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Bimodules over Cartan MASAs in von Neumann algebras, norming algebras, and Mercer's Theorem

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Abstract. In a 1991 paper, R. Mercer asserted that a Cartan bimodule isomorphism between Cartan bimodule algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ extends uniquely to a normal $*$-isomorphism of the von Neumann algebras generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ (Corollary 4.3 of Mercer, 1991). Mercer's argument relied upon the Spectral Theorem for Bimodules of Muhly, Saito and Solel, 1988 (Theorem 2.5, there). Unfortunately, the arguments in the literature supporting their Theorem 2.5 contain gaps, and hence Mercer's proof is incomplete.

In this paper, we use the outline in Pitts, 2008, Remark 2.17, to give a proof of Mercer's Theorem under the additional hypothesis that the given Cartan bimodule isomorphism is σ-weakly continuous. Unlike the arguments contained in the abovementioned papers of Mercer and Muhly–Saito–Solel, we avoid the use of the machinery in Feldman–Moore, 1977; as a consequence, our proof does not require the von Neumann algebras generated by the algebras $\mathcal{A}_i$ to have separable preduals. This point of view also yields some insights on the von Neumann subalgebras of a Cartan pair $(\mathcal{M}, \mathcal{D})$, for instance, a strengthening of a result of Aoi, 2003.

We also examine the relationship between various topologies on a von Neumann algebra $\mathcal{M}$ with a Cartan MASA $\mathcal{D}$. This provides the necessary tools to parameterize the family of Bures-closed bimodules over a Cartan MASA in terms of projections in a certain abelian von Neumann algebra; this result may be viewed as a weaker form of the Spectral Theorem for Bimodules, and is a key ingredient in the proof of our version of Mercer's Theorem. Our results lead to a notion of spectral synthesis for σ-weakly closed bimodules appropriate to our context, and we show that any von Neumann subalgebra of $\mathcal{M}$ which contains $\mathcal{D}$ is synthetic.

We observe that a result of Sinclair and Smith shows that any Cartan MASA in a von Neumann algebra is norming in the sense of Pop, Sinclair and Smith.

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Contents

1. Background and preliminaries

1.1. Background. The following appears in a 1991 paper of R. Mercer:

Assertion 1.1.1 ([13, Corollary 4.3]). *For $i = 1, 2$, let $\mathcal{M}_i$ be a von Neumann algebra with separable predual and let $\mathcal{D}_i \subseteq \mathcal{M}_i$ be a Cartan MASA. Suppose $\mathcal{A}_i$ is a σ-weakly closed subalgebra of $\mathcal{M}_i$ which contains $\mathcal{D}_i$ and which generates $\mathcal{M}_i$ as a von Neumann algebra.*

If $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric algebra isomorphism such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$, then θ extends to a von Neumann algebra isomorphism $\overline{\theta} : \mathcal{M}_1 \to \mathcal{M}_2$. Furthermore, if $\mathcal{M}_i$ is identified with its Feldman–Moore representation, so $\mathcal{M}_i \subseteq \mathcal{B}(L^2(R_i))$, then $\overline{\theta}$ may be taken to be a spatial isomorphism.

Mercer's argument supporting this assertion relies upon the Spectral Theorem for Bimodules of Muhly, Saito and Solel [15, Theorem 2.5]. The purpose of [15, Theorem 2.5] is to characterize σ-weakly closed bimodules over a Cartan MASA in terms of measure-theoretic data. We know of two articles claiming to prove this characterization: the original paper [15] and another paper of Mercer, see [12, Theorem 5.1]. Unfortunately, the proofs in both articles contain gaps, so the validity of [15, Theorem 2.5] in general is uncertain. However, for σ-weakly closed bimodules over a Cartan MASA in a hyperfinite von Neumann algebra, the Spectral Theorem for Bimodules follows from a more general result of Fulman, see [9, Theorem 15.18].

The paper of Aoi [1, pages 724–725] gives a discussion of the gap in the proof presented in [15, Theorem 2.5]. On the other hand, Mercer's argument (see the proof of [12, Theorem 5.1]) claims that if $(\mathcal{M}, \mathcal{D})$ is a pair consisting of a separably-acting von Neumann algebra $\mathcal{M}$ and a Cartan MASA $\mathcal{D}$, and $\mathcal{S} \subseteq \mathcal{M}$ is a σ-weakly closed subspace, then $\mathcal{S}$ is closed in the relative L^2 topology. (This is the topology arising from the norm, $\mathcal{M} \ni T \mapsto \sqrt{\omega(E(T^*T))}$, where ω is a fixed faithful normal state on $\mathcal{D}$ and $E : \mathcal{M} \to \mathcal{D}$ is the faithful normal conditional expectation.) The following example, from Roger Smith, shows this statement is false.

Example 1.1.2. Let $\mathcal{M} = \mathcal{D} = L^\infty[0,1]$ where the measure is Lebesgue measure. In this case, the L^2 topology on $\mathcal{M}$ is the relative topology on $\mathcal{M}$ arising from viewing $\mathcal{M}$ as a subspace of $L^2[0,1]$. Since $\mathcal{M}_*$ may be identified with $L^1[0,1]$, the linear functional ϕ on $\mathcal{M}$ given by

$$\phi(f) := \int_0^1 f(x)x^{-3/4}\,dx$$

is σ-weakly continuous. Let $\mathcal{S} := \ker\phi$. Then $\mathcal{S}$ is σ-weakly closed. But ϕ is not continuous with respect to the L^2-norm, so $\mathcal{S}$ is not L^2-closed [5, Theorem 3.1].

Because of these issues, the question of whether Assertion 1.1.1 is correct in general arises. It is interesting that when Assertion 1.1.1 is valid, θ is necessarily σ-weakly continuous. While Mercer did not explicitly assume θ is σ-weakly continuous (or continuous with respect to another appropriate topology) in his assertion, Mercer tacitly assumes continuity of θ. Indeed, Mercer's argument for Assertion 1.1.1 relies upon [13, Proposition 2.2], and the first paragraph of the proof of that proposition implicitly assumes a continuity hypothesis. Thus, the statement of Assertion 1.1.1 appearing in [13] should also include an appropriate continuity assumption.

A principal goal of this paper is to provide a proof of Assertion 1.1.1, under the additional hypothesis that θ is σ-weakly continuous, which does not use the Spectral Theorem for Bimodules. Our argument uses the notion of norming algebras and follows the outline given in [16, Remark 2.17]. Unlike Mercer's original statement, we do not require that $\mathcal{M}$ have separable predual. We shall require an understanding of two topologies on $\mathcal{M}$, the Bures and L^2 topologies. As a consequence of this analysis, we obtain Theorem 2.5.1, the Spectral Theorem for Bures Closed Bimodules, where the bimodules characterized are those which are closed in the Bures (or, equivalently, the L^2) topology rather than the σ-weak topology. Instead of using measure theoretic data to characterize Bures closed bimodules, our characterization uses projections in a certain abelian von Neumann algebra constructed from the Cartan MASA $\mathcal{D}$ and $\mathcal{M}$. This leads to a notion of synthesis similar to that found in Arveson's seminal paper [2], but appropriate to our context. When $\mathcal{A} \subseteq \mathcal{M}$ is a von Neumann algebra containing $\mathcal{D}$, we give a new proof, and a strengthening, of a result of Aoi [1], which shows that $\mathcal{D}$ is a Cartan MASA in $\mathcal{A}$ and establishes the existence of a conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$. Our methods also show that any von Neumann subalgebra of $\mathcal{M}$ containing $\mathcal{D}$ is Bures closed, from which it follows that the class of von Neumann subalgebras of $\mathcal{M}$ which contain $\mathcal{D}$ is a class of $\mathcal{D}$-bimodules which satisfy synthesis and for which the conclusion of [15, Theorem 2.5] is valid.

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We also wish to acknowledge our indebtedness to the very interesting papers of Muhly–Saito–Solel [15] and Mercer [12] discussed above. Many of the ideas found in those papers provide techniques for the analysis of bimodules in our context. We utilized several of the tools in those papers and the present paper would not have been written without them.

1.2. Some general notation. Because we shall be dealing with certain nonselfadjoint algebras, we use $X^{\#}$ for the dual of the Banach space X; likewise, when X is a complex vector space and τ is a locally convex topology on X, $(X,\tau)^{\#}$ will denote its dual space.

For any unital C^*-algebra $\mathcal{C}$ containing a unital abelian C^*-subalgebra $\mathcal{D}$, let

$$\mathcal{N}(\mathcal{C},\mathcal{D}) := \{v \in \mathcal{C} : v^*\mathcal{D}v \cup v\mathcal{D}v^* \subseteq \mathcal{D}\}.$$

An element $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ is a *normalizer* of $\mathcal{D}$. Finally, if $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ is a partial isometry, then we say that v is a *groupoid normalizer* of $\mathcal{D}$, and write $v \in \mathcal{GN}(\mathcal{C},\mathcal{D})$.

Lemma 1.2.1. *Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{D} \subseteq \mathcal{M}$ be an abelian von Neumann subalgebra (with the same unit) and let $\mathcal{S} \subseteq \mathcal{M}$ be a σ-weakly closed $\mathcal{D}$-bimodule. Given $v \in \mathcal{S} \cap \mathcal{N}(\mathcal{M},\mathcal{D})$, let $v = u|v|$ be the polar decomposition of v. Then $u \in \mathcal{S} \cap \mathcal{GN}(\mathcal{M},\mathcal{D})$.*

Proof. The statement is trivial if $v = 0$, so assume $v \neq 0$. Since $v \in \mathcal{N}(\mathcal{M},\mathcal{D})$, $v^*Iv \in \mathcal{D}$, so $|v| \in \mathcal{D}$. Let S be the spectral measure for $|v|$. For $0 < \varepsilon < \|v\|$, let $f_\varepsilon(t) = t^{-1}\chi_{[\varepsilon,\infty)}(t)$ and $P_\varepsilon = S([\varepsilon, \|v\|])$. Then $|v|f_\varepsilon(|v|) = P_\varepsilon$, so $vf_\varepsilon(|v|) = uP_\varepsilon$ converges σ-strong-$*$ to u as $\varepsilon \to 0$. Also, $vf_\varepsilon(|v|) \in \mathcal{N}(\mathcal{M},\mathcal{D})$ with $\|vf_\varepsilon(|v|)\| \leq 1$. Since multiplication on bounded sets is jointly continuous in the σ-strong topology, we conclude that $u \in \mathcal{GN}(\mathcal{M},\mathcal{D})$.

Since $v \in \mathcal{S}$, $u|v|^n = v|v|^{n-1} \in \mathcal{S}$ for all $n \in \mathbb{N}$, which implies $u|v|^{1/n} \in \mathcal{S}$ for all $n \in \mathbb{N}$. But $u|v|^{1/n} \overset{\sigma\text{-weak}}{\longrightarrow} uu^*u = u$, so $u \in \mathcal{S}$. $\square$

Definition 1.2.2. A MASA $\mathcal{D}$ in a von Neumann algebra $\mathcal{M}$ is called a *Cartan MASA* if there is a faithful, normal conditional expectation $E : \mathcal{M} \to \mathcal{D}$ and $\operatorname{span}\{U \in \mathcal{M} : U \text{ is unitary and } U\mathcal{D}U^* = \mathcal{D}\}$ is σ-weakly dense in $\mathcal{M}$. We will call the pair $(\mathcal{M},\mathcal{D})$ a *Cartan pair*.

Standing Assumption 1.2.3. Unless explicitly stated to the contrary, throughout this paper, $\mathcal{M}$ will denote a von Neumann algebra with a Cartan MASA $\mathcal{D}$.

1.3. Bimodules and normalizers. We now give some properties of the expectation E, and use them to show that bimodules often contain a rich supply of normalizers. We require some notation. Recall that any discrete abelian group G has an invariant mean Λ. This means that Λ is a state on $\ell^\infty(G)$ such that for any $h \in G$ and $F \in \ell^\infty(G)$, $\Lambda(F) = \Lambda(F_h)$, where $F_h(g) = F(gh^{-1})$. We will usually write, $\Lambda_{g\in G} F(g)$ instead of $\Lambda(F)$. We will always assume that Λ has the additional property that it is invariant under inversion, that is,

$$\bigwedge_{g\in G} F(g) = \bigwedge_{g\in G} F(g^{-1});$$

this can be achieved by replacing Λ if necessary with $\tilde{\Lambda}$, where

$$\tilde{\bigwedge}_{g\in G} F(g) = \bigwedge_{g\in G} \frac{F(g) + F(g^{-1})}{2}.$$

We now require two lemmas, the first of which is standard. Throughout, when $\mathcal{C}$ is a unital C^*-algebra, $\mathcal{U}(\mathcal{C})$ denotes the unitary group of $\mathcal{C}$.

Lemma 1.3.1. *Let X be a Banach space and let Λ be an invariant mean on the (discrete) group $\mathcal{U}(\mathcal{D})$. Suppose that $f : \mathcal{U}(\mathcal{D}) \to X^\#$ is a bounded function. Then there exists $T \in \overline{co}^{\text{weak-}*}\{f(U) : U \in \mathcal{U}(\mathcal{D})\}$ such that for every $x \in X$,*

$$\langle x, T\rangle = \bigwedge_{U} \langle x, f(U)\rangle.$$

Proof. The existence of T follows from the fact that the map $X \ni x \mapsto \Lambda_U \langle x, f(U)\rangle$ is a bounded linear functional on X. For every $x \in X$, $\langle x, T\rangle$ belongs to the closed convex hull of $\{\langle x, f(U)\rangle : U \in \mathcal{U}(\mathcal{D})\}$. So a separation theorem shows that $T \in \overline{\text{co}}^{\text{weak-}*}\{f(U) : U \in \mathcal{U}(\mathcal{D})\}$. □

Notation 1.3.2. In the setting of Lemma 1.3.1, we write $T := \Lambda_U f(U)$.

The following well-known fact appears as [3, Theorem 6.2.1]. Since it will be useful in the sequel, we include a proof for the convenience of the reader.

Lemma 1.3.3. *For $T \in \mathcal{M}$,*

$$E(T) = \bigwedge_{U\in\mathcal{U}(\mathcal{D})} UTU^*$$

and

$$\{E(T)\} = \mathcal{D} \cap \overline{co}^{\sigma\text{-weak}}\{UTU^* : U \in \mathcal{U}(\mathcal{D})\}.$$

Proof. For $T \in \mathcal{M}$, set $E_1(T) = \Lambda_{U\in\mathcal{U}(\mathcal{D})} UTU^*$. Given $\rho \in \mathcal{M}_*$, and $W \in \mathcal{U}(\mathcal{D})$, we have

$$\begin{aligned}\rho(WE_1(T)) &= \bigwedge_{U\in\mathcal{U}(\mathcal{D})} \rho(WUTU^*)\\ &= \bigwedge_{U\in\mathcal{U}(\mathcal{D})} \rho((WU)T(WU)^*W)\end{aligned}$$

$$= \bigwedge_{U\in\mathcal{U}(\mathcal{D})} \rho(UTU^*W)$$

$$= \rho(E_1(T)W).$$

Therefore $E_1(T)$ commutes with $\mathcal{U}(\mathcal{D})$. But $\mathcal{D}$ is the linear span of $\mathcal{U}(\mathcal{D})$, so $E_1(T) \in \mathcal{D}' \cap \mathcal{M}$. Since $\mathcal{D}$ is a MASA in $\mathcal{M}$, $E_1(T) \in \mathcal{D}$. The normality of E and the fact that $E(UTU^*) = E(T)$ for each $U \in \mathcal{U}(\mathcal{D})$ yield

$$\begin{aligned}\{E_1(T)\} &\subseteq \mathcal{D} \cap \overline{\text{co}}^{\sigma\text{-weak}}\{UTU^* : U \in \mathcal{U}(\mathcal{D})\} \\ &= E(\mathcal{D} \cap \overline{\text{co}}^{\sigma\text{-weak}}\{UTU^* : U \in \mathcal{U}(\mathcal{D})\}) \\ &\subseteq E(\overline{\text{co}}^{\sigma\text{-weak}}\{UTU^* : U \in \mathcal{U}(\mathcal{D})\}) \\ &= \{E(T)\}.\end{aligned}$$ □

The following result, together with Lemma 1.2.1, shows that any $\mathcal{D}$-bimodule in $\mathcal{M}$ which is closed in an appropriate topology contains an abundance of groupoid normalizers. The technique used here has been employed previously in several articles, for example, see [14, Proposition 4.4] or [7, Proposition 3.10].

Proposition 1.3.4. *Let $\mathcal{S} \subseteq \mathcal{M}$ be a σ-weakly closed $\mathcal{D}$-bimodule. If $v \in \mathcal{N}(\mathcal{M}, \mathcal{D})$ and $T \in \mathcal{S}$, then $vE(v^*T) \in \mathcal{S}$, and when $T \neq 0$, $v \in \mathcal{N}(\mathcal{M}, \mathcal{D})$ may be chosen so that $vE(v^*T) \neq 0$. In particular, if $\mathcal{S}$ is nonzero, then $(\mathcal{S}\backslash\{0\}) \cap \mathcal{N}(\mathcal{M}, \mathcal{D}) \neq \emptyset$.*

Proof. If $v \in N(\mathcal{M}, \mathcal{D})$ and $T \in \mathcal{S}$, then Lemma 1.3.3 shows that

$$\begin{aligned}\{vE(v^*T)\} &\subseteq v\, \overline{co}^{\sigma\text{-weak}}\{Uv^*TU^* : U \in \mathcal{U}(\mathcal{D})\} \\ &\subseteq \overline{co}^{\sigma\text{-weak}}\{(vUv^*)TU^* : U \in \mathcal{U}(\mathcal{D})\} \subseteq \mathcal{S}\end{aligned}$$

(because $vUv^* \in \mathcal{D}$).

If $T \in \mathcal{M}$ satisfies $E(v^*T) = 0$ for every $v \in \mathcal{N}(\mathcal{M}, \mathcal{D})$, then $T = 0$. Indeed, for every $x \in \operatorname{span} \mathcal{N}(\mathcal{M}, \mathcal{D})$, $E(x^*T) = 0$. By normality of E, we conclude that $E(T^*T) = 0$. As E is faithful, $T = 0$.

If $0 \neq T \in \mathcal{M}$ and $v \in \mathcal{N}(\mathcal{M}, \mathcal{D})$ satisfies $E(v^*T) \neq 0$, then $vE(v^*T) \neq 0$. To see this, argue by contradiction. If $vE(v^*T) = 0$, then $(v^*v)^nE(v^*T) = 0$ for every $n \in \mathbb{N}$. Therefore, $(v^*v)^{1/n}E(v^*T) = 0$ for every $n \in \mathbb{N}$. But

$$0 \neq E(v^*T) = \lim_{n\to\infty} E((v^*v)^{1/n}v^*T) = \lim_{n\to\infty} (v^*v)^{1/n}E(v^*T) = 0,$$

which is absurd. Thus $vE(v^*T) \neq 0$, and the proof is complete. □

We now give a slight generalization of a result appearing in [12]. We use it throughout the paper, often without explicit mention. We include the proof because it seems novel.

Lemma 1.3.5 ([12, Lemma 2.1])**.** *Let $v \in \mathcal{N}(\mathcal{M}, \mathcal{D})$. Then for every $x \in \mathcal{M}$,*

$$E(v^*xv) = v^*E(x)v.$$

Proof. We prove this in several steps.

Step 1. First, assume that v is a unitary normalizer. Since $v^*\mathcal{U}(\mathcal{D})v = \mathcal{U}(\mathcal{D})$, Lemma 1.3.3 gives

$$\begin{aligned}\{E(v^*xv)\} &= \overline{co}^{\sigma\text{-weak}}\{U^*v^*xvU : U \in \mathcal{U}(\mathcal{D})\} \cap \mathcal{D} \\ &= \overline{co}^{\sigma\text{-weak}}\{v^*(vU^*v^*)x(vUv^*)v : U \in \mathcal{U}(\mathcal{D})\} \cap \mathcal{D} \\ &= [v^*\left(\overline{co}^{\sigma\text{-weak}}\{(vU^*v^*)x(vUv^*) : U \in \mathcal{U}(\mathcal{D})\}\right)v] \cap v^*\mathcal{D}v \\ &= v^*[\left(\overline{co}^{\sigma\text{-weak}}\{(vU^*v^*)x(vUv^*) : U \in \mathcal{U}(\mathcal{D})\}\right) \cap \mathcal{D}]v \\ &= \{v^*E(x)v\}.\end{aligned}$$

Thus the lemma holds in this case.

Step 2. Next, assume v is a partial isometry. Then

$$V := \begin{pmatrix} v & (I - vv^*) \\ (I - v^*v) & v^* \end{pmatrix}$$

is a unitary element of $M_2(\mathcal{M}) = \mathcal{M} \otimes M_2(\mathbb{C})$. Let

$$D_2(\mathcal{D}) := \left\{ \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} : d_i \in \mathcal{D} \right\}.$$

Then $(M_2(\mathcal{M}), D_2(\mathcal{D}))$ is a Cartan pair, and the conditional expectation is the map E_2 given by

$$M_2(\mathcal{M}) \ni \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \mapsto \begin{pmatrix} E(y_{11}) & 0 \\ 0 & E(y_{22}) \end{pmatrix}.$$

A simple calculation using the fact that $vv^*, v^*v \in \mathcal{D}$ shows that V belongs to $\mathcal{N}(M_2(\mathcal{M}), D_2(\mathcal{D}))$. By Step 1, we have, for $X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $E_2(V^*XV) = V^*E_2(X)V$. The equality of the upper-left corner entries of these matrices yields $E(v^*xv) = v^*E(x)v$.

Step 3. Finally, assume that v is a general normalizer. Let $v = u|v|$ be the polar decomposition of v. Then u is a partial isometry normalizer, by Lemma 1.2.1. Since $|v| \in \mathcal{D}$, we have

$$E(v^*xv) = |v|E(u^*xu)|v| = |v|u^*E(x)u|v| = v^*E(x)v. \qquad \square$$

1.4. A MASA. Here we show that when $(\mathcal{M}, \mathcal{D})$ is in the standard form arising from a suitable weight, then the von Neumann algebra generated by $\mathcal{D}$ and $\mathcal{D}'$ is a MASA. As a corollary, we show that $\mathcal{D}$ norms $\mathcal{M}$, in the sense of Pop–Sinclair–Smith [18]. Note that these observations are implicit in [20] when the von Neumann algebra $\mathcal{M}$ is assumed to be finite and have separable predual.

Fix a faithful, normal, semifinite weight ϕ on $\mathcal{M}$ such that $\phi \circ E = \phi$. (If ω is a faithful, normal, semifinite weight on $\mathcal{D}$, then $\phi = \omega \circ E$ is such a weight on $\mathcal{M}$, see [22, Proposition IX.4.3].) We freely use notation from [22]: in particular,

$$\mathfrak{n}_\phi := \{T \in \mathcal{M} : \phi(T^*T) < \infty\},$$

and $(\pi_\phi, \mathfrak{H}_\phi, \eta_\phi)$ is the semi-cyclic representation associated to ϕ. (See [22, VII.1] for more details.) Since $E(T)^*E(T) \leq E(T^*T)$ for every $T \in \mathcal{M}$, we have $E(\mathfrak{n}_\phi) = \mathfrak{n}_\phi \cap \mathcal{D}$.

Lemma 1.4.1. *Let* $\Gamma = \{d \in \mathfrak{n}_\phi \cap \mathcal{D} : 0 \leq d \leq I\}$ *and view* Γ *as a net indexed by itself. Then for* $x \in \mathfrak{n}_\phi$*,* $\lim_{d\in\Gamma} \eta_\phi(xd) = \eta_\phi(x)$*.*

Proof. Let S be the spectral measure for $E(x^*x)$, and let μ be the (finite) Borel measure on $[0, \infty)$ defined by $\mu(A) = \phi(E(x^*x)S(A))$. Then $\lim_{t\to 0} \mu([0, t)) = \mu(\{0\}) = 0$, so given $\varepsilon > 0$ we may find $t > 0$ so that $\mu([0, t)) < \varepsilon^2$. Since $tS([t, \infty)) \leq E(x^*x)$, we obtain $p := S([t, \infty)) \in \Gamma$. For $d \in \Gamma$ with $d \geq p$, we have

$$\begin{aligned}\|\eta_\phi(x) - \eta_\phi(xd)\|^2 = \phi(E(x^*x)(I - d)^2) &\leq \phi(E(x^*x)(I - p)) \\ &= \mu([0, t)) < \varepsilon^2.\end{aligned}$$

□

Corollary 1.4.2. *Given* $\varepsilon > 0$ *and* $\zeta \in \mathfrak{H}_\phi$*, there exists* $d \in \mathfrak{n}_\phi \cap \mathcal{D}$ *and* $y \in \operatorname{span} \mathcal{N}(\mathcal{M}, \mathcal{D})$ *such that*

$$\|\zeta - \eta_\phi(yd)\| < \varepsilon.$$

Proof. Since $\eta_\phi(\mathfrak{n}_\phi)$ is dense in $\mathfrak{H}_\phi$, we may find $x \in \mathfrak{n}_\phi$ such that

$$\|\zeta - \eta_\phi(x)\| < \varepsilon/3.$$

By Lemma 1.4.1, there exists $d \in \mathfrak{n}_\phi \cap \mathcal{D}$ such that

$$0 \leq d \leq I \quad \text{and} \quad \|\eta_\phi(x) - \eta_\phi(xd)\| < \varepsilon/3.$$

Let $\mathcal{M}_0 := \operatorname{span} \mathcal{N}(\mathcal{M}, \mathcal{D})$. Then $\mathcal{M}_0$ is a unital $*$-algebra which is σ-strongly dense in $\mathcal{M}$. Thus we may find $y \in \operatorname{span} \mathcal{N}(\mathcal{M}, \mathcal{D})$ such that

$$\|\eta_\phi(xd) - \eta_\phi(yd)\| = \sqrt{\langle \pi_\phi((x - y)^*(x - y))\eta_\phi(d), \eta_\phi(d)\rangle} < \varepsilon/3.$$

It follows that $\|\zeta - \eta_\phi(yd)\| < \varepsilon$. □

Since $\phi \circ E = \phi$, $\mathfrak{n}_\phi$ and $\mathfrak{n}_\phi^*$ are $\mathcal{D}$-bimodules and furthermore, for $D \in \mathcal{D}$, $x \in \mathfrak{n}_\phi$ and $y \in \mathfrak{n}_\phi^*$,

$$\max\{\phi((Dx)^*(Dx)), \phi((xD)^*(xD))\} \leq \|D\|^2 \phi(x^*x), \tag{1.1}$$

$$\max\{\phi((Dy^*)^*(Dy^*)), \phi((y^*D)^*(y^*D))\} \leq \|D\|^2 \phi(yy^*). \tag{1.2}$$

In particular, for $D \in \mathcal{D}$, the maps on $\eta_\phi(\mathfrak{n}_\phi)$ given by

$$\pi_\ell(D)\eta_\phi(x) = \eta_\phi(Dx) \quad \text{and} \quad \pi_r(D)\eta_\phi(x) = \eta_\phi(xD)$$

extend to bounded operators $\pi_\ell(D)$ and $\pi_r(D)$ on $\mathfrak{H}_\phi$. This produces $*$-representations π_ℓ and π_r of $\mathcal{D}$ on $\mathfrak{H}_\phi$. Clearly,

$$\pi_\ell = \pi_\phi|_{\mathcal{D}}.$$

The relationship between π_ℓ and π_r is given by Lemma 1.4.3 below, whose proof is joint work with Adam Fuller. The image of $\mathcal{M}$ under π_ϕ acts on $\mathfrak{H}_\phi$ in standard form, and we write J for the modular conjugation operator.

Lemma 1.4.3. *For each $D \in \mathcal{D}$,*

$$J\pi_\ell(D)J = \pi_r(D^*).$$

Proof. Throughout the proof, we will freely use notation from [22], sometimes without explicit mention.

Let $\mathfrak{A}_\phi$ be the full left Hilbert algebra $\eta_\phi(\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)$ (see [22, Thm. VII.2.6]). For $x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$ and $D \in \mathcal{D}$,

$$\pi_\ell(D)(\eta_\phi(x)^\sharp) = \eta_\phi(Dx^*) = \eta_\phi(xD^*)^\sharp = (\pi_r(D^*)\eta_\phi(x))^\sharp. \tag{1.3}$$

The estimates (1.1) and (1.2) combined with [22, Lemma VI.1.4] yield that $\mathfrak{D}^\sharp$ is invariant under $\pi_\ell(D)$ and $\pi_r(D^*)$. Thus, (1.3) implies that for $\xi \in \mathfrak{D}^\sharp$, $\pi_\ell(D)S\xi = S\pi_r(D^*)\xi$; similarly, $S\pi_\ell(D)\xi = \pi_r(D^*)S\xi$. Hence

$$\pi_\ell(D)S = S\pi_r(D^*) \quad \text{and} \quad S\pi_\ell(D) = \pi_r(D^*)S. \tag{1.4}$$

Since $\mathfrak{D}^\flat = \{\zeta \in \mathfrak{H}_\phi : \mathfrak{D}^\sharp \ni \xi \mapsto \langle \zeta, S\xi\rangle \text{ is bounded}\}$, we see that $\mathfrak{D}^\flat$ is also invariant under $\pi_\ell(D^*)$ and $\pi_r(D)$. Next, [22, Lemma VI.1.5(ii)] yields,

$$F\pi_\ell(D^*) = \pi_r(D)F \quad \text{and} \quad \pi_\ell(D^*)F = F\pi_r(D). \tag{1.5}$$

Therefore,

$$\Delta\pi_\ell(D) = FS\pi_\ell(D) = F\pi_r(D^*)S = \pi_\ell(D)FS = \pi_\ell(D)\Delta.$$

We thus obtain,

$$\Delta^{1/2}\pi_\ell(D) = \pi_\ell(D)\Delta^{1/2}.$$

By [22, Lemma VI.1.5(v)], for $\xi \in \mathfrak{D}(\Delta^{1/2}) = \mathfrak{D}^\sharp$,

$$\pi_r(D^*)\xi = S\pi_\ell(D)S\xi = J\Delta^{1/2}\pi_\ell(D)\Delta^{-1/2}J\xi = J\pi_\ell(D)J\xi.$$

Since $\mathfrak{D}^\sharp$ is dense in $\mathfrak{H}_\phi$ and $\{\pi_r(D^*), J\pi_\ell(D)J\} \subseteq \mathcal{B}(\mathfrak{H}_\phi)$, the lemma follows. $\square$

Notation 1.4.4. Let

$$\mathcal{Z} := (\pi_\ell(\mathcal{D}) \cup \pi_r(\mathcal{D}))''.$$

Our first task is to show that $\mathcal{Z}$ is a MASA in $\mathcal{B}(\mathfrak{H}_\phi)$. While this is established in [8, Theorem 1 and Proposition 2.9(1)], we provide an alternate proof (also see [19]). Our proof has the advantage that it avoids some of the measure-theoretic issues of the Feldman–Moore approach, and does not require the separability of $\mathcal{M}_*$.

Notation 1.4.5. Denote by P the projection on $\mathfrak{H}_\phi$ determined by extending the map $\eta_\phi(\mathfrak{n}_\phi) \ni \eta_\phi(x) \mapsto \eta_\phi(E(x))$ by continuity. A calculation shows that for any $D \in \mathcal{D}$,

$$\pi_\ell(D)P = \pi_r(D)P = P\pi_r(D) = P\pi_\ell(D). \tag{1.6}$$

Lemma 1.4.6. *For $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, set*

$$P_v = \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \pi_\ell(vUv^*)\pi_r(U^*) \in \mathcal{B}(\mathfrak{H}_\phi).$$

Then $P_v \in \mathcal{Z}$ and the following statements hold.

(a) $P_v = \pi_\phi(v)P\pi_\phi(v)^*$.

(b) *P_v is the orthogonal projection onto $\overline{\{\eta_\phi(vd) : d \in \mathfrak{n}_\phi \cap \mathcal{D}\}}$, and for $x \in \mathfrak{n}_\phi$,*

$$P_v\eta_\phi(x) = \eta_\phi(vE(v^*x)). \tag{1.7}$$

(c) *If $\xi \in \operatorname{range}(P_v)$, then there exists $h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that P_h is the projection onto $\overline{\mathcal{Z}\xi}$ and $P_h \leq P_v$.*

(d) *If $v, w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, then $P_v \perp P_w$ if and only if $E(v^*w) = 0$.*

Proof. Since $v \in \mathcal{N}(\mathcal{M}, \mathcal{D})$, we have $vUv^* \in \mathcal{D}$ for every $U \in \mathcal{U}(\mathcal{D})$. Hence the function $f(U) = \pi_\ell(vUv^*)\pi_r(U^*)$ maps $\mathcal{U}(\mathcal{D})$ into $\mathcal{Z}$, so Lemma 1.3.1 shows that $P_v \in \mathcal{Z}$.

Let $d \in \mathfrak{n}_\phi \cap \mathcal{D}$ satisfy $0 \leq d \leq I$. For $x, y \in \mathfrak{n}_\phi$,

$$\begin{aligned}
\langle P_v\eta_\phi(x), \eta_\phi(yd)\rangle &= \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \langle \pi_\ell(vUv^*)\pi_r(U^*)\eta_\phi(x), \eta_\phi(yd)\rangle \\
&= \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \langle \eta_\phi(vUv^*xU^*), \pi_r(d)\eta_\phi(y)\rangle \\
&= \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \langle \pi_r(d)\eta_\phi(vUv^*xU^*), \eta_\phi(y)\rangle \\
&= \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \langle \eta_\phi(vUv^*xU^*d), \eta_\phi(y)\rangle \\
&= \bigwedge_{U \in \mathcal{U}(\mathcal{D})} \langle \pi_\phi(vUv^*xU^*)\eta_\phi(d), \eta_\phi(y)\rangle \\
&= \langle \pi_\phi(vE(v^*x))\eta_\phi(d), \eta_\phi(y)\rangle \\
&= \langle \eta_\phi(vE(v^*x)d), \eta_\phi(y)\rangle \\
&= \langle \pi_r(d)\eta_\phi(vE(v^*x)), \eta_\phi(y)\rangle \\
&= \langle \eta_\phi(vE(v^*x)), \pi_r(d)\eta_\phi(y)\rangle \\
&= \langle \eta_\phi(vE(v^*x)), \eta_\phi(yd)\rangle.
\end{aligned}$$

The equality (1.7) of part (b) now follows from Lemma 1.4.1. The remainder of part (b) follows from Equation (1.7), which in turn implies (a).

Turning now to the proof of (c), suppose that $\xi \in \operatorname{range}(P_v)$. Then $\xi = \pi_\phi(v)\zeta$ for some $\zeta \in \operatorname{range}(P)$. For $d_1 \in \mathcal{D}$ and $d_2 \in \mathfrak{n}_\phi \cap \mathcal{D}$, we have that

$$\pi_\ell(d_1)\pi_\phi(v)\eta_\phi(d_2) = \eta_\phi(d_1 v d_2) = \eta_\phi(v d_2 v^* d_1 v) = \pi_r(v^* d_1 v)\pi_\phi(v)\eta_\phi(d_2).$$

Since $\eta(\mathfrak{n}_\phi \cap \mathcal{D})$ is dense in $\operatorname{range}(P)$, it follows that

$$\pi_\ell(d_1)\xi = \pi_\ell(d_1)\pi_\phi(v)\zeta = \pi_r(v^* d_1 v)\pi_\phi(v)\zeta = \pi_r(v^* d_1 v)\xi,$$

so $\pi_\ell(\mathcal{D})\xi \subseteq \pi_r(\mathcal{D})\xi$. Likewise $\pi_r(d_1)\xi = \pi_\ell(v d_1 v^*)\xi$, so $\pi_r(\mathcal{D})\xi \subseteq \pi_\ell(\mathcal{D})\xi$. The fact that $\mathcal{Z}$ is generated by $\pi_\ell(\mathcal{D})$ and $\pi_r(\mathcal{D})$ yields

$$\overline{\pi_\ell(\mathcal{D})\xi} = \overline{\pi_r(\mathcal{D})\xi} = \overline{\mathcal{Z}\xi}.$$

We claim $\pi_r(\mathcal{D})|_{\operatorname{range}(P_v)}$ is a MASA in $\mathcal{B}(\operatorname{range}(P_v))$. First, $\pi_\ell(\mathcal{D})|_{\operatorname{range}(P)}$ is a MASA in $\mathcal{B}(\operatorname{range}(P))$, since $\pi_\ell(\cdot)|_{\operatorname{range}(P)}$ is unitarily equivalent to π_ω, the semi-cyclic representation of $\mathcal{D}$ corresponding to $\omega := \phi|_{\mathcal{D}}$. (The implementing unitary $U : \operatorname{range}(P) \to \mathfrak{H}_\omega$ maps $\eta_\phi(d)$ to $\eta_\omega(d)$ for all $d \in \mathfrak{n}_\phi \cap \mathcal{D}$.) It follows that $\pi_\ell(\mathcal{D})|_{\pi_\ell(v^*v)\operatorname{range}(P)}$ is a MASA in $\mathcal{B}(\pi_\ell(v^*v)\operatorname{range}(P))$. Now $\pi_r(\cdot)|_{\operatorname{range}(P_v)}$ is unitarily equivalent to $\pi_\ell(\cdot)|_{\pi_\ell(v^*v)\operatorname{range}(P)}$. (The implementing unitary $V : \operatorname{range}(P_v) \to \pi_\ell(v^*v)\operatorname{range}(P)$ maps $\eta_\phi(vd)$ to $\pi_\ell(v^*v)\eta_\phi(d)$ for all $d \in \mathfrak{n}_\phi \cap \mathcal{D}$.) This establishes the claim.

Now let $Q \in \mathcal{B}(\operatorname{range}(P_v))$ be the orthogonal projection onto $\overline{\pi_r(\mathcal{D})\xi}$. Then $Q \in (\pi_r(\mathcal{D})|_{\operatorname{range}(P_v)})' = \pi_r(\mathcal{D})|_{\operatorname{range}(P_v)}$, and so there exists a projection $q \in \mathcal{D}$ such that $Q = \pi_r(q)|_{\operatorname{range}(P_v)}$. Define $h = vq$. Then $h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, and we have

$$\begin{aligned} \operatorname{range}(P_h) &= \pi_\phi(h)\operatorname{range}(P) = \pi_\phi(vq)\operatorname{range}(P) = \pi_\phi(v)\pi_\ell(q)\operatorname{range}(P) \\ &= \pi_\phi(v)\pi_r(q)\operatorname{range}(P) = \pi_r(q)\pi_\phi(v)\operatorname{range}(P) = \pi_r(q)\operatorname{range}(P_v) \\ &= \operatorname{range}(Q) = \overline{\pi_r(\mathcal{D})\xi} = \overline{\mathcal{Z}\xi}. \end{aligned}$$

The fact that $P_h \leq P_v$ follows from the facts that both are projections and $\operatorname{range}(P_h) \subseteq \operatorname{range}(P_v)$.

Finally we prove (d). For $v, w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and $d_1, d_2 \in \mathfrak{n}_\phi \cap \mathcal{D}$, we have that

$$\begin{aligned} \langle \eta_\phi(vd_1), \eta_\phi(wd_2)\rangle &= \phi(d_2^* w^* v d_1) = \phi(E(d_2^* w^* v d_1)) = \phi(d_2^* E(w^*v) d_1) \\ &= \omega(d_2^* E(w^*v) d_1) = \langle \pi_\omega(E(w^*v))\eta_\omega(d_1), \eta_\omega(d_2)\rangle, \end{aligned}$$

and so $P_v \perp P_w$ if and only if $E(w^*v) = 0$. □

Theorem 1.4.7. *The algebra $\mathcal{Z}$ is a MASA in $\mathcal{B}(\mathfrak{H}_\phi)$.*

Proof. Let $0 \neq Q \in \mathcal{Z}'$ be a projection. We first show there exists $0 \neq h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ so that $P_h \leq Q$.

Let ζ be a unit vector in the range of Q. Corollary 1.4.2 implies that there exists $w \in \mathcal{N}(\mathcal{M}, \mathcal{D})$ and $d \in \mathfrak{n}_\phi \cap \mathcal{D}$ so that $\langle \zeta, \eta_\phi(wd)\rangle \neq 0$. Writing the polar decomposition, $w = v|w|$, we have $\eta_\phi(wd) = \pi_\phi(v)\eta_\phi(|w|d) \in$

range(P_v). Hence $P_v\zeta \neq 0$. By Lemma 1.4.6, $\overline{\mathcal{Z}P_v\zeta}$ is the range of P_h for some $h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, and as range($Q$) is invariant for $\mathcal{Z}$, we have $P_h \leq Q$.

As $P_h \in \mathcal{Z} \subseteq \mathcal{Z}'$, $Q - P_h \in \mathcal{Z}'$. A Zorn's Lemma argument now yields a maximal family $A \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that (a) $\{P_v : v \in A\}$ is a pairwise orthogonal family of projections; and (b) $P_v \leq Q$ for each $v \in A$. The maximality of A ensures that $\bigvee_{v\in A} P_v = Q$. As each $P_v \in \mathcal{Z}$, we conclude that $Q \in \mathcal{Z}$ as well. Therefore $\mathcal{Z}$ is a MASA. □

The following extends part of [8, Proposition 2.8] to our context.

Corollary 1.4.8. *Let Δ be the modular operator and $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ be the modular automorphism group . Then for each $t \in \mathbb{R}$, $\Delta^{it} \in \mathcal{U}(\mathcal{Z})$. Moreover, $\sigma_t^\phi|_{\mathcal{D}} = \mathrm{id}|_{\mathcal{D}}$ and for $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $h := v^*\sigma_t^\phi(v)$ is a partial isometry in $\mathcal{D}$ and $\sigma_t^\phi(v) = vh$.*

Proof. The proof of Lemma 1.4.3 shows that Δ commutes with each element of $\pi_\ell(\mathcal{D})$, hence for each $t \in \mathbb{R}$, $\Delta^{it} \in \pi_\ell(\mathcal{D})'$. Since $J\Delta J = \Delta^{-1}$ ([22, LemmaVI.1.5(v)]), Lemma 1.4.3 implies that $\Delta^{it} \in \pi_r(\mathcal{D})'$. Hence $\Delta^{it} \in \mathcal{Z}' = \mathcal{Z}$.

For $D \in \mathcal{D}$, $\pi_\phi(\sigma_t^\phi(D)) = \Delta^{it}\pi_\ell(D)\Delta^{-it} = \pi_\phi(D)$, so σ_t^ϕ fixes each element of $\mathcal{D}$. Let $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and fix $t \in \mathbb{R}$. Set $w = \sigma_t^\phi(v)$. We show that $v^*w \in \mathcal{D}$ and that $w = v(v^*w) \in v\mathcal{D}$. To see this, observe that for $d \in \mathcal{D}$ we have,

$$wdw^* = \sigma_t^\phi(vdv^*) = vdv^*.$$

Therefore for $d \in \mathcal{D}$,

$$v^*wd = v^*(wdw^*)w = v^*(vdv^*)w = dv^*w.$$

Since $\mathcal{D}$ is a MASA in $\mathcal{M}$, $v^*w \in \mathcal{D}$. Finally, $w = (ww^*)w = v(v^*w)$, as desired. □

We now turn to showing that $\mathcal{D}$ norms $\mathcal{M}$. We need some general preparation. Recall that if $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra of operators, then $\mathcal{C}$ is *locally cyclic* if, for any $\varepsilon > 0$, $n \in \mathbb{N}$, and vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}$, there is a vector $\zeta \in \mathcal{H}$ and elements $T_1, \ldots, T_n \in \mathcal{C}$ such that for $1 \leq i \leq n$, $\|T_i\zeta - \xi_i\| < \varepsilon$.

In our context, $\pi_\phi(\mathcal{M})$ is locally cyclic. Indeed, we may find $x_i \in \mathfrak{n}_\phi$ with $\|\eta_\phi(x_i) - \xi_i\| < \varepsilon/2$. Lemma 1.4.1 yields $d \in \mathcal{D} \cap \mathfrak{n}_\phi$ with

$$\|\eta_\phi(x_i) - \eta_\phi(x_i d)\| < \varepsilon/2$$

for $1 \leq i \leq n$; then $\|\pi_\phi(x_i)\eta_\phi(d) - \xi_i\| < \varepsilon$.[1] Also, when $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ is a MASA, $\mathcal{C}$ is locally cyclic. This can be proved directly, or one can argue as follows. Decompose $\mathcal{H}$ into an orthogonal sum of cyclic subspaces, $\mathcal{H} = \bigoplus_{i\in I} \overline{\mathcal{C}u_i}$ where $\{u_i\}_{i\in I} \subseteq \mathcal{H}$ is a family of unit vectors. As in the proof of [22, Theorem VII.2.7], define a faithful normal semi-finite weight ϕ on the positive cone of $\mathcal{C}$ by $\phi(T) = \sup\{\sum_{i\in F} \langle Tu_i, u_i\rangle : F \subseteq I \text{ is finite}\}$. Then

[1]A similar argument can be used to show that whenever a von Neumann algebra is in standard form, it is locally cyclic; we do not need that fact here.

the identity representation of $\mathcal{C}$ is unitarily equivalent to the semi-cyclic representation $(\pi_\phi, \mathcal{H}_\phi, \eta_\phi)$ and hence $\mathcal{C}$ is locally cyclic because $(\mathcal{C}, \mathcal{C})$ is a Cartan pair.

The following is the analog of [16, Lemma 2.15] for Cartan pairs.

Corollary 1.4.9. *If* $(\mathcal{M}, \mathcal{D})$ *is a Cartan pair, then* $\mathcal{D}$ *norms* $\mathcal{M}$ *in the sense of Pop–Sinclair–Smith* [18].

Proof. The proof is an adaptation of the proof of [20, Proposition 4.1], with the algebras $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ of [20, Proposition 4.1] taken to be $\pi_\phi(\mathcal{M})$, $\pi_\ell(\mathcal{D})$ and $\pi_r(\mathcal{D})$ respectively.

Since $\mathcal{Z}$ is a MASA in $\mathcal{B}(\mathfrak{H}_\phi)$, it norms $\mathcal{B}(\mathfrak{H}_\phi)$ by [18, Theorem 2.7]. Then $C^*(\mathcal{A}, \mathcal{B})$ norms $\mathcal{B}(\mathfrak{H}_\phi)$ ([18, Lemma 2.3(c)]).

Let $X \in M_n(\pi_\phi(\mathcal{M}))$ satisfy $\|X\| = 1$ and let $\varepsilon > 0$. Then there exist $C_1, C_2 \in M_{n,1}(C^*(\mathcal{A}, \mathcal{B}))$ such that

$$\max\{\|C_1\|, \|C_2\|\} < 1 \quad \text{and} \quad \|C_2^* X C_1\| > 1 - \varepsilon. \tag{1.8}$$

The proof now continues exactly as in the proof of [20, Proposition 4.1]: replace the inequality (4.2) of [20] with (1.8) and continue the Sinclair-Smith argument from there to show that $\pi_\ell(\mathcal{D}) = \pi_\phi(\mathcal{D})$ norms $\pi_\phi(\mathcal{M})$. □

2. A spectral theorem for bimodules

In this section, we provide a description of the support of a $\mathcal{D}$-bimodule in terms of a projection in $\mathcal{Z}$, then use this to characterize $\mathcal{D}$-bimodules closed in an appropriate topology.

2.1. The support of a bimodule.

Definition 2.1.1. For any set $A \subseteq \mathcal{M}$, let $\langle A \rangle$ be the $\mathcal{D}$-bimodule generated by A.

(a) Given a $\mathcal{D}$-bimodule (not necessarily closed) $\mathcal{S} \subseteq \mathcal{M}$, let

$$\operatorname{supp}(\mathcal{S}) \in \mathcal{B}(\mathfrak{H}_\phi)$$

be the orthogonal projection onto $\overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$, a $\mathcal{Z}$-invariant subspace. Because of this $\mathcal{Z}$-invariance, $\operatorname{supp}(\mathcal{S})$ is a projection in $\mathcal{Z}$.

(b) For $T \in \mathcal{M}$, we define the *support of* T, $\operatorname{supp}(T)$, to be the projection $\operatorname{supp}(\langle T \rangle) \in \mathcal{Z}$.

(c) Given a projection $Q \in \mathcal{Z}$, the set

$$\operatorname{bimod}(Q) := \{T \in \mathcal{M} : \operatorname{supp}(T) \leq Q\}$$

is a $\mathcal{D}$-bimodule.

Remark 2.1.2. The purpose of this remark is to outline the relationship between the notion of support of a bimodule given above with the notion of support of a bimodule found in [15]. For this, assume that $\mathcal{M}_*$ is separable, that ϕ is a faithful normal state on $\mathcal{M}$ and use the notation found in [8]. By [8, Theorem 1], there exists a countable, standard equivalence relation R on a finite measure space $(X, \mathcal{B}, \mu)$, a cocycle $\sigma \in H^2(R, \mathbb{T})$, and an isomorphism of $\mathcal{M}$ onto $\mathbf{M}(R,\sigma)$ which carries $\mathcal{D}$ onto the diagonal subalgebra $\mathbf{A}(R,\sigma)$ of $\mathbf{M}(R,\sigma)$. We may therefore assume that $\mathcal{M} = \mathbf{M}(R,\sigma)$ and that $\mathcal{D} = \mathbf{A}(R,\sigma)$. With this identification, $\mathcal{M}$ acts on the separable Hilbert space $L^2(R,\nu)$, where ν is the right counting measure associated with μ. By [8, Proposition 2.9], $J\mathcal{D}J$ is an abelian subalgebra of $\mathcal{M}'$ and $\mathcal{Z} = (J\mathcal{D}J \vee \mathcal{D})''$ is a MASA in $\mathcal{B}(L^2(R,\nu))$, with cyclic vector χ_Δ (here $\Delta = \{(x,x) : x \in X\} \subseteq R$). Each element $a \in \mathbf{M}(R,\sigma)$ determines a measurable function $a\chi_\Delta$ on R, and the support of such a function is a measurable subset of R determined uniquely up to null sets. Projections in $\mathcal{Z}$ are in one-to-one correspondence with ν-measurable subsets of R modulo null sets, so we may as well regard the support of an element of $\mathbf{M}(R,\sigma)$ as a projection in $\mathcal{Z}$. The support of the $\mathcal{D}$-bimodule $\mathcal{S}$ is the join of the support projections of the elements of $\mathcal{S}$. Thus, Definition 2.1.1 is a reformulation of the definition of the support of a $\mathcal{D}$-bimodule from [15], but with the measure-theoretic considerations suppressed.

The following observations will be used in the sequel.

Lemma 2.1.3. *Let $h \in \mathcal{GN}(\mathcal{M},\mathcal{D})$. Then* $\operatorname{supp}(h) = P_h$.

Proof. Clearly

$$\operatorname{range}(P_h) = \overline{\pi_\phi(h)(\mathfrak{n}_\phi \cap \mathcal{D})} \subseteq \overline{\pi_\phi(\langle h\rangle)(\mathfrak{n}_\phi \cap \mathcal{D}))},$$

and so $P_h \le \operatorname{supp}(h)$. Conversely, since $\langle h\rangle = \{hd : d \in \mathcal{D}\}$,

$$\overline{\pi_\phi(\langle h\rangle)(\mathfrak{n}_\phi \cap \mathcal{D}))} \subseteq \overline{\pi_\phi(h)(\mathfrak{n}_\phi \cap \mathcal{D})} = \operatorname{range}(P_h),$$

and so $\operatorname{supp}(h) \le P_h$. □

Lemma 2.1.4. *Let $Q \in \mathcal{Z}$ be a projection. For $T \in \mathcal{M}$, the following are equivalent:*

(a) $T \in \operatorname{bimod}(Q)$.
(b) $\pi_\phi(T)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \operatorname{range}(Q)$.
(c) $Q^\perp \pi_\phi(T)P = 0$.

In particular, if $h \in \mathcal{GN}(\mathcal{M},\mathcal{D})$, then $h \in \operatorname{bimod}(Q)$ if and only if $P_h \le Q$.

Proof. As the equivalence of (b) and (c) is clear, we show only the equivalence of (a) and (b). Suppose $T \in \operatorname{bimod}(Q)$. Then $\pi_\phi(\langle T\rangle)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \operatorname{range}(Q)$, and (b) holds as $T \in \langle T\rangle$.

Conversely, if (b) holds, then for any $h, k \in \mathcal{D}$ and $d \in \mathfrak{n}_\phi \cap \mathcal{D}$, we have

$$\pi_\phi(hTk)\eta_\phi(d) = \pi_\ell(h)\pi_r(k)\pi_\phi(T)\eta_\phi(d) \in \operatorname{range}(Q)$$

because $\operatorname{range}(Q)$ is $\mathcal{Z}$-invariant. So $\overline{\pi_\phi(\langle T\rangle)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})} \subseteq \operatorname{range}(Q)$; hence $T \in \operatorname{bimod}(Q)$. □

The Spectral Theorem for Bimodules from [15] may be reformulated as the following conjecture.

Conjecture 2.1.5 (Spectral Theorem for Bimodules)**.** *If $\mathcal{S}$ is a σ-weakly closed $\mathcal{D}$-bimodule in $\mathcal{M}$, then $\mathcal{S} = \operatorname{bimod}(\operatorname{supp}(\mathcal{S}))$, that is,*

$$\mathcal{S} = \left\{T \in \mathcal{M} : \pi_\phi(T)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}\right\}. \tag{2.1}$$

Remarks 2.1.6. For these remarks, assume ϕ is a faithful normal state, so that $\eta_\phi(I)$ is a cyclic and separating vector for $\pi_\phi(\mathcal{M})$.

(a) Observe that replacing $\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})$ with $\eta_\phi(I)$ in Definition 2.1.1 leaves the definition of $\operatorname{supp}(\mathcal{S})$ unchanged; this replacement may also be made in (2.1). Thus, the Spectral Theorem for Bimodules is the same as the equality

$$\mathcal{S} = \left\{T \in \mathcal{M} : \pi_\phi(T)\eta_\phi(I) \in \overline{\pi_\phi(\mathcal{S})\eta_\phi(I)}\right\}.$$

(b) What is known (see [11, Theorem 2.3]) is that when $\mathcal{S}$ is a σ-weakly closed subspace of $\mathcal{M}$, then because $\pi_\phi(\mathcal{M})$ has a separating vector, $\pi_\phi(\mathcal{S})$ is reflexive, that is,

$$\pi_\phi(\mathcal{S}) = \left\{T \in \mathcal{B}(\mathfrak{H}_\phi) : T\xi \in \overline{\pi_\phi(\mathcal{S})\xi} \text{ for every } \xi \in \mathfrak{H}_\phi\right\}.$$

The faithfulness of π_ϕ then yields

$$\mathcal{S} = \left\{T \in \mathcal{M} : \pi_\phi(T)\xi \in \overline{\pi_\phi(\mathcal{S})\xi} \text{ for every } \xi \in \mathfrak{H}_\phi\right\}.$$

Clearly,

$$\begin{aligned}\mathcal{S} &= \left\{T \in \mathcal{M} : \pi_\phi(T)\xi \in \overline{\pi_\phi(\mathcal{S})\xi} \text{ for all } \xi \in \mathfrak{H}_\phi\right\} \\ &\subseteq \left\{T \in \mathcal{M} : \pi_\phi(T)\eta_\phi(I) \in \overline{\pi_\phi(\mathcal{S})\eta_\phi(I)}\right\}.\end{aligned}$$

Thus, Conjecture 2.1.5 holds if and only if the inclusion is an equality. (This is roughly the approach attempted in [15].)

(c) Since $\eta_\phi(I)$ is a cyclic and separating vector for $\pi_\phi(\mathcal{M})$, it is also cyclic and separating for $\pi_\phi(\mathcal{M})'$. If $T \in \mathcal{M}$ and $\pi_\phi(T)\eta_\phi(I) \in \overline{\pi_\phi(\mathcal{S})\eta_\phi(I)}$, then for each $Y \in \pi_\phi(\mathcal{M})'$, we have

$$\pi_\phi(T)Y\eta_\phi(I) = Y\pi_\phi(T)\eta_\phi(I) \in Y\overline{\pi_\phi(\mathcal{S})\eta_\phi(I)} \subseteq \overline{\pi_\phi(\mathcal{S})Y\eta_\phi(I)}.$$

Hence

$$\begin{aligned}&\left\{T \in \mathcal{M} : \pi_\phi(T)\eta_\phi(I) \in \overline{\pi_\phi(\mathcal{S})\eta_\phi(I)}\right\} \\ &\qquad = \left\{T \in \mathcal{M} : \pi_\phi(T)\xi \in \overline{\pi_\phi(\mathcal{S})\xi} \text{ for all } \xi \in \pi_\phi(\mathcal{M})'\eta_\phi(I)\right\}.\end{aligned}$$

Thus we see that Conjecture 2.1.5 holds if and only if the inclusion

$$\begin{aligned}&\left\{T \in \mathcal{M} : \pi_\phi(T)\xi \in \overline{\pi_\phi(\mathcal{S})\xi} \text{ for all } \xi \in \mathfrak{H}_\phi\right\} \\ &\qquad \subseteq \left\{T \in \mathcal{M} : \pi_\phi(T)\xi \in \overline{\pi_\phi(\mathcal{S})\xi} \text{ for all } \xi \in \pi_\phi(\mathcal{M})'\eta_\phi(I)\right\}\end{aligned}$$

is actually an equality.

2.2. Topologies. In this subsection we discuss the Bures and L^2 topologies on $\mathcal{M}$. We begin with a fact well-known to experts in noncommutative integration.

Lemma 2.2.1. *Let $\mathcal{M}$ be a von Neumann algebra, let ϕ be a faithful, semi-finite, normal weight on $\mathcal{M}$, and let $(\pi_\phi, \mathfrak{H}_\phi, \eta_\phi)$ be the semi-cyclic representation of $\mathcal{M}$ arising from ϕ. If $f \in \mathcal{M}_*$, then there are vectors $\xi_1, \xi_2 \in \mathfrak{H}_\phi$ such that for every $x \in \mathcal{M}$, $f(x) = \langle \pi_\phi(x)\xi_1, \xi_2\rangle$.*

Proof. By the polar decomposition for normal functionals on a von Neumann algebra ([21, Theorem III.4.2(i)]), there exists a partial isometry $v \in \mathcal{M}$ and $\rho \in (\mathcal{M}_*)^+$ such that for each $x \in \mathcal{M}$,

$$f(x) = \rho(xv). \tag{2.2}$$

Since π_ϕ puts $\mathcal{M}$ into standard form, [22, Theorem IX.1.2] shows there exists $\xi_2 \in \mathfrak{H}_\phi$ such that for every $x \in \mathcal{M}$, $\rho(x) = \langle \pi_\phi(x)\xi_2, \xi_2\rangle$. Taking $\xi_1 := \pi_\phi(v)\xi_2$, the lemma follows from (2.2). □

As noted earlier, the semi-cyclic representation of $\mathcal{D}$ induced by $\phi|_\mathcal{D}$ is unitarily equivalent to $(\pi_\ell, \mathrm{range}(P), \eta_\phi|_{\mathfrak{n}_\phi \cap \mathcal{D}})$. Thus, given $f \in \mathcal{D}_*$ there are $\xi_1, \xi_2 \in \mathrm{range}(P)$ so that for every $D \in \mathcal{D}$,

$$f(D) = \langle \pi_\ell(D)\xi_1, \xi_2\rangle .$$

Lemma 2.2.2. *The two families of semi-norms on $\mathcal{M}$,*

$$\left\{\mathcal{M} \ni T \mapsto \sqrt{\tau(E(T^*T))} : \tau \in (\mathcal{D}_*)^+\right\} \text{ and}$$
$$\left\{\mathcal{M} \ni T \mapsto \|\pi_\phi(T)\xi\| : \xi \in \mathrm{range}(P)\right\},$$

coincide.

Proof. Given $\tau \in (\mathcal{D}_*)^+$, there exists $\xi \in \mathrm{range}(P)$ so that

$$\tau(d) = \langle \pi_\ell(d)\xi, \xi\rangle .$$

Choose $h_n \in \mathfrak{n}_\phi \cap \mathcal{D}$ so that $\eta_\phi(h_n) \to \xi$. Then

$$\begin{aligned}\tau(E(T^*T)) &= \langle \pi_\ell(E(T^*T))\xi, \xi\rangle = \lim_{n\to\infty} \langle \pi_\ell(E(T^*T))\eta_\phi(h_n), \eta_\phi(h_n)\rangle \\ &= \lim_{n\to\infty} \|\pi_\phi(T)\eta_\phi(h_n)\|^2 = \|\pi_\phi(T)\xi\|^2 .\end{aligned}$$

It follows that

$$\left\{\mathcal{M} \ni T \mapsto \sqrt{\tau(E(T^*T))} : \tau \in (\mathcal{D}_*)^+\right\} \subseteq \left\{\mathcal{M} \ni T \mapsto \|\pi_\phi(T)\xi\| : \xi \in \mathrm{range}(P)\right\}.$$

The reverse inclusion is left to the reader. □

We require two topologies on $\mathcal{M}$, both discussed in [12], but the second is extended slightly here.

Definition 2.2.3.

(a) The *Bures* topology (see [4, page 48]) on $\mathcal{M}$ is the locally convex topology generated by the family of seminorms

$$\begin{aligned}\mathfrak{T}_B &:= \left\{\mathcal{M} \ni T \mapsto \sqrt{\tau(E(T^*T))} : \tau \in (\mathcal{D}_*)^+\right\}\\ &= \left\{\mathcal{M} \ni T \mapsto \|\pi_\phi(T)\xi\| : \xi \in \text{range}(P)\right\}.\end{aligned}$$

We denote the Bures topology by τ_B.

(b) The L^2 *topology* on $\mathcal{M}$ is the topology on $\mathcal{M}$ induced by the family of seminorms

$$\left\{\mathcal{M} \ni T \mapsto \|\pi_\phi(T)\eta_\phi(d)\| : d \in \mathfrak{n}_\phi \cap \mathcal{D}\right\}.$$

We will use $(\mathcal{M}, L^2)$ to denote $\mathcal{M}$ equipped with the L^2 topology.

Remark 2.2.4. When ϕ is a faithful normal state on $\mathcal{M}$, the L^2 topology is determined by the single seminorm $\mathcal{M} \ni T \mapsto \|\pi_\phi(T)\eta_\phi(I)\| = \|\eta_\phi(T)\|$, and in this case, the L^2 topology was considered by Mercer in [12]. When $\mathcal{D}$ is isomorphic to $L^\infty(X, \mu)$, $\mathfrak{n}_\phi \cap \mathcal{D}$ may be thought of as $L^2 \cap L^\infty$, so it is tempting to use the term "bounded Bures topology" instead of the L^2-topology, but we have chosen to stay with the nomenclature used by Mercer.

Clearly the L^2-topology is coarser than the Bures topology, which in turn is coarser than the norm topology, so the dual spaces of $\mathcal{M}$ equipped with these topologies satisfy

$$(\mathcal{M}, L^2)^\# \subseteq (\mathcal{M}, \tau_B)^\# \subseteq (\mathcal{M}, \text{norm})^\#.$$

Corollary 2.2.5. *For* $\xi \in \text{range}(P)$ *and* $\zeta \in \mathfrak{H}_\phi$*, the functional* $T \mapsto \langle \pi_\phi(T)\xi, \zeta\rangle$ *belongs to* $(\mathcal{M}, \tau_B)^\#$.

Proof. By the Cauchy–Schwarz inequality, $|\langle \pi_\phi(T)\xi, \zeta\rangle| \le \|\pi_\phi(T)\xi\|\,\|\zeta\|$. By Lemma 2.2.2, the first term in the product is one of the seminorms defining the Bures topology. The corollary follows. □

We now show that every Bures-continuous linear functional is of this form.

Lemma 2.2.6. *Let* f *be a linear functional on* $\mathcal{M}$.

(a) *If* f *is* τ_B *continuous, then there exist* $\xi \in \text{range}(P)$ *and* $\zeta \in \mathfrak{H}_\phi$ *such that*

$$f(T) = \langle \pi_\phi(T)\xi, \zeta\rangle.$$

In particular, f *is* σ*-weakly continuous on* $\mathcal{M}$.

(b) *If* $f \in (\mathcal{M}, L^2)^\#$*, then there exists* $d \in \mathfrak{n}_\phi \cap \mathcal{D}$ *and* $\zeta \in \mathfrak{H}_\phi$ *such that*

$$f(T) = \langle \pi_\phi(T)\eta_\phi(d), \zeta\rangle.$$

Moreover, $(\mathcal{M}, \tau_B)^\#$ *and* $(\mathcal{M}, L^2)^\#$ *are norm-dense in* $\mathcal{M}_*$.

Proof. For the first statement, we give a standard argument (see the proof of [21, Lemma II.2.4]). Since f is τ_B continuous, there exist $p_1, \dots, p_n \in \mathfrak{T}_B$ such that for every $T \in \mathcal{M}$, we have

$$|f(T)| \leq \sum_{k=1}^{n} p_k(T)$$

(see [5, Theorem IV.3.1]). Write $p_k(T) = \sqrt{\omega_k(E(T^*T))}$, where the ω_k are positive normal functionals on $\mathcal{D}$. Set $\omega = n \sum_{k=1}^{n} \omega_k$ and let $p(T) = \sqrt{\omega(E(T^*T))}$. By the Cauchy–Schwarz inequality,

$$|f(T)| \leq p(T). \tag{2.3}$$

By Lemma 2.2.2, there is a vector $\xi \in \text{range}(P)$ such that for $T \in \mathcal{M}$,

$$p(T) = \|\pi_\phi(T)\xi\|.$$

By (2.3), the map

$$\pi_\phi(T)\xi \mapsto f(T)$$

is bounded on the subspace $\{\pi_\phi(T)\xi : T \in \mathcal{M}\} \subseteq \mathfrak{H}_\phi$. The Riesz Representation Theorem implies that there exists a vector $\zeta \in \overline{\{\pi_\phi(T)\xi : T \in \mathcal{M}\}} \subseteq \mathfrak{H}_\phi$ such that

$$f(T) = \langle \pi_\phi(T)\xi, \zeta \rangle.$$

Hence f is σ-weakly continuous on $\mathcal{M}$.

The proof of statement (b) is similar and left to the reader.

Suppose $T \in \mathcal{M}$ and $f(T) = 0$ for every $f \in (\mathcal{M}, L^2)^\#$. For every $d \in \mathfrak{n}_\phi \cap \mathcal{D}$, the map $\mathcal{M} \ni S \mapsto \langle \pi_\phi(S)\eta_\phi(d), \pi_\phi(T)\eta_\phi(d)\rangle$ belongs to $(\mathcal{M}, L^2)^\#$, so $\langle \pi_\phi(T)\eta_\phi(d), \pi_\phi(T)\eta_\phi(d)\rangle = 0$. Hence $\langle \pi_\ell(E(T^*T))\eta_\phi(d), \eta_\phi(d)\rangle = 0$ for each $d \in \mathfrak{n}_\phi \cap \mathcal{D}$. This implies that $E(T^*T) = 0$, and hence $T = 0$. It follows that $(\mathcal{M}, L^2)^\#$ is weakly dense in $\mathcal{M}_*$. As $(\mathcal{M}, L^2)^\#$ is a subspace, its weak and norm closures coincide, so

$$\mathcal{M}_* = \overline{(\mathcal{M}, L^2)^\#}^{\sigma(\mathcal{M}_*, \mathcal{M})} = \overline{(\mathcal{M}, L^2)^\#}^{\|\|}.$$

Thus, $(\mathcal{M}, L^2)^\#$ is norm dense in $\mathcal{M}_*$. Since every L^2 continuous linear functional is Bures continuous, the Bures continuous linear functionals are norm dense in $\mathcal{M}_*$ also. □

Corollary 2.2.7. *Let C be a convex set in $\mathcal{M}$. Then*

$$\overline{C}^{\sigma\text{-weak}} \subseteq \overline{C}^{\text{Bures}} \subseteq \overline{C}^{L^2},$$

with equality throughout if C is also a bounded set.

2.3. σ-weakly closed bimodules.

Lemma 2.3.1. *Let $\mathcal{S} \subseteq \mathcal{M}$ be a σ-weakly closed $\mathcal{D}$-bimodule. Then the following statements hold.*

(a) *If $u \in \mathcal{N}(\mathcal{M}, \mathcal{D})$, there exists a projection $Q \in \mathcal{D}$ such that $uQ \in \mathcal{S}$ and $uQ^\perp$ satisfies $E((uQ^\perp)^*S) = 0$ for every $S \in \mathcal{S}$.*

(b) *If $X \in \operatorname{bimod}(\operatorname{supp}(\mathcal{S}))$, then for every $u \in \mathcal{N}(\mathcal{M}, \mathcal{D})$, $uE(u^*X) \in \mathcal{S}$.*

(c) *Let $\mathcal{S}_B$ be the Bures closure of $\mathcal{S}$. Then $\operatorname{supp}(\mathcal{S}) = \operatorname{supp}(\mathcal{S}_B)$.*

Proof. Let $u \in \mathcal{N}(\mathcal{M}, \mathcal{D})$, and set $J := \{d \in \mathcal{D} : ud \in \mathcal{S}\}$. Since $\mathcal{S}$ is a bimodule, J is an ideal in $\mathcal{D}$, and the fact that $\mathcal{S}$ is σ-weakly closed ensures that J is also σ-weakly closed. Therefore, there exists a unique projection $Q \in \mathcal{D}$ such that $J = \mathcal{D}Q$. Obviously, $Q \in J$ and $uQ^\perp \in \mathcal{N}(\mathcal{M}, \mathcal{D})$. Proposition 1.3.4 shows that if $S \in \mathcal{S}$, then $uQ^\perp E((uQ^\perp)^*S) \in \mathcal{S}$. Thus $uQ^\perp E((uQ^\perp)^*S) = uQ^\perp E(u^*S) \in \mathcal{S}$, and hence $Q^\perp E(u^*S) \in J$. It follows that $0 = Q^\perp E(u^*S) = E((uQ^\perp)^*S)$, as desired.

Turning to (b), suppose first $u \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and $X \in \operatorname{bimod}(\operatorname{supp}(\mathcal{S}))$. Then $\pi_\phi(X)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$. Let Q be the projection obtained as in part (a). For any $S \in \mathcal{S}$ and $d \in \mathfrak{n}_\phi \cap \mathcal{D}$, using Lemma 1.4.6(b), we have

$$P_{uQ^\perp}(\pi_\phi(S)\eta_\phi(d)) = \pi_\phi(uQ^\perp E((uQ^\perp)^*S))\eta_\phi(d) = 0.$$

Hence, for any $S \in \mathcal{S}$ and $h \in \mathfrak{n}_\phi \cap \mathcal{D}$,

$$\begin{aligned}\left\|P_{uQ^\perp}(\pi_\phi(X)\eta_\phi(h))\right\| &= \left\|P_{uQ^\perp}(\pi_\phi(X)\eta_\phi(h)) - P_{uQ^\perp}(\pi_\phi(S)\eta_\phi(d))\right\| \\ &\leq \|\pi_\phi(X)\eta_\phi(h) - \pi_\phi(S)\eta_\phi(d)\|.\end{aligned}$$

Holding h fixed and taking the infimum over $S \in \mathcal{S}$ and $d \in \mathfrak{n}_\phi \cap \mathcal{D}$, the hypothesis on X gives

$$\begin{aligned}0 = P_{uQ^\perp}(\pi_\phi(X)\eta_\phi(h)) &= \pi_\phi(uQ^\perp E((uQ^\perp)^*X))\eta_\phi(h) \\ &= \pi_\phi(uE(u^*X)Q^\perp)\eta_\phi(h).\end{aligned}$$

Setting $y := uE(u^*X)Q^\perp$, this shows that for every $h \in \mathfrak{n}_\phi \cap \mathcal{D}$ we have

$$0 = \phi(E(h^*y^*yh)) = \phi(h^*hE(y^*y)).$$

Thus, for every $\tau \in \mathcal{D}_*^+$, $\tau(E(y^*y)) = 0$. This shows that $E(y^*y) = 0$, and by faithfulness of E, $y = 0$; thus, $uE(u^*X)Q^\perp = 0$. Hence

$$uE(u^*X) = uE(u^*X)Q \in \mathcal{S}.$$

Now let $u \in \mathcal{N}(\mathcal{M}, \mathcal{D})$, with $u \neq 0$ (the case when $u = 0$ is trivial). If $u = w|u|$ is the polar decomposition of u, Lemma 1.2.1 gives $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Then $|u|^2 = u^*u \in \mathcal{D}$ and $wE(w^*X) \in \mathcal{S}$. Since

$$uE(u^*X) = wE(w^*X)u^*u \in \mathcal{S},$$

the proof of (b) is complete.

To establish (c), we must show that $\overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})} = \overline{\pi_\phi(\mathcal{S}_B)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$. Since $\mathcal{S} \subseteq \mathcal{S}_B$, we obtain $\overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})} \subseteq \overline{\pi_\phi(\mathcal{S}_B)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$. If $T \in \mathcal{S}_B$, we may find a net (T_λ) in $\mathcal{S}$ converging in the Bures topology to T. Then $T_\lambda \xrightarrow{L^2} T$, and hence given $d \in \mathfrak{n}_\phi \cap \mathcal{D}$, $\pi_\phi(T_\lambda)\eta_\phi(d) \to \pi_\phi(T)\eta_\phi(d)$. Therefore, $\pi_\phi(T)\eta_\phi(d) \in \overline{\pi_\phi(\mathcal{S})\eta_\phi(d)}$. Thus, $\overline{\pi_\phi(\mathcal{S}_B)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})} \subseteq \overline{\pi_\phi(\mathcal{S})\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$ and part (c) follows. $\square$

Corollary 2.3.2. *Let $\mathcal{S} \subseteq \mathcal{M}$ be a σ-weakly closed $\mathcal{D}$-bimodule and $h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Then $h \in \mathcal{S}$ if and only $P_h \leq \operatorname{supp}(\mathcal{S})$. Thus*

$$\operatorname{supp}(\mathcal{S}) = \bigvee_{h \in \mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})} P_h.$$

Proof. Suppose $h \in \mathcal{S}$. Then $h \in \operatorname{bimod}(\operatorname{supp}(\mathcal{S}))$, and so $P_h \leq \operatorname{supp}(\mathcal{S})$, by Lemma 2.1.4. Conversely, suppose $P_h \leq \operatorname{supp}(\mathcal{S})$. Then, again by Lemma 2.1.4, $h \in \operatorname{bimod}(\operatorname{supp}(\mathcal{S}))$. By Lemma 2.3.1(b), $h = hE(h^*h) \in \mathcal{S}$.

By the proof of Theorem 1.4.7, $\operatorname{supp}(\mathcal{S}) = \bigvee_{h \in A} P_h$, for some $A \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$. For $h \in A$, $P_h \leq \operatorname{supp}(\mathcal{S})$, and so $h \in \mathcal{S}$. Thus

$$\operatorname{supp}(\mathcal{S}) = \bigvee_{h \in A} P_h \leq \bigvee_{h \in \mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})} P_h \leq \operatorname{supp}(\mathcal{S}). \qquad \square$$

2.4. $\mathfrak{D}$-orthogonality.

Definition 2.4.1. A nonempty set $\mathcal{E} \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})\backslash\{0\}$ is called $\mathcal{D}$-*orthogonal* if for every $v_1, v_2 \in \mathcal{E}$ with $v_1 \neq v_2$, $E(v_1^* v_2) = 0$ (equivalently, $P_{v_1} \perp P_{v_2}$, by Lemma 1.4.6(d)).

A simple Zorn's Lemma argument shows the existence of a maximal $\mathcal{D}$-orthogonal set.

Remark 2.4.2. Notice that for $v_1, v_2 \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, v_1 and v_2 are $\mathcal{D}$-orthogonal if and only if v_1^* and v_2^* are $\mathcal{D}$-orthogonal. Indeed, $E(v_1^* v_2) = 0$ implies $0 = v_1 E(v_1^* v_2) v_1^* = E(v_1 v_1^* v_2 v_1^*) = E(v_2 v_1^*)$; the converse is similar.

Lemma 2.4.3. *Let $\mathcal{E} \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})\backslash\{0\}$ be a maximal $\mathcal{D}$-orthogonal set. Then $\sum_{u \in \mathcal{E}} P_u = I$, where the sum converges strongly in $\mathcal{B}(\mathfrak{H}_\phi)$.*

Proof. Let $Q = \sum_{u \in \mathcal{E}} P_u \in \mathcal{Z}$. If $I - Q \neq 0$, then by the proof of Theorem 1.4.7, there exists $0 \neq h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $P_h \leq I - Q$. Then $P_h \perp P_u$ for all $u \in \mathcal{E}$, contradicting maximality of $\mathcal{E}$. $\square$

The following is an adaptation of a result of Mercer to our context.

Proposition 2.4.4 (cf. [12, Theorem 4.4])**.** *Let $\mathcal{E} \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})\backslash\{0\}$ be a maximal $\mathcal{D}$-orthogonal set and let Γ be the set of all finite subsets of $\mathcal{E}$ directed by inclusion. Fix $X \in \mathcal{M}$. For $F \in \Gamma$, let*

$$X_F = \sum_{u \in F} uE(u^* X).$$

Then $(X_F)_{F \in \Gamma}$ is a net which converges in the Bures topology to X.

Proof. Let $d \in \mathfrak{n}_\phi \cap \mathcal{D}$. Lemma 1.4.6(b) gives,

$$\pi_\phi(X_F)\eta_\phi(d) = \sum_{u\in F} \eta_\phi(uE(u^*Xd)) = \sum_{u\in F} P_u\eta_\phi(Xd) = \sum_{u\in F} P_u\pi_\phi(X)\eta_\phi(d),$$

and hence for every $\xi \in \overline{\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$,

$$\pi_\phi(X_F)\xi = \sum_{u\in F} P_u\pi_\phi(X)\xi.$$

Since $I = \sum_{u\in\mathcal{E}} P_u$ (where the sum converges strongly in $\mathcal{B}(\mathfrak{H}_\phi)$), for every $\xi \in \overline{\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$,

$$\pi_\phi(X_F)\xi \to \pi_\phi(X)\xi.$$

Therefore, $X_F \overset{\text{Bures}}{\to} X$. □

2.5. A characterization of Bures closed bimodules. The following is a version of the Spectral Theorem for Bimodules, which characterizes Bures (or L^2) closed bimodules.

Theorem 2.5.1. *Let $\mathcal{S} \subseteq \mathcal{M}$ be a $\mathcal{D}$-bimodule. Then the following statements are equivalent:*

(a) $\mathcal{S} = \text{bimod}(\text{supp}(\mathcal{S}))$.
(b) $\mathcal{S}$ *is L^2-closed.*
(c) $\mathcal{S}$ *is Bures-closed.*
(d) $\mathcal{S}$ *is the smallest Bures-closed $\mathcal{D}$-bimodule containing* $\mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$.

Proof. Suppose (a) holds. Let (T_λ) in $\mathcal{S}$ be such that $T_\lambda \xrightarrow{L^2} T \in \mathcal{M}$. Then given $d \in \mathfrak{n}_\phi \cap \mathcal{D}$, $\pi_\phi(T_\lambda)\eta_\phi(d) \to \pi_\phi(T)\eta_\phi(d)$. Therefore, $\pi_\phi(T)\eta_\phi(d) \in \overline{\pi_\phi(\mathcal{S})\eta_\phi(d)}$. Then Lemma 2.1.4 gives $T \in \text{bimod}(\text{supp}(\mathcal{S})) = \mathcal{S}$. Thus $\mathcal{S}$ is L^2-closed.

As the Bures topology is stronger than the L^2-topology, we see that (b) $\Rightarrow$ (c).

We now establish (c) $\Rightarrow$ (a). Suppose $X \in \text{bimod}(\text{supp}(\mathcal{S}))$. Let $\mathcal{E}$ be a maximal $\mathcal{D}$-orthogonal subset of $\mathcal{GN}(\mathcal{M}, \mathcal{D})\backslash\{0\}$. By Lemma 2.3.1(b) and Proposition 2.4.4, $X_F \in \mathcal{S}$ and $X_F \overset{\text{Bures}}{\to} X$; hence $X \in \overline{\mathcal{S}}^{\text{Bures}} = \mathcal{S}$. Thus, $\text{bimod}(\text{supp}(\mathcal{S})) \subseteq \mathcal{S}$. As the reverse inclusion is obvious, (a) holds.

Let $\mathcal{S}_1 = \langle \mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})\rangle_B$, the smallest Bures-closed $\mathcal{D}$-bimodule containing $\mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$. If $\mathcal{S} = \mathcal{S}_1$, then $\mathcal{S}$ is Bures closed, thus (d) $\Rightarrow$ (c). Conversely, suppose $\mathcal{S}$ is Bures-closed. Then $\mathcal{S}_1 \subseteq \mathcal{S}$, clearly. On the other hand, $\mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \subseteq \mathcal{S}_1 \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$, which implies $\text{supp}(\mathcal{S}) \subseteq \text{supp}(\mathcal{S}_1)$, by Corollary 2.3.2, and so $\mathcal{S} = \text{bimod}(\text{supp}(\mathcal{S})) \subseteq \text{bimod}(\text{supp}(\mathcal{S}_1)) = \mathcal{S}_1$, using the equivalence of (a) and (c). □

Corollary 2.5.2. *Let $\mathcal{S} \subseteq \mathcal{M}$ be a σ-weakly closed $\mathcal{D}$-bimodule. Then* $\overline{\mathcal{S}}^{\text{Bures}} = \text{bimod}(\text{supp}(\mathcal{S}))$.

Proof. By Theorem 2.5.1 and Lemma 2.3.1(c),

$$\overline{\mathcal{S}}^{\text{Bures}} = \text{bimod}(\text{supp}(\overline{\mathcal{S}}^{\text{Bures}})) = \text{bimod}(\text{supp}(\mathcal{S})). \qquad \square$$

Let $\mathcal{L}$ be a commutative subspace lattice acting on the Hilbert space $\mathcal{H}$. Consider the family $\mathcal{R}$ of all σ-weakly closed subalgebras $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A} \cap \mathcal{A}^* = \mathcal{L}'$ and $\text{Lat}\,(\mathcal{A}) = \mathcal{L}$. Arveson [2, Theorem 2.1.8] showed that relative to set inclusion, the family $\mathcal{R}$ has a minimal element, $\mathcal{A}_{\min}(\mathcal{L})$, and $\text{Alg}\,(\mathcal{L})$ is the maximal element of $\mathcal{R}$. The following proposition has the same flavor.

Proposition 2.5.3. *Let $Q \in \mathcal{Z}$ and let $\mathfrak{B}$ be the set of all σ-weakly closed $\mathcal{D}$-bimodules $\mathcal{S} \subseteq \mathcal{M}$ with $\text{supp}(\mathcal{S}) = Q$. Then $\overline{\langle \text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \rangle}^{\sigma\text{-weak}}$ is the minimal element of $\mathfrak{B}$ and $\text{bimod}(Q)$ is the maximal element of $\mathfrak{B}$.*

Proof. First we show that $\text{supp}(\text{bimod}(Q)) = Q$. Indeed, for $h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$,

$$h \in \text{bimod}(Q) \iff P_h \le Q,$$

by Lemma 2.1.4. Therefore,

$$\begin{aligned}\text{supp}(\text{bimod}(Q)) &= \bigvee \{P_h : h \in \text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})\} \\ &= \bigvee \{P_h : h \in \mathcal{GN}(\mathcal{M}, \mathcal{D}),\ P_h \le Q\} = Q.\end{aligned}$$

Thus $\text{bimod}(Q) \in \mathfrak{B}$.

Now let $\mathcal{S}_0 = \overline{\langle \text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \rangle}^{\sigma\text{-weak}}$, a σ-weakly closed $\mathcal{D}$-bimodule in $\mathcal{M}$. Then

$$\text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \subseteq \mathcal{S}_0 \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \subseteq \text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}).$$

Corollary 2.3.2 gives $\text{supp}(\mathcal{S}_0) = \text{supp}(\text{bimod}(Q)) = Q$, which implies $\mathcal{S}_0 \in \mathfrak{B}$.

Finally, if $\mathcal{S} \in \mathfrak{B}$, then for $h \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$,

$$h \in \mathcal{S} \iff P_h \le \text{supp}(\mathcal{S}) = Q \iff h \in \text{bimod}(Q),$$

by Corollary 2.3.2 and Lemma 2.1.4. Therefore,

$$\begin{aligned}\mathcal{S}_0 &= \overline{\langle \text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \rangle}^{\sigma\text{-weak}} = \overline{\langle \mathcal{S} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \rangle}^{\sigma\text{-weak}} \\ &\subseteq \mathcal{S} \subseteq \text{bimod}(\text{supp}(\mathcal{S})) = \text{bimod}(Q). \qquad \square\end{aligned}$$

Remark 2.5.4. By Lemma 1.2.1,

$$\langle \text{bimod}(Q) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \rangle = \text{span}(\text{bimod}(Q) \cap \mathcal{N}(\mathcal{M}, \mathcal{D})),$$

and so $\overline{\text{span}}^{\sigma\text{-weak}}(\text{bimod}(Q) \cap \mathcal{N}(\mathcal{M}, \mathcal{D}))$ is another expression for the minimal element of $\mathfrak{B}$.

Remark 2.5.5. Using the failure of spectral synthesis for an appropriate locally compact abelian group, in [2], Arveson constructed a commutative subspace lattice $\mathcal{L}$ for which $\mathcal{A}_{\min}(\mathcal{L}) \subsetneq \text{Alg}\,(\mathcal{L})$. This, together with Proposition 2.5.3, suggests that Conjecture 2.1.5 may not hold in general.

While our context differs from that of [2], the parallels are sufficiently strong that we make the following definition.

Definition 2.5.6. Let $\mathcal{S} \subseteq \mathcal{M}$ be a σ-weakly closed $\mathcal{D}$-bimodule. We say $\mathcal{S}$ is *synthetic*, or satisfies *spectral synthesis*, if the minimal and maximal σ-weakly closed bimodules with $\operatorname{supp}(\mathcal{S})$ coincide, that is, if

$$\overline{\operatorname{span}}^{\sigma\text{-weak}}(\mathcal{S} \cap \mathcal{N}(\mathcal{M}, \mathcal{D})) = \mathcal{S} = \overline{\mathcal{S}}^{\text{Bures}}.$$

Remark 2.5.7. Let $\mathcal{A}$ be a CSL algebra. We wish to point out that when $\mathcal{A}$ contains a Cartan MASA in $\mathcal{B}(\mathcal{H})$, our notion of synthesis and Arveson's notion coincide.

Let $\mathcal{H}$ be a separable Hilbert space, let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$ and let $\mathcal{D}$ be the atomic MASA of all operators diagonal with respect to this basis. Then $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ is a Cartan pair (all Cartan MASAs in $\mathcal{B}(\mathcal{H})$ are of this form). Let $e_i e_j^*$ denote the rank-one operator $\xi \mapsto \langle \xi, e_j \rangle e_i$. Taking ϕ to be the tracial weight on $\mathcal{B}(\mathcal{H})$, $\mathfrak{H}_\phi$ is the set of Hilbert-Schmidt operators. Each minimal projection in $\mathcal{Z} \subseteq \mathcal{B}(\mathfrak{H}_\phi)$ has range $\mathbb{C}\eta_\phi(e_i e_j^*)$ for some $i, j \in \mathbb{N}$, and it follows that $\mathcal{Z} \subseteq \mathcal{B}(\mathfrak{H}_\phi)$ is an atomic MASA.

If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a σ-weakly closed algebra with $\mathcal{D} \subseteq \mathcal{A}$, then for each finite-rank projection $P \in \mathcal{D}$, $P\mathcal{A}P$ is spanned by $\{PvP : v \in \mathcal{A} \cap \mathcal{GN}(\mathcal{B}(\mathcal{H}), \mathcal{D})\}$. Since I may be written as the strong limit of an increasing sequence of such projections, the span of the rank one operators contained in $\mathcal{A}$ is σ-weakly dense in $\mathcal{A}$. Using the description of the atoms of $\mathcal{Z}$ from above, one shows $\mathcal{A}$ is synthetic in the sense of Definition 2.5.6. Moreover, [6, Theorem 23.7] shows $\mathcal{A}$ is a completely distributive CSL algebra. Hence by [6, Corollary 23.9], $\mathcal{A}$ is synthetic in Arveson's sense as well.

The following consequence of Theorem 2.5.1 and the proof of Proposition 2.5.3 is worth noting; we leave the proof to the reader.

Theorem 2.5.8. *Let $\mathfrak{S}$ be the lattice of all Bures-closed $\mathcal{D}$-bimodules of $\mathcal{M}$ (where $\bigwedge$ is intersection and $\bigvee$ is Bures-closed span) and let $\mathfrak{L}$ be the projection lattice of $\mathcal{Z}$. The maps* $\operatorname{bimod} : \mathfrak{L} \to \mathfrak{S}$ *and* $\operatorname{supp} : \mathfrak{S} \to \mathfrak{L}$ *are lattice isomorphisms and* $(\operatorname{bimod})^{-1} = \operatorname{supp}$.

We close this section by showing that the class of von Neumann subalgebras which lie between $\mathcal{D}$ and $\mathcal{M}$ is a class of σ-weakly closed $\mathcal{D}$-bimodules which is well-behaved with respect to the operations of bimod and supp. Suppose $\mathcal{A}$ is a von Neumann algebra with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{M}$. Then $\mathcal{A}_0 := \overline{\operatorname{span}}^{\sigma\text{-weak}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A})$ is also a von Neumann algebra. A much less obvious fact, contained in Theorem 2.5.9 below, is that the Bures-closure, $\mathcal{A}_B := \overline{\mathcal{A}}^{\text{Bures}}$ is also a von Neumann algebra. By Proposition 2.5.3, $\mathcal{A}_0$ and $\mathcal{A}_B$ are the minimal and maximal σ-weakly closed $\mathcal{D}$-bimodules with $\operatorname{supp}(\mathcal{A})$. It is somewhat surprising that actually $\mathcal{A}$ is Bures closed, so that

$$\mathcal{A}_0 = \mathcal{A} = \mathcal{A}_B = \operatorname{bimod}(\operatorname{supp}(\mathcal{A})).$$

Thus, the class of von Neumann algebras which lie between $\mathcal{D}$ and $\mathcal{M}$ is a class of $\mathcal{D}$-bimodules for which Conjecture 2.1.5 (the spectral theorem for bimodules) holds, and for which each element of the class is synthetic.

We now prove these facts, and somewhat more, by extending, and providing a new proof of, a theorem of Aoi [1]. Aoi attributes the statement of his theorem to unpublished work of C. Sutherland. Aoi's proof uses the Feldman–Moore formalism and therefore requires that the von Neumann algebras involved have separable predual. Our proof allows us to eliminate the separability hypothesis, and also to give a description of the conditional expectation.

Theorem 2.5.9 (cf. [1, Theorem 1.1])**.** *Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair, and suppose $\mathcal{A}$ is a von Neumann algebra such that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{M}$. Then $(\mathcal{A}, \mathcal{D})$ is a Cartan pair, and there exists a unique faithful normal conditional expectation Φ of $\mathcal{M}$ onto $\mathcal{A}$. In addition, $\mathcal{A}$ is a synthetic $\mathcal{D}$-bimodule, and the following statements hold.*

(a) *For each $x \in \mathfrak{n}_\phi$, $\Phi(x) \in \mathfrak{n}_\phi$ and*

$$\eta_\phi(\Phi(x)) = \operatorname{supp}(\mathcal{A})\eta_\phi(x). \tag{2.4}$$

(b) *For each $T \in \mathcal{M}$,*

$$\pi_\phi(\Phi(T))P = \operatorname{supp}(\mathcal{A})\pi_\phi(T)P \tag{2.5}$$

and Φ is Bures-continuous.

(c) *If $\mathcal{E} \subseteq \mathcal{GN}(\mathcal{A}, \mathcal{D})$ is a maximal $\mathcal{D}$-orthogonal family, then for every $x \in \mathcal{M}$, $\Phi(x)$ is the Bures-convergent sum,*

$$\Phi(x) = \sum_{u \in \mathcal{E}} uE(u^*x).$$

Proof. Let $\mathcal{A}_0 := \overline{\operatorname{span}}^{\sigma\text{-weak}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A})$. Then $\mathcal{A}_0$ is a von Neumann algebra such that

$$\mathcal{D} \subseteq \mathcal{A}_0 \subseteq \mathcal{A} \subseteq \mathcal{M}.$$

Here is the plan of the proof. Our first step is to show the existence of a unique faithful, normal conditional expectation Φ of $\mathcal{M}$ onto $\mathcal{A}_0$. Step 2 will show that parts (a) and (b) hold. We then show $\mathcal{A}$ is synthetic, that is,

$$\mathcal{A}_0 = \mathcal{A} = \overline{\mathcal{A}}^{\text{Bures}}. \tag{2.6}$$

Afterwards, we show $(\mathcal{A}, \mathcal{D})$ is a Cartan pair, and then conclude the proof by verifying (c) holds.

Let σ_t^ϕ be the modular automorphism group arising from ϕ. Since the span of $\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}_0$ is σ-weakly dense in $\mathcal{A}_0$, Corollary 1.4.8 implies that for every $t \in \mathbb{R}$,

$$\sigma_t^\phi(\mathcal{A}_0) = \mathcal{A}_0. \tag{2.7}$$

Next we show that $\phi|_{\mathcal{A}_0}$ is semi-finite on $\mathcal{A}_0$. Let

$$\mathcal{GN}_{\text{sf}}(\mathcal{M}, \mathcal{D}) := \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathfrak{n}_\phi \quad \text{and} \quad \mathcal{GN}_{\text{sf}}(\mathcal{A}_0, \mathcal{D}) := \mathcal{GN}_{\text{sf}}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}_0.$$

Since $\mathfrak{n}_\phi \cap \mathcal{D}$ is σ-weakly dense in $\mathcal{D}$, we may find $d_\lambda \in \mathcal{D} \cap \mathfrak{n}_\phi$ which converges to $I_{\mathcal{D}}$ σ-weakly. Thus for any $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}_0$, $v = \lim_\lambda v d_\lambda$ which gives,

$$\overline{\mathcal{GN}_{\mathrm{sf}}(\mathcal{A}_0, \mathcal{D})}^{\sigma\text{-weak}} \supseteq \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}_0.$$

As $\operatorname{span}(\mathcal{GN}_{\mathrm{sf}}(\mathcal{A}_0, \mathcal{D})) \subseteq \mathfrak{n}_\phi$, we conclude that ϕ is semi-finite on $\mathcal{A}_0$.

An application of [22, Theorem IX.4.2] yields the existence and uniqueness of a normal conditional expectation $\Phi : \mathcal{M} \to \mathcal{A}_0$ such that $\phi \circ \Phi = \phi$. Note that Φ is faithful because ϕ is.

We now show the formulas (2.4) and (2.5) hold for the conditional expectation Φ just constructed. If $x \in \mathfrak{n}_\phi$, then $\phi(\Phi(x)^*\Phi(x)) \leq \phi(\Phi(x^*x)) = \phi(x^*x) < \infty$, so $\mathfrak{n}_\phi$ is invariant under Φ. This calculation also shows the map $\eta_\phi(\mathfrak{n}_\phi) \ni \eta_\phi(x) \mapsto \eta_\phi(\Phi(x))$ is norm decreasing on $\eta_\phi(\mathfrak{n}_\phi)$. The facts that Φ is an idempotent linear map and $\Phi(x)^* = \Phi(x^*)$ for every $x \in \mathcal{M}$ imply that this map extends to a projection $Q \in \mathcal{B}(\mathfrak{H}_\phi)$ such that $Q\eta_\phi(x) = \eta_\phi(\Phi(x))$ for every $x \in \mathfrak{n}_\phi$.

Notice $\pi_\phi(\mathcal{A}_0)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D}) \subseteq \operatorname{range}(Q)$. By definition,

$$\operatorname{range}(\operatorname{supp}(\mathcal{A}_0)) = \overline{\pi_\phi(\mathcal{A}_0)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})},$$

so $\operatorname{supp}(\mathcal{A}_0) \leq Q$. On the other hand, for $x \in \mathfrak{n}_\phi$, $\Phi(x) \in \mathfrak{n}_\phi \cap \mathcal{A}_0$. Lemma 1.4.1 gives $\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{A}_0) \subseteq \overline{\pi_\phi(\mathcal{A}_0)\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{D})}$, so

$$\eta_\phi(\Phi(\mathfrak{n}_\phi)) \subseteq \operatorname{range}(\operatorname{supp}(\mathcal{A}_0)).$$

But $\operatorname{range}(Q) = \overline{\eta_\phi(\Phi(\mathfrak{n}_\phi))}$, which yields $Q \leq \operatorname{supp}(\mathcal{A}_0)$, so $Q = \operatorname{supp}(\mathcal{A}_0)$. By Lemma 2.3.1(c), $\operatorname{supp}(\mathcal{A}_0) = \operatorname{supp}(\mathcal{A}) = \operatorname{supp}(\mathcal{A}_B)$, so (2.4) holds.

Let $T \in \mathcal{M}$ and $d \in \mathcal{D} \cap \mathfrak{n}_\phi$. Using (2.4), we have,

$$\pi_\phi(\Phi(T))\eta_\phi(d) = \eta_\phi(\Phi(Td)) = \operatorname{supp}(\mathcal{A})\eta_\phi(Td) = \operatorname{supp}(\mathcal{A})\pi_\phi(T)\eta_\phi(d).$$

Since $\eta_\phi(\mathcal{D} \cap \mathfrak{n}_\phi)$ is dense in $\operatorname{range}(P)$, we obtain (2.5).

By (2.5), Φ is Bures continuous. Let $T \in \overline{\mathcal{A}}^{\text{Bures}}$. Theorem 2.5.1 ensures that $T \in \overline{\mathcal{A}_0}^{\text{Bures}}$, so there exists a net T_λ in $\mathcal{A}_0$ which Bures-converges to T. Since Φ is Bures continuous,

$$T = \lim_\lambda T_\lambda = \lim_\lambda \Phi(T_\lambda) = \Phi(\lim_\lambda T_\lambda) = \Phi(T),$$

so $T \in \mathcal{A}_0$. The equality (2.6) now follows.

Next we show $(\mathcal{A}, \mathcal{D})$ is a Cartan pair. Obviously, $E|_{\mathcal{A}}$ is a faithful normal conditional expectation of $\mathcal{A}$ onto $\mathcal{D}$, so we need only prove that the span of $\mathcal{U}(\mathcal{A}) \cap \mathcal{N}(\mathcal{M}, \mathcal{D})$ is σ-weakly dense in $\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}_0$, it suffices to show that

$$\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A} \subseteq \overline{\operatorname{span}}^{\sigma\text{-weak}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{U}(\mathcal{A})). \tag{2.8}$$

Recall that the maximal ideal space of $\mathcal{D}$, $\hat{\mathcal{D}}$, is a compact, extremally disconnected space (see [21, Theorem III.1.18]). In particular, the Gelfand transform determines a bijection between the set of projections in $\mathcal{D}$ and the family of clopen subsets of $\hat{\mathcal{D}}$. Also, each nonzero $v \in \mathcal{GN}(\mathcal{A}, \mathcal{D})$ determines a partial homeomorphism β_v of $\hat{\mathcal{D}}$ with domain $\{\rho \in \hat{\mathcal{D}} : \rho(v^*v) = 1\}$ and range $\{\rho \in \hat{\mathcal{D}} : \rho(vv^*) = 1\}$, via the formula,

$$\beta_v(\rho)(d) := \rho(v^*dv).$$

When $\beta_v(\rho) = \rho$ for all $\rho \in \mathrm{dom}(\beta_v)$, it is not difficult to see that v commutes with $\mathcal{D}$ and hence $v \in \mathcal{D}$. Finally, when $Q \in \mathcal{D}$ is a projection such that $Qv^*v \neq 0$, β_{vQ} is the restriction of β_v to $\{\rho \in \hat{\mathcal{D}} : \rho(Q) = 1\} \cap \mathrm{dom}(\beta_v)$.

Given $v \in \mathcal{GN}(\mathcal{A}, \mathcal{D})$ with $v \neq 0$, applying a variant of Frolík's Theorem (see [17, Proposition 2.7]) to β_v yields projections $Q_0, Q_1, Q_2, Q_3 \in \mathcal{D}$ such that $vQ_0 \in \mathcal{D}$, $(vQ_j)^2 = 0$ for $j = 1, 2, 3$ and

$$v = \sum_{j=0}^{3} vQ_j. \tag{2.9}$$

A calculation shows that when $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}$ satisfies $w^2 = 0$, then

$$U := w + w^* + (I - w^*w - ww^*) \in \mathcal{U}(\mathcal{A}) \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) \quad \text{and} \quad w = Uw^*w.$$

Since $\mathrm{span}(\mathcal{U}(\mathcal{D}))$ is σ-weakly dense in $\mathcal{D}$, we obtain

$$w \in \overline{\mathrm{span}}^{\sigma\text{-weak}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{U}(\mathcal{A})).$$

Hence,

$$\overline{\mathrm{span}}^{\sigma\text{-weak}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{U}(\mathcal{A})) \supseteq \mathrm{span}\{w \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A} : w^2 = 0\}.$$

This together with (2.9) implies that

$$\overline{\mathrm{span}}^{\sigma\text{-weak}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{U}(\mathcal{A})) \supseteq \mathrm{span}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}).$$

Thus (2.8) holds and $(\mathcal{A}, \mathcal{D})$ is a Cartan pair.

To obtain (c), let $\mathcal{E} \subseteq \mathcal{GN}(\mathcal{A}, \mathcal{D})$ be a maximal $\mathcal{D}$-orthogonal family. For each $u \in \mathcal{E}$, Corollary 2.3.2 gives $P_u \leq \mathrm{supp}(\mathcal{A})$. For any $u \in \mathcal{E}$, $x \in \mathcal{M}$ and $d \in \mathcal{D} \cap \mathfrak{n}_\phi$ we obtain,

$$\begin{aligned}
\pi_\phi(uE(u^*\Phi(x)))\eta_\phi(d) &= P_u\pi_\phi(\Phi(x))\eta_\phi(d) \\
&= P_u\eta_\phi(\Phi(xd)) \quad \text{(now apply part (a))} \\
&= P_u \,\mathrm{supp}(\mathcal{A})\eta_\phi(xd) = P_u\eta_\phi(xd) = P_u\pi_\phi(x)\eta_\phi(d) \\
&= \pi_\phi(uE(u^*x))\eta_\phi(d).
\end{aligned}$$

This holds for every $d \in \mathcal{D} \cap \mathfrak{n}_\phi$. Thus (using the fact that $\phi = \phi \circ E$ is faithful) for every $x \in \mathcal{M}$ and $u \in \mathcal{E}$, we have

$$uE(u^*\Phi(x)) = uE(u^*x). \tag{2.10}$$

By Proposition 2.4.4 applied to the Cartan pair $(\mathcal{A}, \mathcal{D})$,

$$\Phi(x) = \sum_{u \in E} uE(u^*\Phi(x)) = \sum_{u \in E} uE(u^*x),$$

where the sums are Bures convergent. This completes the proof. □

3. An extension theorem

In this section, we prove our main result about extending isometric algebra isomorphisms. We begin with two definitions.

Definition 3.1.1.

(a) Given a Cartan pair $(\mathcal{M}, \mathcal{D})$, a *Cartan bimodule algebra* is a σ-weakly closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfying $\mathcal{D} \subseteq \mathcal{A}$ and which generates $\mathcal{M}$ as a von Neumann algebra. We will sometimes write $\mathcal{A} \subseteq (\mathcal{M}, \mathcal{D})$ to indicate that $\mathcal{A}$ is a Cartan bimodule algebra for the pair $(\mathcal{M}, \mathcal{D})$.

(b) For $i = 1, 2$, let $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$ be Cartan bimodule algebras. An (algebraic) isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a *Cartan bimodule isomorphism* if θ is isometric and $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

Remark 3.1.2. In view of Theorem 2.5.9, when $\mathcal{A}$ is a σ-weakly closed subalgebra of $\mathcal{M}$ containing $\mathcal{D}$, $\mathcal{A}$ is a Cartan bimodule algebra relative to the Cartan pair, $(W^*(\mathcal{A}), \mathcal{D})$.

Lemma 3.1.3. *For $i = 1, 2$, let $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$ be Cartan bimodule algebras and suppose $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a Cartan bimodule isomorphism. Then*

$$\theta(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{A}_1) = \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{A}_2.$$

Proof. Let $v \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{A}_1$. Obviously $\theta(v) \in \mathcal{A}_2$. For all $h \in \mathcal{D}_1$, we have that

$$\theta(h)\theta(v) = \theta(hv) = \theta(hvv^*v) = \theta(vv^*hv) = \theta(v)\theta(v^*hv).$$

Since $\theta|_{\mathcal{D}_1}$ is a $*$-isomorphism, $\theta(v)^*\theta(h) = \theta(v^*hv)\theta(v)^*$. Hence, for all $d \in \mathcal{D}_1$,

$$\begin{aligned}\theta(h)\theta(v)\theta(d)\theta(v)^* &= \theta(v)\theta(v^*hv)\theta(d)\theta(v)^* = \theta(v)\theta(d)\theta(v^*hv)\theta(v)^* \\ &= \theta(v)\theta(d)\theta(v)^*\theta(h);\end{aligned}$$

thus $\theta(v)\theta(d)\theta(v)^* \in \mathcal{M}_2 \cap \mathcal{D}_2' = \mathcal{D}_2$. Likewise $\theta(v)^*\theta(d)\theta(v) \in \mathcal{D}_2$, and so $\theta(v) \in \mathcal{N}(\mathcal{M}_2, \mathcal{D}_2)$. We now show that $\theta(v)$ is a partial isometry. Note that $p := \theta(v)^*\theta(v)$ belongs to the unit ball of $\mathcal{D}_2$; we must show that p is a projection. To do this, we show that the spectrum of p is $\{0, 1\}$. If not, let $0 < \lambda < 1$ belong to the spectrum of p, and $\delta > 0$ be such that $0 < \lambda - \delta < \lambda + \delta < 1$ and let q be the spectral projection for p corresponding to the interval $(\lambda - \delta, \lambda + \delta)$. Then $\theta(v)q \neq 0$, and $\theta^{-1}(q)$ is a projection in $\mathcal{D}_1$. Then $0 \neq v\theta^{-1}(q) \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{A}_1$. As θ is isometric,

$$1 = \left\|v\theta^{-1}(q)\right\|^2 = \|\theta(v)q\|^2 = \|pq\| < 1,$$

which is absurd. Therefore, the spectrum of p equals $\{0, 1\}$, so p is a projection. Hence $\theta(v) \in \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{A}_2$. The lemma follows. □

Proposition 3.1.4. *Let $\mathcal{A} \subseteq (\mathcal{M}, \mathcal{D})$ be a Cartan bimodule algebra. Define $\mathcal{A}^0 = \overline{\text{span}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A})$ (norm closure) and $\mathcal{C} = C^*(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A})$. Then:*

(a) $\mathcal{C} = C^*(\mathcal{A}^0)$ *and* $\mathcal{D} \subseteq \mathcal{A}^0 \subseteq \mathcal{C}$.
(b) $\mathcal{C} = \overline{\operatorname{span}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{C})$.
(c) $\overline{\mathcal{C}}^{\sigma\text{-weak}} = \mathcal{M}$.

In particular, the pair $(\mathcal{C}, \mathcal{D})$ *is a* C^**-diagonal in the sense of Kumjian* [10].

Proof. (a) and (b) are routine. We turn now to (c). Since $\mathcal{A}^0 \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) = \mathcal{A} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$, we have that $\overline{\mathcal{A}^0}^{\sigma\text{-weak}} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) = \mathcal{A} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$, and so $\operatorname{supp}(\overline{\mathcal{A}^0}^{\sigma\text{-weak}}) = \operatorname{supp}(\mathcal{A})$, by Corollary 2.3.2. By Corollary 2.5.2,

$$\overline{\mathcal{A}^0}^{\text{Bures}} = \operatorname{bimod}(\operatorname{supp}(\overline{\mathcal{A}^0}^{\sigma\text{-weak}})) = \operatorname{bimod}(\operatorname{supp}(\mathcal{A})) = \overline{\mathcal{A}}^{\text{Bures}}.$$

Theorem 2.5.9 gives $\overline{\mathcal{C}}^{\text{Bures}} = \overline{\mathcal{C}}^{\sigma\text{-weak}}$. Thus,

$$\mathcal{A} \subseteq \overline{\mathcal{A}}^{\text{Bures}} = \overline{\mathcal{A}^0}^{\text{Bures}} \subseteq \overline{\mathcal{C}}^{\text{Bures}} = \overline{\mathcal{C}}^{\sigma\text{-weak}} = \mathcal{M},$$

with the last equality holding because $W^*(\mathcal{A}) = \mathcal{M}$. Hence $\mathcal{M} = \overline{\mathcal{C}}^{\sigma\text{-weak}}$.

Now (b) says that $(\mathcal{C}, \mathcal{D})$ is a regular inclusion. Moreover, as $\mathcal{D}$ is a MASA in $\mathcal{M}$, it is a MASA in $\mathcal{C}$. Since $\mathcal{D}$ is injective and $E|_{\mathcal{C}}$ is a faithful conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$, an application of [17, Theorem 2.10] shows $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. □

Corollary 3.1.5. *Let* $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ *be a Cartan bimodule isomorphism. Then:*

(a) *There exists a unique* $*$*-isomorphism* $\Theta : \mathcal{C}_1 \to \mathcal{C}_2$ *such that* $\Theta(x) = \theta(x)$ *for all* $x \in \mathcal{A}_1^0$ *(notation as in Proposition* 3.1.4*).*
(b) $\Theta(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{C}_1) = \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{C}_2$.

Proof. By Lemma 3.1.3, $\theta(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{A}_1) = \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{A}_2$. It follows that $\theta(\mathcal{A}_1^0) = \mathcal{A}_2^0$. By Proposition 3.1.4, the pair $(\mathcal{C}_i, \mathcal{D}_i)$ is a C^*-diagonal and $C^*(\mathcal{A}_i^0) = \mathcal{C}_i$, for $i = 1, 2$. An application of [16, Theorem 2.16] establishes (a).

Since Θ is a $*$-isomorphism and $\Theta(\mathcal{D}_1) = \mathcal{D}_2$, (b) holds. □

The following gives most of Assertion 1.1.1.

Theorem 3.1.6. *For* $i = 1, 2$*, let* $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$ *be Cartan bimodule algebras and let* $E_i : \mathcal{M}_i \to \mathcal{D}_i$ *be the faithful normal conditional expectations. Let* $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ *be a Cartan bimodule isomorphism. Then there exists a unique* $*$*-isomorphism* $\overline{\theta} : \mathcal{M}_1 \to \mathcal{M}_2$ *such that*

$$\overline{\theta}|_{\mathcal{A}_1^0} = \theta|_{\mathcal{A}_1^0};$$

and $\theta|_{\mathcal{A}_1^0}$ *is a homeomorphism of* $(\mathcal{A}_1^0, \tau_B)$ *onto* $(\mathcal{A}_2^0, \tau_B)$*.*

Furthermore, suppose ω_1 *is a faithful normal semi-finite weight on* $\mathcal{D}_1$ *and let* $\omega_2 = \omega_1 \circ (\theta|_{\mathcal{D}_1})^{-1}$*. Set* $\phi_i := \omega_i \circ E_i$ *and let* $(\pi_{\phi_i}, \mathfrak{H}_i, \eta_{\phi_i})$ *be the semicyclic representation of* $\mathcal{M}_i$ *corresponding to* ϕ_i*. Then there exists a unitary* $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ *such that for every* $X \in \mathcal{M}_1$*,*

$$U\pi_{\phi_1}(X) = \pi_{\phi_2}(\overline{\theta}(X))U.$$

Proof. We use the notation of Corollary 3.1.5. Since $(\mathcal{C}_i, \mathcal{D}_i)$ are C^*-diagonals, $\mathcal{C}_i$ has the extension property relative to $\mathcal{D}_i$. In particular, the expectations $E_i|_{\mathcal{C}_i}$ are unique. Since $\Theta \circ E_1|_{\mathcal{C}_1} \circ \Theta^{-1}$ is a conditional expectation of $\mathcal{C}_2$ onto $\mathcal{D}_2$, we obtain

$$E_2|_{\mathcal{C}_2} \circ \Theta = \Theta \circ E_1|_{\mathcal{C}_1}.$$

Hence

$$(\omega_2 \circ E_2)|_{\mathcal{C}_2} = (\omega_1 \circ E_1)|_{\mathcal{C}_1} \circ \Theta^{-1}.$$

For $Y \in \mathcal{C}_1$ and $d \in \mathfrak{n}_{\phi_1} \cap \mathcal{D}_1$, we have

$$\begin{aligned}\|\pi_{\phi_2}(\Theta(Y))\eta_{\phi_2}(\theta(d))\|^2 &= \omega_2(E_2(\Theta(d^*Y^*Yd))) = \omega_1(E_1(d^*Y^*Yd)) \\ &= \|\pi_{\phi_1}(Y)\eta_{\phi_1}(d)\|^2 .\end{aligned}$$

As $\pi_{\phi_i}(\mathcal{C}_i)\eta_{\phi_i}(\mathfrak{n}_{\phi_i} \cap \mathcal{D}_i)$ is dense in $\mathfrak{H}_i$ by Lemma 1.4.1, we find that the map $\pi_{\phi_1}(Y)\eta_{\phi_1}(d) \mapsto \pi_{\phi_2}(\Theta(Y))\eta_{\phi_2}(\theta(d))$ extends to a unitary $U \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Moreover, for any $X \in \mathcal{C}_1$, we obtain

$$U\pi_{\phi_1}(X) = \pi_{\phi_2}(\Theta(X))U.$$

For $X \in \mathcal{M}_1$ we now define

$$\overline{\theta}(X) := \pi_{\phi_2}^{-1}(U\pi_{\phi_1}(X)U^*).$$

Then $\overline{\theta}$ is a $*$-isomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$ and by construction, $\overline{\theta}|_{\mathcal{A}_1^0} = \Theta|_{\mathcal{A}_1^0} = \theta|_{\mathcal{A}_1^0}$.

The uniqueness of $\overline{\theta}$ follows from the facts that $\mathcal{C}_i$ are σ-weakly dense in $\mathcal{M}_i$ and Θ is the unique extension of $\theta|_{\mathcal{A}_1^0}$ to a $*$-isomorphism of $\mathcal{C}_1$ onto $\mathcal{C}_2$.

It is easy to see that $\overline{\theta} \circ E_1 = E_2 \circ \overline{\theta}$, which implies $\overline{\theta}$ and $(\overline{\theta})^{-1}$ are Bures continuous. Thus the restriction of $\overline{\theta}$ to $\mathcal{A}_1^0$ is a Bures homeomorphism onto $\mathcal{A}_2^0$. □

We now strengthen Theorem 3.1.6 by showing that when θ is σ-weakly continuous, $\overline{\theta}|_{\mathcal{A}_1} = \theta$. (Note: If we knew that $\mathcal{A}_1^0$ was σ-weakly dense in $\mathcal{A}_1$, this would be trivial. Unfortunately, all we know is that $\mathcal{A}_1^0$ is Bures dense in $\mathcal{A}_1$.) We require some preparation. The notation will be as in Theorem 3.1.6.

Lemma 3.1.7. *For $i = 1, 2$, let $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$ be Cartan bimodule algebras and let $E_i : \mathcal{M}_i \to \mathcal{D}_i$ be the faithful normal conditional expectations. Let $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ be a σ-weakly continuous Cartan bimodule isomorphism. If $x \in \mathcal{A}_1$ and $v \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1)$, then*

$$\theta(vE_1(v^*x)) = \overline{\theta}(v)E_2(\overline{\theta}(v)^*\theta(x)). \tag{3.1}$$

Before giving the proof, notice that Lemma 2.3.1(b) gives $vE(v^*x) \in \mathcal{A}_1$, so the left side of (3.1) is defined.

Proof. Let $x \in \mathcal{A}_1$ and $v \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1)$. By Lemma 1.3.3,

$$\{vE_1(v^*x)\} = v\mathcal{D}_1 \cap \overline{\text{co}}^{\sigma\text{-weak}}\{vUv^*xU^* : U \in \mathcal{U}(\mathcal{D}_1)\}. \tag{3.2}$$

We then have,

$$\begin{aligned}
\theta(vE_1(v^*x)) &\in \theta(\overline{\text{co}}^{\sigma\text{-weak}}\{vUv^*xU^* : U \in \mathcal{U}(\mathcal{D}_1)\}) \\
&\subseteq \overline{\text{co}}^{\sigma\text{-weak}}\{\theta(vUv^*)\theta(x)\theta(U^*) : U \in \mathcal{U}(\mathcal{D}_1)\} \\
&= \overline{\text{co}}^{\sigma\text{-weak}}\{\overline{\theta}(vUv^*)\theta(x)\theta(U)^* : U \in \mathcal{U}(\mathcal{D}_1)\} \\
&= \overline{\text{co}}^{\sigma\text{-weak}}\{\overline{\theta}(v)\theta(U)\overline{\theta}(v)^*\theta(x)\theta(U)^* : U \in \mathcal{U}(\mathcal{D}_1)\} \\
&= \overline{\text{co}}^{\sigma\text{-weak}}\{\overline{\theta}(v)W\overline{\theta}(v)^*\theta(x)W^* : W \in \mathcal{U}(\mathcal{D}_2)\}.
\end{aligned}$$

Since $vE_1(v^*x) \in \mathcal{A}_1^0$ (cf. Lemma 1.2.1), we have

$$\theta(vE_1(v^*x)) = \overline{\theta}(vE_1(v^*x)) \in \overline{\theta}(v)\mathcal{D}_2.$$

Thus,

$$\begin{aligned}
\theta(vE_1(v^*x)) &\in \overline{\theta}(v)\mathcal{D}_2 \cap \overline{\text{co}}^{\sigma\text{-weak}}\{\overline{\theta}(v)W\overline{\theta}(v)^*\theta(x)W^* : W \in \mathcal{U}(\mathcal{D}_2)\} \\
&= \{\overline{\theta}(v)E_2(\overline{\theta}(v)^*\theta(x))\}.
\end{aligned}$$

The lemma follows. □

Theorem 3.1.8. *In addition to the hypotheses of Theorem* 3.1.6*, assume* θ *is* σ*-weakly continuous. Then*

$$\theta = \overline{\theta}|_{\mathcal{A}_1}.$$

Proof. Let $\mathcal{E} \subseteq \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1)$ be a maximal $\mathcal{D}_1$-orthogonal set. Then $\overline{\theta}(\mathcal{E}) \subseteq \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2)$ is a maximal $\mathcal{D}_2$-orthogonal set.

Let $X \in \mathcal{A}_1$ and suppose $F \subseteq \mathcal{E}$ is a finite set. Then, with the notation of Proposition 2.4.4 and using Lemma 3.1.7, we have

$$\theta(X_F) = \sum_{v\in F} \theta(vE_1(v^*X)) = \sum_{v\in F} \overline{\theta}(v)E_2(\overline{\theta}(v)^*\theta(X)).$$

It then follows from Proposition 2.4.4 that $\theta(X_F)$ Bures converges to $\theta(X)$. On the other hand, since $X_F \in \mathcal{A}_1^0$, we have $\theta(X_F) = \overline{\theta}(X_F)$. As we noted in the proof of Theorem 3.1.6, $\overline{\theta}$ is Bures continuous. Therefore,

$$\overline{\theta}(X) = \text{Bures-}\lim_F \overline{\theta}(X_F) = \text{Bures-}\lim_F \theta(X_F) = \theta(X).$$ □

Remark 3.1.9. Without a continuity hypothesis, we have been unable to obtain Assertion 1.1.1, even when the Cartan pairs $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$ are assumed synthetic. Suppose $\mathcal{A}_i$ are synthetic. With the notation of Theorem 3.1.6, let $\alpha := \overline{\theta}^{-1} \circ \theta$. The hypothesis of synthesis implies α is an isometric automorphism of $\mathcal{A}_1$ such that $\alpha|_{\mathcal{A}_1^0} = \text{id}|_{\mathcal{A}_1^0}$. We have not been able to show $\alpha = \text{id}_{\mathcal{A}}$ without making a continuity hypothesis, and we suspect such a hypothesis may in general be necessary.

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