon}_1^1(g) = r(g)$; while the degeneracy maps $\tilde{\eta}_i^n : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_{n+1}, \rho_{n+1})$ are given by:

$$\begin{aligned} \tilde{\eta}_0^n(g_1, g_2, \ldots, g_n) &= (r(g_1), g_1, \ldots, g_n), \\ \tilde{\eta}_i^n(g_1, g_2, \ldots, g_n) &= (g_1, \ldots, s(g_i), g_{i+1}, \ldots, g_n),\ 1 \le i \le n, \end{aligned} \tag{2.4}$$

and $\tilde{\eta}_0^0 : \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ is the unit map of the Real groupoid.

Now for $n \in \mathbb{N}$, we define the space $(E\mathcal{G})_n$ of $(n+1)$-tuples of elements of $\mathcal{G}$ that map to the same unit; *i.e.*,

$$(E\mathcal{G})_n := \{(\gamma_0, \ldots, \gamma_n) \in \mathcal{G}^{n+1} \mid r(\gamma_0) = r(\gamma_1) = \cdots = r(\gamma_n)\}.$$

Suppose we are given $(g_1, \ldots, g_n) \in \mathcal{G}_n$. Then we can choose an $(n+1)$-tuple $(\gamma_0, \ldots, \gamma_n) \in (E\mathcal{G})_n$ such that $g_i = \gamma_{i-1}^{-1}\gamma_i$ for each $i = 1, \ldots, n$. If $(\gamma'_0, \ldots \gamma'_n)$ is another $(n+1)$-tuple satisfying these identities, then

$$s(\gamma'_i) = s((\gamma'_{i-1})^{-1}\gamma'_i) = s(\gamma_{i-1}^{-1}\gamma_i) = s(\gamma_i),$$

for all $i = 1, \ldots, n$, and that means that there exists a unique $g \in \mathcal{G}$, such that $s(g) = r(\gamma_i)$ and $\gamma'_i = g \cdot \gamma_i$. This hence gives us a well-defined injective map

$$\mathcal{G}_n \longrightarrow (E\mathcal{G})_n/_\sim,\ (g_1, \ldots, g_n) \longmapsto [\gamma_0, \ldots, \gamma_n],$$

where $(\gamma_0, \ldots, \gamma_n) \sim (g \cdot \gamma_0, \ldots, g \cdot \gamma_n)$. Moreover, this map is surjective, for if $(\gamma_0, \ldots, \gamma_n) \in (E\mathcal{G})_n$, one can consider morphisms g_i from $s(\gamma_i)$ to $s(\gamma_{i-1})$, $i = 1, \ldots, n$, so that we have

$$\gamma_1 = \gamma_0 g_1,\ \gamma_2 = \gamma_1 g_2 = \gamma_0 g_1 g_2, \ldots, \gamma_n = \gamma_0 g_1 \cdots g_n,$$

and then

$$[\gamma_0, \ldots, \gamma_n] = [r(g_1), g_1, g_1 g_2, \ldots, g_1 \cdots g_n]$$

which gives the inverse $(E\mathcal{G})_n/_\sim \ni [\gamma_0, \ldots, \gamma_n] \longmapsto (g_1, \ldots, g_n) \in \mathcal{G}_n$. It hence turns out that we can identify $\mathcal{G}_n$ with the quotient $(E\mathcal{G})_n$. Note that the quotient space $(E\mathcal{G})_n/_\sim$ naturally inherits the Real structure ρ_{n+1} and that the isomorphism defined above is compatible with the Real structures.

Henceforth, an element of $\mathcal{G}_n$ will be represented by a vector

$$\overrightarrow{g} = (g_1, \ldots, g_n),$$

where we view $\overrightarrow{g}$ as a morphism $[n] \times [n] \longrightarrow \mathcal{G}$, and $g_i = \overrightarrow{g}(i-1, i)$, $i = 1, \ldots, n$, or $\overrightarrow{g} = [\gamma_0, \ldots, \gamma_n]$ as a class in $(E\mathcal{G})_n/_\sim$. For the first picture, if $f \in \mathrm{Hom}_\Delta([m], [n])$, then the Real face/degeneracy map $\tilde{f} : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_m, \rho_m)$ is given by:

$$\tilde{f}(\overrightarrow{g}) = \left(\overrightarrow{g}\,(f(0), f(1))\,, \ldots, \overrightarrow{g}\,(f(m-1), f(m))\right). \tag{2.5}$$

For instance, if f in injective, then

$$\overrightarrow{g}\,(f(i-1), f(i)) = \overrightarrow{g}\,(f(i-1), f(i-1)+1) \cdots \overrightarrow{g}\,(f(i)-1, f(i))$$

for $f(i) \ge 1$, and thus

$$\tilde{f}(\overrightarrow{g}) = (g_{f(0)+1} \cdots g_{f(1)}, \ldots, g_{f(m-1)+1} \cdots g_{f(m)}). \tag{2.6}$$

However, the second picture offers a more general formula for the face and degeneracy maps; roughly speaking, for any $f \in \mathrm{Hom}_{\Delta}([m],[n])$, we have $\overrightarrow{g}(i,j) = \gamma_i^{-1}\gamma_j$ for every $(i,j) \in [n]\times[n]$. In particular,

$$\overrightarrow{g}(f(k-1),f(k)) = \gamma_{f(k-1)}^{-1}\gamma_{f(k)},$$

for every $k \in [m]$; then (2.5) gives:

$$\tilde{f}(\overrightarrow{g}) = [\gamma_{f(0)},\ldots,\gamma_{f(m)}]. \tag{2.7}$$

2.2. *Real* sheaves on *Real* simplicial spaces. In this subsection we closely follow [21, §3] to study Real sheaves on Real (pre-)simplicial spaces. We start by introducing some preliminary notions.

Let $\mathcal{C}$ be a topological category. We define the category $\mathcal{C}_R$ by setting:

- $\mathrm{Ob}(\mathcal{C}_R)$ consists of triples (A,σ_A,A'), where $A,\ A' \in \mathrm{Ob}(\mathcal{C})$ and $\sigma_A \in \mathrm{Hom}_{\mathcal{C}}(A,A')$;
- $\mathrm{Hom}_{\mathcal{C}_R}((A,\sigma_A,A'),(B,\sigma_B,B'))$ consists of pairs $(f,\tilde{f})$ of morphisms $f: A \longrightarrow B,\ \tilde{f}: A' \longrightarrow B'$ in $\mathcal{C}$ such that the diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma_A"'] & B \arrow[d, "\sigma_B"] \\ A' \arrow[r, "\tilde{f}"] & B' \end{tikzcd}$$

commute.

Now, let $\phi: \mathcal{C} \longrightarrow \mathcal{C}$ be a functor. Then we define the subcategory $\mathcal{C}_\phi$ of $\mathcal{C}_R$ whose objects are pairs $(A,\phi(A))$, where $A \in \mathrm{Ob}(\mathcal{C})$, and in which a morphism from $(A,\phi(A))$ to $(B,\phi(B))$ is a pair $(f,\tilde{f})$ of morphisms $f: A \longrightarrow B,\ \tilde{f}: \phi(A) \longrightarrow \phi(B)$ such that $\tilde{f}\circ\phi = \phi\circ f$. A fundamental example of this is the category $\mathfrak{OB}(X)$ of open subsets of a given topological Real space (X,ρ). Recall that objects of this category are the collection of the open sets $U \subset X$, and morphisms are the canonical injections $V \hookrightarrow U$ when $V \subset U$. Given such a Real space (X,ρ), the map ρ induces a functor (which is an isomorphism) $\rho: \mathfrak{OB}(X) \longrightarrow \mathfrak{OB}(X)$ given by

$$\left(V \xhookrightarrow{\iota} U \right) \longmapsto \left(\rho(V) \xhookrightarrow{\rho\circ\iota\circ\rho} \rho(U) \right).$$

Definition 2.4 (Real presheaves)**.** Let (X,ρ) be a topological Real space, and let $\mathcal{C}$ be a topological category. A *Real presheaf* $(\mathfrak{F},\sigma)$ on (X,ρ) with values in $\mathcal{C}$ is a contravariant functor from $\mathfrak{OB}(X)_\rho$ to $\mathcal{C}_R$; a morphism of Real presheaves is a morphism of such functors.

Specifically, from the fact that $\rho: X \longrightarrow X$ is a homeomorphism and from the canonical properties of the injections $V \hookrightarrow U$ of open sets $V \subset U \subset X$, a Real presheaf on (X,ρ) with values in $\mathcal{C}$ assigns to each open subset $U \subset X$ a triple $(\mathfrak{F}(U),\sigma_U,\mathfrak{F}(\rho(U)))$, where $\mathfrak{F}(U),\ \mathfrak{F}(\rho(U))$ are objects of $\mathcal{C}$, and $\sigma_U \in \mathrm{Isom}_{\mathcal{C}}(\mathfrak{F}(U),\mathfrak{F}(\rho(U)))$, and for $V \subset U$ we are given two morphisms

$\varphi_{V,U} : \mathfrak{F}(U) \longrightarrow \mathfrak{F}(V)$ and $\varphi_{\rho(V),\rho(U)} : \mathfrak{F}(\rho(U)) \longrightarrow \mathfrak{F}(\rho(V))$, called the restriction morphisms, such that:

- $\varphi_{U,U} = \mathrm{Id}_{\mathfrak{F}(U)}$.
- $\sigma_V \circ \varphi_{V,U} = \varphi_{\rho(V),\rho(U)} \circ \sigma_U$.
- $\varphi_{W,U} = \varphi_{W,V} \circ \varphi_{V,U}$, and $\varphi_{\rho(W),\rho(U)} = \varphi_{\rho(W),\rho(V)} \circ \varphi_{\rho(V),\rho(U)}$.

A morphism of Real presheaves $\phi : (\mathfrak{F}, \sigma^{\mathfrak{F}}) \longrightarrow (\mathfrak{G}, \sigma^{\mathfrak{G}})$ is then a family of $\phi_U \in \mathrm{Hom}_{\mathcal{C}}(\mathfrak{F}(U), \mathfrak{G}(U))$ such that, for all pairs of open sets U, V with $V \subset U$, the diagrams below commute:

$$\begin{tikzcd}
\mathfrak{F}(\rho(U)) \arrow[d, "\phi_{\rho(U)}"] & \mathfrak{F}(U) \arrow[l, "\sigma_U^{\mathfrak{F}}"'] \arrow[r, "\varphi_{V,U}^{\mathfrak{F}}"] \arrow[d, "\phi_U"] & \mathfrak{F}(V) \arrow[d, "\phi_V"] \\
\mathfrak{G}(\rho(U)) & \mathfrak{G}(U) \arrow[l, "\sigma_U^{\mathfrak{G}}"'] \arrow[r, "\varphi_{V,U}^{\mathfrak{G}}"] & \mathfrak{G}(V).
\end{tikzcd} \tag{2.8}$$

As in the standard case, if $(\mathfrak{F}, \sigma)$ is a Real presheaf over X, and if U is an open subset of X, an element $\mathsf{s} \in \mathfrak{F}(U)$ is called a *section of* $(\mathfrak{F}, \sigma)$ *on* U, and for $x \in X$. If V is an open subset of U, and $\mathsf{s} \in \mathfrak{F}(U)$, one often writes $\mathsf{s}_{|V}$ for $\varphi_{V,U}(\mathsf{s})$.

Definition 2.5 ([10, Definition 2.2])**.** A *Real sheaf* over (X, ρ) with values in $\mathcal{C}$ is a Real presheaf $(\mathfrak{F}, \sigma)$ satisfying the following conditions:

(i) For any open set $U \subset X$, any open cover $U = \bigcup_{i \in I} U_i$, any section $\mathsf{s} \in \mathfrak{F}(U)$, $\mathsf{s}_{|U_i} = 0$ for all i implies $\mathsf{s} = 0$.

(ii) For any open set $U \subset X$, any open cover $U = \bigcup_{i \in I} U_i$, any family of sections $\mathsf{s}_i \in \mathfrak{F}(U_i)$ satisfying $\mathsf{s}_{i|U_{ij}} = \mathsf{s}_{j|U_{ij}}$ for all nonempty intersection U_{ij}, there exists $\mathsf{s} \in \mathfrak{F}(U)$ such that $\mathsf{s}_{|U_i} = \mathsf{s}_i$ for all i.

A morphism of Real sheaves is a morphism of the underlying presheaves. We denote by $\mathcal{C}_R(X)$ (or simply by $\mathsf{Sh}_\rho(X)$ if there is no risk of confusion) for the category of Real sheaves on (X, ρ) with values in $\mathcal{C}$.

Notice that if $(\mathfrak{F}, \sigma)$ is a Real sheaf (resp. presheaf) on (X, ρ), then $\mathfrak{F}$ is a sheaf (resp. presheaf) on X in the usual sense. Recall that the *stalk* of $\mathfrak{F}$ at a point $x \in X$, denoted by $\mathfrak{F}_x$, is the direct limit of the direct system $(\mathfrak{F}(U), \varphi_{V,U})$ where U runs along the family of open neighborhoods of x; *i.e.*,

$$\mathfrak{F}_x := \varinjlim_{x \in U} \mathfrak{F}(U),$$

The image of a section $\mathsf{s} \in \mathfrak{F}(U)$ in $\mathfrak{F}_x$ by the canonical morphism

$$\mathfrak{F}(U) \longrightarrow \mathfrak{F}_x$$

(where $x \in U$) is called the *germ* of s at x and denoted by s_x.

Note that if U is an open neighborhood of x, $\rho(U)$ is an open neighborhood of $\rho(x)$, and the isomorphism $\sigma_U : \mathfrak{F}(U) \ni \mathsf{s} \longmapsto \sigma_U(\mathsf{s}) \in \mathfrak{F}(\rho(U))$ extends to an isomorphism $\sigma_x : \mathfrak{F}_x \longrightarrow \mathfrak{F}_{\rho(x)}$, defined by $\sigma_x(\mathsf{s}_x) = (\sigma_U(\mathsf{s}))_{\rho(x)}$, whose inverse is $\sigma_{\rho(x)}$. We thus have a well-defined 2-periodic isomorphism, also denoted by σ, on the topological [2] space $\mathcal{F} := \coprod_{x\in X} \mathfrak{F}_x$, given by

$$\sigma : \mathcal{F} \longrightarrow \mathcal{F},\ (x, \mathsf{s}_x) \longmapsto (\rho(x), \sigma_x(\mathsf{s}_x)) \tag{2.9}$$

which gives a Real space $(\mathcal{F}, \sigma)$.

Example 2.6. Let (X, ρ) be a Real space. Then the space $C(X)$ of continuous complex values functions on X defines a Real sheaf of abelian groups on (X, ρ) by $(U, \rho(U)) \longmapsto (C(U), \tilde{\rho}_U, C(\rho(U)))$, where $\tilde{\rho}_U(f)(\rho(x)) := \overline{f(x)}$.

Definition 2.7 (Pushforward, pullback)**.** Let (X, ρ), (Y, ϱ) be topological Real spaces, $f : (Y, \varrho) \longrightarrow (X, \rho)$ a continuous Real map. Suppose that $(\mathfrak{F}, \sigma)$ and $(\mathfrak{G}, \varsigma)$ are Real sheaves on (X, ρ) and (Y, ϱ) respectively, with values in the same category $\mathcal{C}$.

(i) The *pushforward* of $(\mathfrak{G}, \varsigma)$ by f, denoted by $(f_*\mathfrak{G}, f_*\varsigma)$, is the Real sheaf on (X, ϱ) defined by the contravariant functor:

$$\mathfrak{O}\mathfrak{B}(X)_\rho \longrightarrow \mathcal{C}_R,\ (U, \rho(U)) \longmapsto (f_*\mathfrak{G}(U), f_*\varsigma_U, f_*\mathfrak{G}(\rho(U)))\,, \tag{2.10}$$

where $f_*\mathfrak{G}(U) := \mathfrak{G}(f^{-1}(U))$, $f_*\varsigma_U := \varsigma_{f^{-1}(U)}$, and

$$f_*\mathfrak{G}(\rho(U)) = \mathfrak{G}(f^{-1}(\rho(U))) \cong \mathfrak{G}(\varrho(f^{-1}(U))).$$

(ii) The *pullback* of $(\mathfrak{F}, \sigma)$ along f, denoted by $(f^*\mathfrak{F}, f^*\sigma)$, is the Real sheaf on (Y, ϱ) associated to the Real presheaf defined by:

$$\mathfrak{O}\mathfrak{B}(Y)_\varrho \longrightarrow \mathcal{C}_R,\ (V, \varrho(V)) \longmapsto (f^*\mathfrak{F}(V), f^*\sigma_V, f^*\mathfrak{F}(\varrho(V))), \tag{2.11}$$

where $f^*\mathfrak{F}(V) := \varinjlim_{\substack{f(V)\subset U\subset X\\ U \text{ open}}} \mathfrak{F}(U)$, and $f^*\sigma_V : f^*\mathfrak{F}(V) \longrightarrow f^*\mathfrak{F}(\varrho(V))$ is the morphism in $\mathcal{C}$ extending functorially $\sigma_U : \mathfrak{F}(U) \longrightarrow \mathfrak{F}(\rho(U))$ along the family of open neighborhoods of $f(V)$ in X.

It immediately follows from this definition that we have a covariant functor

$$\mathfrak{RTop} \longrightarrow \mathfrak{RSh}, \tag{2.12}$$

$$\left((Y, \varrho) \xrightarrow{f} (X, \rho) \right) \longmapsto \left(\mathsf{Sh}_\varrho(Y) \xrightarrow{f_*} \mathsf{Sh}_\rho(X) \right),$$

and a contravariant functor

$$\mathfrak{RTop} \longrightarrow \mathfrak{RSh}, \tag{2.13}$$

$$\left((Y, \varrho) \xrightarrow{f} (X, \rho) \right) \longmapsto \left(\mathsf{Sh}_\rho(X) \xrightarrow{f^*} \mathsf{Sh}_\varrho(Y) \right),$$

---

[2] Recall that if $\mathfrak{F}$ is a presheaf over X, any section $\mathsf{s} \in \mathfrak{F}(U)$ induces a map $[\mathsf{s}] : U \longrightarrow \coprod_x \mathfrak{F}_x,\ y \longmapsto \mathsf{s}_y$. We give $\mathcal{F} := \coprod_{x\in X} \mathfrak{F}_x$ the largest topology such that all the maps $[\mathsf{s}]$ are continuous. On the other hand, associated to $\mathfrak{F}$, there is a sheaf $\widehat{\mathfrak{F}}$ given by $\widehat{\mathfrak{F}}(U) := \Gamma(U, \mathcal{F})$, and we have that $\mathfrak{F}(U) \cong \Gamma(U, \mathcal{F})$ if and only if $\mathfrak{F}$ is a sheaf. Then, given a Real presheaf $(\mathfrak{F}, \sigma)$, one can define its associated Real sheaf in the same fashion.

where $\mathfrak{RSh}$ is the category whose objects are the categories of Real sheaves on given Real spaces and morphisms are functors of such categories.

We will also need the following proposition.

Proposition 2.8. *Let $f : (Y, \varrho) \longrightarrow (X, \rho)$ be a a continuous Real map. Suppose that $(\mathfrak{F}, \sigma)$ and $(\mathfrak{G}, \varsigma)$ are Real sheaves on (X, ρ) and on (Y, ϱ) respectively, with values in the same category $\mathcal{C}$. Then*

$$\mathrm{Hom}_{\mathsf{Sh}_\rho(X)}((\mathfrak{F}, \sigma), (f_*\mathfrak{G}, f_*\varsigma)) \cong \mathrm{Hom}_{\mathsf{Sh}_\varrho(Y)}((f^*\mathfrak{F}, f^*\sigma), (\mathfrak{G}, \varsigma)). \tag{2.14}$$

Proof. The proof is the same as in the general case where Real structures are not concerned (see for instance [10, Proposition 2.3.3]). □

Definition 2.9. Given a continuous Real map $f : (Y, \varrho) \longrightarrow (X, \rho)$ and Real sheaves $(\mathfrak{F}, \sigma)$ and $(\mathfrak{G}, \varsigma)$ as above, we define the set $\mathrm{Hom}_f(\mathfrak{F}, \mathfrak{G})_{\sigma,\varsigma}$ of *Real f-morphisms* from $(\mathfrak{F}, \sigma)$ to $(\mathfrak{G}, \varsigma)$ to be

$$\mathrm{Hom}_{\mathsf{Sh}_\rho(X)}((\mathfrak{F}, \sigma), (f_*\mathfrak{G}, f_*\varsigma)) = \mathrm{Hom}_{\mathsf{Sh}_\varrho(Y)}((f^*\mathfrak{F}, f^*\sigma), (\mathfrak{G}, \varsigma)).$$

Definition 2.10. Let $(X_\bullet, \rho_\bullet)$ be a Real simplicial (resp. pre-simplicial) space. A Real sheaf on $(X_\bullet, \rho_\bullet)$ is a family $(\mathfrak{F}^n, \sigma^n)_{n\in\mathbb{N}}$ such that $(\mathfrak{F}^n, \sigma^n)$ is a Real sheaf on (X_n, ρ_n) for all n, and such that for each morphism $f : [m] \longrightarrow [n]$ in Δ (resp. Δ') we are given Real $\tilde{f}$-morphisms $\tilde{f}^* \in \mathrm{Hom}_{\tilde{f}}(\mathfrak{F}^m, \mathfrak{F}^n)_{\sigma^m,\sigma^n}$ such that

$$\widetilde{f \circ g}^* = \tilde{f}^* \circ \tilde{g}^*, \tag{2.15}$$

whenever f and g are composable.

One can use the definition of the push-forward to give a concrete interpretation of this definition. Roughly speaking, a sequence $(\mathfrak{F}^n, \sigma^n)_{n\in\mathbb{N}}$ is a Real sheaf on a Real simplicial (resp. pre-simplicial, ...) space $(X_\bullet, \rho_\bullet)$, if for a given morphism $f : [m] \longrightarrow [n]$ in Δ (resp. Δ', ...), then for any pair of open sets $U \subset X_n$ and $V \subset X_m$ such that $\tilde{f}(U) \subset V$ there is a *restriction map* $\tilde{f}^* : \mathfrak{F}^m(V) \longrightarrow \mathfrak{F}^n(U)$ such that the diagram

$$\begin{tikzcd} \mathfrak{F}^m(V) \arrow[r, "\tilde{f}^*"] \arrow[d, "\sigma^m_V"'] & \mathfrak{F}^n(U) \arrow[d, "\sigma^n_U"] \\ \mathfrak{F}^m(\rho(V)) \arrow[r, "\tilde{f}^*"] & \mathfrak{F}^n(\rho(U)) \end{tikzcd} \tag{2.16}$$

commutes, and $\tilde{f}^* \circ \tilde{g}^* = \widetilde{f \circ g}^* : \mathfrak{F}^k(W) \longrightarrow \mathfrak{F}^n(U)$ whenever $\tilde{g}(V) \subset W \subset X_k$. Morphisms of Real sheaves over $(X_\bullet, \rho_\bullet)$ are defined in the obvious way; we denote by $\mathsf{Sh}_{\rho_\bullet}(X_\bullet)$ for the category of Real sheaves over $(X_\bullet, \rho_\bullet)$.

2.3. *Real* $\mathcal{G}$-sheaves and reduced *Real* sheaves.

Definition 2.11.

(i) A Real space (Y,ϱ) is said to be *étale* over (X,ρ) if there exists an *étale* Real map $f:(Y,\varrho)\longrightarrow(X,\rho)$; that is to say, every point $y\in Y$ has an open neighborhood V such that $f_{|_V}:V\longrightarrow U$ is homeomorphism, where U in an open neighborhood of $f(y)$ in X.
(ii) A Real groupoid $(\mathcal{G},\rho)$ is étale if the range (equivalently the source) map is étale.
(iii) A morphism $\pi_\bullet:(Y_\bullet,\varrho_\bullet)\longrightarrow(X_\bullet,\rho_\bullet)$ of Real (pre-)simplicial spaces is étale if for all n, $\pi_n:(Y_n,\varrho_n)\longrightarrow(X_n,\rho_n)$ is étale.

Example 2.12. Any Real sheaf $(\mathfrak{F},\sigma)$ on (X,ρ) can be viewed as an étale Real space over (X,ρ). Indeed, considering the underlying topological Real space $(\mathcal{F},\sigma)$, it is easy to check that the canonical projection

$$\mathcal{F}\longrightarrow X,\ (x,\mathsf{s}_x)\longmapsto x$$

is an étale Real map.

Definition 2.13. Let $(\mathcal{G},\rho)$ be a topological Real groupoid. A Real $\mathcal{G}$-sheaf (or an étale Real $\mathcal{G}$-space) is an étale Real space $(\mathcal{E}_0,\nu_0)$ over (X,ρ) equipped with a continuous Real $\mathcal{G}$-action.

We say that $(\mathcal{E}_0,\nu_0)$ is an Abelian Real $\mathcal{G}$-sheaf if in addition it is an Abelian Real sheaf on (X,ρ) such that the action $\alpha_g:(\mathcal{E}_0)_{s(g)}\longrightarrow(\mathcal{E}_0)_{r(g)}$ is a group homomorphism, for any $g\in\mathcal{G}$.

A morphism of Real $\mathcal{G}$-sheaves $(\mathcal{E}_0,\nu_0)$ and $(\mathcal{E}'_0,\nu'_0)$ is a $\mathcal{G}$-equivariant continuous Real map $\psi:(\mathcal{E}_0,\nu_0)\longrightarrow(\mathcal{E}'_0,\nu'_0)$ such that $p'\circ\psi=p$.

The category of Real $\mathcal{G}$-sheaves is denoted by $\mathfrak{B}_\rho\mathcal{G}$, and is called the *classifying topos* of $(\mathcal{G},\rho)$.

Examples 2.14.

(1) Considering a Real space (X,ρ) as a Real groupoid, a Real X-sheaf is the same thing as a Real sheaf over (X,ρ); in other words we have that $\mathfrak{B}_\rho X\cong\mathsf{Sh}_\rho(X)$.
(2) If $(\mathcal{G},\rho)$ is a Real group, then a Real $\mathcal{G}$-sheaf is just a Real space equipped with a continuous Real $\mathcal{G}$-action.

Lemma 2.15. *Any generalized Real morphism $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ induces a morphism of toposes*

$$Z^*:\mathfrak{B}_\rho(\mathcal{G})\longrightarrow\mathfrak{B}_\varrho(\Gamma).$$

Consequently, there is a contravariant functor

$$\mathfrak{B}:\mathfrak{RG}\longrightarrow\mathfrak{RBG},$$

defined by

$$(\ (\Gamma,\varrho)\xrightarrow{(Z,\tau)}(\mathcal{G},\rho)\ )\longmapsto(\ \mathfrak{B}_\rho\mathcal{G}\xrightarrow{Z^*}\mathfrak{B}_\varrho\Gamma\ ),$$

where $\mathfrak{RBG}$ is the category whose objects are classifying toposes of Real groupoids.

Proof. As noted in [15, 2.2] for the usual case, any Real morphism $f : (\Gamma, \varrho) \longrightarrow (\mathcal{G}, \rho)$ gives rise to a functor $f^* : \mathfrak{B}_\rho\mathcal{G} \longrightarrow \mathfrak{B}_\varrho\Gamma$. Indeed, if $(\mathcal{E}_0, \nu_0)$ is a Real $\mathcal{G}$-sheaf through an étale Real $\mathcal{G}$-map $p : (\mathcal{E}_0, \nu_0) \longrightarrow (X, \rho)$, then we obtain a Real Γ-sheaf $(f^*\mathcal{E}_0, f^*\nu_0)$ by pulling back $(\mathcal{E}_0, \nu_0)$ along f; *i.e.*, $f^*\mathcal{E}_0 = Y \times_{f,X,p} \mathcal{E}_0$, $f^*\nu_0 = \varrho \times \nu_0$, $f^*p(y, e) := y$, and the right Real Γ-action is $\gamma \cdot (s(\gamma), e) := (r(\gamma), f(\gamma) \cdot e)$ when $p(e) = s(f(\gamma))$. If $\psi : (\mathcal{E}_0, \nu_0) \longrightarrow (\mathcal{E}'_0, \nu'_0)$ is a morphism of Real $\mathcal{G}$-sheaves, then the map $f^*\psi : (f^*\mathcal{E}_0, f^*\nu_0) \longrightarrow (f^*\mathcal{E}'_0, f^*\nu'_0)$ defined by $f^*\psi(y, e) := (y, \psi(e))$ is obviously a morphism a Real Γ-sheaves. It follows that any $(\mathcal{U}, f_\mathcal{U}) \in \mathrm{Hom}_{\mathfrak{RG}_\Omega}((\Gamma, \varrho), (\mathcal{G}, \rho))$ gives rise to a covariant functor $f^*_\mathcal{U} : \mathfrak{B}_\rho\mathcal{G} \longrightarrow \mathfrak{B}_\varrho\Gamma[\mathcal{U}]$. Now if (Z, τ) corresponds to $(\mathcal{U}, f_\mathcal{U})$, and if as in the previous chapter, $\iota : \Gamma[\mathcal{U}] \longrightarrow \Gamma$ is the canonical Real morphism, then we can push forward $(f^*_\mathcal{U}\mathcal{E}_0, f^*_\mathcal{U}\nu_0)$ through ι to get a Real Γ-sheaf $(Z^*\mathcal{E}_0, Z^*\nu_0)$; i.e.,

$$Z^*\mathcal{E}_0 := \iota_* f^*_\mathcal{U}\mathcal{E}_0, \tag{2.17}$$

and the Real structure $Z^*\nu_0$ is the obvious one. □

Lemma 2.16. *Let $(\mathcal{G}, \rho)$ be a topological Real groupoid. Then, a Real $\mathcal{G}$-sheaf canonically defines a Real sheaf over the Real simplicial space $(\mathcal{G}_n, \rho_n)_{n\in\mathbb{N}}$.*

To prove this Lemma, we need some more preliminary notions.

Definition 2.17. [21]] A morphism $\pi_\bullet : (\mathcal{E}_\bullet, \nu_\bullet) \longrightarrow (X_\bullet, \rho_\bullet)$ of Real simplicial spaces is called *reduced* if for all m, n and for all $f \in \mathrm{Hom}_\Delta([m], [n])$, the morphism $\tilde{f}$ induces an isomorphism

$$(\mathcal{E}_n, \nu_n) \cong (X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{E}_m, \rho_n \times \nu_m).$$

In this case, we say that $(\mathcal{E}_\bullet, \nu_\bullet)$ is a reduced Real simplicial space over $(X_\bullet, \rho_\bullet)$.

Morphisms of reduced Real simplicial spaces over $(X_\bullet, \rho_\bullet)$ are defined in the obvious way.

Definition 2.18 ([21])**.** We say that a Real sheaf $(\mathfrak{F}^\bullet, \sigma^\bullet)$ over a Real simplicial space $(X_\bullet, \rho_\bullet)$ is *reduced* if for all m, n and all $f \in \mathrm{Hom}_\Delta([m], [n])$, $\tilde{f}^* \in \mathrm{Hom}\Big((\tilde{f}^*\mathfrak{F}^m, \tilde{f}^*\sigma^m), (\mathfrak{F}^n, \sigma^n)\Big)$ is an isomorphism.

Lemma 2.19 ([21, Lemma 3.5])**.** *Let $(X_\bullet, \rho_\bullet)$ be a Real simplicial space. Then, there is a one-to-one correspondence between reduced Real sheaves over $(X_\bullet, \rho_\bullet)$ and reduced étale Real simplicial spaces over $(X_\bullet, \rho_\bullet)$.*

Proof. Suppose that we are given a Real sheaf $(\mathfrak{F}^\bullet, \sigma^\bullet)$ over the Real simplicial space $(X_\bullet, \rho_\bullet)$, and let $(\mathcal{F}_n, \sigma_n)_{n\in\mathbb{N}}$ be its underlying sequence of topological Real spaces. We already know from Example 2.12 that each of the canonical projection maps $\pi_n : (\mathcal{F}_n, \sigma_n) \longrightarrow (X_n, \rho_n)$ is étale. Now suppose that

$(\mathfrak{F}^\bullet, \sigma^\bullet)$ is reduced; that is to say that for any morphism $f \in \mathrm{Hom}_\Delta([m],[n])$, and every open set $V \subset X_m$, $\tilde{f}^* : \mathfrak{F}^m(V) \longrightarrow \mathfrak{F}^n(\tilde{f}^{-1}(V))$ is an isomorphism, so that we have a commutative diagram

$$\begin{tikzcd} \mathfrak{F}^m(V) \arrow[r, "\tilde{f}^*", "\cong"'] \arrow[d, "\sigma^m_V"'] & \mathfrak{F}^n(\tilde{f}^{-1}(V)) \arrow[d, "\sigma^n_{\tilde{f}^{-1}(V)}"] \\ \mathfrak{F}^m(\rho^m(V)) \arrow[r, "\tilde{f}^*", "\cong"'] & \mathfrak{F}^n(\rho^n(\tilde{f}^{-1}(V))). \end{tikzcd} \tag{2.18}$$

Let $x \in X_n$, $y \in X_m$ such that $\tilde{f}(x) = y$, and let $U \subset X_n$ and $V \subset X_m$ be open neighborhoods of x and y respectively such that $\tilde{f}(U) \subset V$. Then, for a section $\mathsf{s}^m \in \mathfrak{F}^m(V)$, we have an element $(x, (y, \mathsf{s}^m_y)) \in X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{F}_m$ to which we assign an element $(x, \mathsf{s}^n_x) \in \mathcal{F}_n$ as follows: since $U \subset \tilde{f}^{-1}(V)$, the section $\mathsf{s}^m \in \mathfrak{F}^m(V) \cong \mathfrak{F}^n(\tilde{f}^{-1}(V))$ has a restriction $\mathsf{s}^n := \mathsf{s}^m_U \in \mathfrak{F}^n(U)$. In this way we get a well-defined map $X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{F}_m \longrightarrow \mathcal{F}_n$. Moreover, it is easy to check that this map is an isomorphism; the inverse is the map

$$\mathcal{F}_n \ni (x, \mathsf{s}^n_x) \longmapsto (x, (\tilde{f}(x), (\tilde{f}^*\mathsf{s}^n)_{\tilde{f}(x)})) \in X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{F}_m,$$

where if $x \in U \subset X_n$ and $\tilde{f}(U) \subset V \subset X_m$, $\tilde{f}^*\mathsf{s}^n$ is any section in $\mathfrak{F}^m(V) \cong \mathfrak{F}^n(\tilde{f}^{-1}(V))$ that has the same class as s^n at the point x when restricted to $\mathfrak{F}^n(U)$ through the restriction map $\mathfrak{F}^n(\tilde{f}^{-1}(V)) \longrightarrow \mathfrak{F}^n(U)$. Furthermore, for every $f \in \mathrm{Hom}_\Delta([m],[n])$, there is a face/degeneracy map $\tilde{f} : (\mathcal{F}_n, \sigma_n) \longrightarrow (\mathcal{F}_m, \sigma_m)$ given by $\tilde{f}(x, \mathsf{s}_x) := (\tilde{f}(x), (\tilde{f}^*\mathsf{s})_{\tilde{f}(x)})$; hence $(\mathcal{F}_\bullet, \sigma_\bullet)$ is a reduced étale Real simplicial space over $(X_\bullet, \rho_\bullet)$.

Conversely, if $\pi_\bullet : (\mathcal{E}_\bullet, \nu_\bullet) \longrightarrow (X_\bullet, \rho_\bullet)$ is a reduced étale morphism of Real simplicial spaces, we let $\mathfrak{F}^n(U)$ be the space $C(U, \mathcal{E}_n)$ of continuous sections over U (where U is an open subset of X_n) of the projection $\pi_n : (\mathcal{E}_n, \nu_n) \longrightarrow (X_n, \rho_n)$. Next we define $\sigma^n_U : \mathfrak{F}^n(U) \longrightarrow \mathfrak{F}^n(\rho^n(U))$ by $\sigma^n_U(\mathsf{s})(\rho^n(x)) := \nu_n(\mathsf{s}(x))$. Notice that since the π_n's are étale, one can recover the Real spaces $(\mathcal{E}_n, \nu_n)$ by considering the underlying Real spaces of the Real sheaves $(\mathfrak{F}^n, \sigma^n)$. Now for any $f \in \mathrm{Hom}_\Delta([m],[n])$ and for any open set $V \subset X_m$, we have an isomorphism

$$\begin{aligned} \tilde{f}^* : \mathfrak{F}^m(V) &\longrightarrow \mathfrak{F}^n(\tilde{f}^{-1}(V)), \\ \mathsf{s} &\longmapsto \tilde{f}^*\mathsf{s}, \end{aligned}$$

where $(\tilde{f}^*\mathsf{s})(x) = (x, \mathsf{s}(\tilde{f}(x))) \in X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{E}_m \cong \mathcal{E}_m$. □

Using the same construction as in the second part of this proof, we deduce the following:

Lemma 2.20. *Any reduced Real simplicial space over $(X_\bullet, \rho_\bullet)$, étale or not, determines a Real sheaf over $(X_\bullet, \rho_\bullet)$.*

Proof of Lemma 2.16. Let (Z,τ) be a Real $\mathcal{G}$-sheaf, and let

$$\pi:(Z,\tau)\longrightarrow(X,\rho)$$

be an étale Real map. Put for all $n\geq 0$, $\mathcal{E}_n:=(\mathcal{G}\ltimes Z)_n:=\mathcal{G}_n\times_{\tilde{\pi}_n,X,\pi}Z$, where $\tilde{\pi}_n(g_1,\dots,g_n)=\tilde{\pi}_n[\gamma_0,\dots,\gamma_n]=s(\gamma_n)=s(g_n)$. Define $\nu_n:=\rho_n\times\tau$. We thus obtain a Real simplicial space $(\mathcal{E}_n,\nu_n)$: the simplicial structure is given by

$$\mathcal{E}_n\ni([\gamma_0,\dots,\gamma_n],z)\longmapsto\Big((\gamma_{f(0)},\dots,\gamma_{f(m)}),\gamma_{f(m)}^{-1}\gamma_n\cdot z\Big)\in\mathcal{E}_m, \tag{2.19}$$

for $f\in\mathrm{Hom}_\Delta([m],[n])$. Furthermore, it is straightforward to see that the projections $\pi_n:\mathcal{E}_n\longrightarrow\mathcal{G}_n$ are compatible with the Real structures ν_n and ρ_n, and that they define a morphism of Real simplicial spaces. If $f\in\mathrm{Hom}_\Delta([m],[n])$, then the assignment

$$([\gamma_0,\dots,\gamma_n],z)\longmapsto\Big([\gamma_0,\dots,\gamma_n],([\gamma_{f(0)},\dots,\gamma_{f(m)}],\gamma_{f(m)}^{-1}\gamma_n\cdot z)\Big)$$

obviously defines a Real homeomorphism $\mathcal{E}_n\cong\mathcal{G}_n\times_{\tilde{f},\mathcal{G}_m,\pi_m}\mathcal{E}_m$ which shows that $(\mathcal{E}_\bullet,\nu_\bullet)$ is a reduced Real simplicial space over $(\mathcal{G}_n,\rho_n)$. It follows from Lemma 2.20 that $(\mathcal{E}_\bullet,\nu_\bullet)$ determines an object of $\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet)$. □

Remark 2.21. Notice that in the proof above we did not use the fact that (Z,τ) is étale. In fact, the Real $\mathcal{G}$-action suffices for (Z,τ) to give rise to a Real sheaf over $(\mathcal{G}_\bullet,\rho_\bullet)$. However, the property of being étale will be necessary to show that the Real sheaf obtained is reduced (as it is mentioned in the following corollary).

Corollary 2.22. *Let $(\mathcal{G},\rho)$ be a topological Real groupoid. Then there is a functor*

$$\mathcal{E}:\mathfrak{B}_\rho\mathcal{G}\longrightarrow\mathfrak{red}\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet),$$

where $\mathfrak{red}\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet)$ is the full subcategory of $\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet)$ consisting of all reduced Real sheaves over $(\mathcal{G}_\bullet,\rho_\bullet)$.

Proof. Let us keep the same notations as in the proof of Lemma 2.16. Since π is étale, so is π_n for all n. The reduced Real simplicial space $(\mathcal{E}_\bullet,\nu_\bullet)$ is then étale over $(\mathcal{G}_\bullet,\nu_\bullet)$. Now, it suffices to apply Lemma 2.19. □

2.4. *Real* $\mathcal{G}$-modules.

Definition 2.23 (Cf. [21, Definition 3.9])**.** Let $(\mathcal{G},\rho)$ be a topological Real groupoid. A Real $\mathcal{G}$-module is a topological Real groupoid $(\mathcal{M},\ ^-)$, with unit space (X,ρ), and with source and range maps equal to a Real map $\pi:(\mathcal{M},\ ^-)\longrightarrow(X,\rho)$, such that:

- $\mathcal{M}_x$ $(=\mathcal{M}^x=\mathcal{M}_x^x)$ is an abelian group for all $x\in X$.
- For all $x\in X$, the map $(^-):\mathcal{M}_x\longrightarrow\mathcal{M}_{\rho(x)}$ is a group morphism.
- As a Real space, $(\mathcal{M},\ ^-)$ is endowed with a Real $\mathcal{G}$-action

$$\alpha:\mathcal{G}\times_{s,\pi}\mathcal{M}\longrightarrow\mathcal{M}.$$

- For each $g \in \mathcal{G}$, the map $\alpha_g : \mathcal{M}_{s(g)} \longrightarrow \mathcal{M}_{r(g)}$ given by the action is a group morphism.

By Remark 2.21, any Real $\mathcal{G}$-module $(\mathcal{M}, \,^{-})$ determines an abelian Real sheaf $(\mathfrak{F}^\bullet, \sigma^\bullet)$ on $(\mathcal{G}_\bullet, \rho_\bullet)$ constructed as follows: consider the reduced Real simplicial space $(\mathcal{E}_\bullet, \nu_\bullet) = ((\mathcal{G} \ltimes \mathcal{M})_n, \rho_n \times (^{-}))$, where the Real simplicial structure is given by:

$$\tilde{f}\left([\gamma_0, \ldots, \gamma_n], t\right) = \left([\gamma_{f(0)}, \ldots, \gamma_{f(m)}], \gamma_{f(m)}^{-1}\gamma_n \cdot t\right),$$

for any $f \in \mathrm{Hom}_\Delta([m],[n])$. Next, $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is defined as the sheaf of germs of continuous sections of the projections $\pi_\bullet : (\mathcal{E}_\bullet, \nu_\bullet) \longrightarrow (\mathcal{G}_\bullet, \rho_\bullet)$.

Example 2.24. Let $(\mathcal{G}, \rho)$ be a topological Real groupoid. Let $\mathcal{M} = X \times \mathbb{S}^1$ be endowed with the canonical Real structure $\overline{(x, \lambda))} := (\rho(x), \bar{\lambda})$, and Real $\mathcal{G}$-action $g \cdot (s(g), \lambda) = (r(g), \lambda)$. Then $(\mathcal{M}, ^{-})$ is a Real $\mathcal{G}$-module. The corresponding Real sheaf is called the constant sheaf of germs of $\mathbb{S}^1$-valued functions and denoted (abusively) $\mathbb{S}^1$. More generally, if S is any Real group, $X \times S$ is a Real $\mathcal{G}$-module, and the induced Real sheaf over $(\mathcal{G}_\bullet, \rho_\bullet)$ is denoted by S.

2.5. Pre-simplicial *Real* covers.

Definition 2.25 (Cf. [21, Definition 4.1])**.** Let $(X_\bullet, \rho_\bullet)$ be a Real pre-simplicial space. A *Real open cover* of $(X_\bullet, \rho_\bullet)$ is a sequence $\mathcal{U}_\bullet = (\mathcal{U}_n)_{n\in\mathbb{N}}$ such that $\mathcal{U}_n = (U_j^n)_{j\in J_n}$ is a Real open cover of (X_n, ρ_n).

We say that $\mathcal{U}_\bullet$ is *pre-simplicial* if $(J_\bullet, \,^{-}) = (J_n, \,^{-})_{n\in\mathbb{N}}$ is a Real pre-simplicial set such that for all $f \in \mathrm{Hom}_{\Delta'}([m],[n])$ and for all $j \in J_n$, one has $\tilde{f}(U_j^n) \subseteq U^m_{\tilde{f}(j)}$. In the same way, one defines the notions of simplicial Real cover and N-simplicial Real cover.

We will use the same construction as in [21, §4.1] to show the following lemma.

Lemma 2.26. *Any Real open cover $\mathcal{U}_\bullet$ of a Real (pre-)simplicial space $(X_\bullet, \rho_\bullet)$ gives rise to a pre-simplicial Real open cover $_\natural\mathcal{U}_\bullet$.*

Proof. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \bigcup_{k=0}^n \mathcal{P}_n^k$, where $\mathcal{P}_n^k = \mathrm{Hom}_{\Delta'}([k],[n])$. Let $\mathcal{P} = \bigcup_n \mathcal{P}_n$, and let Λ_n (or $\Lambda_n(J_\bullet)$ if there is a risk of confusion) be the set of maps

$$\lambda : \mathcal{P} \longrightarrow \bigcup_k J_k \text{ such that } \lambda(\mathcal{P}_n^k) \in J_k, \text{ for all } k. \tag{2.20}$$

It is immediate to see that Λ_n is non-empty; indeed, for each $k \in \mathbb{N}$, we fix a map $\overrightarrow{\jmath}^k : [n] \longrightarrow J_k$ which can be written as $\overrightarrow{\jmath}^k = (j_0^k, \ldots, j_n^k)$. Next, we define $\overrightarrow{\jmath} = (\overrightarrow{\jmath}^k)_{k\in\mathbb{N}}$. Then the map $\lambda : \mathcal{P} \longrightarrow \bigcup_k J_k$ given by $\lambda(\varphi) := \overrightarrow{\jmath} \circ \varphi$ lies in Λ_n. Moreover, Λ_n has a Real structure defines as follows: if $\varphi \in \mathcal{P}_n^k$, then we set

$$\bar{\lambda}(\varphi) := \overline{\lambda(\varphi)} \in J_k. \tag{2.21}$$

Now, for all $\lambda \in \Lambda_n$, we let

$$U^n_\lambda := \bigcap_{k\le n} \bigcap_{\varphi\in \mathcal{P}^k_n} \tilde{\varphi}^{-1}(U^k_{\lambda(\varphi)}). \tag{2.22}$$

Let $x \in X_n$. For each $k \le n$ and $\varphi \in \mathcal{P}^k_n$, there is $j^k_\varphi \in J_k$ such that $\tilde{\varphi}(x) \in U^k_{j^k_\varphi} \subset X_k$. Define the map $\lambda_x : \mathcal{P} \longrightarrow \bigcup_k J_k$ by $\lambda_x(\varphi) := (j^k_\varphi)_k$. Then, one can see that $x \in \bigcap_{k\le n}\bigcap_{\varphi\in\mathcal{P}^k_n} \tilde{\varphi}^{-1}(U^k_{\lambda_x(\varphi)}) = U^n_{\lambda_x}$. Furthermore, $\rho_n(U^n_\lambda) = U^n_{\bar\lambda}$; hence, $(U^n_\lambda)_{\lambda\in\Lambda_n}$ is a Real open cover of (X_n, ρ_n). If for any $f \in \mathrm{Hom}_{\Delta'}([m],[n])$, we define a map $\tilde{f} : \Lambda_n \longrightarrow \Lambda_m$ by

$$(\tilde{f}\lambda)(\varphi) := \lambda(f\circ\varphi), \text{ for all } \lambda\in\Lambda_n, \text{ and } \varphi\in\mathcal{P}^k_n,$$

one sees that $\tilde{f}(U^n_\lambda) \subseteq U^m_{\tilde{f}(\lambda)}$. Thus, ${}_\natural\mathcal{U}_\bullet = ((U^n_\lambda)_{\lambda\in\Lambda_n})_{n\in\mathbb{N}}$ is a pre-simplicial Real open cover of $(X_\bullet, \rho_\bullet)$. □

In the same way, for $N \in \mathbb{N}$ and $n \le N$, we denote by Λ^N_n the set of all maps

$$\lambda : \bigcup_{k\le n} \mathrm{Hom}_\Delta([k],[n]) \longrightarrow \bigcup_{k\le n} J_k$$

that satisfy $\lambda(\mathrm{Hom}_\Delta([k],[n])) \subset J_k$, and we set

$$U^n_\lambda := \bigcap_{k\le n} \bigcap_{\varphi\in\mathrm{Hom}_\Delta([k],[n])} \tilde{\varphi}^{-1}(U^n_{\lambda(\varphi)}).$$

Then we equip $\Lambda^N_\bullet$ with the Real structure defined in the same fashion, and we give it the N-simplicial structure defined as follows: for any $f \in \mathrm{Hom}_{\Delta^N}([m],[n])$, the map $\tilde{f} : \Lambda^N_n \longrightarrow \Lambda^N_m$ is given by $(\tilde{f}\lambda)(\varphi) := \lambda(f\circ\varphi)$. We thus obtain a N-simplicial Real cover ${}_{\natural^N}\mathcal{U}_\bullet = ({}_{\natural^N}\mathcal{U}_n)_{n\in\mathbb{N}}$ of the N-skeleton of $(X_\bullet, \rho_\bullet)$, where ${}_{\natural^N}\mathcal{U}_n = (U^n_\lambda)_{\lambda\in\Lambda^N_n}$.

We endow the collection of Real open covers of $(X_\bullet, \rho_\bullet)$ with the partial pre-order given by the following definition.

Definition 2.27. Let $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ be Real open covers of a Real simplicial space $(X_\bullet, \rho_\bullet)$, with $\mathcal{U}_n = (U^n_j)_{j\in J_n}$ and $\mathcal{V}_n = (V^n_i)_{i\in I_n}$. We say that $\mathcal{V}_\bullet$ is *finer* than $\mathcal{U}_\bullet$ if for each $n \in \mathbb{N}$, there exists a Real map

$$\theta_n : (I_n, \,^-) \longrightarrow (J_n, \,^-)$$

such that $V^n_i \subseteq U^n_{\theta_n(i)}$ for every $i \in I_n$. The Real map $\theta_\bullet = (\theta_n)_{n\in\mathbb{N}}$ is required to be pre-simplicial (resp. N-simplicial) if $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are pre-simplicial (resp. N-simplicial).

2.6. "Real" Čech cohomology.

Definition 2.28 (Real local sections)**.** Let $(\mathfrak{F}, \sigma)$ be an abelian Real (pre-)sheaf over (X, ρ) and let $\mathcal{U} = (U_j)_{j\in J}$ be a Real open cover of (X, ρ). We say that a family $\mathsf{s}_j \in \mathfrak{F}(U_j)$ is a *globally Real family* of local sections of $(\mathfrak{F}, \sigma)$ over $\mathcal{U}$ if for every $j \in J$, $\mathsf{s}_{\bar{j}}$ is the image of s_j in $\mathfrak{F}(U_{\bar{j}})$ by σ_{U_j}.

We define $CR_{ss}(\mathcal{U},\mathfrak{F})_{\rho,\sigma}$ to be the set of all globally Real families of local sections of $(\mathfrak{F},\sigma)$ relative to $\mathcal{U}$; *i.e.*,

$$CR_{ss}(\mathcal{U},\mathfrak{F})_{\rho,\sigma} := \left\{ (\mathsf{s}_j)_{j\in J} \subset \prod_{j\in J}\mathfrak{F}(U_j) \;\middle|\; \mathsf{s}_{\bar{j}} = \sigma_{U_j}(\mathsf{s}_j),\ \forall j\in J \right\}.$$

To avoid irksome notations, we will write $CR_{ss}(\mathcal{U},\mathfrak{F})$ or $CR_{ss}(\mathcal{U},\mathfrak{F})_\sigma$ instead of $CR_{ss}(\mathcal{U},\mathfrak{F})_{\rho,\sigma}$. It is clear that $CR_{ss}(\mathcal{U},\mathfrak{F})$ is an abelian group.

Now let $(X_\bullet,\rho_\bullet)$ be a Real simplicial space, and let $\mathcal{U}_\bullet$ be a pre-simplicial Real open cover of $(X_\bullet,\rho_\bullet)$. Suppose $(\mathfrak{F}^\bullet,\sigma^\bullet)$ is a (pre-simplicial) abelian Real (pre-)sheaf over $(X_\bullet,\rho_\bullet)$.

Definition 2.29. We define the complex $CR^*_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)_{\rho_\bullet,\sigma^\bullet}$ by

$$CR^n_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) := CR_{ss}(\mathcal{U}_n,\mathfrak{F}^n)_{\rho_n,\sigma^n}, \tag{2.23}$$

for $n\in\mathbb{N}$. We will also write $CR^*_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$ if there is no risk of confusion.

A *Real* n*-cochain* of $(X_\bullet,\rho_\bullet)$ relative to a pre-ssimplicial Real open cover $\mathcal{U}_\bullet$ with coefficients in $(\mathfrak{F}^\bullet,\sigma^\bullet)$ is an element in $CR^n_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$.

Let us consider again the maps $\varepsilon_k : [n] \longrightarrow [n+1]$ defined by (2.1), for $k=0,\ldots,n+1$. We have Real maps $\tilde{\varepsilon}_k : (J_{n+1},\ ^-) \longrightarrow (J_n,\ ^-)$, $\tilde{\varepsilon}_k : (X_{n+1},\rho_{n+1}) \longrightarrow (X_n,\rho_n)$, and $\tilde{\varepsilon}_k : (\mathfrak{F}^{n+1},\sigma^{n+1}) \longrightarrow (\mathfrak{F}^n,\sigma^n)$; and since $\tilde{\varepsilon}_k(U^{n+1}_j) \subseteq U^n_{\tilde{\varepsilon}_k(j)}$ for every $j\in J_{n+1}$, we have a restriction map

$$\tilde{\varepsilon}^*_k : \mathfrak{F}^n(U^n_{\tilde{\varepsilon}(j)}) \longrightarrow \mathfrak{F}^{n+1}(U^{n+1}_j)$$

such that $\sigma^{n+1}_{U^{n+1}_j}\circ\tilde{\varepsilon}^*_k = \tilde{\varepsilon}^*_k\circ\sigma^n_{U^n_{\tilde{\varepsilon}_k(j)}}$.

Definition 2.30. Let $\mathcal{U}_\bullet$ be a pre-simplicial Real open cover of $(X_\bullet,\rho_\bullet)$. For $n\ge 0$, we define the *differential map*

$$d^n : CR^n_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \longrightarrow CR^{n+1}_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \tag{2.24}$$

also denoted by d, by setting for $c=(c_j)_{j\in J_n}\in CR^n_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$ and for $j\in J_{n+1}$:

$$(dc)_j := \sum_{k=0}^{n+1}(-1)^k\tilde{\varepsilon}^*_k(c_{\tilde{\varepsilon}_k(j)}). \tag{2.25}$$

Remark 2.31. The differential d of (2.25) does indeed map $CR^n_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$ to $CR^{n+1}_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$; combining the fact that the $\tilde{\varepsilon}_k$ are Real maps and the discussion preceeding the last definition, one has

$$(dc)_{\bar{j}} = \sum_{k=0}^{n+1}(-1)^k\tilde{\varepsilon}^*_k(c_{\tilde{\varepsilon}_k(\bar{j})}) = \sum_{k=0}^{n+1}(-1)^k\tilde{\varepsilon}^*_k(\sigma^n_{U^n_{\tilde{\varepsilon}_k(j)}}c_{\tilde{\varepsilon}_k(j)}) = \sigma_{U^{n+1}_j}((dc)_j).$$

Lemma 2.32. *The differential maps d are group homomorphisms that satisfy $d^n\circ d^{n-1}=0$ for $n\ge 1$.*

Proof. That for any $n \in \mathbb{N}$, d^n is a group homomorphism is straightforward. Let $(c_{j'})_{j' \in J_{n-1}} \in CR^{n-1}_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$. Then, for $j \in J_{n+1}$ one has

$$\begin{aligned}
(d^n d^{n-1} c)_j &= \sum_{l=0}^{n+1} (-1)^l (\tilde{\varepsilon}_l^{n+1})^* \left(\sum_{k=0}^{n} (-1)^k (\tilde{\varepsilon}_k^n)^* (c_{\tilde{\varepsilon}_k^n \circ \tilde{\varepsilon}_l^{n+1}(j)}) \right) \\
&= \sum_{l=0}^{n+1} \sum_{k=0}^{n} (-1)^{l+k} (\tilde{\varepsilon}_l^{n+1})^* \circ (\tilde{\varepsilon}_k^n)^* (c_{\tilde{\varepsilon}_k^n \circ \tilde{\varepsilon}_l^{n+1}(j)}) \\
&= \sum_{p=0}^{n} \sum_{k=0, k \leq 2p}^{n} (\tilde{\varepsilon}_{2p-k}^{n+1})^* (\tilde{\varepsilon}_k^n)^* (c_{\tilde{\varepsilon}_k^n \circ \tilde{\varepsilon}_{2p-k}^{n+1}(j)}) \\
&\quad - \sum_{p=0}^{n} \sum_{k=0, k \leq 2p+1}^{n} (\tilde{\varepsilon}_{2p+1-k}^{n+1})^* \circ (\tilde{\varepsilon}_k^n)^* (c_{\tilde{\varepsilon}_k^n \circ \tilde{\varepsilon}_{2p+1-k}^{n+1}(j)}) \\
&= 0,
\end{aligned}$$

since $\varepsilon_r^{n+1} \circ \varepsilon_q^n = \varepsilon_{r+1}^{n+1} \circ \varepsilon_q^n$, for any $r, q \leq n$. □

We thus can give the following:

Definition 2.33. A Real n-cochain c in the kernel of d^n is called a *Real n-cocycle* relative to the pre-simplicial Real open cover $\mathfrak{U}_\bullet$ with coefficients in $(\mathfrak{F}^\bullet, \sigma^\bullet)$; the Real n-cocyles form a subgroup $ZR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ of $CR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$. The Real n-cochains belonging to the image of d^{n-1} are called *Real n-coboundaries* relative to $\mathfrak{U}_\bullet$ and form a subgroup $BR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ (since $d^2 = 0$). The n^{th} *Real cohomology group* of the pre-simplicial Real open cover $\mathfrak{U}_\bullet$ with coefficients in $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is defined by the n^{th} cohomology group of the complex

$$\cdots \xrightarrow{d^{n-2}} CR^{n-1}_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \xrightarrow{d^{n-1}} CR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \xrightarrow{d^n} CR^{n+1}_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \xrightarrow{d^{n+1}} \cdots$$

That is,

$$HR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) := \frac{ZR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)}{BR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)} := \frac{\ker d^n}{Im\ d^{n-1}}.$$

Example 2.34 (Cf. [21, Example 4.3])**.** Let $(X_\bullet, \rho_\bullet)$ be the constant Real simplicial space associated with a topological Real space (X, ρ); that is $(X_n, \rho_n) = (X, \rho)$ for every $n \geq 0$. Suppose $\mathfrak{U} =: \mathfrak{U}_0 = (U^0_j)_{j \in J_0}$ is a Real open cover of (X, ρ). Define $J_n := J_0^{n+1}$ together with the obvious Real structure. Then $(J_n, \ ^-)$ is admits a simplicial structure by

$$\tilde{f}(j_0, \ldots, j_n) := (j_{f(0)}, \ldots, j_{f(m)}), \text{ for all } f \in \mathrm{Hom}_\Delta([m], [n]).$$

Let $U^n_{(j_0, .., j_n)} := U^0_{j_0} \cap \cdots \cap U^0_{j_n}$ and $\mathfrak{U}_n = (U^n_j)_{j \in J_n}$. Of course $\mathfrak{U}_n$ is a Real open cover of (X_n, ρ_n), and for any $f \in \mathrm{Hom}_\Delta([m], [n])$ one has $\tilde{f}(U^n_{(j_0, \ldots, j_n)}) = U^n_{(j_0, \ldots, j_n)} \subseteq U^0_{f(0)} \cap \cdots \cap U^0_{f(m)} = U^m_{\tilde{f}(j_0, \ldots, j_n)}$; hence $\mathfrak{U}_\bullet$ is a simplicial Real open cover of $(X_\bullet, \rho_\bullet)$.

Let $(\mathcal{F},\sigma)$ be an Abelian Real sheaf on (X,ρ) and let $(\mathfrak{F}^n,\sigma^n) := (\mathfrak{F},\sigma)$ for all $n \geq 0$. Then, $HR^*_{ss}(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$ can be viewed as the "Real" analogue of the usual (*i.e.*, when all the Real structures are trivial) cohomology group $H^*(\mathcal{U}_0,\mathfrak{F})$ and is denoted by $HR^*(\mathcal{U},\mathfrak{F})$. A Real 0-cochain is a globally Real family $(\mathsf{s}_j)_{j\in J}$ of local sections. Given such a family, the differential d^0 gives: $(d^0\mathsf{s})_{(j_0,j_1)} = \mathsf{s}_{j_1|U_{j_0j_1}} - \mathsf{s}_{j_0|U_{j_0j_1}}$; it hence defines a Real 0-cocycle if there exists a Real global section $f \in \Gamma(X,\mathfrak{F})$ such that $\mathsf{s}_j = f_{U_j}$ for all $j \in J$.

A Real 1-coboundary is then a family $(c_{j_0j_1})_{j_0,j_1\in J}$ of sections $c_{j_0j_1} \in \mathfrak{F}(U_{j_0j_1}) \cong \Gamma(U_{j_0j_1},\mathcal{F})$ verifying $c_{\bar{j}_0\bar{j}_1}(\rho(x)) = \sigma(c_{j_0j_1}(x))$ for every $x \in U_{j_0j_1}$, and such that there exists a globally Real family $(\mathsf{s}_j)_{j\in J}$ of sections $\mathsf{s}_j \in \Gamma(U_j,\mathcal{F})$ such that $c_{j_0j_1} = \mathsf{s}_{j_1} - \mathsf{s}_{j_0}$ over all non-empty intersection $U_{j_0j_1}$.

Finally, a Real 1-cochain $c = (c_{j_0j_1}) \in CR^1_{ss}(\mathcal{U},\mathfrak{F})$ can be seen as a family of sections $c_{j_0j_1} \in \Gamma(U_{j_0j_1},\mathcal{F})$ satisfying $c_{\bar{j}_0\bar{j}_1}(\rho(x)) = \sigma(c_{j_0j_1}(x))$. Such a cocyle is 1-cocyle if and only if one has $(dc)_{j_0j_1j_2} = 0$ for all $j_0,j_1,j_2 \in J$; in other words, $c_{j_0j_1} + c_{j_1j_2} = c_{j_0j_2}$ over all non-empty intersection $U_{j_0j_1j_2}$.

We can apply Lemma 2.26 to generalize the definition of the Real cohomology groups relative to pre-simplicial Real open covers to arbitrary Real open covers of $(X_\bullet,\rho_\bullet)$.

Definition 2.35. Let $(X_\bullet,\rho_\bullet)$ be a Real (pre-)simplicial space and let $(\mathfrak{F}^\bullet,\sigma^\bullet) \in \mathrm{Ob}(\mathsf{Sh}_{\rho_\bullet}(X_\bullet))$. For any Real open cover $\mathcal{U}_\bullet$ of $(X_\bullet,\mathfrak{F}^\bullet)$, we let

$$CR^*(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) := CR^*_{ss}(\natural\mathcal{U}_\bullet,\mathfrak{F}^\bullet), \tag{2.26}$$

and we define the *Real cohomology* groups of $\mathcal{U}_\bullet$ with coefficients in $(\mathfrak{F}^\bullet,\sigma^\bullet)$ by

$$HR^*(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) := HR^*_{ss}(\natural\mathcal{U}_\bullet,\mathfrak{F}^\bullet). \tag{2.27}$$

We head now toward the definition of the *Real Čech cohomology*; roughly speaking, given an Abelian Real (pre-)sheaf $(\mathfrak{F}^\bullet,\sigma^\bullet)$ over a Real simplicial space $(X_\bullet,\rho_\bullet)$, we want to define the Real cohomology groups $HR^n(X_\bullet,\mathfrak{F}^\bullet)$ as the inductive limit of the groups $HR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$ over some category of Real open covers of $(X_\bullet,\rho_\bullet)$. To do this, we need some preliminaries elements.

Lemma 2.36. *Let $(X_\bullet,\rho_\bullet)$ and $(\mathfrak{F}^\bullet,\sigma^\bullet)$ be as above. Assume $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are Real open covers of $(X_\bullet,\rho_\bullet)$, with $\mathcal{U}_n = (U^n_j)_{j\in J_n}$ and $\mathcal{V}_n = (V^n_i)_{i\in I_n}$. Then all refinements $\theta_\bullet : (I_\bullet,\ ^-) \longrightarrow (J_\bullet,\ ^-)$ induces group homomorphisms*

$$\theta^*_n : HR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet). \tag{2.28}$$

Proof. In virtue of Lemma 2.26, one can assume that $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are pre-simplicial, and so that $\theta_\bullet$ is a pre-simplicial Real map. Define

$$\theta^*_n : CR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \longrightarrow CR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$$

as follows: for any $c = (c_j)_{j\in J_n} \in CR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet)$, we put

$$(\theta^*_n c)_i := c_{\theta_n(i)|V^n_i};$$

i.e., $(\theta_n^* c)_i$ is the image of $c_{\theta_n(i)}$ by the canonical restriction

$$\mathfrak{F}^n(U^n_{\theta_n(i)}) \longrightarrow \mathfrak{F}^n(V_i^n).$$

A straightforward calculation shows that this does define an element in $CR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$. Moreover, it is clear that θ_n^* is a group homomorphism for any n. Moreover, since $\theta_\bullet$ is pre-simplicial, $\tilde{\varepsilon}_k \circ \theta_{n+1} = \theta_n \circ \tilde{\varepsilon}_k$. Then, for $i \in I_{n+1}$, one has

$$\begin{aligned}(d\theta_n^*(c))_i &= \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^*(c_{\theta_n \circ \tilde{\varepsilon}_k(i)|V^n_{\tilde{\varepsilon}_k(i)}}) \\ &= \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^*(c_{\tilde{\varepsilon}_k \circ \theta_{n+1}(i)})_{|V_i^{n+1}} \\ &= (\theta_{n+1}^* d(c))_i,\end{aligned}$$

then $d^n \circ \theta_n^* = \theta_{n+1}^* \circ d^n$ for all $n \in \mathbb{N}$. It turns out that θ_n^* maps $ZR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$ into $ZR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ and maps $BR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$ into $BR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$. Consequently, θ_n^* passes through the quotients: $\theta_n^*([c]) := [\theta_n^*(c)]$, for $c \in ZR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$. $\square$

As noted in [21], the map $HR^*(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^*(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ may depend on the choice of the given refinement.

Definition 2.37. Let $(X_\bullet, \rho_\bullet)$ and $(\mathfrak{F}^\bullet, \sigma^\bullet)$ be as previously. Let $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ be Real open covers of $(X_\bullet, \rho_\bullet)$. Let $\phi_n, \psi_n : CR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ be two families of group homomorphisms commuting with d. We say that $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ are *equivalent* (resp. N-*equivalent*, for a given $N \in \mathbb{N}$ such that the N-keleton of $\mathcal{V}_\bullet$ admits an N-simplicial Real structure) if for all $n \in \mathbb{N}$ (resp. for all $n \leq N$), there exists a group homomorphism $h^n : CR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^{n-1}(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$, with the convention that $CR^{-1}(\mathcal{V}_\bullet, \mathfrak{F}^\bullet) = \{0\}$ (and $h^{N+1} = h^N$ in case of N-equivalence), such that

$$\phi_n - \psi_n = d^{n-1} \circ h^n + h^{n+1} \circ d^n, \ \forall n \in \mathbb{N} \text{ (resp. } \forall n \leq N). \tag{2.29}$$

Observe that such N-equivalent families $\phi_\bullet$ and $\psi_\bullet$ induces group homomorphisms

$$HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet),$$

also denoted by ϕ_n and ψ_n respectively, and given by $\phi_n([c]) := [\phi_n(c)]$, and $\psi_n([c]) := [\psi_n(c)]$ for all $c \in ZR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$. Assume

$$h^n : CR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^{n-1}(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$$

is such that (2.29) holds for all $n \leq N$, then for all $c \in ZR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$, one has

$$(\phi_n - \psi_n)([c]) = [d^{n-1}(h^n c)] + [h^{n+1}(d^n c)] = 0;$$

in other words, ϕ_n and ψ_n define the same homomorphism from $HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$ to $HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ when $n \leq N$.

It is clear that (N-)equivalence of morphisms

$$\phi_n : CR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$$

is an equivalence relation. We also denote by $\phi_\bullet$ for the (N-)class of $\phi_\bullet$.

Definition 2.38. Denote by $\mathfrak{N}$ the collection of all Real open covers of $(X_\bullet, \rho_\bullet)$. Let $\mathcal{U}_\bullet$, $\mathcal{V}_\bullet \in \mathfrak{N}$. We say that $\mathcal{V}_\bullet$ is *h-finer* than $\mathcal{U}_\bullet$ if $\mathcal{V}_\bullet$ is finer than $\mathcal{U}\bullet$ in the sense of Definition 2.27, and if there exists $N \in \mathbb{N}$ such that the N-skeleton of $\mathcal{V}_\bullet$ admits an N-simplicial Real strucutre. In this case, we will write $\mathcal{U}_\bullet \preceq_N \mathcal{V}_\bullet$ or $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet$.

We refer to [21, Lemma 4.5]) for the proof of the following:

Lemma 2.39. *Let $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ be Real open covers of $(X_\bullet, \rho_\bullet)$ such that $\mathcal{U}_\bullet \preceq_N \mathcal{V}_\bullet$. If $\theta_\bullet, \theta'_\bullet : (I_\bullet, \,^-) \longrightarrow (J_\bullet, \,^-)$ are two arbitrary refinements, then their induced group homomorphisms $\theta^*_\bullet$ and $(\theta'_\bullet)^*$ are N-equivalent. Consequently, there is a canonical morphism*

$$HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$$

for each $n \leq N$.

Example 2.40. By Lemma 2.26, from anyy Real open cover $\mathcal{U}_\bullet$ of $(X_\bullet, \rho_\bullet)$ and anyy $N \in \mathbb{N}$, one can form an N-simplicial Real open cover $_{\natural^N}\mathcal{U}_\bullet$ of the N-skeleton of $(X_\bullet, \rho_\bullet)$. Next, we define a new Real open cover $_\natural\mathcal{U}^N_\bullet$ by setting

$$_\natural\mathcal{U}^N_n := \begin{cases} _{\natural^N}\mathcal{U}_n, & \text{if } n \leq N, \\ \mathcal{U}_n, & \text{if } n \geq N+1. \end{cases} \tag{2.30}$$

It is clear that the N-skeleton of $_\natural\mathcal{U}^N_\bullet$ admits an N-simplicial Real structure. Recall that $_\natural\mathcal{U}^N_\bullet$ is indexed by $I_\bullet$, with $I_n = \Lambda^N_n$ if $n \leq N$ and $I_n = J_n$ if $n \geq N+1$. Now we get a refinement $_N\theta_\bullet : (I_\bullet, \,^-) \longrightarrow (J_\bullet, \,^-)$ by setting

$$_N\theta_n := \begin{cases} \Lambda^N_n \longrightarrow J_n, \ \lambda \longmapsto \lambda(\mathrm{Id}_{[n]}), & \text{if } n \leq N, \\ \mathrm{Id} : J_n \longrightarrow J_n, & \text{if } n \geq N+1, \end{cases} \tag{2.31}$$

hence $\mathcal{U}_\bullet \preceq_N {}_\natural\mathcal{U}^N_\bullet$ for all $N \in \mathbb{N}$. In particular, $\mathcal{U}_\bullet \preceq_0 \mathcal{U}_\bullet$.

We deduce from the example above that "$\preceq_h$" is a pre-order in the collection $\mathfrak{N}$. Suppose that $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet \preceq_h \mathcal{W}$ and $K_\bullet \xrightarrow{\theta'_\bullet} I_\bullet \xrightarrow{\theta_\bullet} J_\bullet$ are refinements. Then it is easy to check that the maps $\theta^*_\bullet$ and $(\theta'_\bullet)^*$ defined by (2.28) verify the relation $(\theta_n \circ \theta'_n)^* = (\theta'_n)^* \circ \theta^*_n$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we denote by $\mathfrak{N}(n)$ the collection of all elements $\mathcal{U}_\bullet \in \mathfrak{N}$ such that $\mathcal{U}_\bullet \preceq_N \mathcal{U}_\bullet$ for some $N \geq n+1$; *i.e.*, $\mathcal{U}_\bullet \in \mathfrak{N}(n)$ if there is $N \geq n+1$ such that the N-skeleton of $\mathcal{U}_\bullet$ admits an N-simplicial Real structure. It is obvious that "$\preceq_h$" is also a preorder in $\mathfrak{N}(n)$. Furthermore, Lemma 2.39, states that if $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet$ in $\mathfrak{N}(n)$, there is a canonical map $HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$. It follows that for all $n \in \mathbb{N}$, the collection

$$\{HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \mid \mathcal{U}_\bullet \in \mathfrak{N}(n)\}$$

is a directed system of groups; this allows us to give the following definition.

Definition 2.41. We define the n^{th} *Čech cohomology group* of $(X_\bullet,\rho_\bullet)$ with coefficients in $(\mathfrak{F}^\bullet,\sigma^\bullet)$ to be the direct limit

$$\check{H}R^n(X_\bullet,\mathfrak{F}^\bullet) := \varinjlim_{\mathcal{U}_\bullet\in\mathfrak{N}(n)} HR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet). \tag{2.32}$$

Lemma 2.42. *For every $\mathcal{U}_\bullet\in\mathfrak{N}$, pre-simplicial or not, there is a canonical group homomorphism*

$$\theta_{\mathcal{U}_\bullet} : HR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \longrightarrow \check{H}R^n(X_\bullet,\mathfrak{F}^\bullet),$$

for all $n\in\mathbb{N}$.

Proof. For every $\mathcal{U}_\bullet\in\mathfrak{N}$ (simplicial or not), and for every $n\in\mathbb{N}$, we define the map

$$\theta_{\mathcal{U}_\bullet} : HR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \longrightarrow \check{H}R^n(X_\bullet,\mathfrak{F}^\bullet)$$

by composing the canonical homomorphism

$$_N\theta_n^* : HR^n(\mathcal{U}_\bullet,\mathfrak{F}^\bullet) \longrightarrow HR^n(_\natural\mathcal{U}_\bullet^N,\mathfrak{F}^\bullet)$$

with the canonical projection

$$p_{\mathcal{U}_\bullet}^N : HR^n(_\natural\mathcal{U}_\bullet^N,\mathfrak{F}^\bullet) \longrightarrow \check{H}R^n(X_\bullet,\mathfrak{F}^\bullet),$$

for some $N\geq n+1$; *i.e.*, $\theta_{\mathcal{U}_\bullet} = p_{\mathcal{U}_\bullet}^N\circ {_N\theta_n^*}$ (recall that $_N\theta_n$ is defined by (2.31)). □

Let $(\mathfrak{F}^\bullet,\sigma^\bullet)$ and $(\mathfrak{G}^\bullet,\varsigma^\bullet)$ be Abelian Real sheaves on a Real simplicial space $(X_\bullet,\rho_\bullet)$. Suppose that $\phi_\bullet = (\phi_n)_{n\in\mathbb{N}} : (\mathfrak{F}^\bullet,\sigma^\bullet) \longrightarrow (\mathfrak{G}^\bullet,\varsigma^\bullet)$ is a morphism of Abelian Real (pre)sheaves, and that $\mathcal{U}_\bullet$ is a Real open cover of $(X_\bullet,\rho_\bullet)$. Consider the pre-simplicial Real open cover $_\natural\mathcal{U}_\bullet$ associated to $\mathcal{U}_\bullet$. Then for any $n\in\mathbb{N}$, and any $\lambda\in\Lambda_n$, there is a morphism of Abelian groups

$$\tilde{\phi}_n : \mathfrak{F}^n(U_\lambda^n) \longrightarrow \mathfrak{G}^n(U_\lambda^n), \mathsf{s}_\lambda \longmapsto \phi_{n|U_\lambda^n}(\mathsf{s}_\lambda), \tag{2.33}$$

satisfying $\varsigma^n_{U_\lambda^n}\circ\tilde{\phi}_n = \tilde{\phi}_n\circ\sigma_{U_{\bar\lambda}^n}$. This gives a group homomorphism

$$\tilde{\phi}_n : CR^n_{ss}(_\natural\mathcal{U}_\bullet,\mathfrak{F}^\bullet)_{\sigma^\bullet} \longrightarrow CR^n_{ss}(_\natural\mathcal{U}_\bullet,\mathfrak{G}^\bullet)_{\varsigma^\bullet}.$$

Moreover, for any $\lambda\in\Lambda_{n+1}$ and any $k\in[n+1]$, one has a commutative diagram

$$\begin{tikzcd}
\mathfrak{F}^n(U^n_{\tilde\varepsilon_k(\lambda)}) \arrow[r, "\phi_{n|U^n_{\tilde\varepsilon_k(\lambda)}}"] \arrow[d, "\tilde\varepsilon_k^*"'] & \mathfrak{G}^n(U^n_{\tilde\varepsilon_k(\lambda)}) \arrow[d, "\tilde\varepsilon_k^*"] \\
\mathfrak{F}^{n+1}(U^{n+1}_\lambda) \arrow[r, "\phi_{n+1|U^{n+1}_\lambda}"] & \mathfrak{G}^{n+1}(U^{n+1}_\lambda).
\end{tikzcd}$$

Thus, $d^n \circ \tilde{\phi}_n = \tilde{\phi}_{n+1} \circ d^n$; i.e., one has a commutative diagram

$$\begin{array}{ccc} CR^n_{ss}(\natural \mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} & \xrightarrow{d^n} & CR^{n+1}_{ss}(\natural \mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \\ \downarrow \tilde{\phi}_n & & \downarrow \tilde{\phi}_{n+1} \\ CR^n_{ss}(\natural \mathcal{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet} & \xrightarrow{d^n} & CR^{n+1}_{ss}(\natural \mathcal{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet} \end{array} \tag{2.34}$$

that shows that ϕ gives rise to a homomorphism of Abelian groups

$$\begin{aligned} (\phi_{\mathcal{U}_\bullet})_* : HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} &\longrightarrow HR^n(\mathcal{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet}, \\ [c] &\longmapsto [\tilde{\phi}_n(c)]; \end{aligned} \tag{2.35}$$

and therefore a group homomorphism

$$\phi_* : \check{H}R^n(X_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \longrightarrow \check{H}R^n(X_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet}$$

defined in the obvious way. We thus have shown that $\check{H}R^*$ is functorial in the category $\mathsf{Sh}_{\rho_\bullet}(X_\bullet)$.

Proposition 2.43. *Suppose $(X_\bullet, \rho_\bullet)$ is a Real simplicial space such that each X_n is paracompact. If*

$$0 \longrightarrow (\mathfrak{F}'^\bullet, \sigma'^\bullet) \xrightarrow{\phi'_\bullet} (\mathfrak{F}^\bullet, \sigma^\bullet) \xrightarrow{\phi_\bullet} (\mathfrak{F}"^\bullet, \sigma"^\bullet) \longrightarrow 0$$

is an exact sequence of Real (pre-)sheaves over $(X_\bullet, \rho_\bullet)$, then there is a long exact sequence of Abelian groups

$$\begin{aligned} 0 \longrightarrow \check{H}R^0(X_\bullet, \mathfrak{F}'^\bullet) \xrightarrow{\phi'_*} \check{H}R^0(X_\bullet, \mathfrak{F}^\bullet) \xrightarrow{\phi_*} \check{H}R^0(X_\bullet, \mathfrak{F}"^\bullet) \\ \xrightarrow{\partial} \check{H}R^1(X_\bullet, \mathfrak{F}'^\bullet) \xrightarrow{\phi'_*} \cdots \end{aligned}$$

The proof of this proposition is almost the same as in [21, §4].

2.7. Comparison with usual groupoid cohomologies. In this subsection we compare our cohomology with the usual cohomology theory in some special cases, especially with that developed in [21].

Proposition 2.44. *Suppose S is an Abelian Real group. Let ${}^r\mathrm{S}$ be the fixed point subgroup of S. Let $(\mathcal{G}, \rho)$ be a Real groupoid. Then if ρ is trivial , we have*

$$\check{H}R^*(\mathcal{G}_\bullet, \mathrm{S}) = \check{H}^*(\mathcal{G}_\bullet, {}^r\mathrm{S}).$$

In particular, if S has no non-trivial fixed point, we have $\check{H}R^*(\mathcal{G}_\bullet, \mathrm{S}) = 0$.

Notice that this result generalizes easily to the Real cohomology with coefficients in a Real sheaf induced from a Real $\mathcal{G}$-module.

Proof. Let $(c_\lambda) \in ZR^n(\mathcal{U}_\bullet, \mathrm{S})$. Since $\rho = \mathrm{Id}$, we may take the involution on $J_\bullet$ to be trivial. For every $\overrightarrow{g} \in U^n_\lambda$, we have

$$c_\lambda(\overrightarrow{g}) = c_\lambda(\overline{\overrightarrow{g}}) = \overline{c_\lambda(\overrightarrow{g})} \in {}^r\mathrm{S}.$$

Thus $c_\lambda \in ZR^n(\mathcal{U}_\bullet, {}^r\mathrm{S})$.

Conversely, we obviously have $\check{H}^n(\mathcal{G}_\bullet, {}^r\mathrm{S}) \subset \check{H}R^n(\mathcal{G}_\bullet, \mathrm{S})$ since ρ is trivial. $\square$

Corollary 2.45. *If ρ and the Real structure of* S *are trivial, then*

$$\check{H}^*(\mathcal{G}_\bullet, \mathrm{S}) = \check{H}^*(\mathcal{G}_\bullet, \mathrm{S}).$$

Focus now on the case where $\mathcal{G}$ reduces to a Real space (X, τ) and $\mathrm{S} = \mathbb{Z}^{0,1}$. Then τ induces an action of $\mathbb{Z}_2$ on X by $(-1)\cdot x := \tau(x), (+1)\cdot x := x$.

Proposition 2.46. *We have the following group isomorphisms:*

(i) $\check{H}R^*(X, \mathbb{Z}^{0,1}) \cong \check{H}^*_{(\mathbb{Z}_2,-)}(X, \mathbb{Z})$*, where the sign "$-$" stands for the $\mathbb{Z}_2$-equivariant cohomology with respect to the action of $\mathbb{Z}_2$ on $\mathbb{Z}$ given by $(-1)\cdot n := -n$, $(+1)\cdot n := n$.*

(ii) $\check{H}^*(X, \mathbb{Z}) \cong_{\mathbb{Q}} \check{H}^*_{(\mathbb{Z}_2,-)}(X, \mathbb{Z}) \oplus \check{H}^*_{(\mathbb{Z}_2,+)}(X, \mathbb{Z})$*, where the sign "$+$" means the trivial $\mathbb{Z}_2$-action on $\mathbb{Z}$.*

Proof. (i) Let $c \in \check{H}R^n(X, \mathbb{Z}^{0,1})$ be represented on the Real open cover (U_j) of X. Then $c_{\bar{\jmath}_0\cdots\bar{\jmath}_n}(\tau(x)) = -c_{j_0\cdots j_n}(x)$ implies $\tau^* c_{j_0\cdots j_n}(x) = -c_{j_0\cdots j_n}(x)$, $\forall x \in X$; in other words, c is $\mathbb{Z}_2$-equivariant with respect to the $\mathbb{Z}_2$-action "$-$" on $\mathbb{Z}$. The converse is easy to check.

(ii) We define the involution $\tilde{\tau}$ on $\check{H}^n(X, \mathbb{Z})$ by $\tilde{\tau}(c) := -\tau^* c$. Then it is straightforward that the Real part ${}^r\check{H}^n(X, \mathbb{Z}) \cong \check{H}R^n(X, \mathbb{Z}^{0,1})$, while the imaginary part ${}^{\mathfrak{I}}\check{H}^n(X, \mathbb{Z})$ is exactly $\check{H}^n_{(\mathbb{Z}_2,+)}(X, \mathbb{Z})$. $\square$

2.8. The group $\check{H}R^0$. We shall recall the notations of [21, Section 4] that we will use throughout the rest of the section. Let $\mathcal{U}_\bullet$ be a Real open cover of a Real simplicial space $(X_\bullet, \rho_\bullet)$ and let ${}_\natural\mathcal{U}_\bullet$ be its associated pre-simplicial Real open cover. Recall that any $\varphi \in \mathcal{P}_n^k$ is represented by its image in $[n]$; *i.e.*, $\varphi = \{\varphi(0), \ldots, \varphi(k)\}$. Then $\mathcal{P}_n$ is nothing but the collection of all non empty subsets of $[n]$. Henceforth, any subset $S = \{i_0, \ldots, i_k\} \subseteq [n]$, with $i_0 \le \cdots \le i_k$, designates the maps $\varphi \in \mathcal{P}_n^k$ such that $\varphi(0) = i_0, \ldots, \varphi(k) = i_k$.

Notations 2.47. *With the above observations, any element $\lambda \in \Lambda_n$ is represented by a $(2^{n+1}-1)-$tuple $(\lambda_S)_{\emptyset \ne S \subseteq [n]}$, where the subsets S are ordered first by cardinality, then by lexicographic order;* i.e.,

$$S \in \big\{\{0\}, \ldots, \{n\}, \{0,1\}, \ldots, \{0,n\}, \{1,2\}, \ldots, \{1,n\}, \\ \ldots, \{0,1,2\}, \ldots, \{0,\ldots,n\}\big\},$$

and $\lambda_S := \lambda(S)$. For instance, any element $\lambda \in \Lambda_1$ is represented by a triple $(\lambda_0, \lambda_1, \lambda_{01})$, with $\lambda_0 = \lambda(\{0\})$, $\lambda_1 = \lambda(\{1\})$ and $\lambda_{01} = \lambda(\{0,1\})$.

Recall that if $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is an abelian Real sheaf over $(X_\bullet, \rho_\bullet)$, we are given two "restriction" maps on the space of global Real sections

$$\tilde{\varepsilon}_0^*, \tilde{\varepsilon}_1^* : \mathfrak{F}^0(X_0)_{\sigma^0} \longrightarrow \mathfrak{F}^1(X_1)_{\sigma^1}.$$

Let us set

$$\Gamma_{\mathrm{inv}}(\mathfrak{F}^\bullet)_{\sigma^\bullet} := \ker\Big(\mathfrak{F}^0(X_0)_{\sigma^0} \underset{\tilde{\varepsilon}_1^*}{\overset{\tilde{\varepsilon}_0^*}{\rightrightarrows}} \mathfrak{F}^1(X_1)_{\sigma^1} \Big)$$
$$= \left\{ \mathsf{s} \in \mathfrak{F}^0(X_0)_{\sigma^0} \mid \tilde{\varepsilon}_0^*(\mathsf{s}) = \tilde{\varepsilon}_1^*(\mathsf{s}) \right\}.$$

Proposition 2.48 ([21, Proposition 5.1])**.** *Let $(\mathfrak{F}^\bullet, \sigma^\bullet)$ be an abelian Real sheaf over $(X_\bullet, \rho_\bullet)$ and let $\mathcal{U}_\bullet$ be a Real open cover of $(X_\bullet, \rho_\bullet)$. Then*

$$\check{H}R^0(X_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \cong HR^0(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \cong \Gamma_{\mathrm{inv}}(\mathfrak{F}^\bullet)_{\sigma^\bullet}. \tag{2.36}$$

Proof. One identifies Λ_0 with J_0. Note that $\mathcal{P}_1 = \{\varepsilon_0^1, \varepsilon_1^1, \mathrm{Id}_{[1]}\}$, and that for any $\lambda = (\lambda_0, \lambda_1, \lambda_{01})$ in Λ_1 one has $\tilde{\varepsilon}_0(\lambda) = \lambda(\varepsilon_0) = \lambda_1$, $\tilde{\varepsilon}_1(\lambda) = \lambda(\varepsilon_1) = \lambda_0$. We thus have $U_\lambda^1 = U_{\lambda_{01}}^1 \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_1}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_0}^0)$. Now, let $(\mathsf{s}_{\lambda_0})_{\lambda_0 \in J_0} \in ZR^0(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet}$. Then

$$0 = (d\mathsf{s})_{(\lambda_0, \lambda_1, \lambda_{01})} = \tilde{\varepsilon}_0^*(\mathsf{s}_{\lambda_1}) - \tilde{\varepsilon}_1^*(\mathsf{s}_{\lambda_0}), \text{ on } U_\lambda^1, \tag{2.37}$$

Therefore, $\tilde{\varepsilon}_0^*(\mathsf{s}_{\lambda_1}) = \tilde{\varepsilon}_1^*(\mathsf{s}_{\lambda_0})$ on $\tilde{\varepsilon}_0^{-1}(U_{\lambda_1}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_0}^0)$, and $\tilde{\varepsilon}_0^*(\mathsf{s}_{\bar{\lambda}_1}) = \tilde{\varepsilon}_1^*(\mathsf{s}_{\bar{\lambda}_0})$ on $\tilde{\varepsilon}_0^{-1}(U_{\bar{\lambda}_1}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\bar{\lambda}_0}^0)$, for all $\lambda_0, \lambda_1 \in J_0$. Applying $\tilde{\eta}_0^*$ to both sides of the above identity, we get that $\mathsf{s}_{\lambda_0} = \mathsf{s}_{\lambda_1}$ and $\mathsf{s}_{\bar{\lambda}_0} = \mathsf{s}_{\bar{\lambda}_1}$; in other words, $\mathsf{s}_{\lambda_0} = \mathsf{s}_{\lambda_1}$ on $U_{\lambda_0}^0 \cap U_{\lambda_1}^0$ for all $\lambda_0, \lambda_0 \in J_0$. Since $(\mathfrak{F}^0, \sigma^0)$ is a Real sheaf on (X_0, ρ_0), there exists a global Real sections $\mathsf{s} \in \mathfrak{F}^0(X_0)_{\sigma^0}$ such that $\mathsf{s}_{U_{\lambda_0}^0} = \mathsf{s}_{\lambda_0}$ for all $\lambda_0 \in J_0$. Now, equation (2.37) is equivalent to $\tilde{\varepsilon}_0^*(\mathsf{s}) = \tilde{\varepsilon}_1^*(\mathsf{s})$; *i.e.*, $\mathsf{s} \in \Gamma_{\mathrm{inv}}(\mathfrak{F}^\bullet)_{\sigma^\bullet}$ and this ends the proof. □

2.9. $\check{H}R^1$ and the Real Picard group. Let us consider the same data as in the previous subsection. Let $\mathcal{U}_\bullet$ be a Real open cover of $(X_\bullet, \rho_\bullet)$. For $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{012}) \in \Lambda_2$, one has

$$U_\lambda^2 = \tag{2.38}$$
$$\tilde{\varphi}_{00}^{-1}(U_{\lambda_0}^0) \cap \tilde{\varphi}_{01}^{-1}(U_{\lambda_1}^0) \cap \tilde{\varphi}_{02}^{-1}(U_{\lambda_2}^0) \cap \tilde{\varepsilon}_2^{-1}(U_{\lambda_{01}}^1) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_{02}}^1) \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_{12}}^1) \cap U_{\lambda_{012}}^2,$$

where $\varphi_{00} = \varepsilon_1^2 \circ \varepsilon_1^1, \varphi_{01} = \varepsilon_0^2 \circ \varepsilon_0^1$ and $\varphi_{02} = \varepsilon_1^2 \circ \varepsilon_0^1$.
Let $c = (c_\lambda)_{\lambda \in \Lambda_1} \in ZR^1(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet}$. Then

$$0 = (dc)_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}\lambda_{012}} = \tilde{\varepsilon}_0^* c_{\lambda_1\lambda_2\lambda_{12}} - \tilde{\varepsilon}_1^* c_{\lambda_0\lambda_2\lambda_{02}} + \tilde{\varepsilon}_2^* c_{\lambda_0\lambda_1\lambda_{02}}, \tag{2.39}$$

on U_λ^2, and of course we get a similar identities for $(dc)_{\bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_{01}\bar{\lambda}_{02}\bar{\lambda}_{12}\bar{\lambda}_{012}}$ on $U_{\bar{\lambda}}^2$. Now applying $\tilde{\eta}_1^*$ to (2.39), we obtain

$$c_{\lambda_0\lambda_1\lambda_{01}} = c_{\lambda_0\lambda_1\lambda_{02}} - c_{\lambda_1\lambda_2\lambda_{12}}$$

on $\tilde{\varepsilon}_1^{-1}(U_{\lambda_0}^0) \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_1}^0) \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_2}^0) \cap U_{\lambda_{01}}^1 \cap U_{\lambda_{02}}^1 \cap U_{\lambda_{12}}^1 \cap \tilde{\eta}_1^{-1}(U_{\lambda_{012}}^2)$, which means that for any $\lambda_0, \lambda_1, \lambda_{01} \in J_0$, $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}$ does not depends on the choice of λ_{01}. Therefore, there exists a Real family

$$(f_{\lambda_0\lambda_1}) \in \prod_{\lambda_0, \lambda_1 \in \Lambda_0} \mathfrak{F}^1\left(\tilde{\varepsilon}_1^{-1}(U_{\lambda_0}^0) \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_1}^0)\right)$$

such that $f_{\lambda_0\lambda_1|U^1_{\lambda_0\lambda_1\lambda_{01}}} = c_{\lambda_0\lambda_1\lambda_{01}}$ for any $(\lambda_0,\lambda_1,\lambda_{01}) \in \Lambda_1$. Now, the cocycle relation (2.39) becomes

$$\tilde{\varepsilon}_0^* f_{\lambda_1\lambda_2} - \tilde{\varepsilon}_1^* f_{\lambda_0\lambda_2} + \tilde{\varepsilon}_2^* f_{\lambda_0\lambda_1} \tag{2.40}$$

on $U^1_{\lambda_0\lambda_1\lambda_{01}} \cap U^1_{\lambda_{02}} \cap U^1_{\lambda_{12}}$.

Let $(\mathcal{G},\rho)$ be a locally compact Hausdorff Real groupoid. We are interested in the 1^{st} Real Čech cohomology group of $(\mathcal{G}_\bullet,\rho_\bullet)$ with coefficients in the Abelian Real sheaf $(\mathcal{S}^\bullet,\sigma^\bullet) = (\mathrm{S},\sigma)$ over $(\mathcal{G}_\bullet,\rho_\bullet)$ associated to the Real $\mathcal{G}$-module $(X\times \mathrm{S}, \rho\times\ ^-)$, where $(\mathrm{S},\ ^-)$ is an Abelian group endowed with the trivial $\mathcal{G}$-action. Note that in this case, for any pre-simplicial Real open cover $\mathcal{U}_\bullet \in \mathfrak{N}(n)$ of $(\mathcal{G}_\bullet,\rho_\bullet)$, elements of the group $CR^n(\mathcal{U}_\bullet,\mathcal{S}^\bullet)$ are of the form $(c_\lambda)_{\lambda\in\Lambda_n}$, where $c_\lambda \in \Gamma(U^n_\lambda,\mathrm{S})$ are such that $c_{\bar\lambda}(\rho_n(\overrightarrow{g})) = \overline{c_\lambda(\overrightarrow{g})} \in \mathrm{S}$ for any $\overrightarrow{g} \in U^n_\lambda \subset \mathcal{G}_n$.

Proposition 2.49. *With the above notations, the Real Čech cohomology group* $\check{H}R^1(\mathcal{G}_\bullet,\mathrm{S})$ *is isomorphic to the group* $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathrm{S})$ *of isomorphism classes of Real generalized homomorphisms* $(\mathcal{G},\rho) \longrightarrow (\mathrm{S},\ ^-)$.

Proof. The operations in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},S)$ are defined as follows. If

$$(Z,\tau),(Z',\tau') : (\mathcal{G},\rho) \longrightarrow (S,\ ^-)$$

are Real generalized homomorphisms, their sum is

$$(Z,\tau)+(Z',\tau') := Z\times_X Z'/_\sim \tag{2.41}$$

where $(z,z') \sim (z\cdot t^{-1}, z'\cdot t)$ for all $t\in \mathrm{S}$, together with the obvious Real structure $\tau\times\tau'$. The inverse of (Z,τ) is (Z^{-1},τ), where Z^{-1} is Z as a topological space, and if $\flat : Z \hookrightarrow Z^{-1}$ is the identity map, then the S-action on Z^{-1} is defined by $\flat(z)\cdot t := \flat(z\cdot t^{-1})$ and the $\mathcal{G}$-action is defined as follows: $(g,\flat(z)) \in \mathcal{G}\ltimes Z^{-1}$ if and only if $(g,z)\in\mathcal{G}\ltimes Z$, in which case we set

$$g\cdot\flat(z) := \flat(g\cdot z).$$

Finally, the Real structure on Z^{-1} is $\tau(\flat(z)) := \flat(\tau(z))$. Then we define the sum in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},S)$ by $[Z,\tau]+[Z',\tau'] := [(Z,\tau)+(Z',\tau')]$, and we put $[Z,\tau]^{-1} := [(Z^{-1},\tau)]$. It is not hard to check that subject to these operations, $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},S)$ is an Abelian group.

Now, suppose we are given a Real open cover $\mathcal{U}_0 = (U^0_j)_{j\in J_0}$ of (X,ρ) trivializing the Real generalized homomorphism $(Z,\tau) : (\mathcal{G},\rho) \longrightarrow (S,\ ^-)$. Let $(\mathsf{s}_j)_{j\in J_0}$ be a Real family of local sections of the S-principal Real bundle $\mathfrak{r} : (Z,\tau) \longrightarrow (X,\rho)$. Form a pre-simplicial Real open cover $\mathcal{U}_\bullet$ of the Real simplicial space $(\mathcal{G}_\bullet,\rho_\bullet)$ by setting $J_n := J_0^{n+1}$, $\mathcal{U}_n := (U^n_{(j_0,\dots,j_n)})_{(j_0,\dots,j_n)\in J_n}$, where

$$U^n_{(j_0,\dots,j_n)} := \left\{(g_1,\dots,g_n)\in\mathcal{G}_n \;\middle|\; r(g_1)\in U^0_{j_0},\dots,r(g_n)\in U^0_{j_{n-1}}, s(g_n)\in U^0_{j_n}\right\}. \tag{2.42}$$

Then, for all $g \in U^1_{(j_0,j_1)}$, $\mathfrak{r}(g \cdot \mathsf{s}_{j_1}(s(g))) = r(g) = \mathfrak{r}(\mathsf{s}_{j_0}(r(g)))$; hence, there exists a unique element $c_{j_0j_1}(g) \in S$ such that $g\cdot\mathsf{s}_{j_1}(s(g)) = \mathsf{s}_{j_0}(r(g))\cdot c_{j_0j_1}(g)$. We then obtain a family of continuous functions $c_{j_0j_1} : U^1_{(j_0,j_1)} \longrightarrow S$ such that

$$g \cdot \mathsf{s}_{j_1}(s(g)) = \mathsf{s}_{j_0}(r(g)) \cdot c_{j_0j_1}(g),\ \forall g \in U^1_{(j_0,j_1)}. \tag{2.43}$$

Note further that $U^1_{(j_0,j_1)} = \tilde{\varepsilon}_0^{-1}(U^0_{j_1}) \cap \tilde{\varepsilon}_1^{-1}(U^0_{j_0})$. Let $(g_1, g_2) \in U^2_{(j_0,j_1,j_2)}$. Then

$$\begin{aligned}(g_1g_2)\cdot \mathsf{s}_{j_2}(s(g_2)) &= g_1 \cdot \mathsf{s}_{j_1}(r(g_2)) \cdot c_{j_1j_2}(g_2) = g_1 \cdot \mathsf{s}_{j_1}(s(g_1)) \cdot c_{j_1j_2}(g_2)\\ &= \mathsf{s}_{j_0}(r(g_1)) \cdot c_{j_0j_1}(g_1) \cdot c_{j_1j_2}(g_2);\end{aligned}$$

hence $c_{j_0j_2}(g_1g_2) = c_{j_0j_1}(g_1) \cdot c_{j_1j_2}(g_2)$. In other words,

$$\tilde{\varepsilon}_0^* c_{\tilde{\varepsilon}_0(j_0,j_1,j_2)} \cdot (\tilde{\varepsilon}_1^* c_{\tilde{\varepsilon}_1(j_0,j_1,j_2)})^{-1} \cdot \tilde{\varepsilon}_2^* c_{\tilde{\varepsilon}_2(j_0,j_1,j_2)} = 1$$

over all $U^2_{(j_0,j_1,j_2)}$. Moreover, we clearly have $c_{\bar{j}_0\bar{j}_1}(\rho(g)) = \overline{c_{j_0j_1}(g)} \in S$. This gives us a Real 1-cocycle $(c_{j_0j_1})_{(j_0,j_1)\in J_1} \in ZR^1(\mathcal{U}_\bullet, \mathcal{S}^\bullet)$.

Suppose $f : (Z,\tau) \longrightarrow (Z',\tau')$ is an isomorphism of Real generalized morphisms (see chapter 2). Up to a refinement, we can choose $\mathcal{U}_0$ in such a way that we have two Real families $(\mathsf{s}_j)_{j\in J_0},\ (\mathsf{s}')_{j\in J_0}$ of local sections of the Real projections $\mathfrak{r} : (Z,\tau) \longrightarrow (X,\rho)$ and $\mathfrak{r}' : (Z',\tau') \longrightarrow (X,\rho)$ respectively. Since for all $j \in J_0$ and $x \in U_j$, $\mathfrak{r}'(f_{U_j}(\mathsf{s}_j)(x)) = \mathfrak{r}(\mathsf{s}_j(x)) = x = \mathfrak{r}'(\mathsf{s}'_j(x))$, there exists a unique element $\varphi_j(x) \in S$ such that $\mathsf{s}'_j(x) = f_{U_j}(\mathsf{s}_j(x)) \cdot \varphi_j(x)$, and this gives a Real family of continuous functions $\varphi_j : U_j \longrightarrow S$. It follows that if $c = (c_{j_0j_1})$ and $c' = (c'_{j_0j_1})$ are the Real 1-cocycle associated to (Z,τ) and (Z',τ') respectively. Then, over $U^1_{(j_0,j_1)}$, one has

$$g \cdot f_{U_{j_1}}(\mathsf{s}_{j_1}(s(g))) \cdot \varphi_{j_1} = f_{U_{j_0}}(\mathsf{s}_{j_0}(r(g))) \cdot \varphi_{j_0}(r(g)) \cdot c'_{j_0j_1}(g);$$

But, since f is $\mathcal{G}$-S-equivariant, we get

$$f_{U_{j_0}(\mathsf{s}_{j_0}(r(g)))} \cdot c_{j_0j_1}(g) \cdot \varphi_{j_1}(s(g)) = f_{U_{j_0}}(\mathsf{s}_{j_0}(r(g))) \cdot \varphi_{j_0}(r(g)) \cdot c'_{j_0j_1}(g);$$

thus $c'_{j_0j_1}(g)\cdot c^{-1}_{j_0j_1}(g) = \varphi_{j_1}(s(g))\cdot\varphi_{j_0}(r(g))^{-1}$, or $(c'\cdot c^{-1})_{(j_0,j_1)} = \tilde{\varepsilon}_0^*\varphi_{\tilde{\varepsilon}_0(j_0,j_1)}\cdot \tilde{\varepsilon}_1^*\varphi^{-1}_{\tilde{\varepsilon}_1(j_0,j_1)}$ for all $(j_0, j_1) \in J_1$. This shows that $c'.c^{-1} \in BR^1(\mathcal{U}_\bullet, \mathrm{S})$. We then deduce a well-defined group homomorphism

$$c_1 : \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathrm{S}) \longrightarrow \check{H}R^1(\mathcal{G}_\bullet,\mathrm{S}),\ c_1([Z,\tau]) := [c_{j_0j_1}] \in HR^1(\mathcal{U}_\bullet,\mathrm{S}), \tag{2.44}$$

where $\mathcal{U}_\bullet$ is the Real open cover defined from any Real local trivialization of (Z,τ).

Conversely, given a Real Čech 1-cocycle $c = (c_{\lambda_0\lambda_1})$ over a pre-simplicial Real open cover $\mathcal{U}_\bullet \in \mathfrak{N}(1)$, we let $Z := \coprod_{\lambda_0\in\Lambda_0} U_{\lambda_0} \times \mathrm{S}$, together with the Real structure ν defined by $\nu(x,t) := (\rho(x),\bar{t})$, and equipped with the Real $\mathcal{G}$-action $g\cdot(s(g),t) := (r(g), c_{\lambda_0\lambda_1}(g)\cdot t)$ for any $g \in U^1_{\lambda_0\lambda_1\lambda_{01}}$, $t \in S$, and the obvious Real S-action. It is easy to see that the canonical projections define a Real generalized morphism $(Z, \nu) : (\mathcal{G}, \rho) \longrightarrow (\mathrm{S}, \bar{\ })$. One can check that if $[c] = [c']$ then $(Z, \tau) \cong (Z', \tau')$ by working backwards. $\square$

Remark 2.50. Suppose that (S, σ) is a non-abelian Real group. Then we still can talk about Čech Real 1-cocycles on $(\mathcal{G}_\bullet, \rho_\bullet)$ with coefficients on the non-Abelian Real sheaf $(\mathcal{S}^\bullet, \sigma^\bullet)$, and then form in the same way $\check{H}R^1(\mathcal{G}_\bullet, \mathcal{S}^\bullet)$ as a set. However, there is no reason for $\check{H}R^1(\mathcal{G}_\bullet, \mathrm{S})$ to be an Abelian group, it is not even a group since the sum of a Real 1-cocycle is not necessarily a Real 1-cocycle. Nevertheless, the result above remains valid in the sense that there is a bijection between the set $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathrm{S})$ of isomorphism classes of generalized Real morphism $(\mathcal{G}, \rho) \longrightarrow (\mathrm{S}, \sigma)$ and the set $\check{H}R^1(\mathcal{G}_\bullet, \mathrm{S})$.

A particular example of Proposition 2.49 is when $S = \mathbb{S}^1$ together with the complex conjugation as Real structure; in this case, the associated Real sheaf is denoted by $\mathbb{S}^1$ as mentioned earlier. It is well known that the *Picard* group $\mathrm{Pic}(X)$ of a locally compact topological space X is isomorphic to the 1^{st} sheaf cohomology group $H^1(X, \underline{\mathbb{S}^1}_X)$ (see for instance [3, Chap. 2]). In the Real case, we shall introduce the Real Picard group $\mathrm{PicR}(\mathcal{G})$ of a Real groupoid, and we will apply Proposition 2.49 to get an analogous result.

Definition 2.51 (Real line $\mathcal{G}$-bundle)**.**

(1) By a *Real line $\mathcal{G}$-bundle* we mean a Real $\mathcal{G}$-space $(\mathcal{L}, \nu)$, and a continuous surjective Real map $\pi : (\mathcal{L}, \nu) \longrightarrow (X, \rho)$ such that $\pi : \mathcal{L} \longrightarrow X$ is a complex vector bundle of rank 1, and such that for every $x \in X$, the induced isomorphism $\nu_x : \mathcal{L}_x \longrightarrow \mathcal{L}_{\rho(x)}$ is $\mathbb{C}$-anti-linear in the sense that $\nu_x(v \cdot z) = \nu_x(v) \cdot \bar{z}$.

(2) A homomorphism from a Real line $\mathcal{G}$-bundle $(\mathcal{L}, \nu)$ to a Real line $\mathcal{G}$-bundle $(\mathcal{L}', \nu')$ is a homormophism of complex vector bundles $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ intertwining the Real structures and which is $\mathcal{G}$-equivariant; *i.e.*, $\phi(g \cdot v) = g \cdot \phi(v)$ for any $(g, v) \in \mathcal{G} \ltimes \mathcal{L}$.

(3) We say that a Real line $\mathcal{G}$-bundle $(\mathcal{L}, \nu)$ is *locally trivial* if there exists a Real open cover $\mathcal{U}$ of (X, ρ), and a family of isomorphisms of complex vector bundles $\varphi_j : U_j \times \mathbb{C} \longrightarrow \mathcal{L}_{|U_j}$ such that:
   - $\varphi_{\bar{j}}(\rho(x), \bar{z}) = \nu_{U_j}(\varphi_j(x, z))$ for all $x \in U_j$ and $(x, z) \in U_j \times \mathbb{C}$.
   - If $r(g) \in U_{j_0}$ and $s(g) \in U_{j_1}$, then one has
   $$g.\varphi_{j_1}(s(g), z) = \varphi_{j_0}(r(g), z).$$

Example 2.52. The trivial action $\mathcal{G}$ on $X \times \mathbb{C}$ (*i.e.*, $g \cdot (s(g), z) := (r(g), z)$) is Real; moreover, the canonical projection $X \times \mathbb{C} \longrightarrow X$ defines a Real line $\mathcal{G}$-bundle that we call *trivial*.

Definition 2.53 (Real hermitian $\mathcal{G}$-metric)**.** Let $(\mathcal{L}, \nu)$ be a locally trivial Real line $\mathcal{G}$-bundle. A *Real hermitian $\mathcal{G}$-metric* on $(\mathcal{L}, \nu)$ is a continuous function $\mathrm{h} : \mathcal{L} \longrightarrow \mathbb{R}_+$ such that:

- $\mathrm{h}(\nu(v)) = \mathrm{h}(v)$, and $\mathrm{h}(v \cdot z) = \mathrm{h}(v) \cdot |z|^2$, for all $v \in \mathcal{L},\ z \in \mathbb{C}$.
- $\mathrm{h}(g \cdot v) = \mathrm{h}(v)$, for all $(g, v) \in \mathcal{G} \ltimes \mathcal{L}$.

- $\mathrm{h}(v) > 0$ whenever $v \in \mathcal{L}^+ := \mathcal{L} \smallsetminus \mathrm{o}$, where $\mathrm{o} : X \hookrightarrow \mathcal{L}$ is the zero-section.

If such h exists, $(\mathcal{L}, \nu, \mathrm{h})$ is called a *hermitian Real line $\mathcal{G}$-bundle* (we will often omit the metric).

Definition 2.54 (The Real Picard group)**.** The *Real Picard group* of $(\mathcal{G}, \rho)$ is defined as the set of isomorphism classes of locally trivial hermitian Real line $\mathcal{G}$-bundles. This "group" is denoted by $\mathrm{PicR}(\mathcal{G})$.

Theorem 2.55. *(compare with* [3, Theorem 2.1.8]*). Let $(\mathcal{G}, \rho)$ be a locally compact Hausdorff Real groupoid. Then $\mathrm{PicR}(\mathcal{G})$ is an Abelian group. Furthermore,*

$$\mathrm{PicR}(\mathcal{G}) \cong \check{H}R^1(\mathcal{G}_\bullet, \mathbb{S}^1).$$

Proof. Associated to any hermitian Real line $\mathcal{G}$-bundle $\pi : (\mathcal{L}, \nu) \longrightarrow (X, \rho)$, there is a Real generalized morphism $(\mathcal{L}^1, \nu) : (\mathcal{G}, \rho) \longrightarrow (\mathbb{S}^1, \bar{\ })$ obtained by setting

$$\mathcal{L}^1 := \{v \in \mathcal{L} \mid \mathrm{h}(v) = 1\}. \tag{2.45}$$

$\pi : (\mathcal{L}^1, \nu) \longrightarrow (X, \rho)$ is indeed an $\mathbb{S}^1$-principal Real bundle, and $\mathcal{L}^1$ is invariant under the action of $\mathcal{G}$. Hence $(\mathcal{L}^1, \nu)$ is indeed a Real generalized morphism. Conversely, if $(\tilde{\mathcal{L}}, \tilde{\nu}) : (\mathcal{G}, \rho) \longrightarrow (\mathbb{S}^1, \bar{\ })$ is a Real generalized morphism, define $\mathcal{L} := \tilde{\mathcal{L}} \times_{\mathbb{S}^1} \mathbb{C}$, where $\mathbb{S}^1$ acts by multiplication on $\mathbb{C}$; $\nu(v, z) := (\tilde{\nu}(v), \bar{z})$, $g \cdot (v, z) := (g \cdot v, z)$ for $(g, v) \in \mathcal{G} \ltimes \tilde{\mathcal{L}}$, and $\mathrm{h}(v, z) := |z|^2$. Then $(\mathcal{L}, \nu, \mathrm{h})$ is a hermitian Real line $\mathcal{G}$-bundle. Moreover, it is not hard to check that if $(\mathcal{L}, \nu, \mathrm{h})$ and $(\mathcal{L}', \nu', \mathrm{h}')$ are isomorphic hermitian Real line $\mathcal{G}$-bundles, then their associated Real generalized homomorphisms $(\mathcal{L}^1, \nu)$ and $((\mathcal{L}')^1, \nu')$ are isomorphic. We then have a map

$$\mathrm{PicR}(\mathcal{G}) \longrightarrow H^1(\mathcal{G}, \mathbb{S}^1)_\rho, \ [(\mathcal{L}, \nu, \mathrm{h})] \longmapsto [\mathcal{L}^1, \nu] \tag{2.46}$$

which is clearly an isomorphism of Abelian groups. Now, applying Proposition 2.49, we get the desired result. ☐

2.10. $\check{H}R^2$ and ungraded Real extensions. Let us consider the subgroup $\widehat{\mathrm{extR}}^+(\Gamma, \mathrm{S})$ of ungraded Real S-twists of the Real groupoid Γ; that is $(\widetilde{\Gamma}, \delta)$ is ungraded if $\delta = 0$. Similarly, we define the subgroup $\widehat{\mathrm{ExtR}}^+(\mathcal{G}, S)$ of $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathrm{S})$ of ungraded Real S-central extensions over $\mathcal{G}$. Elements of $\widehat{\mathrm{ExtR}}^+(\mathcal{G}, \mathrm{S})$ will then be denoted by pairs of the form $(\widetilde{\Gamma}, \Gamma)$.

Let $\mathcal{T} = \ \mathrm{S} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G}[\mathcal{U}_0] \ \in \widehat{\mathrm{extR}}^+(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$ be an ungraded Real S-twist, for a fixed Real open cover $\mathcal{U}_0 = (U_j^0)_{j \in J_0}$. Consider again the pre-simplicial Real open cover $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet, \rho_\bullet)$ defined by (2.42). Recall that the groupoid $\mathcal{G}[\mathcal{U}_0]$ is defined by

$$\mathcal{G}[\mathcal{U}_0] = \left\{(j_0, g, j_1) \in J_0 \times \mathcal{G} \times J_0 \mid g \in U^1_{(j_0, j_1)}\right\}.$$

Suppose that the S-principal Real bundle $\pi : (\tilde{\mathcal{G}}, \tilde{\rho}) \longrightarrow (\mathcal{G}[\mathcal{U}_0], \rho)$ admits a Real family of local continuous sections $\mathsf{s}_{j_0 j_1}$ relative to the Real open cover $\mathcal{V}_1$ of $(\mathcal{G}[\mathcal{U}_0], \rho)$ given by $\mathcal{V}_1 = (V^1_{(j_0,j_1)})_{(j_0,j_1)\in J_1}$, where

$$V^1_{(j_0,j_1)} := \{j_0\} \times U^1_{(j_0,j_1)} \times \{j_1\}.$$

Then, for any $(g_1, g_2) \in U^2_{(j_0,j_1,j_2)}$, we have that

$$\begin{aligned}\pi(\mathsf{s}_{j_0 j_1}(j_0, g_1, j_1) \cdot \mathsf{s}_{j_1 j_2}(j_1, g_2, j_2)) &= \pi(\mathsf{s}_{j_0 j_1}(j_0, g_1, j_1)) \cdot \pi(\mathsf{s}_{j_1 j_2}(j_1, g_2, j_2)) \\ &= (j_0, g_1 g_2, j_2) = \pi(\mathsf{s}_{j_0 j_2}(j_0, g_1 g_2, j_2));\end{aligned}$$

thus, there exists a unique element $\omega_{(j_0,j_1,j_2)}(g_1, g_2) \in S$ such that

$$\mathsf{s}_{j_0 j_2}(j_0, g_1 g_2, j_2) = \omega_{(j_0,j_1,j_2)}(g_1, g_2) \cdot \mathsf{s}_{j_0 j_1}(j_0, g_1, j_1).\mathsf{s}_{j_1 j_2}(j_1, g_2, j_2). \tag{2.47}$$

This provides a family of continuous functions $\omega_{(j_0,j_1,j_2)} : U^2_{(j_0,j_1,j_2)} \longrightarrow \mathrm{S}$ determined by (2.47) and that clearly verifies

$$\omega_{(\bar{j}_0,\bar{j}_1,\bar{j}_2)}(\rho(g_1), \rho(g_2)) = \overline{\omega_{(j_0,j_1,j_2)}(g_1, g_2)}, \forall (g_1, g_2) \in U^2_{(j_0,j_1,j_2)} \subset \mathcal{G}_2.$$

It is straightforward that the family $(\omega_{(j_0,j_1,j_2)})$ verifies the cocycle condition; hence we obtain a Real Čech 2-cocycle

$$\omega(\mathcal{T}) := (\omega_{(j_0,j_1,j_2)})_{(j_0,j_1,j_2)\in J_2} \in ZR^2(\mathcal{U}_\bullet, \mathrm{S}) \tag{2.48}$$

associated to $\mathcal{T}$.

In fact, this construction generalizes to arbitrary Real open covers $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet, \rho_\bullet)$.

Lemma 2.56 (Cf. Proposition 5.6 in [21])**.** *Let $(\mathcal{G}, \rho)$ be a topological Real groupoid. Given a Real open cover $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet, \rho_\bullet)$, let $\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$ denote the subgroup of all twists $\mathrm{S} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G}[\mathcal{U}_0] \in \widehat{\mathrm{extR}}^+(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$ such that π admits a Real family of local continuous sections*

$$\mathsf{s}_\lambda : \{\lambda_0\} \times U_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}$$

relative to the Real open cover

$$\mathcal{V}_1 := (\{\lambda_0\} \times U^1_{(\lambda_0,\lambda_1,\lambda_{01})} \times \{\lambda_1\})_{(\lambda_0,\lambda_1,\lambda_{01})\in\Lambda_1}$$

of $(\mathcal{G}[\mathcal{U}_0], \rho)$. Then the canonical map

$$\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S}) \longrightarrow HR^2(\mathcal{U}_\bullet, \mathrm{S}),\ [\mathcal{T}] \longmapsto [\omega(\mathcal{T})], \tag{2.49}$$

is a group isomorphism.

Proof. First, we prove $\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$ is a subgroup of $\widehat{\mathrm{extR}}^+(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$. Let

$$\mathcal{T} = (\ \mathrm{S} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G}[\mathcal{U}_0]\ ),\ \mathcal{T}' = (\ S \longrightarrow \tilde{\mathcal{G}}' \xrightarrow{\pi'} \mathcal{G}[\mathcal{U}_0]\ )$$

be representatives in $\widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0], \mathrm{S})$. Then their tensor product (see (1.8)) is

$$\mathcal{T}\hat{\otimes}\mathcal{T}' := (\ \mathrm{S} \longrightarrow \tilde{\mathcal{G}}\hat{\otimes}\tilde{\mathcal{G}}' \xrightarrow{\pi} \mathcal{G}[\mathfrak{U}_0]\ , 0),$$

where $\tilde{\mathcal{G}}\hat{\otimes}\tilde{\mathcal{G}}' = \tilde{\mathcal{G}} \times_{\mathcal{G}[\mathfrak{U}_0]} \tilde{\mathcal{G}}'/\mathrm{S}$. Let

$$f_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}},$$
$$f'_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}'$$

be Real families of continuous local sections of π and π' respectively. Then we get a Real family of continuous local sections

$$\mathsf{s}_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}\hat{\otimes}\tilde{\mathcal{G}}'$$

for π by setting

$$\mathsf{s}_\lambda(\lambda_0, g, \lambda_1) := \left[(f_\lambda(\lambda_0, g, \lambda_1), f'_\lambda(\lambda_0, g, \lambda_1))\right],$$

which implies that $\mathcal{T}\hat{\otimes}\mathcal{T}' \in \widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0], \mathrm{S}$.

Now let $\mathcal{T}$ be an (ungraded) Real twist of $(\mathcal{G}[\mathfrak{U}_0], \rho)$ such that π verifies the condition of the lemma. Assume that $\mathcal{T}'$ is any Real twist of $(\mathcal{G}[\mathfrak{U}_0], \rho)$ isomorphic to $\mathcal{T}$. Let $f : \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}'$ be a Real S-equivariant isomorphism that makes the following diagram

(2.50)
$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[r, "\pi"] \arrow[d, "f"] & \mathcal{G}[\mathfrak{U}_0] \\ \tilde{\mathcal{G}}' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

commute. Thus, given a Real family $\mathsf{s}_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}$, the maps $f \circ \mathsf{s}_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}'$ define a Real family of local continuous sections for π'; hence the class $[\mathcal{T}] \in \widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0], \mathbb{S}^1)$.

Suppose we are given a representative

$$\mathcal{T} = \ \mathrm{S} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G}[\mathfrak{U}_0]$$

in $\widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0], \mathrm{S})$. Recall that for $(\lambda_0, \lambda_1, \lambda_{01}) \in \Lambda_1$,

$$U^1_{\lambda_0\lambda_1\lambda_{01}} = U^1_{\lambda_{01}} \cap r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_1}),$$

and for $\lambda = (\lambda_0, \lambda_1, \lambda_2\lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{012}) \in \Lambda_2$, we have from (2.38) that

$$U^2_\lambda = \tilde{\varepsilon}_1^{-1} \circ r^{-1}(U^0_{\lambda_0}) \cap \tilde{\varepsilon}_0^{-1} \circ s^{-1}(U^0_{\lambda_1}) \cap \tilde{\varepsilon}_1^{-1} \circ s^{-1}(U^0_{\lambda_2})$$
$$\cap\, \tilde{\varepsilon}_2^{-1}(U^1_{\lambda_{01}}) \cap \tilde{\varepsilon}^{-1}(U^1_{\lambda_{02}}) \cap \tilde{\varepsilon}_0^{-1}(U^1_{\lambda_{12}}) \cap U^2_{\lambda_{012}}.$$

Then, for all $(g_1, g_2) \in U^2_\lambda$, one has:

- $g_1g_2 = \tilde{\varepsilon}_1(g_1, g_2) \in r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_2}) \cap U^1_{\lambda_{02}} = U^1_{\lambda_0\lambda_2\lambda_{02}}$;

- $g_1 = \tilde{\varepsilon}_2(g_1,g_2) \in U^1_{\lambda_{01}}$, $g_2 = \tilde{\varepsilon}_0(g_1,g_2) \in s^{-1}(U^0_{\lambda_1}) \cap U^1_{\lambda_{12}}$, and hence
$$g_1 \in r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_1}) \cap U^1_{\lambda_{01}} = U^1_{\lambda_0\lambda_1\lambda_{01}},$$
$$g_2 \in r^{-1}(U^0_{\lambda_1}) \cap s^{-1}(U^0_{\lambda_2}) \cap U^1_{\lambda_{12}} = U^1_{\lambda_1\lambda_2\lambda_{12}}.$$

Then as in the discussion before the lemma (see (2.48)), there exists a Real family of functions $\omega_\lambda : U^2_\lambda \longrightarrow \mathbb{S}^1$ such that

$$\mathsf{s}_{\lambda_0\lambda_2\lambda_{02}}(\lambda_0, g_1g_2, \lambda_2) = \omega_\lambda(g_1,g_2) \cdot \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(\lambda_0, g_1, \lambda_1) \cdot \mathsf{s}_{\lambda_1\lambda_2\lambda_{12}}(\lambda_1, g_2, \lambda_2) \tag{2.51}$$

and $\omega_{\bar\lambda}(\rho(g_1),\rho(g_2)) = \overline{\omega_\lambda(g_1,g_2)}$, for all $(g_1,g_2) \in U^2_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}\lambda_{012}}$. Moreover, it is easy to verify by a routine calculation that $(\omega_\lambda)_{\lambda\in\Lambda_2}$ verify the cocycle condition on

$$U^3_{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_{01}\lambda_{02}\lambda_{03}\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{0123}} \subset \mathcal{G}_2;$$

thus, we have constructed a Real Čech 2-cocyle $(\omega_\lambda)_{\lambda\in\Lambda_2} \in ZR^2(\mathfrak{U}_\bullet, \mathrm{S})$ associated to $\mathcal{T}$.

Assume that $(\tilde{\mathsf{s}}_\lambda)_{\lambda\in\Lambda_2}$ is another Real family of continuous local sections of π, and that $(\tilde\omega_\lambda)_{\lambda\in\Lambda_2} \in ZR^2(\mathfrak{U}_\bullet,\mathrm{S})$ is its associated Real Čech 2-cocycle. Then for any $(\lambda_0,\lambda_1,\lambda_{01}) \in \Lambda_1$ and $g \in U^1_{\lambda_0\lambda_1\lambda_{01}}$, there exists a unique $c_{\lambda_0\lambda_1\lambda_{01}}(g) \in \mathrm{S}$ such that

$$\tilde{\mathsf{s}}_{\lambda_0\lambda_1\lambda_{01}}(g) = c_{\lambda_0\lambda_1\lambda_{01}}(g) \cdot \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g), \tag{2.52}$$

where we abusively write, for instance, $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g)$ for $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(\lambda_0,g,\lambda_1)$. Since $(\tilde{\mathsf{s}}_{\lambda_0\lambda_1\lambda_{01}})$ and $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}$ are Real families, we have that

$$c_{\bar\lambda_0\bar\lambda_1\bar\lambda_{01}}(\rho(g)) = \overline{c_{\lambda_0\lambda_1\lambda_{01}}(g)} \text{ for all } g \in U^1_{\lambda_0\lambda_1\lambda_{01}}.$$

It turns out that the $c_{\lambda_0\lambda_1\lambda_{01}}$'s define an element in $CR^1(\mathfrak{U}_\bullet,\mathrm{S})$. Moreover, for $\lambda \in \Lambda_2$ as previously, and for $(g_1,g_2) \in U^2_\lambda$, we obtain from (2.51) and (2.52)

$$\begin{aligned}\mathsf{s}_{\lambda_0\lambda_2\lambda_{02}}(g_1g_2) = c_{\lambda_0\lambda_2\lambda_{02}}(g_1g_2)^{-1} \cdot c_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot c_{\lambda_1\lambda_2\lambda_{12}}(g_2) \cdot \tilde\omega_\lambda(g_1,g_2)\\ \cdot \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot \mathsf{s}_{\lambda_1\lambda_2\lambda_{12}}(g_2);\end{aligned}$$

and

$$\begin{aligned}(\omega_\lambda \cdot \tilde\omega_\lambda^{-1})(g_1,g_2) &= c_{\lambda_0\lambda_2\lambda_{02}}(g_1g_2)^{-1} . c_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot c_{\lambda_1\lambda_2\lambda_{12}}(g_2)\\ &= (dc)_\lambda(g_1,g_2);\end{aligned}$$

hence $((\omega \cdot \tilde\omega^{-1})_\lambda)_{\lambda\in\Lambda_2} \in BR^2(\mathfrak{U}_\bullet,\mathbb{S}^1)$. I.e., the class in $HR^2(\mathfrak{U}_\bullet,\mathrm{S})$ of the Real 2-cocycle (ω_λ) does not depend on the choice of the Real family of local sections of π.

We want now to check that the map (2.49) is well-defined. To do so, suppose that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent in $\widehat{\mathrm{extR}}_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathrm{S})$, and that $(\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}})$ and $\mathsf{s}'_{\lambda_0\lambda_1\lambda_{01}}$ are Real family of local continuous sections of π and π'. Let us keep the diagram (2.50). Let $(\omega_\lambda)_{\lambda\in\Lambda_2}$ and $(\omega'_\lambda)_{\lambda\in\Lambda_2}$ be the associated Real 2-cocycles in $ZR^2(\mathfrak{U}_\bullet,\mathrm{S})$ of $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then we define an element $(b_{\lambda_0\lambda_1\lambda_{01}}) \in CR^1(\mathcal{U}_\bullet, \mathrm{S})$ as follows: for any $g \in U^1_{\lambda_0\lambda_1\lambda_{01}}$, $b_{\lambda_0\lambda_1\lambda_{01}}(g)$ is the unique element of S such that

$$\mathsf{s}'_{\lambda_0\lambda_1\lambda_{01}}(g) = b_{\lambda_0\lambda_1\lambda_{01}}(g) \cdot f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g). \tag{2.53}$$

This is well-defined since

$$\pi'(\mathsf{s}'_{\lambda_0\lambda_1\lambda_{01}}(g)) = \pi(\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g)) = \pi'(f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g)).$$

Furthermore, the functions $f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}, (\lambda_0, \lambda_1, \lambda_{01}) \in \Lambda_1$, defines a globally Real family of local continuous sections of π. Then, for all $\lambda \in \Lambda_2$ and all $(g_1, g_2) \in U^2_\lambda$, we can write

$$f \circ \mathsf{s}_{\lambda_0\lambda_2\lambda_{02}}(g_1 g_2) = \omega_\lambda(g_1, g_2) \cdot f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot f \circ \mathsf{s}_{\lambda_1\lambda_2\lambda_{12}}(g_2),$$

up to a multiplication of ω_λ by a Real 2-coboundary. It then follows that

$$\begin{aligned}\omega_\lambda(g_1, g_2) \cdot \omega'_\lambda(g_1, g_2)^{-1} &= b_{\lambda_0\lambda_2\lambda_{02}}(g_1 g_2)^{-1} \cdot b_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot b_{\lambda_1\lambda_2\lambda_{12}}(g_2) \\ &= (db)_\lambda(g_1, g_2).\end{aligned}$$

Consequently, $(\omega_\lambda)_{\lambda\in\Lambda_2}$ depends only on the class of $\mathcal{T}$ in $\widehat{\mathrm{extR}}_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$. The fact that $(\delta_{\lambda_0\lambda_1\lambda_{01}})$ also depends only on the class of $\mathcal{T}$ is straightforward. We then have proved that any element $[\mathcal{T}]$ in $\widehat{\mathrm{extR}}_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$ determines a unique cohomology class

$$[\omega(\mathcal{T})] \in HR^2(\mathcal{U}_\bullet, \mathrm{S}). \tag{2.54}$$

Conversely, given a pair $(\omega_\lambda)_{\lambda\in\Lambda_2} \in ZR^2(\mathcal{U}_\bullet, \mathrm{S})$, we want to construct an ungraded Real extension of $(\mathcal{G}[\mathcal{U}_0], \rho)$ which is in $\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$. For this we proceed as in the proof of Proposition 5.6 in [21]. For $\lambda \in \Lambda_2$, put

$$\begin{aligned}\mu_{01} &:= (\lambda_0, \lambda_{01}, \lambda_1), \\ \mu_{02} &:= (\lambda_0, \lambda_{02}, \lambda_2), \\ \mu_{12} &:= (\lambda_1, \lambda_{12}, \lambda_2).\end{aligned}$$

Let $\mathfrak{c}_{\mu_{01}\mu_{02}\mu_{12}} := \omega_\lambda$. We have $\mathcal{V}_1 = (V^1_{\mu_{01}})_{i\in I_1}$, where I_1 consists of triples $\mu_{01} = (\lambda_0, \lambda_{01}, \lambda_1)$ and $V^1_{\mu_{01}} := \{\lambda_0\} \times U^1_{\lambda_0\lambda_1\lambda_{01}} \times \{\lambda_1\}$. I_1 is equipped with the obvious involution, so that $\mathcal{V}_1$ is a Real open cover of $\mathcal{G}[\mathcal{U}_0]$. We set

$$\widetilde{\Gamma}^\omega := \coprod_{\mu_{01}\in I_1} \{(t, g, \mu_{01}) \mid t \in \mathrm{S}, g \in V^1_{\mu_{01}}\}/\sim,$$

subject to the product law

$$[t_1, g_1, \mu_{01}] \cdot [t_1, g_2, \mu_{12}] = [t_1 \cdot t_2 \cdot \mathfrak{c}_{\mu_{01}\mu_{02}\mu_{12}}(g_1, g_2), g_1 g_2, \mu_{02}],$$

where

$$(t, g, \mu_{12}) \sim (\mathfrak{c}_{\mu_{01}\mu_{01}\mu_{01}}(r(g), r(g))^{-1} \cdot t \cdot \mathfrak{c}_{\mu_{01}\mu_{02}\mu_{12}}(r(g), g), g, \mu_{02}). \tag{2.55}$$

The projection $\pi : \widetilde{\Gamma}^\omega \longrightarrow \mathcal{G}[\mathcal{U}_0]$ is defined by $\pi([t, g, \mu_{01}]) := g$, and the Real structure is

$$\overline{[t, g, \mu_{01}]} := [\bar{t}, \rho(g), \overline{\mu_{01}}].$$

It is straightforward to see that these operations give $\widetilde{\Gamma}^\omega$ the structure of ungraded Real S-twist of $\mathcal{G}[\mathcal{U}_0]$; what is more, the maps $\mathsf{s}_{\mu_{01}} : V^1_{\mu_{01}} \longrightarrow \widetilde{\Gamma}^\omega$ defined by $\mathsf{s}_{\mu_{01}}(g) := [0, g, \mu_{01}]$ are a Real family of continuous sections of π, so that the Real extension

$$\mathcal{T} = \ \mathrm{S} \longrightarrow \widetilde{\Gamma}^\omega \xrightarrow{\pi} \mathcal{G}[\mathcal{U}_0]$$

is in $\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathrm{S})$. It is also clear that $[\omega(\mathcal{T})] = [\omega]$. □

Corollary 2.57. *We have* $\widehat{\mathrm{ExtR}}^+(\mathcal{G}, \mathrm{S}) \cong \check{H}R^2(\mathcal{G}_\bullet, \mathrm{S})$.

2.11. The cup-product $\check{H}R^1(\cdot, \mathbb{Z}_2) \times \check{H}R^1(\cdot, \mathbb{Z}_2) \to \check{H}R^2(\cdot, \mathbb{S}^1)$. Let $\delta, \delta' \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2)$, and let L and L' be representatives of their corresponding classes in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathbb{Z}_2)$ (see Proposition 2.49). Then by viewing $\mathbb{Z}_2 = \{\mp 1\}$ as a Real subgroup of $\mathbb{S}^1$ (identifying -1 with $(-1, 0)$ and $+1$ with $(1, 0)$), we define the tensor product $r^*L \otimes \overline{s^*L'} \longrightarrow \mathcal{G}$, and and using the same reasoning as in Example 1.45, we see that this is clearly a Real $\mathbb{Z}_2$-principal bundle; thus we have an ungraded Real $\mathbb{Z}_2$-central extension

$$\mathbb{Z}_2 \longrightarrow r^*L \otimes \overline{s^*L'} \longrightarrow \mathcal{G}.$$

Therefore, we get an ungraded Real $\mathbb{S}^1$-central extension $(L \smile L', \mathcal{G})$ given by

$$L \smile L' := (r^*L \otimes \overline{s^*L'}) \times_{\mathbb{Z}_2} \mathbb{S}^1, \tag{2.56}$$

together with the evident Real structure and Real $\mathbb{S}^1$-action.

Definition 2.58. We define the cup product

$$\smile \ : \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \longrightarrow \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$$

by

$$\delta \smile \delta' := \omega(L \smile L'),$$

where $L \smile L'$ is determined by equation (2.56).

Lemma 2.59. *The cup product $\smile$ defined above is a well-defined bilinear map;* i.e.,

$$(\delta_1 + \delta_2) \smile (\delta'_1 + \delta'_2) = \delta_1 \smile \delta'_1 + \delta_1 \smile \delta'_2 + \delta_2 \smile \delta'_1 + \delta_2 \smile \delta'_2.$$

Proof. If δ_i is realized by the generalized Real homomorphism

$$L_i : \mathcal{G} \longrightarrow \mathbb{Z}_2,$$

then $\delta_1 + \delta_2$ is realized by $L_1 + L_2$. The result follows from the easy to check bilinearity of the tensor product $r^*L \otimes \overline{s^*L'}$ with respect to the sum in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathbb{Z}_2)$. □

2.12. Cohomological picture of the group $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$. Let

$$\mathcal{T}=(\tilde{\mathcal{G}},\delta)\in\widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}_0],\mathbb{S}^1),$$

where as usual $\mathcal{U}_0$ is a Real open cover of X. Let $\mathcal{U}_\bullet$ be the pre-simplicial Real open cover of $(\mathcal{G}_\bullet,\rho_\bullet)$ defined as in (2.42).

Define a continuous map $\delta_{j_0j_1}:U^1_{(j_0,j_1)}\longrightarrow\mathbb{Z}_2$ over all $U^1_{(j_0,j_1)}\in\mathcal{U}_1$ by $\delta_{j_0j_1}(g):=\delta(j_0,g,j_1)$. Then, over all $U^2_{(j_0,j_1,j_2)}$, we have that

$$\delta_{j_0j_2}(g_1g_2)=\delta((j_0,g_1,j_1)\cdot(j_1,g_2,j_2))=\delta_{j_0j_1}(g_1)\cdot\delta_{j_1j_2}(g_2).$$

Moreover, since δ is a Real morphism, we have that $\delta_{\bar{j}_0\bar{j}_1}(\rho(g))=\delta_{j_0j_1}(g)$; hence $\mathcal{T}$ determines a Real Čech 1-cocycle

$$\delta(\mathcal{T}):=(\delta_{j_0j_1})_{(j_0,j_1)\in J_1}\in ZR^1(\mathcal{U}_\bullet,\mathbb{Z}_2), \tag{2.57}$$

Then, (2.57) gives a Real Čech 1-cocycle $(\delta_{\lambda_0\lambda_1\lambda_{01}})\in ZR^1(\mathcal{U}_\bullet,\mathbb{Z}_2)$ defined by $\delta_{\lambda_0\lambda_1\lambda_{01}}(g):=\delta(\lambda_0,g,\lambda_1)$ for any $g\in U^1_{\lambda_0\lambda_1\lambda_{01}}$; this does make sense, for we know from Section 2.9 that Real Čech 1-cocycles do not depend on λ_{01}.

If $\mathcal{T}'$ is another Rg $\mathbb{S}^1$-central extension over $\mathcal{G}$, we may suppose it is represented by a Rg $\mathbb{S}^1$-twisted $(\tilde{\mathcal{G}}',\delta')$ of $\mathcal{G}[\mathcal{U}_0]$. Then by definition of the grading of $\mathcal{T}\hat{\otimes}\mathcal{T}'$, we have $\delta(\mathcal{T}\hat{\otimes}\mathcal{T}')=\delta(\mathcal{T})+\delta(\mathcal{T}')$.

Theorem 2.60 (Cf. [6, Proposition 2.13])**.** *Let $(\mathcal{G},\rho)$ be a locally compact Hausdorff Real groupoid. There is a set-theoretic split-exact sequence*

$$0\longrightarrow\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)\hookrightarrow\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)\xrightarrow{\delta}\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\longrightarrow 0 \tag{2.58}$$

so that we have a canonical group isomorphism

$$dd\colon\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)\cong\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\ltimes\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1), \tag{2.59}$$

where the semi-direct product $\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\ltimes\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)$ is defined by the operation

$$(\delta,\omega)+(\delta',\omega'):=(\delta+\delta',(\delta\smile\delta')\cdot\omega\cdot\omega').$$

The image of a Real graded extension $\mathbb{E}$ by dd is called the Dixmier–Douady class *of* $\mathbb{E}$.

Proof. The first arrow is the canonical inclusion

$$\widehat{\mathrm{ExtR}}^+(\mathcal{G},\mathbb{S}^1)\subset\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1),$$

and hence is injective. The exactness of the sequence (2.58) is obvious, by definition of δ and $\widehat{\mathrm{ExtR}}^+(\mathcal{G},\mathbb{S}^1)$.

The map δ is well-defined; indeed, if $\mathcal{T}\sim\mathcal{T}'$ in $\widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}_0],\mathbb{S}^1)$, they differ from a twist coming from an element of $\mathrm{PicR}(\mathcal{G}[\mathcal{U}_0])$, and hence by construction of δ, one has $\delta(\mathcal{T})=\delta(\mathcal{T}')$. Moreover, δ is surjective, for if $L\in\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathbb{Z}_2)$ represents the Real 1-cocycle $(\varepsilon_{j_0j_1})\in ZR^1(\mathcal{U}_\bullet,\mathbb{Z}_2)$, then $L\smile L$ is graded as follows:

$$L\smile L:=(\mathbb{S}^1\longrightarrow(r^*L\otimes\overline{s^*L})\times_{\mathbb{Z}_2}\mathbb{S}^1\longrightarrow\mathcal{G}[\mathcal{U}_0],\delta'),$$

where

$$\delta'((j_0,\gamma,j_1)) := \varepsilon_{j_0j_1}(\gamma).$$

We see that $\delta(L \smile L) = \varepsilon$. Finally, note that the operation law comes from the definition of the sum in $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$. $\square$

2.13. The proper case. In this subsection, we are interested in some particular Abelian Real sheaves on $(\mathcal{G}_\bullet,\rho_\bullet)$, where $(\mathcal{G},\rho)$ is a proper groupoid. More precisely, we aim to generalize a result by Crainic (see [4, Proposition 1]) stating that for a proper Lie groupoid $\mathcal{G}$, and "*representation*" E of $\mathcal{G}$ ([4, 1.2]), the *differentiable* cohomology $H^n_d(\mathcal{G},E)=0$ for all $n\geq 1$. Let us first introduce some few notions and properties.

Definition 2.61 (Real Haar measure)**.** Let $(\mathcal{G},\rho)$ be a locally compact Real groupoid, and let $\{\mu^x\}_{x\in X}$ be a (left) *Haar system* for $\mathcal{G}$ (see [19, §.2]). Define a new family $\{\mu^x_\rho\}_{x\in X}$ of measures μ^x_ρ, with support $\mathcal{G}^x$ for all $x\in X$, defined by

$$\mu^x_\rho(C) := \mu^{\rho(x)}(\rho(C)),\ \text{for all measurable subset } C\subset \mathcal{G}^x. \tag{2.60}$$

We say that $\{\mu^x\}_{x\in X}$ is *Real* if

$$\mu^x = \mu^x_\rho,\ \forall x\in X. \tag{2.61}$$

Lemma 2.62. *Any Haar system for $\mathcal{G}$ gives rise to a Real one.*

Proof. Assume $\{\mu^x\}$ is a Haar system for $\mathcal{G}$. For every $x\in X$, we set

$$\tilde{\mu}^x := \frac{1}{2}(\mu^x+\mu^x_\rho). \tag{2.62}$$

It is clear that $\{\tilde{\mu}^x\}_{x\in X}$ is a Haar system for $\mathcal{G}$; measurable subsets for $\tilde{\mu}^x$ being exactly those for μ^x. Moreover, one has

$$\tilde{\mu}^x_\rho = \frac{1}{2}\left(\mu^{\rho(x)}\circ\rho + \mu^{\rho(x)}_\rho\circ\rho\right) = \frac{1}{2}\left(\mu^x_\rho+\mu^x\right) = \tilde{\mu}^x,\ \forall x\in X. \qquad \square$$

Remark 2.63. From the lemma above, we will always assume Haar systems for $\mathcal{G}$ to be Real.

In what follows, the Real group $\mathbb{K}$ is either the additive group $\mathbb{R}$ equipped with the Real structure $t\longmapsto \bar{t} := -t$, or the additive group $\mathbb{C}$ equipped with the complex conjugation $z\longmapsto \bar{z}$ as Real structure.

Definition 2.64. Let $(\mathcal{G},\rho)$ be a locally compact Real groupoid. A *Real representation* of $(\mathcal{G},\rho)$ is a locally trivial Real $\mathbb{K}$-vector bundle

$$\pi : (E,\nu)\longrightarrow (X,\rho)$$

endowed with a (left) continuous Real $\mathcal{G}$-action; that is a Real open cover (U_j) of (X,ρ) and isomorphisms $\phi_j : U_j\times\mathbb{K}^r \longrightarrow E_{|U_j}$ such that

$$\nu(\phi_j(x,(a_1,\ldots,a_r))) = \phi_{\bar{j}}(\rho(x),(\bar{a}_1,\ldots,\bar{a}_r)),\ \forall x\in U_j, (a_1,\ldots,a_r)\in\mathbb{K}^r,$$

and

- $\forall x \in X$, the induced isomorphism $\nu_x : E_x \longrightarrow E_{\rho(x)}$ is $\mathbb{K}$-antilinear:
$$\nu_x(\xi \cdot a) = \nu_x(\xi) \cdot \bar{a},\ \forall \xi \in E_x, a \in \mathbb{K};$$
- $\forall g \in \mathcal{G}$, the isomorphism $E_{s(g)} \longrightarrow E_{r(g)}$, induced by the $\mathcal{G}$-action, is linear.

Note that such a Real representation (E, ν) can be viewed as a Real $\mathcal{G}$-module in the following way: E is the groupoid $E \rightrightarrows X$ with $r_E(\xi) = s_E(\xi) := \pi(\xi)$ for every $\xi \in E$, for any $x \in X$, $E_x = E^x = E^x_x$ is isomorphic to the group $\mathbb{K}$, then the product in E is defined by the sum on the fibres. The Real sheaf on $(\mathcal{G}_\bullet, \rho_\bullet)$ associated to the Real $\mathcal{G}$-module (E, ν) will be denoted $(E^\bullet, \nu^\bullet)$.

Remark 2.65. More generally, we define a Real representation of of type $\mathbb{R}^{p,q}$ as a locally trivial real vector bundle $E \longrightarrow X$ of rank $p+q$, together with a Real structure $\nu : E \longrightarrow E$, and a Real $\mathcal{G}$-action on E with respect to the projection map, such that locally, the Real space (E, ν) identifies with $\mathbb{R}^{p,q}$; that is there is a Real open cover (U_j) of X and commutative diagrams

$$\begin{array}{ccc} U_j \times \mathbb{R}^{p,q} & \xrightarrow{\phi_j} & E_{|U_j} \\ \downarrow{\scriptstyle \rho\times \mathrm{bar}} & & \downarrow{\scriptstyle \nu} \\ U_{\bar{j}} \times \mathbb{R}^{p,q} & \xrightarrow{\phi_{\bar{j}}} & E_{|U_{\bar{j}}} \end{array}$$

where $\mathrm{bar} : \mathbb{R}^{p,q} \longrightarrow \mathbb{R}^{p,q}$ is the Real structure defined in the first section.

Definition 2.66 ([25, Definition 2.20])**.** A locally compact Real groupoid $(\mathcal{G}, \rho)$ is said to be *proper* if either of the following equivalent conditions is satisfied:

(i) The Real map $(s, r) : \mathcal{G} \longrightarrow X \times X$ is proper.
(ii) For every $K \subset X$ compact, $\mathcal{G}^K_K$ is compact.

Proper Real groupoids can be characterized by the following (we refer to Propositions 6.10 and 6.11 in [22] for a proof).

Proposition 2.67. *Let $(\mathcal{G}, \rho)$ be a locally compact Real groupoid with a Haar system $\{\mu^x\}_{x\in X}$. Then $(\mathcal{G}, \rho)$ is proper if and only it admits a* cutoff *Real function; that is, a function $x : X \longrightarrow \mathbb{R}_+$ such that:*

(i) *$\forall x \in X$, $c(\rho(x)) = c(x)$.*
(ii) *$\forall x \in X$, $\int_{\mathcal{G}^x} c(s(g))d\mu^x(g) = 1$.*
(iii) *The map $r : supp(\mathsf{c} \circ s) \longrightarrow X$ is proper;* i.e., *for every $K \subset X$ compact,* $\mathrm{supp}(c) \cap s(\mathcal{G}^K)$ *is compact.*

Theorem 2.68. *Suppose $(\mathcal{G}, \rho)$ is a locally compact proper Real groupoid with a Haar system. Then, for any Real representation (E, ν) of $(\mathcal{G}, \rho)$, we have*
$$\check{H}R^n(\mathcal{G}_\bullet, E^\bullet) = 0,\ \forall n \geq 1.$$

To prove this result, we shall recall fundamentals of vector-valued integration exposed, for instance, in [26, Appendix B.1], and then adapt them to the case when we deal with Real structures. Let X be a locally compact Hausdorff space, and let B be a separable Banach space. Let μ be a Radon measure on X. Then measurable functions $f : X \longrightarrow B$ are defined as usual, and such function is *integrable* if

$$\|f\|_1 := \int_X \|f(x)\| d\mu(x) < \infty.$$

The collection of all B-valued integrable functions on X is denoted by $\mathcal{L}^1(X, B)$, and the set of equivalence classes of functions in $\mathcal{L}^1(X, B)$ is a Banach space denoted by $L^1(X, B)$ ([26, Proposition B.31]). Furthermore, $\mathcal{C}_c(X, B)$ is dense in $L^1(X, B)$. The B-valued integration of elements of $L^1(X, B)$ is defined as a linear map $I : \mathcal{C}_c(X, B) \longrightarrow B$ given by

$$I(f) := \int_X f(x)d\mu(x), \text{ and } \|I(f)\| \leq \|f\|_1. \tag{2.63}$$

Moreover, this integral is characterized by the following:

Proposition 2.69 (Cf. Proposition B.34 [26])**.** *Let μ be a Radon measure on X, and let B be a Banach space. Then, the integral is characterized by:*

(a) *For all $f \in \mathcal{C}_c(X, B)$ and $\varphi \in B^*$,*

$$\varphi\left(\int_X f(x)d\mu(x)\right) = \int_X \varphi(f(x))d\mu(x).$$

(b) *If $L : B \longrightarrow B'$ is any bounded linear map between two Banach spaces, than*

$$L\left(\int_X f(x)d\mu(x)\right) = \int_X L(f(x))d\mu(x).$$

Now suppose (X, ρ) is a locally compact Hausdorff Real space, μ is a Real Radon measure; *i.e.*, $\mu(\rho(C)) = \rho(C)$ for every measurable set $C \subset X$. Let (B, ς) be a separable Real Banach space. Then from the above, we deduce the

Lemma 2.70. *Let $\mathcal{C}_c(X, B)$ be equipped with the Real structure denoted by $\tilde{\rho} : \mathcal{C}_c(X, B) \longrightarrow \mathcal{C}_c(X, B)$, and given by $\rho(f)(x) := \varsigma(f(\rho(x)))$. Then, under the above assumption, the integral $\int : \mathcal{C}_c(X, B) \longrightarrow B$ is Real, in that it commutes with the Real structures ς and $\tilde{\rho}$; i.e.,*

$$\int_X \varsigma(f(\rho(x)))d\mu(x) = \varsigma\left(\int_X f(x)d\mu(x)\right), \forall f \in \mathcal{C}_c(X, B). \tag{2.64}$$

Proof. For any $\varphi \in B^*$, define $\bar{\varphi} \in B^*$ by $\bar{\varphi}(b) := \overline{\varphi(\varsigma(b))}$. Then, from Proposition 2.69(a) and the definition of $\bar{\varphi}$, one has

$$\overline{\varphi\left(\varsigma\left(\int_X f(x)d\mu(x)\right)\right)} = \int_X \overline{\varphi(\varsigma(f(x)))}d\mu(x) = \overline{\int_X \varphi(\varsigma(f(x)))d\mu(x)}.$$

Thus,

$$\varphi\left(\varsigma\left(\int_X f(x)d\mu(x)\right)\right) = \int_X \varphi(\varsigma(f(x)))d\mu(x).$$

Again from (b) of Proposition 2.69 and from the fact that μ is Real, we then get

$$\varphi\left(\varsigma\left(\int_X f(x)d\mu(x)\right)\right) = \varphi\left(\int_X \varsigma(f(\rho(x)))d\mu(x)\right), \forall\varphi \in B^*,$$

and the result holds. □

Let us investigate the case of a Real groupoid $(\mathcal{G}, \rho)$ together with a Real representation (E, ν). Let $\mu = \{\mu^x\}_{x\in X}$ be a Real Haar system for $(\mathcal{G}, \rho)$. For any $x \in X$, we can apply (2.63) to E_x and get the integral $\int_{\mathcal{G}^x} : \mathcal{C}_c(\mathcal{G}^x, E_x) \longrightarrow E_x$. Further, it is very easy to check that

$$\nu_x\left(\int_{\mathcal{G}^x} f(\gamma)d\mu^x(\gamma)\right) = \int_{\mathcal{G}^{\rho(x)}} \nu_x(f(\rho(\gamma)))d\mu^{\rho(x)}(\gamma),\ \forall f \in \mathcal{C}_c(\mathcal{G}^x, E_x). \tag{2.65}$$

Proof of Theorem 2.68. Fix a Real Haar system $\{\mu^x\}_{x\in X}$ for $(\mathcal{G}, \rho)$ and a cutoff Real function $c : X \longrightarrow \mathbb{R}_+$. Let $\mathcal{U}_\bullet$ be a Real open cover of $(\mathcal{G}_\bullet, \rho_\bullet)$. Let $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_{01\ldots n}) \in \Lambda_n$ and $U^n_\lambda \in \natural\mathcal{U}_n$. Denote by $\Lambda_{n+1|\lambda}$ the subset of Λ_{n+1} consisting of those $\tilde\lambda \in \Lambda_{n+1}$ such that $\tilde\lambda(S) = \lambda_S$ for all $\emptyset \neq S \subseteq [n]$. Then, if for any $x \in U^0_{\lambda_n}$, we denote

$$\begin{aligned}(U^n_\lambda \star \mathcal{G}^x) &\cap \operatorname{supp}(\mathsf{c}\circ s)\\ &:= \{(g_1, \ldots, g_n, \gamma) \in U^n_\lambda \times (\mathcal{G}^x \cap \operatorname{supp}(\mathsf{c}\circ s)) \mid s(g_n) = r(\gamma) = x\},\end{aligned}$$

we have that

$$(U^n_\lambda \star \mathcal{G}^x) \cap \operatorname{supp}(\mathsf{c}\circ s) \subset \bigcup_{\tilde\lambda\in\Lambda_{n+1|\lambda}} U^{n+1}_{\tilde\lambda}. \tag{2.66}$$

Notice that for $\tilde\lambda$ running over $\Lambda_{n+1|\lambda}$, only its images $\tilde\lambda_S \in \Lambda_{\#S-1}$, for $S \subseteq [n+1]$ containing $n+1$, are led to vary. On the other hand, since $\mathcal{G}^x \cap \operatorname{supp}(\mathsf{c}\circ s)$ is compact in $\mathcal{G}$ (by (iii) of Proposition 2.67), the union (2.66) is finite. In particular, for every $S \in S(n+1) := \{S \subseteq [n+1] \mid n+1 \in S \neq \emptyset\}$, where elements of $S(n+1)$ are ranged in cardinality and in lexicographic order, there is $\tilde\lambda^{l_S}_S \in \Lambda_{\#S-1}$, $l_S = 0, \ldots, m_S$, such that

$$(U^n_\lambda \star \mathcal{G}^x) \cap \operatorname{supp}(\mathsf{c}\circ s) \subset \bigcup_{l=(l_S)_{S\in S(n+1)}} U^{n+1}_{\lambda^l}, \tag{2.67}$$

where for any $l = (l_S)_{S\in S(n+1)} \in \mathbb{N}^{2^{n+1}}$ written as

$$l = \left(l_{\{n+1\}}, l_{\{0,n+1\}}, l_{\{1,n+1\}}, \ldots, l_{\{n,n+1\}}, \ldots, l_{\{1,\ldots,n+1\}}, l_{\{0,1,\ldots,n+1\}}\right),$$

the element $\lambda^l \in \Lambda_{n+1|\lambda}$ is given by the following

$$\begin{cases} \lambda^l(S) := \lambda_S, & \text{for any } S \subseteq [n]; \\ \lambda^l(S) := \lambda_S^{l_S}, & \text{for any } S \in S(n+1). \end{cases} \tag{2.68}$$

Now for each $S \in S(n+1)$, $\varepsilon_S^{n+1} =: \varepsilon_S : [\#S-1] \longrightarrow [n+1]$ denotes the unique morphism in $\mathrm{Hom}_{\Delta'}([\#S-1],[n+1])$ whose range is exactly S. It is then clear that

$$\tilde{\varepsilon}_S((U_\lambda^n \star \mathcal{G}^x) \cap \mathrm{supp}(\mathsf{c} \circ s)) \subset \bigcup_{l_S} U^{\#S-1}_{\lambda_S^{l_S}},\ \forall S \in S(n+1). \tag{2.69}$$

Next, choose for every $S \in S(n+1)$, a partition of unity

$$\varphi_{\lambda_S^{l_S}} : \tilde{\varepsilon}_S((U_\lambda^n \star \mathcal{G}^x) \cap \mathrm{supp}(\mathsf{c} \circ s)) \longrightarrow \mathbb{R}_+$$

subordinate to the open covering $\left(U^{\#S-1}_{\lambda_S^{l_S}}\right)_{l_S=0}^{m_S}$.

For all $n \geq 1$, we define the map

$$h^n : CR_{ss}^{n+1}(\mathcal{U}_\bullet, E^\bullet) \longrightarrow CR_{ss}^n(\mathcal{U}_\bullet, E^\bullet)$$

by

$$\begin{aligned}(h^n f)_\lambda(g_1,\ldots,g_n) &:= \\ &(-1)^{n+1}\int_{\mathcal{G}^{s(g_n)}} \sum_{l=(l_S)_{S\in S(n+1)}} f_{\lambda^l}(g_1,\ldots,g_n,\gamma) \\ &\cdot \prod_{S\in S(n+1)} \prod_{l_S} \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S(g_1,\ldots,g_n,\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma).\end{aligned} \tag{2.70}$$

Observe that

$$(U_{\bar{\lambda}}^n \star \mathcal{G}^{\rho(x)}) \cap \mathrm{supp}(\mathsf{c} \circ s \circ \rho) \subset \bigcup_{l=(l_S)_{S\in S(n+1)}} U^{n+1}_{\bar{\lambda}^l},$$

where the $\bar{\lambda}^l$'s are defined in the obvious way. Hence, we get a partition of unity of $\tilde{\varepsilon}_S((U_{\bar{\lambda}}^n \star \mathcal{G}^{\rho(x)}) \cap \mathrm{supp}(\mathsf{c} \circ s \circ \rho))$ subordinate to the open covering $\left(U^{\#S-1}_{\bar{\lambda}_S^{l_S}}\right)_{l_S=0}^{m_S}$ by setting $\varphi_{\bar{\lambda}_S^{l_S}}(\tilde{\varepsilon}_S(\rho(g_1),\ldots,\rho(g_n))) := \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S(g_1,\ldots,g_n))$. Next, using (2.65), it is straightforward that

$$(h^n f)_{\bar{\lambda}}(\rho(g_1),\ldots,\rho(g_n)) = \nu_{|U_\lambda^n} \circ (h^n f)_\lambda(g_1,\ldots,g_n),$$

which means that $((h^n f)_\lambda)_{\lambda\in\Lambda_n} \in CR_{ss}^n(\mathcal{U}_\bullet, E^\bullet)$.

Assume now that $(f_\lambda)_{\lambda\in\Lambda_n} \in CR_{ss}^n(\mathcal{U}_\bullet, E^\bullet)$. Then, for every $U_\lambda^n \in {}_\natural\mathcal{U}_n$ and $(g_1,\ldots,g_n) \in U_\lambda^n$, one has

$$(h^n d^n f)_\lambda(g_1,\dots,g_n) \tag{2.71}$$
$$= (-1)^{n+1} \int_{\mathcal{G}^{s(g_n)}} \sum_{(l_S)_{S\in S(n+1)}} (d^n f)_{\lambda^l}(g_1,\dots,g_n,\gamma)$$
$$\cdot \prod_{S\in S(n+1)} \prod_{l_S} \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S^{n+1}(g_1,\dots,g_n,\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma)$$
$$= f_\lambda(g_1,\dots,g_n) - A_\lambda(g_1,\dots,g_n),$$

where

$$A_\lambda(g_1,\dots,g_n)$$
$$:= (-1)^n \sum_{k=0}^{n} (-1)^k \int_{\mathcal{G}^{s(g_n)}} \sum_{(l_S)_{S\in S(n+1)}} f_{\tilde{\varepsilon}_k^{n+1}(\lambda^l)}(\tilde{\varepsilon}_k^{n+1}(g_1,\dots,g_n,\gamma))$$
$$\cdot \prod_{S\in S(n+1)} \prod_{l_S} \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S^{n+1}(g_1,\dots,g_n,\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma).$$

We want to show that

$$A_\lambda(g_1,\dots,g_n) = (d^{n-1}h^{n-1}f)_\lambda(g_1,\dots,g_n). \tag{2.72}$$

One has

$$(d^{n-1}h^{n-1}f)_\lambda(g_1,\dots,g_n) \tag{2.73}$$
$$= (-1)^n \sum_{k=0}^{n-1} \int_{\mathcal{G}^{s(g_n)}} \sum_{r_k:=(r_{k,T})_{T\in S(n)}} f_{\tilde{\varepsilon}_k^n(\lambda)^{r_k}}(\tilde{\varepsilon}_k^n(g_1,\dots,g_n),\gamma)$$
$$\cdot \prod_{T\in S(n)} \prod_{r_{k,T}} \varphi_{\tilde{\varepsilon}_k^n(\lambda)_T^{r_{k,T}}}(\tilde{\varepsilon}_T^n(\tilde{\varepsilon}_k^n(g_1,\dots,g_n),\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma)$$
$$+ \int_{\mathcal{G}^{s(g_{n-1})}} \sum_{r_n:=(r_{n,T})_{T\in S(n)}} f_{\tilde{\varepsilon}_n^n(\lambda)^{r_n}}(g_1,\dots,g_{n-1},\gamma)$$
$$\cdot \prod_{T\in S(n)} \prod_{r_{n,T}} \varphi_{\tilde{\varepsilon}_n^n(\lambda)_T^{r_{n,T}}}(\tilde{\varepsilon}_T^n(g_1,\dots,g_{n-1},\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma)$$
$$= B_\lambda(g_1,\dots,g_n) + C_\lambda(g_1,\dots,g_n).$$

Notice that by the left-invariance of $\{\mu^x\}_{x\in X}$, the second integral C_λ in the right hand side of (2.73) can be written as

$$C_\lambda(g_1,\dots,g_n)$$
$$= \int_{\mathcal{G}^{s(g_n)}} \int_{(r_{n,T})_{T\in S(n)}} f_{\tilde{\varepsilon}_n^n(\lambda)^{r_n}}(g_1,\dots,g_{n-1},g_n\gamma)$$
$$\cdot \prod_{T\in S(n)} \prod_{r_{n,T}} \varphi_{\tilde{\varepsilon}_n^n(\lambda)_T^{r_{n,T}}}(\tilde{\varepsilon}_T^n(g_1,\dots,g_{n-1},g_n\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma)$$

$$= \int_{\mathcal{G}^{s(g_n)}} \sum_{(r_{n,T})_{T\in S(n)}} f_{\tilde{\varepsilon}^n_n(\lambda)^{r_n}}(\tilde{\varepsilon}^{n+1}_n(g_1,\ldots,g_n,\gamma))$$
$$\cdot \prod_{T\in S(n)} \prod_{r_{n,T}} \varphi_{\tilde{\varepsilon}^n_n(\lambda)^{r_{n,T}}_T}(\tilde{\varepsilon}^n_T(\tilde{\varepsilon}^{n+1}_n(g_1,\ldots,g_{n-1},g_n,\gamma)))\cdot c(s(\gamma))d\mu^{s(g_{n-1})}(\gamma).$$

On the other hand, for all $k=0,\ldots,n-1$, one has

$$(\tilde{\varepsilon}^n_k(g_1,\ldots,g_n),\gamma) = \tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma);$$

hence

$$B_\lambda(g_1,\ldots,g_n)$$
$$= (-1)^n \sum_{k=0}^{n-1}(-1)^k \int_{\mathcal{G}^{s(g_n)}} \sum_{(r_{k,T})_{T\in S(n)}} f_{\tilde{\varepsilon}^n_k(\lambda)^{r_k}}(\tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma))$$
$$\cdot \prod_{T\in S(n)} \prod_{r_{k,T}} \varphi_{\tilde{\varepsilon}^n_k(\lambda)^{r_{k,T}}_T}(\tilde{\varepsilon}^n_T(\tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma)))\cdot c(s(\gamma))d\mu^{s(g_{n-1})}(\gamma).$$

Thus, (2.73) becomes

$$(d^{n-1}h^{n-1}f)_\lambda(g_1,\ldots,g_n) \tag{2.74}$$
$$= (-1)^n \sum_{k=0}^{n}(-1)^k \int_{\mathcal{G}^{s(g_n)}} \sum_{(r_{k,T})_{T\in S(n)}} f_{\tilde{\varepsilon}^n_k(\lambda)^{r_k}}(\tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma))$$
$$\cdot \prod_{T\in S(n)} \prod_{r_{k,T}} \varphi_{\tilde{\varepsilon}^n_k(\lambda)^{r_{k,T}}_T}(\tilde{\varepsilon}^n_T(\tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma)))\cdot c(s(\gamma))d\mu^{s(g_{n-1})}(\gamma).$$

Now, for any $k=0,\ldots,n$, $r_k=(r_{k,T})_{T\in S(n)}$, let $\gamma\in\mathcal{G}^{s(g_n)}$ such that

$$\tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma)\in U^n_{\tilde{\varepsilon}^n_k(\lambda)^{r_k}}.$$

Then, there exists $l=(l_S)_{S\in S(n+1)}$ such that $(g_1,\ldots,g_n,\gamma)\in U^{n+1}_{\lambda^l}$, so that

$$\tilde{\varepsilon}^{n+1}_k(g_1,\ldots,g_n,\gamma)\in U^n_{\tilde{\varepsilon}^n_k(\lambda)^{r_k}} \bigcup U^n_{\tilde{\varepsilon}^{n+1}_k(\lambda^l)}.$$

One can then suppose that for any $k\in[n]$ and any family $r_k=(r_{k,T})_{T\in S(n)}$, there exists a family $l=(l_S)_{S\in S(n+1)}$ such that $\tilde{\varepsilon}^n_k(\lambda)^{r_k}=\tilde{\varepsilon}^{n+1}_k(\lambda^l)$. Moreover, in virtue to the identities (2.2), it is straightforward that for each $k\in[n]$ and any $T\in S(n)$, there exists a unique $S\in S(n+1)$ such that $\varepsilon^{n+1}_S=\varepsilon^{n+1}_k\circ\varepsilon^n_T$, so that $\tilde{\varepsilon}^{n+1}_S=\tilde{\varepsilon}^n_T\circ\tilde{\varepsilon}^{n+1}_k$. Therefore, we obtain from (2.74) that

$$
\begin{aligned}
&(d^{n-1}h^{n-1}f)_\lambda(g_1,\dots,g_n)\\
&= (-1)^n\sum_{k=0}^{n}(-1)^k\int_{\mathcal{G}^{s(g_n)}}\sum_{(l_S)_{S\in S(n+1)}} f_{\tilde{\varepsilon}^{n+1}(\lambda^l)}(\tilde{\varepsilon}_k^{n+1}(g_1,\dots,g_n,\gamma))\\
&\quad\cdot\prod_{S\in S(n+1)}\prod_{l_S}\varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S^{n+1}(g_1,\dots,g_n,\gamma)).c(s(\gamma))d\mu^{s(g_n)}(\gamma)\\
&= A_\lambda(g_1,\dots,g_n).
\end{aligned}
$$

Combining with (2.71), we thus have shown that

$$
h^n\circ d^n + d^{n-1}\circ h^{n-1} = \mathrm{Id}_{CR^n_{ss}(\mathcal{U}_\bullet,E^\bullet)},\ \forall n\geq 1; \tag{2.75}
$$

i.e., $h^\star$ defines a contraction of $CR^\star_{ss}(\mathcal{U}_\bullet,E^\bullet)$ for any Real open cover $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet,\rho_\bullet)$ and this ends our proof. □

Remark 2.71. It is straightforward, using the same arguments, that Theorem 2.68 remains true for a Real representation of type $\mathbb{R}^{p,q}$ (see Remark 2.65).

Corollary 2.72. *Let $\mathcal{G}$ be a proper groupoid. Let $E\longrightarrow X$ be a representation of $\mathcal{G}$ in the sense of Crainic [4]; that is, a real $\mathcal{G}$-equivariant vector bundle of rank p. Then $\check{H}^n(\mathcal{G}_\bullet,E^\bullet)=0,\forall n\geq 1$.*

Proof. Let $\mathcal{G}$ be endowed with the trivial Real structure. Form the Real representation (F,ν) of type $\mathbb{R}^{p,p}$ of $(\mathcal{G},\mathrm{Id})$ by $F:=E\oplus E$ endowed with the diagonal $\mathcal{G}$-action and the Real structure $\nu(e_1,e_2):=(e_1,-e_2)$. Then by Theorem 2.68, we have $\check{H}R^n(\mathcal{G}_\bullet,F^\bullet)=0$ for all $n\geq 1$. But since the Real structure is trivial, we have $\check{H}R^n(\mathcal{G}_\bullet,F^\bullet)=\check{H}^n(\mathcal{G}_\bullet,{}^rF^\bullet)$, thanks to the discussion following Proposition 2.44. Moreover, we obviously have ${}^rF^\bullet=E^\bullet$. □

Acknowledgements. This work was done during my Ph.D. at the universities of Metz and Paderborn, while I was suported by the German Research Foundation (DFG) via the IRTG 1133 "*Geometry and Analysis of Symmetries*". I am greatly indebted to my supervisor Jean-Louis Tu, and I am grateful to Camille Laurent for many helpful comments and suggestions on an earlier version of this manuscript.

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This paper is available via http://nyjm.albany.edu/j/2013/19-37.html.