New York Journal of Mathematics

New York J. Math. **19** (2013) 647–655.

Integral Hopf–Galois structures for tame extensions

Paul J. Truman

Abstract. We study the Hopf–Galois module structure of algebraic integers in some Galois extensions of p-adic fields L/K which are at most tamely ramified, generalizing some of the results of the author's 2011 paper cited below. If $G = \mathrm{Gal}(L/K)$ and $H = L[N]^G$ is a Hopf algebra giving a Hopf–Galois structure on L/K, we give a criterion for the $\mathfrak{O}_K$-order $\mathfrak{O}_L[N]^G$ to be a Hopf order in H. When $\mathfrak{O}_L[N]^G$ is Hopf, we show that it coincides with the associated order $\mathfrak{A}_H$ of $\mathfrak{O}_L$ in H and that $\mathfrak{O}_L$ is free over $\mathfrak{A}_H$, and we give a criterion for a Hopf–Galois structure to exist at integral level. As an illustration of these results, we determine the commutative Hopf–Galois module structure of the algebraic integers in tame Galois extensions of degree qr, where q and r are distinct primes.

Contents

1. Introduction

Nonclassical Hopf–Galois structures can provide a variety of contexts in which we can ask module-theoretic questions about a given finite separable extension of fields L/K and, in the case of local or global fields, study the structure of valuation rings or rings of algebraic integers. In this paper, we shall focus on finite Galois extensions of p-adic fields L/K (for some prime number p) with Galois group G and valuation rings $\mathfrak{O}_L, \mathfrak{O}_K$ respectively. Classically, we view L as a module over the group algebra $K[G]$ and $\mathfrak{O}_L$ as a module over its associated order $\mathfrak{A}_{K[G]}$ in $K[G]$. This situation is

Received July 29, 2013.
2010 *Mathematics Subject Classification.* 11R33 (primary), 11S23 (secondary).
Key words and phrases. Hopf–Galois structures, Hopf–Galois module theory, Hopf order, tame ramification.

ISSN 1076-9803/2013

generalized by replacing $K[G]$ with one of a (finite) number of different K-Hopf algebras H which act on the extension in a "Galois-like" way, each giving a *Hopf–Galois structure* on the extension (we also say that L is an *H-Galois extension of K*; see [6, Definition 2.7]). A theorem of Greither and Pareigis [6, Theorem 6.8] shows that there is a bijection between the Hopf–Galois structures admitted by a given finite Galois extension L/K and the regular subgroups N of $\operatorname{Perm} G$ that are stable under the action of G by conjugation via the left regular embedding; the Hopf algebra corresponding to the subgroup N is $H = L[N]^G$, and the action of an element of H on an element $x \in L$ is given by:

$$\left(\sum_{n\in N} c_n n\right) \cdot x = \sum_{n\in N} c_n (n^{-1}(1_G))x. \tag{1}$$

To study the structure of $\mathfrak{O}_L$ relative to a Hopf algebra H giving a Hopf–Galois structure on L/K, we define its associated order $\mathfrak{A}_H$ in H, and the principal question is to determine whether $\mathfrak{O}_L$ is free over $\mathfrak{A}_H$. An account of this theory appears in [6]. There are examples of wildly ramified Galois extensions L/K for which $\mathfrak{O}_L$ is not free over its associated order in the group algebra $K[G]$, but is free over its associated order in a Hopf algebra H giving a nonclassical Hopf–Galois structure on the extension [2]. Examples such as these illustrate the value of using nonclassical Hopf–Galois structures to study wildly ramified extensions.

However, in [9] we investigated the nonclassical Hopf–Galois module structure of valuation rings in extensions of p-adic fields L/K which are at most tamely ramified. In particular, we studied the $\mathfrak{O}_K$-order $\mathfrak{O}_L[N]^G$ (henceforth denoted Λ^G) within a Hopf algebra $H = L[N]^G$ giving a Hopf–Galois structure on the extension. We showed [9, Theorem 3.4] that if L/K is unramified then Λ^G is a Hopf order in H and $\mathfrak{A}_H = \Lambda^G$. We then showed that in this case $\mathfrak{O}_L$ is a *Λ^G-tame extension of* $\mathfrak{O}_K$ (see [6, Definition 13.1]) and used a result of Childs ([6, Theorem 13.4]) to conclude that $\mathfrak{O}_L$ is a free Λ^G-module. In Section 2 of this paper we generalize these results. Let L/K be a Galois extension of p-adic fields with group G and let $H = L[N]^G$ be a Hopf algebra giving a Hopf–Galois structure on the extension.

Theorem 1.1. *The $\mathfrak{O}_K$-order Λ^G is a Hopf order in $H = L[N]^G$ if and only if the kernel of the action of G on N contains the inertia group of L/K.*

Theorem 1.2. *Suppose that L/K is at most tamely ramified and that Λ^G is a Hopf order in H. Then $\mathfrak{O}_L$ is a Λ^G-tame extension of $\mathfrak{O}_K$. Hence $\mathfrak{A}_H = \Lambda^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module.*

As an application of these results, in Section 3 we study Galois extensions of p-adic fields which are at most tamely ramified and have degree qr, where q, r are primes and $q < r$. We prove the following:

Theorem 1.3. *Suppose that $[L:K]=qr$, where q,r are prime and $q<r$, that L/K is at most tamely ramified, and that H is commutative. Then $\mathfrak{A}_H=\Lambda^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module.*

In the final section, we return to the more general setting where L/K is a Galois extension of p-adic fields which is at most tamely ramified. Under the assumption that that Λ^G is a Hopf order in H, we determine a criterion for a Hopf–Galois structure to exist at integral level:

Theorem 1.4. *The valuation ring $\mathfrak{O}_L$ is a Λ^G-Galois extension of $\mathfrak{O}_K$ if and only if L/K is unramified.*

Since this paper continues the investigations from [9], we refer the reader to that paper for further information about the background to, and context of, these results.

Acknowledgements. I am grateful to the referee for many helpful suggestions regarding the exposition.

2. The fixed points of the integral group ring

In this section we prove Theorem 1.1 and Theorem 1.2. We continue to denote by L/K a finite Galois extension of p-adic fields with group G and valuation rings $\mathfrak{O}_L,\mathfrak{O}_K$ respectively, and by H a Hopf algebra giving a Hopf–Galois structure on the extension. By the theorem of Greither and Pareigis [6, Theorem 6.8], $H=L[N]^G$ for some regular subgroup N of $\operatorname{Perm} G$ that is stable under the action of G by conjugation via the left regular embedding $\lambda: G\to \operatorname{Perm} G$. We shall denote the integral group ring $\mathfrak{O}_L[N]$ by Λ, so that the $\mathfrak{O}_K$-order $\mathfrak{O}_L[N]^G$ is Λ^G. This order is contained in $\mathfrak{A}_H$, the associated order of $\mathfrak{O}_L$ in H (see [9, Proposition 2.5]). In [1, Lemma 2.1], Boltje and Bley determine an $\mathfrak{O}_K$-basis of Λ^G as follows:

Let $N_1,\ldots,N_r$ be the orbits of G in N. For each $i=1,\ldots,r$, let $n_i\in N_i$ be a generator of the orbit N_i, and let $S_i=\operatorname{Stab}_G(n_i)$. Now let $L_i=L^{S_i}$, and let $\{x_{i,j}\mid j=1,\ldots,[L_i:K]\}$ be an integral basis of L_i over K. For each $i=1,\ldots,r$ and $j=1,\ldots,[L_i:K]$, define

$$a_{i,j}=\sum_{g\in G/S_i} g(x_{i,j})\,{}^g n_i,$$

where the sum is taken over a set of left coset representatives of S_i in G (in general S_i need not be normal in G). Then the set

$$\{a_{i,j}\mid i=1,\ldots,r\ \ j=1,\ldots,[L_i:K]\}$$

is an $\mathfrak{O}_K$-basis of Λ^G. In [1, Proposition 4.6] it is shown that Λ^G is a Hopf order in H if and only if each of the fields L_i is unramified over K. We can now restate and prove Theorem 1.1:

Theorem 2.1. *The $\mathfrak{O}_K$-order Λ^G is a Hopf order in H if and only if the kernel of the action of G on N contains the inertia group of L/K.*

Proof. Let G_0 be the inertia group of L/K, so that $L_0 = L^{G_0}$ is the maximal unramified subextension of L/K. Let $H \trianglelefteq G$ be the kernel of the action of G on N. If $G_0 \trianglelefteq H$ then $G_0 \trianglelefteq S_i$ for each i, so by Galois theory we have $L_i = L^{S_i} \subseteq L^{G_0} = L_0$, and so each L_i is unramified over K. Conversely, if each L_i is unramified over K then $G_0 \trianglelefteq S_i$ for each i. Now let $n \in N$. Then $n = {}^g n_i$ for some $g \in G$ and some $i = 1, \ldots, r$. If $\sigma \in G_0$ then, since G_0 is normal in G, there exists $\tau \in G_0$ such that $\sigma = g\tau g^{-1}$. Now we have

$${}^\sigma n = {}^{\sigma g} n_i = {}^{g\tau g^{-1} g} n_i = {}^{g\tau} n_i = {}^g n_i = n,$$

so ${}^\sigma n = n$, and so $\sigma \in H$. Therefore $G_0 \leq H$, as claimed. □

Under the bijection established by the theorem of Greither and Pareigis [6, Theorem 6.8], the classical Hopf–Galois structure on L/K, with Hopf algeba $K[G]$, corresponds to the image of G under the right regular embedding $\rho : G \to \operatorname{Perm} G$. The action of G on $\rho(G)$ by conjugation via the left regular embedding is trivial, so $H = K[\rho(G)]$ and $\Lambda^G = \mathfrak{O}_K[\rho(G)]$. In this case, the kernel of the action of G on $\rho(G)$ is all of G, the inertia subgroup of L/K is certainly contained in this kernel, and we recover the fact that $\mathfrak{O}_K[\rho(G)]$ is always a Hopf order in $K[\rho(G)]$. If G is nonabelian, then L/K has a canonical nonclassical Hopf–Galois structure, whose Hopf algebra H_λ corresponds to the regular subgroup $\lambda(G)$. In this case, we have:

Corollary 2.2. *The $\mathfrak{O}_K$-order $\mathfrak{O}_L[\lambda(G)]^G$ is a Hopf order in $H_\lambda = L[\lambda(G)]^G$ if and only if the inertia subgroup of L/K is contained in the centre of G.*

Proof. In this case the orbits of G in $\lambda(G)$ correspond to the conjugacy classes of G, so the stabilizer of a given element is its centralizer, and the kernel of the action of G on $\lambda(G)$ is the centre of G. Apply Theorem 2.1. □

For any Hopf order A in H for which $\mathfrak{O}_L$ is a module, we say that $\mathfrak{O}_L$ is an *A-tame extension of $\mathfrak{O}_K$* if there exists a left integral θ of A satisfying $\theta \cdot \mathfrak{O}_L = \mathfrak{O}_K$ (see [6, Definition 13.1]). A consequence of a result of Childs ([6, Theorem 13.4]) is that if $\mathfrak{O}_L$ is an A-tame extension of $\mathfrak{O}_K$, then $\mathfrak{O}_L$ is a free A-module of rank one. Using this, we restate and prove Theorem 1.2:

Theorem 2.3. *Suppose that L/K is at most tamely ramified, that $H = L[N]^G$ is a Hopf algebra giving a Hopf–Galois structure on the extension L/K, and that Λ^G is a Hopf order in H. Then $\mathfrak{O}_L$ is a Λ^G-tame extension of $\mathfrak{O}_K$. Hence $\mathfrak{A}_H = \Lambda^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module.*

Proof. Note that the trace element

$$\theta = \sum_{n \in N} n$$

is a left integral of Λ^G. Using the formula for the action of H on L given in Equation (1), we have:

$$\theta \cdot x = \sum_{n \in N} (n^{-1}(1_G))x = \sum_{g \in G} g(x) = \operatorname{Tr}_{L/K}(x) \text{ for all } x \in \mathfrak{O}_L,$$

and since L/K is tame there exists an element $t \in \mathfrak{O}_L$ such that $\theta \cdot t = 1$. Thus $\mathfrak{O}_L$ is an Λ^G-tame extension of $\mathfrak{O}_K$, and so by [6, Theorem 13.4] $\mathfrak{O}_L$ is a free Λ^G-module. Since $\mathfrak{A}_H$ is the only order in H over which $\mathfrak{O}_L$ can possibly be free (see [6, Proposition 12.5]), this implies that $\mathfrak{A}_H = \Lambda^G$. □

Note that if L/K is wildly ramified then $\Lambda^G \subsetneq \mathfrak{A}_H$, since in this case $\theta \cdot x = \mathrm{Tr}_{L/K}(x) \in \pi_K \mathfrak{O}_K$ for all $x \in \mathfrak{O}_L$ (where π_K is a uniformizer of K), and so the element $\pi_K^{-1}\theta$ is in $\mathfrak{A}_H$ but not in Λ^G.

3. Applications to tame extensions of degree *qr*

Let p, q and r be prime numbers, with $q < r$. In this section we study commutative Hopf–Galois structures on Galois extensions of p-adic fields L/K which have degree qr and are at most tamely ramified, culminating in a proof of Theorem 1.3. We restrict our attention to commutative structures since for these we have $\mathfrak{A}_H = \mathfrak{O}_L[N]^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module whenever $p \nmid qr$ [9, Theorem 4.4]. We do not have an analogue of this result for noncommutative structures, and so these will require more detailed analysis, which we intend to complete in a forthcoming paper.

There are two possibilities for the structure of the group $G = \mathrm{Gal}(L/K)$: it may be cyclic or metacyclic. If $r \not\equiv 1 \pmod{q}$ then G must be cyclic, and by [4, Theorem 1] L/K admits only the classical Hopf–Galois structure with Hopf algebra $K[G]$ and its usual action on L. Since L/K is at most tamely ramified, Noether's Theorem implies that $\mathfrak{A}_{K[G]} = \mathfrak{O}_K[G]$ and $\mathfrak{O}_L$ is a free $\mathfrak{O}_K[G]$-module. Having dealt with this case, we shall assume that $r \equiv 1 \pmod{q}$ from now on.

In this case, the extension L/K does admit nonclassical Hopf–Galois structures. If $H = L[N]^G$ is a Hopf algebra giving a Hopf–Galois structure on L/K then we refer to the isomorphism class of N as the *type* of the Hopf algebra. Byott has shown [3, Theorem 6.1 and Theorem 6.2] that:

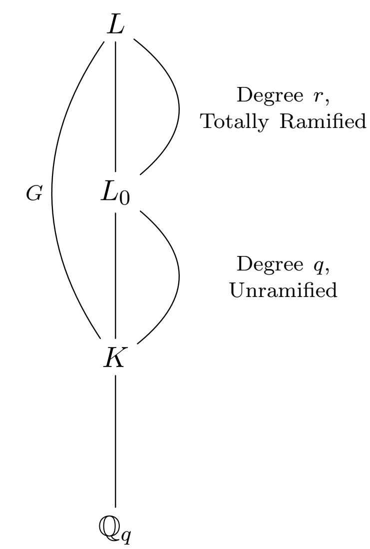
- If L/K is cyclic then it admits precisely $2q-1$ Hopf–Galois structures. The classical structure is of cyclic type, and the other $2(q-1)$ structures are of metacyclic type.
- If L/K is metacyclic then it admits precisely $2 + r(2q-3)$ Hopf–Galois structures. Of these, r are of cyclic type and the remainder are of metacyclic type.

Since we are presently concerned with commutative Hopf–Galois structures, we shall say nothing more about cyclic extensions. If G is metacyclic then we may present it as

$$G = \langle \sigma, \tau \mid \sigma^r = \tau^q = 1, \tau\sigma\tau^{-1} = \sigma^d \rangle,$$

where d is a fixed natural number whose order modulo r is q.

Consider the residue characteristic p of K. If $p \nmid qr$ then, as noted above, we have $\mathfrak{A}_H = \mathfrak{O}_L[N]^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module. The two remaining cases are $p = q$ and $p = r$. Write L_0 for the maximal unramified subextension of L/K, and let $1 \neq G_0 \triangleleft G$ be the Galois group of L/L_0 (the inertia subgroup of G). Since L/K is tamely ramified, we must have $p \nmid |G_0|$. If $p = r$, then this forces $|G_0| = q$. But this is impossible, since G does not have a normal subgroup of order q. So we are left with the case where $p = q$, and G_0 is the unique normal subgroup of G of order r, generated by σ.



In [3, Lemma 4.1], Byott gives an explicit description of the p subgroups of $\mathrm{Perm}(G)$ corresponding to the commutative Hopf–Galois structures on L/K. They are the groups N_c for $0 \leq c \leq r-1$, where N_c is generated by the two permutations:

$$\begin{aligned} \alpha &: \sigma^u \tau^t \mapsto \sigma^{u+1} \tau^v \\ \eta &: \sigma^u \tau^t \mapsto \sigma^{u-cd^v} \tau^{v+1}. \end{aligned}$$

(Here $\sigma^u \tau^v$ denotes an arbitrary element of G.) Using this explicit description, we can examine the relationship between the kernel of the action of G on any of the subgroups N_c and the inertia group $G_0 = \langle \sigma \rangle$:

Lemma 3.1. *For each $0 \leq c \leq r-1$, the inertia subgroup G_0 is contained in the kernel of the action of G on N_c.*

Proof. Let $0 \leq c \leq r-1$. For all $g \in G$ and $s, t \in \mathbb{Z}$, we have

$$ {}^g(\alpha^s \eta^t) = {}^g(\alpha^s)\, {}^g(\eta^t) = ({}^g\alpha)^s ({}^g\eta)^t, $$

so it is sufficient to show that $({}^g\alpha) = \alpha$ and $({}^g\eta) = \eta$ for each $g \in G_0 = \langle \sigma \rangle$. Let σ^i be a typical element of G_0 and $\sigma^u \tau^v$ a typical element of G. Then we have

$$\begin{aligned}[\lambda(\sigma^i)\alpha\lambda(\sigma^{-i})](\sigma^u\tau^v) &= [\lambda(\sigma^i)\alpha](\sigma^{u-i}\tau^v)\\ &= [\lambda(\sigma^i)](\sigma^{u-i+1}\tau^v)\\ &= \sigma^{u+1}\tau^v\\ &= \alpha(\sigma^u\tau^v),\end{aligned}$$

so ${}^{\sigma^i}\alpha = \alpha$. Similarly, we have

$$\begin{aligned}[\lambda(\sigma^i)\eta\lambda(\sigma^{-i})](\sigma^u\tau^v) &= [\lambda(\sigma^i)\eta](\sigma^{u-i}\tau^v)\\ &= [\lambda(\sigma^i)]\sigma^{u-i-cd^v}\tau^{v+1}\\ &= \sigma^{u-cd^v}\tau^{v+1}\\ &= \eta(\sigma^u\tau^v),\end{aligned}$$

so ${}^{\sigma^i}\eta = \eta$. Therefore $\langle\sigma\rangle = G_0$ is contained in the kernel of the action of G on N_c. □

Finally we use Theorems 2.1 and 2.3 to describe the associated order in the Hopf algebra corresponding to each regular subgroup N_c and the structure of $\mathfrak{O}_L$ over each of these associated orders:

Theorem 3.2. *Let $0 \leq c \leq r-1$, and let $H_c = L[N_c]^G$ be the commutative Hopf algebra corresponding to the group N_c and giving a Hopf–Galois structure on L/K. Then $\Lambda_c^G = \mathfrak{O}_L[N_c]^G$ is a Hopf order in H_c, and $\mathfrak{O}_L$ is a free Λ_c^G-module.*

Proof. We have shown in Lemma 3.1 that the inertia subgroup G_0 is contained in the kernel of the action of G on N_c, and by Theorem 2.1 this implies that Λ_c^G is a Hopf order in H_c. Since L/K is tamely ramified, we can apply Theorem 2.3 and conclude that $\mathfrak{O}_L$ is a free Λ_c^G-module. □

We summarise the results of this section by restating and proving Theorem 1.3:

Theorem 3.3. *Suppose that L/K is a Galois extension of p-adic fields of degree qr, where q, r are prime and $q < r$, that L/K is at most tamely ramified, and that $H = L[N]^G$ is a commutative Hopf algebra giving a Hopf–Galois structure on the extension. Then $\mathfrak{A}_H = \Lambda^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module.*

Proof. If $r \not\equiv 1 \pmod{q}$ then by [4, Theorem 1] L/K admits only the classical Hopf–Galois structure with Hopf algebra $K[G]$ and its usual action on L. Since L/K is at most tamely ramified, Noether's Theorem implies that $\mathfrak{A}_{K[G]} = \mathfrak{O}_K[G]$ and $\mathfrak{O}_L$ is a free $\mathfrak{O}_K[G]$-module. If $r \equiv 1 \pmod{q}$ then we must have $p \neq r$ since L/K is tamely ramified. If $p \neq q$ then by [9, Theorem 4.4] we have $\mathfrak{A}_H = \mathfrak{O}_L[N]^G$ and $\mathfrak{O}_L$ is a free $\mathfrak{A}_H$-module. If $p = q$ then Theorem 3.2 yields the same conclusions. □

4. Integral Hopf–Galois structures

In this section we return to the setting of Section 2: L/K denotes a finite Galois extension of p-adic fields which has group G and which is at most tamely ramified. The extension of commutative rings $\mathfrak{O}_L/\mathfrak{O}_K$ is a Galois extension with group G in the sense of [5, Definition 1.4], that is, an $\mathfrak{O}_K[G]$-Galois extension of $\mathfrak{O}_K$, if and only if L/K is unramified. In this section we shall consider a Hopf algebra $H = L[N]^G$ giving a nonclassical Hopf–Galois structure on L/K, and investigate when $\mathfrak{O}_L$ is a Λ^G-Galois extension of $\mathfrak{O}_K$. Obviously it is necessary that Λ^G be a Hopf order in H (see Theorem 2.1). To give a criterion, we shall consider linear duals. The linear dual $H^* = \mathrm{Hom}_K(H, K)$ is also a K-Hopf algebra (see [6, (1.4)]), and if A is a Hopf order in H, then $A^* = \mathrm{Hom}_{\mathfrak{O}_K}(A, \mathfrak{O}_K)$ is a Hopf order in H^*. We can now restate and prove Theorem 1.4:

Theorem 4.1. *Suppose that Λ^G is a Hopf order in H. Then $\mathfrak{O}_L$ is a Λ^G-Galois extension of $\mathfrak{O}_K$ if and only if L/K is unramified.*

Proof. By a result of Greither ([6, Proposition 22.13)] or [8]), $\mathfrak{O}_L$ is a Λ^G-Galois extension of $\mathfrak{O}_K$ if and only if $\mathfrak{O}_L$ is a Λ^G-module algebra [6, §2] and $\mathfrak{d}(\mathfrak{O}_L) = \mathfrak{d}((\Lambda^G)^*)$. Since H gives a Hopf–Galois structure on the extension L/K, the field L is an H-module algebra, so $\mathfrak{O}_L$ is a Λ^G-module algebra. We shall use results of Boltje and Bley to show that $\mathfrak{d}((\Lambda^G)^*) = \mathfrak{O}_K$. Note that H^* is a commutative Hopf algebra since H is cocommutative and, since K has characteristic zero, H^* is also separable (see [10, (§11.4)]). Therefore H^* has a unique maximal order. In [1, Corollary 4.7] it is shown that Λ^G is a Hopf order in H if and only if $(\Lambda^G)^*$ is the unique maximal order in H^*. It is also shown ([1, Lemma 3.1]) that the discriminant of this maximal order is

$$\prod_{i=1}^{r} \mathfrak{d}(\mathfrak{O}_{L_i}),$$

where the fields L_i are as described in Section 2 above. But by [1, Proposition 4.6], Λ^G is a Hopf order in H if and only if each of the fields L_i is unramified over K, that is, if and only if $\mathfrak{d}(\mathfrak{O}_{L_i}) = \mathfrak{O}_K$ for each $i = 1, \ldots, r$. So we have $\mathfrak{d}((\Lambda^G)^*) = \mathfrak{O}_K$ in this case. Now by Greither's result ([6, Proposition 22.13)] or [8]) we have that $\mathfrak{O}_L$ is a Λ^G-Galois extension of $\mathfrak{O}_K$ if and only if

$$\mathfrak{d}(\mathfrak{O}_L) = \mathfrak{d}((\Lambda^G)^*) = \mathfrak{O}_K,$$

that is, if and only if L/K is unramified. □

By applying Theorem 4.1 to the extensions considered in Section 3, we see that the only circumstance under which we have a Hopf–Galois structure at integral level is when L/K is unramified of degree qr and $H = K[G]$ gives the classical structure on the extension.

References

[1] Bley, Werner; Boltje, Robert. Lubin–Tate formal groups and module structure over Hopf orders. *J. Théor. Nombres Bordeaux* **11** (1999), no. 2, 269–305. MR1745880 (2001b:11110), Zbl 0979.11053, doi: 10.5802/jtnb.251.

[2] Byott, Nigel P. Galois structure of ideals in wildly ramified abelian p-extensions of a p-adic field, and some applications. *J. Théor. des Nombres Bordeaux* **9** (1997), no. 1, 201–219. MR1469668 (98h:11152), Zbl 0889.11040, doi: 10.5802/jtnb.196.

[3] Byott, Nigel P. Hopf–Galois structures on Galois field extensions of degree pq. *J. Pure Appl. Algebra* **188** (2004), no. 1–3, 45–57. MR2030805 (2004j:16041), Zbl 1047.16022, doi: 10.1016/j.jpaa.2003.10.010.

[4] Byott, N. P. Uniqueness of Hopf Galois structure for separable field extensions. *Comm. Algebra* **24** (1996), no. 10, 3217–3228. MR1402555 (97j:16051a), Zbl 0878.12001, doi: 10.1080/00927879608825743. Corrigendum. *Comm. Algebra* **24** (1996), no. 11, 3705. MR1405283 (97j:16051b).

[5] Chase, S.U.; Harrison, D.K.; Rosenberg, Alex. Galois theory and Galois cohomology of commutative rings. *Mem. Amer. Math. Soc. No.* **52** (1965), 15–33. MR0195922 (33 #4118), Zbl 0143.05902.

[6] Childs, Lindsay N. Taming wild extensions: Hopf algebras and local Galois module theory. Mathematical Surveys and Monographs, 80. *American Mathematical Society, Providence, RI*, 2000. viii+215 pp. ISBN: 0-8218-2131-8. MR1767499 (2001e:11116), Zbl 0944.11038.

[7] Fröhlich, Albrecht. Galois module structure of algebraic integers. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 1. *Springer-Verlag, Berlin*, 1983. x+262 pp. ISBN: 3-540-11920-5. MR0717033 (85h:11067), Zbl 0501.12012, doi: 10.1007/978-3-642-68816-4.

[8] Greither, C. Extensions of finite group schemes, and Hopf–Galois theory over a discrete valuation ring. *Math. Z.* **210** (1992), no. 1, 37–67. MR1161169 (93f:14024), Zbl 0737.11038, doi: 10.1007/BF02571782.

[9] Truman, Paul J. Towards a generalisation of Noether's theorem to nonclassical Hopf–Galois structures. *New York J. Math.* **17** (2011), 799–810. MR2862153, Zbl 1250.11098, arXiv:1001.1639, http://nyjm.albany.edu/j/2011/17-34v.pdf.

[10] Waterhouse, William C. Introduction to affine group schemes. Graduate Texts in Mathematics, 66. *Springer-Verlag, New York-Berlin*, 1979. xi+164 pp. ISBN: 0-387-90421-2. MR0547117 (82e:14003), Zbl 0442.14017.

School of Computing and Mathematics, Keele University, UK
P.J.Truman@Keele.ac.uk

This paper is available via http://nyjm.albany.edu/j/2013/19-32.html.