Isomorphisms of lattices of Bures-closed bimodules over Cartan MASAs

Adam H. Fuller and David R. Pitts

ABSTRACT. For $i = 1, 2$, let $(\mathcal{M}_i, \mathcal{D}_i)$ be pairs consisting of a Cartan MASA $\mathcal{D}_i$ in a von Neumann algebra $\mathcal{M}_i$, let $\mathrm{atom}(\mathcal{D}_i)$ be the set of atoms of $\mathcal{D}_i$, and let $\mathfrak{S}_i$ be the lattice of Bures-closed $\mathcal{D}_i$ bimodules in $\mathcal{M}_i$. We show that when $\mathcal{M}_i$ have separable preduals, there is a lattice isomorphism between $\mathfrak{S}_1$ and $\mathfrak{S}_2$ if and only if the sets

$$\{(Q_1, Q_2) \in \mathrm{atom}(\mathcal{D}_i) \times \mathrm{atom}(\mathcal{D}_i) : Q_1 \mathcal{M}_i Q_2 \neq (0)\}$$

have the same cardinality. In particular, when $\mathcal{D}_i$ is nonatomic, $\mathfrak{S}_i$ is isomorphic to the lattice of projections in $L^\infty([0,1], m)$ where m is Lebesgue measure, regardless of the isomorphism classes of $\mathcal{M}_1$ and $\mathcal{M}_2$.

CONTENTS

1. Introduction

Let $\mathcal{M}$ be a von Neumann algebra containing a Cartan MASA $\mathcal{D}$; we call $(\mathcal{M}, \mathcal{D})$ a Cartan pair. Feldman and Moore [6, 7] gave a construction of Cartan pairs with separable preduals based on Borel measurable equivalence relations and showed that all (separably acting) Cartan pairs arise from their construction. Building on the work of Feldman and Moore [6, 7], and Arveson [2], Muhly, Solel and Saito [11] introduced the Spectral Theorem for Bimodules. They claimed that if $\mathcal{S}$ is a σ-weakly closed $\mathcal{D}$-bimodule of $\mathcal{M}$, then there is a Borel subset $\mathcal{B}$ of the Feldman-Moore relation R such that $\mathcal{S}$ consists of all those operators in $\mathcal{M}$ whose "matrices" are supported in $\mathcal{B}$. That is, the σ-weakly closed $\mathcal{D}$-bimodule $\mathcal{S}$ is determined precisely by its support $\mathcal{B}$.

Unfortunately, there is a gap in the proof of the Spectral Theorem for Bimodules; consult the "added in proof" portion of Aoi's paper [1] for details. While we are not aware of any complete proof of the Spectral Theorem for Bimodules, Fulman [8] has established it when $\mathcal{M}$ is hyperfinite and $\mathcal{M}_*$ is separable.

In a recent paper, Cameron, the second author and Zarikian [4], introduced a new perspective to the study of $\mathcal{D}$-bimodules in a Cartan pair $(\mathcal{M}, \mathcal{D})$. The approach in [4] is operator theoretic and avoids the measure theoretic tools of Feldman and Moore. In [4], a version of the Spectral Theorem for Bimodules is proved, not for σ-weakly closed bimodules but for *Bures*-closed $\mathcal{D}$-bimodules. In fact, it is shown that the Spectral Theorem for Bimodules as introduced in [11] is true if and only if every σ-weakly closed $\mathcal{D}$-bimodule is itself Bures-closed.

In this paper, we continue the study of the Bures-closed $\mathcal{D}$-bimodules in a Cartan pair $(\mathcal{M}, \mathcal{D})$. Our main result, Theorem 4.3, shows that the lattice of Bures-closed bimodules for a separably acting Cartan pair $(\mathcal{M}, \mathcal{D})$ depends upon: i) whether $\mathcal{D}$ contains a diffuse part, and ii) the cardinality of the restriction of the Murray–von Neumann equivalence relation for projections of $\mathcal{M}$ to the atoms of $\mathcal{D}$. In this sense, the lattice of Bures-closed bimodules depends surprisingly little on the Cartan pair $(\mathcal{M}, \mathcal{D})$, when $\mathcal{M}$ is separably acting. In particular, if $(\mathcal{M}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{D}_2)$ are any two Cartan pairs in which $\mathcal{D}_1$ and $\mathcal{D}_2$ are separably acting diffuse algebras, then they share the same lattice structure of Bures-closed $\mathcal{D}$-bimodules.

Along the way, in Section 3, we give a fuller description of the supports of partial isometry normalizers of $\mathcal{D}$. In particular, we describe a pre-order on $\mathcal{GN}(\mathcal{M}, \mathcal{D})$, the set consisting of all partial isometry normalizers of $\mathcal{D}$, which is induced by their supports.

2. Background and preliminaries

Let $\mathcal{M}$ be a von Neumann algebra. A MASA (maximal abelian self-adjoint subalgebra) $\mathcal{D}$ in $\mathcal{M}$ is *Cartan* if:

  (a) There is a faithful, normal conditional expectation $E : \mathcal{M} \to \mathcal{D}$.
  (b) span$\{U \in \mathcal{M} : U$ is unitary and $U\mathcal{D}U^* = \mathcal{D}\}$ is σ-weakly dense in $\mathcal{M}$.

If $\mathcal{D}$ is Cartan in $\mathcal{M}$ we call $(\mathcal{M}, \mathcal{D})$ a *Cartan pair*. The set of *normalizers* for $\mathcal{D}$ is the set

$$\mathcal{N}(\mathcal{M}, \mathcal{D}) = \{v \in \mathcal{M} : v^*\mathcal{D}v \cup v\mathcal{D}v^* \subseteq \mathcal{D}\}.$$

The *groupoid normalizers*, denoted $\mathcal{GN}(\mathcal{M}, \mathcal{D})$, are the elements of $\mathcal{N}(\mathcal{M}, \mathcal{D})$ which are partial isometries. Clearly, $\mathcal{N}(\mathcal{M}, \mathcal{D})$ and $\mathcal{GN}(\mathcal{M}, \mathcal{D})$ are σ-weakly dense in $\mathcal{M}$ when $(\mathcal{M}, \mathcal{D})$ is a Cartan pair.

Notation 2.1. For any abelian von Neumann algebra $\mathcal{W}$, atom$(\mathcal{W})$ will denote the set of atoms in $\mathcal{W}$. Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair, and let R_a be the

restriction of the Murray–von Neumann equivalence relation on projections of $\mathcal{M}$ to $\mathrm{atom}(\mathcal{D})$. For $A_1, A_2 \in \mathrm{atom}(\mathcal{D})$ write $A_1 \sim A_2$ when $(A_1, A_2) \in R_a$.

Notice that if $v \in \mathcal{M}$ is a partial isometry such that $v^*v, vv^* \in \mathrm{atom}(\mathcal{D})$, then $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Indeed, for $d \in \mathcal{D}$, $dv^*v \in \mathbb{C}v^*v$, so $vdv^* = vdv^*vv^* \in \mathbb{C}vv^* \subseteq \mathcal{D}$. Likewise, $v^*dv \in \mathcal{D}$.

Proposition 2.2. *Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair, and set*

$$X := \sum_{Q \in \mathrm{atom}(\mathcal{D})} Q.$$

Then X is a central projection of $\mathcal{M}$.

Proof. Let $U \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ be a unitary normalizer. For each $Q \in \mathrm{atom}(\mathcal{D})$, $UQU^* \in \mathrm{atom}(\mathcal{D})$, and the map $Q \mapsto UQU^*$ is a permutation of $\mathrm{atom}(\mathcal{D})$. Hence $UXU^* = X$, and hence X commutes with U. As $\mathcal{M}$ is generated by its unitary normalizers, X is in the center of $\mathcal{M}$. $\qquad\square$

It follows that any Cartan pair decomposes as a direct sum of two Cartan pairs, $(\mathcal{M}, \mathcal{D}) = (\mathcal{M}_d, \mathcal{D}_d) \oplus (\mathcal{M}_a, \mathcal{D}_a)$, where $(\mathcal{M}_d, \mathcal{D}_d) = (\mathcal{M}X^\perp, \mathcal{D}X^\perp)$ and $(\mathcal{M}_a, \mathcal{D}_a) = (\mathcal{M}X, \mathcal{D}X)$. Clearly, $\mathrm{atom}(\mathcal{D}_c) = \emptyset$ and $\mathcal{D}_a$ is generated by its atoms. We shall call $(\mathcal{M}_d, \mathcal{D}_d)$ and $(\mathcal{M}_a, \mathcal{D}_a)$ the *diffuse* and *atomic* parts of $(\mathcal{M}, \mathcal{D})$ respectively.[1]

Henceforth, let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair with conditional expectation E. Fix a faithful normal semi-finite weight ϕ on $\mathcal{M}$ such that $\phi \circ E = \phi$. We shall freely use notation from [13] (see pages 41–42 for discussion of $\mathfrak{n}_\phi := \{x \in \mathcal{M} : \phi(x^*x) < \infty\}$, Definition VII.1.5 for a discussion of the semi-cyclic representation $(\pi_\phi, \mathfrak{H}_\phi, \eta_\phi)$, etc.). The following follows from the fact that $\phi \circ E = \phi$; details of the proof are left for the reader.

Lemma 2.3. *With this notation, $\mathfrak{n}_\phi$ and $\mathfrak{n}_\phi^*$ are $\mathcal{D}$-bimodules, and for $d \in \mathcal{D}$, $x \in \mathfrak{n}_\phi$, and $y \in \mathfrak{n}_\phi^*$, we have*

$$\max\{\phi((dx)^*(dx)), \phi((xd)^*(xd))\} \le \|d\|^2 \, \phi(x^*x),$$

$$\max\{\phi((dy^*)^*(dy^*)), \phi((y^*d)^*(y^*d))\} \le \|d\|^2 \, \phi(yy^*).$$

Definition 2.4. Modifying [13, Definition IX.1.13] very slightly, we will say that a quadruple $\{\pi, \mathfrak{H}, J, \mathfrak{P}\}$ is a *standard form for $\mathcal{M}$* if π is a faithful normal representation of $\mathcal{M}$ on $\mathfrak{H}_\pi$ and $\{\pi(\mathcal{M}), \mathfrak{H}, J, \mathfrak{P}\}$ is a standard form for $\pi(\mathcal{M})$ as in [13, Definition IX.1.13]. Due to the uniqueness of the standard form (see [13, Theorem IX.1.14]), we may, and sometimes will, assume without loss of generality that $\{\pi(\mathcal{M}), \mathfrak{H}, J, \mathfrak{P}\} = \{\pi_\phi(\mathcal{M}), \mathfrak{H}_\phi, J_\phi, \mathfrak{P}_\phi\}$, where ϕ is a faithful, semi-finite, normal weight on $\mathcal{M}$ such that $\phi \circ E = \phi$.

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[1]In a previous version of this paper, we used different terminology for the diffuse part of $(\mathcal{M}, \mathcal{D})$. We thank the referee for alerting us that our previous terminology conflicted with terminology found in [5, Part 1, Chapter 8].

When $(\mathcal{M}, \mathcal{D})$ is a Cartan pair and $\{\pi, \mathfrak{H}, J, \mathfrak{P}\}$ is a standard form for $\mathcal{M}$, define representations π_ℓ and π_r of $\mathcal{D}$ on $\mathfrak{H}_\pi$ by

$$(2.1) \qquad\qquad \pi_\ell(d) = \pi(d) \quad \text{and} \quad \pi_r(d) = J\pi(d^*)J,$$

and set

$$(2.2) \qquad\qquad \mathcal{Z} = (\pi_\ell(\mathcal{D}) \cup \pi_r(\mathcal{D}))''.$$

The purpose of the following is to observe that $\mathcal{Z}$ is uniquely determined. The proof is an immediate consequence of [4, Theorem 1.4.7] and [13, Theorem IX.1.14].

Proposition 2.5. *Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. For $i = 1, 2$, suppose $\{\pi_i, \mathfrak{H}_i, J_i, \mathfrak{P}_i\}$ are standard forms for $\mathcal{M}$, and let π_{ri} and $\mathcal{Z}_i$ be as in (2.1) and (2.2). Then there exists a unique unitary operator $U \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ such that:*

(a) $\pi_2(x) = U\pi_1(x)U^*$, *for all* $x \in \mathcal{M}$.
(b) $J_2 = UJ_1U^*$.
(c) $\mathfrak{P}_2 = U\mathfrak{P}_1$.
(d) $\pi_{r2}(d) = U\pi_{r1}(d)U^*$, *$d \in \mathcal{D}$.*
(e) $\mathcal{Z}_i$ *is a MASA in* $\mathcal{B}(\mathfrak{H}_i)$.
(f) $\mathcal{Z}_1 \ni z \mapsto UzU^*$ *is an isomorphism of* $\mathcal{Z}_1$ *onto* $\mathcal{Z}_2$.

2.6. Bimodules. The *Bures topology*, see [3], on $\mathcal{M}$ is the locally convex topology generated by the family of seminorms

$$\{T \mapsto \sqrt{\tau(E(T^*T))} : \tau \in (\mathcal{D}_*)^+\}.$$

In this note we are primarily interested in the Bures-closed $\mathcal{D}$-bimodules in $\mathcal{M}$. When the Cartan MASA is understood, we will sometimes simply say "bimodule" in place of "$\mathcal{D}$-bimodule." Any Bures-closed $\mathcal{D}$-bimodule is necessarily σ-weakly closed.

It is shown in [4, Theorem 2.5.1] that if $\mathcal{S} \subseteq \mathcal{M}$ is a nonzero Bures-closed $\mathcal{D}$-bimodule, then $\mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$ generates $\mathcal{S}$ as a Bures-closed bimodule. We will make frequent use of this fact.

Given a σ-weakly closed $\mathcal{D}$-bimodule $\mathcal{S}$ in $\mathcal{M}$, the *support of $\mathcal{S}$*, denoted by $\text{supp}(S)$, is the orthogonal projection onto the $\mathcal{Z}$-invariant subspace, $\overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$; as $\mathcal{Z}$ is a MASA in $\mathcal{B}(\mathfrak{H})$, $\text{supp}(S)$ is a projection in $\mathcal{Z}$. For an operator $T \in \mathcal{M}$, define $\text{supp}(T)$ to be the support of the Bures-closed bimodule generated by T. The definition of the support of a bimodule given here is as introduced in [4]. The original concept was introduced in [11].

For a partial isometry $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ we denote $\text{supp}(w)$ by P_w. Picking and choosing results from [4] we have the following alternative descriptions of P_w.

Lemma 2.7 ([4, Lemma 1.4.6 and Lemma 2.1.3]). *Given any operator $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ the following hold.*

(a) *Let Λ be an invariant mean on the (discrete) group of unitaries in $\mathcal{D}$, $\mathcal{U}(\mathcal{D})$ (which we may assume to satisfy*

$$\bigwedge_{U \in \mathcal{U}(\mathcal{D})} f(U) = \bigwedge_{U \in \mathcal{U}(\mathcal{D})} f(U^*)$$

for every $f \in \ell^\infty(\mathcal{U}(\mathcal{D}))$. Then

$$P_w = \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \pi_\ell(wUw^*)\pi_r(U^*).$$

(b) *P_w is the orthogonal projection onto $\overline{\{\eta_\phi(wd) : \ d \in \mathfrak{n}_\phi \cap \mathcal{D}\}}$, and for $x \in \mathfrak{n}_\phi$,*

$$P_w\eta_\phi(x) = \eta_\phi(wE(w^*x)).$$

We may view $\mathcal{S} \mapsto \mathrm{supp}(\mathcal{S})$ as a map from the set of $\mathcal{D}$-bimodules of $\mathcal{M}$ into the projection lattice of $\mathcal{Z}$. Conversely, given a projection Q in $\mathcal{Z}$, define a $\mathcal{D}$-bimodule, $\mathrm{bimod}(Q)$, by

$$\mathrm{bimod}(Q) = \{T \in \mathcal{M} : \ \mathrm{supp}(T) \le Q\}.$$

It follows from [4, Lemma 2.1.4(c)] that $\mathrm{bimod}(Q)$ is Bures-closed. The operations bimod and supp satisfy the following "reflexivity-type" condition.

Theorem 2.8 ([4, Theorem 2.5.1]). *A $\mathcal{D}$-bimodule $\mathcal{S} \subseteq \mathcal{M}$ is Bures-closed if and only if*

$$\mathcal{S} = \mathrm{bimod}(\mathrm{supp}(\mathcal{S})).$$

3. Projections and relations

Throughout this section, let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair with conditional expectation E. Let $\{\pi, \mathfrak{H}, J, \mathfrak{P}\}$ be a standard form of $\mathcal{M}$, and construct the maximal abelian algebra $\mathcal{Z}$ in $\mathcal{B}(\mathfrak{H})$ as discussed in Section 2. We do not impose any condition of separable predual in this section.

Our aim in this section is to better describe how the projections in $\mathcal{D}$ relate to each other, in terms of the normalizers in $\mathcal{N}(\mathcal{M}, \mathcal{D})$. This in turn will provide us with a better description of some of the projections in $\mathcal{Z}$. In Proposition 3.3, we will describe exactly when $\pi_\ell(Q_1)\pi_r(Q_2) = 0$ for projections $Q_1, Q_2 \in \mathcal{D}$. This will be determined by the existence of certain normalizers in $\mathcal{N}(\mathcal{M}, \mathcal{D})$. In the case of atomic projections in $\mathcal{Z}$ we will be able to go further. In Proposition 3.7 we will show that the atomic projections of $\mathcal{Z}$ are completely determined by the atomic projections in $\mathcal{D}$. This is a key tool in proving our main result.

Lemma 3.1. *Let Q_1, Q_2 be projections in $\mathcal{D}$. If $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ then $ww^* \le Q_1$ and $w^*w \le Q_2$ if and only if $w \in \mathrm{bimod}(\pi_\ell(Q_1)\pi_r(Q_2))$.*

Proof. Suppose $ww^* \le Q_1$ and $w^*w \le Q_2$. By [4, Lemma 2.1.4] it suffices to show that $\pi_\phi(w)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \mathrm{range}(\pi_\ell(Q_1)\pi_r(Q_2))$. Take any $d \in \mathfrak{n}_\phi \cap \mathcal{D}$.

Then

$$\pi_\ell(Q_1)\pi_r(Q_2)\pi_\phi(w)\eta_\phi(d) = \eta_\phi(Q_1 w d Q_2)$$
$$= \eta_\phi(Q_1 w Q_2 d)$$
$$= \eta_\phi(w d)$$
$$= \pi_\phi(w)\eta_\phi(d).$$

Hence we have $\pi_\phi(w)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \operatorname{range}(\pi_\ell(Q_1)\pi_r(Q_2))$ and thus $w \in \operatorname{bimod}(\pi_\ell(Q_1)\pi_r(Q_2))$.

Conversely, suppose $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \operatorname{bimod}(\pi_\ell(Q_1)\pi_r(Q_2))$. Let

$$v = w - Q_1 w Q_2.$$

We will be done once we show that $v = 0$. Note that, again by [4, Lemma 2.1.4], $\pi_\phi(w)\eta_\phi(d) \in \operatorname{range}(\pi_\ell(Q_1)\pi_r(Q_2))$. Thus for $d \in \mathfrak{n}_\phi \cap \mathcal{D}$ we have

$$\pi_\phi(v)\eta_\phi(d) = \pi_\phi(w)\eta_\phi(d) - \pi_\phi(Q_1 w Q_2)\eta_\phi(d)$$
$$= \pi_\phi(w)\eta_\phi(d) - \eta_\phi(Q_1 w d Q_2)$$
$$= \pi_\phi(w)\eta_\phi(d) - \pi_\ell(Q_1)\pi_r(Q_2)\pi_\phi(w)\eta_\phi(d)$$
$$= 0$$

Hence $\eta_\phi(vd) = 0$ for all $d \in \mathfrak{n}_\phi \cap \mathcal{D}$. By the faithfulness of ϕ it follows that $vd = 0$ for all $d \in \mathfrak{n}_\phi \cap \mathcal{D}$. Since $\mathfrak{n}_\phi \cap \mathcal{D}$ is weak-$*$ dense in $\mathcal{D}$, it follows that $vd = 0$ for every $d \in \mathcal{D}$. Hence $v = 0$. $\square$

We will give a more complete description of the relationship between the projections $\{P_v : v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and the partial isometries in $\mathcal{GN}(\mathcal{M}, \mathcal{D})$ in Lemma 3.6. For the time being, Lemma 3.1 gives the following statement.

Corollary 3.2. *If $u, v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and $P_u = P_v$, then $vv^* = uu^*$ and $v^*v = u^*u$.*

Proof. This follows immediately from Lemma 3.1: if $P_u \leq P_v$, then, since $P_v \leq \pi_\ell(vv^*)\pi_r(v^*v)$, $uu^* \leq vv^*$ and $u^*u \leq v^*v$. $\square$

Proposition 3.3. *For any two projections Q_1 and Q_2 in $\mathcal{D}$, the following are equivalent:*

(a) *$\pi_\ell(Q_1)\pi_r(Q_2) \neq 0$.*
(b) *there is a nonzero $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $vv^* \leq Q_1$ and $v^*v \leq Q_2$.*
(c) *$Q_1\mathcal{M}Q_2 \neq \{0\}$.*
(d) *there exists a σ-weakly closed $\mathcal{D}$-bimodule $\mathcal{S} \subseteq \mathcal{M}$ such that*

$$Q_1\mathcal{S}Q_2 \neq \{0\}.$$

Proof. Since $\pi_\ell(Q_1)\pi_r(Q_2) \neq 0$ implies that $\operatorname{bimod}(\pi_\ell(Q_1)\pi_r(Q_2)) \neq \{0\}$, the equivalence of (a) and (b) follows immediately from Lemma 3.1.

Suppose (b) holds. Given $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $vv^* \leq Q_1$ and $v^*v \leq Q_2$, we have $Q_1 v Q_2 = v$ and so $Q_1\mathcal{M}Q_2 \neq \{0\}$. This gives (c).

That (c) implies (d) is obvious. Finally suppose (d) holds. Clearly, $Q_1 \mathcal{S} Q_2$ is a σ-weakly closed $\mathcal{D}$-bimodule. By [4, Proposition 1.3.4], there exists $0 \neq v \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap Q_1 \mathcal{S} Q_2$. Then $v = Q_1 v Q_2$, $vv^* \leq Q_1$ and $v^*v \leq Q_2$. Hence (b) holds, and we are done. $\square$

The support projections $\{P_w \in \mathcal{Z} : w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})\}$ have a natural partial ordering on them, induced from the partial ordering of the projections in $\mathcal{Z}$. This ordering imposes a pre-order on $\mathcal{GN}(\mathcal{M}, \mathcal{D})$, which we now describe.

Definition 3.4. For $u, v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ we write $u \leq_{\mathcal{D}} v$ if there is a d in $\mathcal{D}$ such that $u = vd$.

It is not hard to see that $\leq_{\mathcal{D}}$ is a pre-order on $\mathcal{GN}(\mathcal{M}, \mathcal{D})$. For any scalar λ with $|\lambda| = 1$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ we have $v \leq_{\mathcal{D}} \lambda v$ and $\lambda v \leq_{\mathcal{D}} v$ so $\leq_{\mathcal{D}}$ is indeed a pre-order and not a partial order. Recall that partial isometries come already equipped with a partial ordering $\leq$ (see, for example, [9]). In this order $u \leq v$ if and only if there is a projection P such that $u = vP$. In fact, P can be chosen to be u^*u. It follows that if $u, v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and $u \leq v$ then $u \leq_{\mathcal{D}} v$. Hence the ordering $\leq_{\mathcal{D}}$ is coarser than the usual ordering $\leq$ on partial isometries.

Lemma 3.5. *Let $u, v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. If $u \leq_{\mathcal{D}} v$, then $v^*u \in \mathcal{D}$ and $u = v(v^*u)$.*

Proof. As $u \leq_{\mathcal{D}} v$ there is a $d \in \mathcal{D}$ such that $u = vd$. Then $v^*u = v^*vd$ is in $\mathcal{D}$. Let

$$a = vd - vv^*u.$$

We will show $a = 0$. We have,

$$\begin{aligned}
a^*a &= (vd - vv^*u)^*(vd - vv^*u) \\
&= d^*v^*vd - d^*v^*u - u^*vd + u^*vv^*u \\
&= d^*v^*vd - d^*v^*vd - d^*v^*vd + d^*v^*vv^*vd \qquad \text{(since } u = vd) \\
&= 0. \hspace{8cm} \square
\end{aligned}$$

Now we relate the pre-ordering to supports of elements of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$.

Lemma 3.6. *Let $u, v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. The following are equivalent:*

(a) $u \leq_{\mathcal{D}} v$.
(b) $u = vE(v^*u)$.
(c) $P_u \leq P_v$.

Proof. The previous lemma shows that (a) implies (b).
Suppose next that $u = vE(v^*u)$. For any $x \in \mathfrak{n}_\phi$ we see that

$$\begin{aligned}
P_v P_u \eta_\phi(x) &= P_v \eta_\phi(uE(u^*x)) \\
&= \eta_\phi(vE(v^*u)E(u^*x)) \\
&= \eta_\phi(uE(u^*x)) = P_u \eta_\phi(x).
\end{aligned}$$

Thus $P_u \leq P_v$. So (b) implies (c).

Now assume that $P_u \leq P_v$ for some $u, v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. We aim to show that $u = vE(v^*u)$. As $P_u \leq P_v$ a similar calculation to above shows that, for all $x \in \mathfrak{n}_\phi$

$$\eta_\phi(vE(v^*u)E(u^*x)) = \eta_\phi(uE(u^*x)).$$

As ϕ is faithful, it follows that for all $x \in \mathfrak{n}_\phi$ we have

$$vE(v^*u)E(u^*x) = uE(u^*x).$$

As $\mathfrak{n}_\phi$ is weak-$*$ dense in $\mathcal{M}$ and E is normal, the above equation holds for all $x \in \mathcal{M}$. In particular it holds when $x = u$, and hence $u = vE(v^*u) \in v\mathcal{D}$. Therefore $u \leq_\mathcal{D} v$. $\qquad\square$

We now classify the atomic projections of $\mathcal{Z}$ in terms of the atomic projections in $\mathcal{D}$.

Proposition 3.7. *Let* $(\mathcal{M}, \mathcal{D})$ *be a Cartan pair and let* $\{\pi, \mathfrak{H}, J, \mathfrak{P}\}$ *be a standard form for* $\mathcal{M}$. *The following statements hold.*

(a) *If* $A \in \mathrm{atom}(\mathcal{Z})$, *then there exist unique* $Q_1, Q_2 \in \mathrm{atom}(\mathcal{D})$ *such that* $A = \pi_\ell(Q_1)\pi_r(Q_2)$. *In addition,* $Q_1 \sim Q_2$.

(b) *For* $i = 1, 2$, *suppose* $Q_i \in \mathrm{atom}(\mathcal{D})$ *and* $Q_1 \sim Q_2$. *Then*

$$\pi_\ell(Q_1)\pi_r(Q_2) \in \mathrm{atom}(\mathcal{Z}).$$

(c) *The algebras* $\mathcal{Z}$ *and* $\mathcal{D}$ *satisfy,*

$$\sum_{Q \in \mathrm{atom}(\mathcal{D})} Q < I_\mathcal{D} \quad \textit{if and only if} \quad \sum_{A \in \mathrm{atom}(\mathcal{Z})} A < I_\mathcal{Z}.$$

Proof. (a) Take any nonzero $A \in \mathrm{atom}(\mathcal{Z})$. As $A \neq 0$ there is a nonzero $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathrm{bimod}(A)$. Hence P_v is a nonzero projection satisfying $P_v \leq A$. As A is atomic it follows that $P_v = A$. Let $Q_1 = vv^*$ and $Q_2 = v^*v$. Obviously, $Q_1 \sim Q_2$. We aim to show that Q_1 and Q_2 are atoms of $\mathcal{D}$.

Suppose that $P \leq Q_1$ is a nonzero projection in $\mathcal{D}$. Let $u = Pv$. Then $uu^* = P$ and $u^*u \leq Q_2$. We also have that $u \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and $u \leq v$. Hence $u \leq_\mathcal{D} v$. By Lemma 3.6, we have that $P_u \leq P_v$. As u is nonzero and P_v is atomic it follows that $P_u = P_v$. By Corollary 3.2, $P = Q_1$, and thus $Q_1 \in \mathrm{atom}(\mathcal{D})$. A similar argument shows $Q_2 \in \mathrm{atom}(\mathcal{D})$.

Recall that a projection B in an abelian von Neumann algebra $\mathcal{W}$ belongs to $\mathrm{atom}(\mathcal{W})$ if and only if $\mathcal{W}B$ is one-dimensional. Thus, for any $h, k \in \mathcal{D}$, we have

$$\pi_\ell(h)\pi_r(k)\pi_\ell(Q_1)\pi_r(Q_2) \in \mathbb{C}\pi_\ell(Q_1)\pi_r(Q_2).$$

Since $\mathcal{Z} = \overline{\mathrm{span}}^{\mathrm{weak}\text{-}*}\{\pi_\ell(h)\pi_r(k) : h, k \in \mathcal{D}\}$, it follows that $\pi_\ell(Q_1)\pi_r(Q_2)\mathcal{Z}$ is one-dimensional and hence $\pi_\ell(Q_1)\pi_r(Q_2) \in \mathrm{atom}(\mathcal{Z})$. As

$$A = P_v \leq \pi_\ell(Q_1)\pi_r(Q_2),$$

it follows that $A = \pi_\ell(Q_1)\pi_r(Q_2)$. Uniqueness of Q_1 and Q_2 follows from Corollary 3.2.

(b) Suppose that for $i = 1, 2$, $Q_i \in \text{atom}(\mathcal{D})$ and $Q_1 \sim Q_2$. Let $v \in \mathcal{M}$ be a partial isometry so that $vv^* = Q_1$ and $v^*v = Q_2$. As observed earlier, $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Hence, $\pi_\ell(Q_1)\pi_r(Q_2) \neq 0$ by Proposition 3.3. An argument similar to that used in part (a) shows $\pi_\ell(Q_1)\pi_r(Q_2) \in \text{atom}(\mathcal{Z})$.

(c) Parts (a) and (b) show that $\text{atom}(\mathcal{D})$ and $\text{atom}(\mathcal{Z})$ are both empty or both nonempty. If both are empty, part (c) holds trivially.

Assume then that $\text{atom}(\mathcal{Z})$ and $\text{atom}(\mathcal{D})$ are both nonempty. Let

$$X := \sum_{Q \in \text{atom}(\mathcal{D})} Q \quad \text{and} \quad Y := \sum_{A \in \text{atom}(\mathcal{Z})} A.$$

Suppose $X < I_\mathcal{D}$. Then $\pi_\ell(I_\mathcal{D} - X) \neq 0$, and if $A \in \text{atom}(\mathcal{Z})$,

$$A\pi_\ell(I_\mathcal{D} - X) = 0$$

by part (a). So $\pi_\ell(I_\mathcal{D} - X) < I_\mathcal{Z} - Y$; hence $Y < I_\mathcal{Z}$.

Conversely, suppose $Y < I_\mathcal{Z}$. Then $0 \neq \text{bimod}(I_\mathcal{Z} - Y)$, so there exists $0 \neq v \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \text{bimod}(I_\mathcal{Z} - Y)$. Hence $P_v \leq (I_\mathcal{Z} - Y)$.

Suppose there is a $Q \in \text{atom}(\mathcal{D})$ such that $vQ \neq 0$. Let $w = vQ \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Clearly $w \leq_\mathcal{D} v$, and so by Lemma 3.6 we have $P_w \leq P_v$. Note that $w^*w = Q$ is in $\text{atom}(\mathcal{D})$ and hence ww^* is in $\text{atom}(\mathcal{D})$. By part (b) and Proposition 3.3 we have that $\pi_\ell(ww^*)\pi_r(w^*w)$ is a nonzero projection in $\text{atom}(\mathcal{Z})$. However, as $P_w \leq \pi_\ell(ww^*)\pi_r(w^*w)$ and $\pi_\ell(ww^*)\pi_r(w^*w)$ is atomic, it follows that $P_w = \pi_\ell(ww^*)\pi_r(w^*w)$. Hence we have

$$\pi_\ell(ww^*)\pi_r(w^*w) = P_w \leq P_v \leq (I_\mathcal{Z} - Y).$$

This is a contradiction. Hence $vQ = 0$ for every $Q \in \text{atom}(\mathcal{D})$. As v is nonzero it follows that $X < I_\mathcal{D}$. $\qquad\square$

The following description of R_a is an immediate consequence of Propositions 3.3 and 3.7.

Corollary 3.8.

$$R_a = \{(Q_1, Q_2) \in \text{atom}(\mathcal{D}) \times \text{atom}(\mathcal{D}) : Q_1\mathcal{M}Q_2 \neq (0)\}.$$

4. Main result

In this section, we prove our main result, Theorem 4.3. This result shows when the Cartan pair $(\mathcal{M}, \mathcal{D})$ has separable predual, the isomorphism class of the family of Bures-closed bimodules for $(\mathcal{M}, \mathcal{D})$ depends mostly upon the atomic part $(\mathcal{M}_a, \mathcal{D}_a)$ of $(\mathcal{M}, \mathcal{D})$.

Notation 4.1. If S is any set, $\text{card}(S)$ will denote the cardinality of S.

Lemma 4.2. *Let $(\mathcal{M}, \mathcal{D})$ be a Cartan inclusion. Then*

$$\text{card}(R_a) = \text{card}(\text{atom}(\mathcal{Z})).$$

Proof. Let $q : \mathrm{atom}(\mathcal{D}) \to \mathrm{atom}(\mathcal{D})/R_a$ be the quotient map. Define a map $\phi : \mathrm{atom}(\mathcal{Z}) \to \mathrm{atom}(\mathcal{D})/R_a$ as follows. For $A \in \mathrm{atom}(\mathcal{Z})$, let $Q_1, Q_2 \in \mathrm{atom}(\mathcal{D})$ be the unique atoms of $\mathcal{D}$ such that $A = \pi_\ell(Q_1)\pi_r(Q_2)$, see Proposition 3.7. Now set $\phi(A) = q(Q_1)$.

Observe that ϕ is onto: if $x \in \mathrm{atom}(\mathcal{D})/R_a$ and $Q \in q^{-1}(x)$, then

$$A := \pi_\ell(Q)\pi_r(Q) \in \mathrm{atom}(Z)$$

(see Proposition 3.7) and $\phi(A) = x$.

Fixing $x \in \mathrm{atom}(\mathcal{D})/R_a$, Proposition 3.7 implies that there is a bijection

$$\alpha_x : q^{-1}(x) \times q^{-1}(x) \to \phi^{-1}(x),$$

where α_x is the map given by

$$q^{-1}(x) \times q^{-1}(x) \ni (Q_1, Q_2) \mapsto \pi_\ell(Q_1)\pi_r(Q_2) \in \phi^{-1}(x).$$

Since $\mathrm{atom}(\mathcal{Z})$ and R_a are the disjoint unions,

$$\mathrm{atom}(\mathcal{Z}) = \bigcup_{x \in \mathrm{atom}(\mathcal{D})/R_a} \phi^{-1}(x),$$

$$R_a = \bigcup_{x \in \mathrm{atom}(\mathcal{D})/R_a} (q^{-1}(x) \times q^{-1}(x)),$$

there exists a bijection

$$\alpha : R_a \to \mathrm{atom}(\mathcal{Z})$$

given by $\alpha(Q_1, Q_2) = \alpha_x(Q_1, Q_2)$ if $(Q_1, Q_2) \in q^{-1}(x) \times q^{-1}(x)$. $\qquad\square$

For the following result we restrict our attention to the separably acting case.

Theorem 4.3. *For $i = 1, 2$, let $(\mathcal{M}_i, \mathcal{D}_i)$ be Cartan pairs where $\mathcal{M}_i$ has separable predual, and let $\mathfrak{S}_i$ be the lattice of all Bures-closed $\mathcal{D}_i$-bimodules contained in $\mathcal{M}_i$. The following statements are equivalent.*

(a) *There is a lattice isomorphism α of $\mathfrak{S}_1$ onto $\mathfrak{S}_2$.*
(b) *There is a lattice isomorphism α' from the projection lattice of $\mathcal{Z}_1$ onto the projection lattice of $\mathcal{Z}_2$.*
(c) *There is a von Neumann algebra isomorphism Θ of $\mathcal{Z}_1$ onto $\mathcal{Z}_2$.*
(d) *The relations $R_{a,i}$ for $(\mathcal{M}_i, \mathcal{D}_i)$ satisfy $\mathrm{card}(R_{a,1}) = \mathrm{card}(R_{a,2})$.*

Proof. Statements (a) and (b) are equivalent by [4, Theorem 2.5.8]. The equivalence of (b) and (c) is a piece of folklore about abelian von Neumann algebras. (Here is a sketch of the nontrivial direction. Suppose α' is an isomorphism of the projection lattices. For every finite Boolean algebra, $\mathcal{A} \subseteq \mathrm{proj}(\mathcal{Z}_1)$, $\alpha'|_\mathcal{A}$ extends uniquely to a C^*-algebra isomorphism $\Theta_\mathcal{A}$ of $C^*(\mathcal{A})$ onto $C^*(\alpha'(\mathcal{A}))$. As $\mathcal{Z}_1$ is the C^*-inductive limit of the family $\{C^*(\mathcal{A}) : \mathcal{A}$ a finite Boolean algebra of $\mathrm{proj}(\mathcal{Z}_1)\}$ (with inclusion maps), the inductive limit Θ of the maps $\Theta_\mathcal{A}$ is an isomorphism of $\mathcal{Z}_1$ onto $\mathcal{Z}_2$. But every isomorphism between von Neumann algebras is weak-$*$ continuous, so $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are isomorphic von Neumann algebras.)

If (c) holds, then $\mathrm{atom}(\mathcal{Z}_1)$ and $\mathrm{atom}(\mathcal{Z}_2)$ have the same cardinality, so Lemma 4.2 implies (d) holds. Conversely, if (d) holds, then the atomic parts of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are isomorphic. As $\mathcal{Z}_i$ are MASAs acting on separable Hilbert spaces, Proposition 3.7(c) implies the nonatomic parts of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are isomorphic. Therefore, $\mathcal{Z}_1$ is unitarily equivalent to $\mathcal{Z}_2$, and (c) holds. □

Theorem 4.3 is perhaps initially most remarkable when we consider von Neumann algebras without atoms. For example, let $(\mathcal{M}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{D}_2)$ be Cartan pairs where $\mathcal{M}_1$ is of type II_1 and $\mathcal{M}_2$ is of type III_λ for some λ. Theorem 4.3 tells us that, while $\mathcal{M}_1$ and $\mathcal{M}_2$ are quite different as von Neumann algebras, the lattice of Bures-closed $\mathcal{D}_1$-bimodules of $\mathcal{M}_1$ is isomorphic to the lattice of Bures-closed $\mathcal{D}_2$-bimodules of $\mathcal{M}_2$. The following simple example illustrates the situation regarding the atoms in Theorem 4.3.

Example 4.4. For any $n \in \mathbb{N}$, let $D_n \subseteq M_n(\mathbb{C})$ be the set of diagonal $n \times n$ matrices. Let $(\mathcal{M}_1, \mathcal{D}_1) = (M_2(\mathbb{C}), D_2)$, and let $(\mathcal{M}_2, \mathcal{D}_2) = (D_4, D_4)$. Then $\mathcal{Z}_1$ is isomorphic to D_4, so these Cartan pairs have isomorphic lattices of bimodules.

Remark 4.5. The nonseparable case is complicated by the fact that there are many isomorphism classes of nonatomic abelian von Neumann algebras. Indeed, if $\mathcal{H}$ is nonseparable and $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$ is a nonatomic MASA with a unit cyclic vector ξ, then there is a countable set I such that $\mathcal{D}$ is isomorphic to the direct sum, $\bigoplus_{i \in I} L^\infty(X_i, \mu_i)$, where $X_i = [0, 1]^{A_i}$ is a Cartesian product of the unit interval, μ_i is product measure, and for at least one i, A_i is a set with $\mathrm{card}(A_i) > \aleph_0$ (see [10] and [12]). A general MASA decomposes into a direct sum of cyclic MASAs, hence there is a family $\{Q_\alpha\}_{\alpha \in \mathfrak{I}}$ of projections in $\mathcal{D}$, for which $Q_\alpha \mathcal{D}$ is isomorphic to $L^\infty([0, 1]^{A_\alpha})$. Since Q_α is not minimal, the arguments of Proposition 3.7 do not seem to apply, and it is not clear how the statement of Theorem 4.3 should be modified in the nonseparable case.

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DEPT. OF MATHEMATICS, UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN, NE, 68588-0130
`afuller7@math.unl.edu`

DEPT. OF MATHEMATICS, UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN, NE, 68588-0130
`dpitts2@math.unl.edu`