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Automorphisms of free groups. I

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Abstract. We describe, up to degree n, the Lie algebra associated with the automorphism group of a free group of rank n. We compute in particular the ranks of its homogeneous components, and their structure as modules over the linear group.

Along the way, we infirm (but confirm a weaker form of) a conjecture by Andreadakis, and answer a question by Bryant–Gupta–Levin–Mochizuki.

Contents

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1. Introduction

Let F denote a free group of rank r. The group-theoretical structure of the automorphism group A of F is probably exceedingly difficult to describe, but A may be 'graded', following Andreadakis [1], into a more manageable object. Let F_n denote the nth term of the lower central series of F, and let A_n denote the kernel of the natural map $\mathsf{Aut}(F)\to\mathsf{Aut}(F/F_{n+1})$. Then $A_0/A_1=\mathsf{GL}_r(\mathbb{Z})$, and A_n/A_{n+1} are finite-rank free $\mathbb{Z}$-modules; furthermore, $[A_n,A_m]\subseteq A_{m+n}$, and therefore

$$\mathscr{L}=\bigoplus_{n\geq 1}A_n/A_{n+1}$$

has the structure of a Lie algebra.

Let, by comparison, $\widehat{F}$ denote the limit of the quotients $(F/F_n)_{n\geq 1}$; it is a pronilpotent group, and $\widehat{F}/\widehat{F}_n$ is naturally isomorphic to F/F_n. Let B denote the automorphism group of $\widehat{F}$, and let similarly B_n denote the kernel of the natural map $\mathsf{Aut}(\widehat{F})\to\mathsf{Aut}(\widehat{F}/\widehat{F}_{n+1})$. Then $B_0/B_1=\mathsf{GL}_r(\mathbb{Z})$ and $\mathscr{M}=\bigoplus_{n\geq 1}B_n/B_{n+1}$ is also a Lie algebra; furthermore, B_n/B_{n+1} are also finite-rank free $\mathbb{Z}$-modules. In contrast to $\mathscr{L}$, the structure of $\mathscr{M}$ is well understood: it acts on the completion of the free $\mathbb{Z}$-Lie algebra $\bigoplus_{n\geq 1}\widehat{F}_n/\widehat{F}_{n+1}$, and its elements may be described as "polynomial noncommutative first-order differential operators", that is expressions

$$\sum_{\underline{i}}\alpha_{\underline{i}}X_{i_1}\dots X_{i_n}\frac{\partial}{\partial X_{i_0}}$$

in the noncommuting variables $X_1,\dots,X_r$. The embedding $F\to\widehat{F}$ with dense image induces a natural map $\mathscr{L}\to\mathscr{M}$, which is injective but not surjective.

The following problems appear naturally:

(1) Describe the closure of the image of $\mathscr{L}$ in $\mathscr{M}$.
(2) Relate A_n to the lower central series $(\gamma_n(A_1))_{n\geq 1}$ of A_1.
(3) Compute the ranks of $\mathscr{L}_n=A_n/A_{n+1}$ and $\mathscr{M}_n=B_n/B_{n+1}$.

Ad (1), Andreadakis observes that $\mathscr{L}_1=\mathscr{M}_1$ and $\mathscr{L}_2=\mathscr{M}_2$, while $\mathscr{L}_3\neq\mathscr{M}_3$ for $r\leq 3$.

Ad (2), Andreadakis conjectures [1, page 253] that $A_n=\gamma_n(A_1)$, and proves his assertion for $r=3$, $n\leq 3$ and for $r=2$. This is further developed by Pettet [27], who proves that $\gamma_3(A_1)$ has finite index in A_3 for all r, building her work on Johnson's homomorphism [15]. Further results have been obtained by Satoh [28, 29] and, in particular, what amounts to our

Theorem C under a slightly stronger restriction on the parameter n. The arguments in [7] let one deduce Theorems B and D from Theorem A.

Ad (3), Andreadakis proves

$$\operatorname{rank}\mathscr{M}_n = \frac{r}{n+1}\sum_{d|n+1}\mu(d)r^{(n+1)/d},$$

where μ denotes the Möbius function, and computes for $r=3$ the ranks $\operatorname{rank}(\mathscr{L}_n) = 9, 18, 44$ for $n = 2, 3, 4$ respectively. Pettet [27] generalizes these calculations to

$$\operatorname{rank}(\mathscr{L}_2) = \frac{r^2(r-1)}{2}, \qquad \operatorname{rank}(\mathscr{L}_3) = \frac{r^2(r^2-4)}{3} + \frac{r(r-1)}{2}.$$

1.1. Main results. In this paper, we prove:

Theorem A. *For all r, n, we have $\gamma_n(A_1) \le A_n$, and $A_n/\gamma_n(A_1)$ is a finite group. Moreover,*

$$A_n = \sqrt{\gamma_n(A_1)},$$

that is, A_n is the set of all $g \in A$ such that $g^k \in \gamma_n(A_1)$ for some $k \neq 0$.

On the other hand, for $r=3$, $n=7$, we have $A_n/\gamma_n(A_1) = \mathbb{Z}/3$.

Therefore, Andreadakis's conjecture is false, but barely so.

Theorem B. *If $r \ge n > 1$, then we have the rank formula*

$$\operatorname{rank}\mathscr{L}_n = \frac{r}{n+1}\sum_{d|n+1}\mu(d)r^{(n+1)/d} - \frac{1}{n}\sum_{d|n}\phi(d)r^{n/d}, \tag{1}$$

where ϕ denotes the Euler totient function.

As a byproduct, Andreadakis's above calculations for $r=3$ should be corrected to $\operatorname{rank}(\mathscr{L}_4) = 43$.

In studying the structure of $\mathscr{L}$, I found it useful to consider $\mathscr{L}_n = A_n/A_{n+1}$ not merely as an abelian group, but rather as a $\mathsf{GL}_r(\mathbb{Z})$-module under the conjugation action of A_0/A_1, and then to appeal to the classification of $\mathsf{GL}_r(\mathbb{Q})$-representations by tensoring with $\mathbb{Q}$. Theorem B is a consequence of the following description of $\mathscr{L}_n \otimes \mathbb{Q}$ as a $\mathsf{GL}_r(\mathbb{Q})$-module.

We start by a $\mathsf{GL}_r(\mathbb{Q})$-module decomposition of $\mathscr{M}_n \otimes \mathbb{Q}$. It turns out that $\mathscr{M}_n \otimes \mathbb{Q}$ naturally fits into an exact sequence

$$0 \longrightarrow \mathscr{T}_n \longrightarrow \mathscr{M}_n \otimes \mathbb{Q} \xrightarrow{\text{tr}} \mathscr{A}_n \longrightarrow 0,$$

whose terms we now describe. Let $\{x_1, \dots, x_r\}$ denote a basis of F. The first subspace $\mathscr{T}_n$ consists of the $\mathsf{GL}_r(\mathbb{Q})$-orbit in $\mathscr{M}_n \otimes \mathbb{Q}$ of the automorphisms

$$T_w : x_i \mapsto x_i \text{ for all } i < r, \quad x_r \mapsto x_r w$$

for all choices of $w \in F_{n+1} \cap \langle x_1, \dots, x_{r-1}\rangle$, and are so called because of their affinity to 'transvections'. The second subspace $\mathscr{A}_n$ may be identified with the $\mathsf{GL}_r(\mathbb{Q})$-orbit in $\mathscr{M}_n \otimes \mathbb{Q}$ of

$$A_{a_1\dots a_n} : x_i \mapsto x_i[x_i, a_1, \dots, a_n] \text{ for all } i,$$

for all choices of $a_1, \dots, a_n \in F$; here and below $[u, v]$ denotes the commutator $u^{-1}v^{-1}uv$, and $[u_1, \dots, u_n]$ denotes the left-normed iterated commutator $[[u_1, \dots, u_{n-1}], u_n]$.

For $r \geq n$, the space $\mathscr{A}_n$ is r^n-dimensional, and is isomorphic qua $\mathsf{GL}_r(\mathbb{Q})$-module with $H_1(F, \mathbb{Q})^{\otimes n}$, via $A_{a_1 \dots a_n} \leftrightarrow a_1 \otimes \cdots \otimes a_n$; hence the name reminding the $A_{a_1 \dots a_n}$ of their 'associative' origin.

Again for $r \geq n$, we may *define* $\mathscr{A}_n$ as $H_1(F, \mathbb{Q})^{\otimes n}$, and then the 'trace map' $\mathrm{tr} : \mathscr{M}_n \to \mathscr{A}_n$ sends an automorphism to the trace of its Jacobian matrix; compare [23].

Theorem C. *Assume $r \geq n > 1$, and identify $\mathscr{A}_n$ with $H_1(F, \mathbb{Q})^{\otimes n}$. Let $\mathbb{Z}/n = \langle \gamma \rangle$ act on $\mathscr{A}_n$ by cyclic permutation:*

$$(a_1 \otimes \cdots \otimes a_n)\gamma = a_2 \otimes \cdots \otimes a_n \otimes a_1.$$

Then $\mathscr{L}_n \otimes \mathbb{Q}$ contains $\mathscr{T}_n$, and its image in $\mathscr{A}_n$ is the subspace of "cyclically balanced" elements $\mathscr{A}_n(1 - \gamma)$ spanned by all

$$a_1 \otimes \cdots \otimes a_n - a_2 \otimes \cdots \otimes a_n \otimes a_1.$$

The $\mathsf{GL}_r(\mathbb{Q})$-decomposition of $V_n = H_1(F, \mathbb{Q})^{\otimes n}$ mimicks that of the regular representation of the symmetric group $\mathfrak{S}_n$, and is well described through Young diagrams (see §2 for the definitions of Young diagram, tableaux and major index). For example, the decomposition of V_n in irreducibles is given by all standard tableaux with n boxes. Lie elements in V_n, which correspond to inner automorphisms in $\mathscr{A}_n$, correspond to standard tableaux with major index $\equiv 1 \pmod{n}$, as shown by Klyashko [17]. We show:

Theorem D. *If $r \geq n$, the decomposition of $\mathscr{L}_n \otimes \mathbb{Q}$ in irreducibles is given as follows:*

- *All standard tableaux with $n+1$ boxes, major index $\equiv 1 \pmod{n+1}$, and at most $r - 1$ rows, to which a column of length $r - 1$ is added at the right.*
- *All standard tableaux with n boxes, at most r rows, and major index $\not\equiv 0 \pmod{n}$.*

The first class corresponds to $\mathscr{T}_n$, and the second one to $\mathscr{A}_n$.

In fact, numerical experiments show that Theorem B should remain true under the weaker condition $r \geq n - 1$. Illustrations appear in §8.

1.2. Main points. The proofs of Theorems A, B, D follow from classical results in the representation theory of $\mathsf{GL}_r(\mathbb{Q})$. The proof of Theorem C uses results of Birman and Bryant–Gupta–Levin–Mochizuki to the respective effects that a endomorphism is invertible if and only if its Jacobian matrix is invertible, and that in that case the trace of its Jacobian matrix is cyclically balanced.

In fact, these last authors ask whether that condition is sufficient for an endomorphism to be invertible; I give in §6 an example showing that it is not so.

1.3. Plan. §2 briefly summarizes the representation theory of $\mathsf{GL}_r(\mathbb{Q})$.

§3 recalls some facts about the automorphism group of a free group in the language of representation theory and free differential calculus.

§4 recalls elementary properties of free differential calculus.

§5 and §6 describe the Lie algebras $\mathscr{M}$ and $\mathscr{L}$ respectively, both as algebras and as $\mathsf{GL}_r(\mathbb{Q})$-modules.

§7 proves the theorems stated above.

Finally, §8 provides some examples and illustrations of the main results. Depending on the reader's familiarity with the subject, she/he may skip to §5.

1.4. Thanks. I greatly benefited from discussions with André Henriques, Joel Kamnitzer and Chenchang Zhu, and wish to thank them for their patience and generosity. I am also grateful to Steve Donkin, Donna Testerman, Takao Satoh and Naoya Enomoto for remarks and references that improved an earlier version of the text, and to the anonymous referee for his/her valuable remarks.

Some decompositions were checked using the computer software system GAP [11], and in particular its implementation of the "meataxe". Extensive calculations led to the second statement of Theorem A.

There has been a big gap between the beginning and the end of my writing this text, and I am very grateful to Benson Farb for having: (1) encouraged me to finish the writeup; (2) given me the opportunity of doing it at the University of Chicago in a friendly and stimulating atmosphere.

2. $\mathsf{GL}_r(\mathbb{Q})$-modules

Throughout this section we denote by V the natural $\mathsf{GL}_r(\mathbb{Q})$-module $\mathbb{Q}^r$.

We consider only algebraic representations, i.e., those linear representations whose matrix entries are polynomial functions of the matrix entries of $\mathsf{GL}_r(\mathbb{Q})$. The *degree* of such a representation is the degree of these polynomial functions. If W is a representation of degree n, then the scalar matrix $\mu\mathbf{1}$ acts by μ^n on W.

A fundamental construction by Weyl (see [10, § 15.3]) is as follows. The tensor algebra of V decomposes as

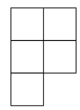
$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \bigoplus_{\lambda \text{ partition of } n} U_\lambda \otimes W_\lambda,$$

where U_λ and W_λ are respectively irreducible $\mathfrak{S}_n$- and $\mathsf{GL}_r(\mathbb{Q})$-modules. Each irreducible $\mathfrak{S}_n$-representation appears exactly once in this construction, and those W_λ which are nonzero, i.e., for which λ has at most r lines, describe all representations of $\mathsf{GL}_r(\mathbb{Q})$ exactly once, up to tensoring with a power of the one-dimensional determinant representation.

Therefore, degree-n representations of $\mathsf{GL}_r(\mathbb{Q})$ are indexed by irreducible representations of $\mathfrak{S}_n$, i.e., by conjugacy classes of $\mathfrak{S}_n$, i.e., by partitions of $\{1, \ldots, n\}$, the parts corresponding to cycle lengths in the conjugacy class.

Partitions with more than r parts yield $W_\lambda = 0$, and therefore do not appear in the decomposition of $V^{\otimes n}$.

It is convenient to represent partitions as Young diagrams, i.e. diagrams of boxes. The lengths of the rows, assumed to be weakly decreasing, give the parts in a partition. Thus



is the partition $5 = 2 + 2 + 1$. The natural representation V is described by a single box, and its symmetric and exterior powers are represented by a single row and a single column of boxes respectively. A *standard tableau* with shape λ, for λ a partition of n, is a filling-in of the Young diagram of λ with each one of the numbers $\{1, \ldots, n\}$ in such a way that rows and columns are strictly increasing rightwards and downwards respectively. For example,

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$$

are the standard tableaux with shape $2 + 2 + 1$. For λ a partition of n, the multiplicity of the $\mathsf{GL}_r(\mathbb{Q})$-module W_λ in $V^{\otimes n}$ is the dimension of U_λ, and is the number of standard tableaux with shape λ. The module W_λ may be written as $V^{\otimes n} c_\lambda$ for some idempotent $c_\lambda : \mathbb{Q}\mathfrak{S}_n \to \mathbb{Q}\mathfrak{S}_n$, called the *Schur symmetrizer*. This amounts to writing

$$W_\lambda = V^{\otimes n} \otimes_{\mathbb{Q}\mathfrak{S}_n} U_\lambda, \tag{2}$$

and c_λ for the projection from $\mathbb{Q}\mathfrak{S}_n$ to U_λ.

The *major index* of a tableau T is the sum of those entries $j \in T$ such that $j + 1$ lies on a lower row that j in T. For example, the major indices in the example above are respectively 4, 5, 6, 7, 8.

The tensor algebra $T(V)$ contains as a homogeneous subspace the Lie algebra generated by V; it is isomorphic to the free Lie algebra $\mathscr{L}(F)$. The homogeneous components $\mathscr{L}_n(F)$ are naturally $\mathsf{GL}_r(\mathbb{Q})$-modules, and their decomposition in irreducibles is described by Klyashko [17]:

Proposition 2.1. *The decomposition in irreducibles of $\mathscr{L}_n(F)$ is given by those tableaux with n boxes whose major index is $\equiv 1 \pmod{n}$.*

2.1. Inflation. We use a construction of $\mathsf{GL}_r(\mathbb{Q})$-modules from $\mathsf{GL}_{r-1}(\mathbb{Q})$-modules, called *inflation*. Consider a $\mathsf{GL}_{r-1}(\mathbb{Q})$-module S. It is naturally a $\mathbb{Q}^{r-1} \rtimes \mathsf{GL}_{r-1}(\mathbb{Q})$-module, via the projection $\mathbb{Q}^{r-1} \rtimes \mathsf{GL}_{r-1}(\mathbb{Q}) \to \mathsf{GL}_{r-1}(\mathbb{Q})$. We may embed the affine group $\mathsf{Aff} := \mathbb{Q}^{r-1} \rtimes \mathsf{GL}_{r-1}(\mathbb{Q})$ in $\mathsf{GL}_r(\mathbb{Q})$ as the matrices with last row $(0, \ldots, 0, 1)$. For an algebraic group G, let $\mathcal{P}(G)$ denote the Hopf algebra of polynomial functions on G. We may then induce S to a $\mathsf{GL}_r(\mathbb{Q})$-module $\widetilde{S} := S \otimes_{\mathcal{P}(\mathsf{Aff})} \mathcal{P}(\mathsf{GL}_r(\mathbb{Q}))$. In fact, the inverse operation is easier to describe: restrict the $\mathsf{GL}_r(\mathbb{Q})$-module $\widetilde{S}$ to Aff, and consider then the fixed points S of $\mathbb{Q}^{r-1}$; this is an irreducible $\mathsf{GL}_{r-1}(\mathbb{Q})$-module. See [14, II.2.11] for details.

This inflated module $\widetilde{S}$ has the same degree as S, and moreover its decomposition in irreducibles admits the same Young diagrams as S's: indeed it is immediate to check that $\mathbb{Q}^{r-1} \otimes_{\mathcal{P}(\mathsf{Aff})} \mathcal{P}(\mathsf{GL}_r(\mathbb{Q})) = \mathbb{Q}^r$. Every irreducible submodule of S may be seen as a submodule of $(\mathbb{Q}^{r-1})^{\otimes n}$ for some n, using (2). We may then describe S as $(\mathbb{Q}^{r-1})^{\otimes n} \otimes_{\mathbb{Q}\mathfrak{S}_n} U$ for some $\mathfrak{S}_n$-module U. We get

$$\widetilde{S} = (\mathbb{Q}^r)^{\otimes n} \otimes_{\mathbb{Q}\mathfrak{S}_n} U.$$

3. Free groups, their Lie algebras, and their automorphisms

Let G be a group. We recall a standard construction due to Magnus [20]. Let $(G_n)_{n\geq 1}$ be a chain of normal subgroups of G, with $G_{n+1} \subseteq G_n$ and $[G_m, G_n] \subseteq G_{m+n}$ for all $m, n \geq 1$.

Definition 3.1. The Lie ring associated with the series (G_n) is

$$\mathscr{L} = \mathscr{L}(G) = \bigoplus_{n=1}^{\infty} \mathscr{L}_n,$$

with $\mathscr{L}_n = G_n/G_{n+1}$.

Addition within the homogeneous component $\mathscr{L}_n$ is inherited from group multiplication in G_n, and the Lie bracket on $\mathscr{L}$ is defined among homogeneous elements by

$$\mathscr{L}_m \times \mathscr{L}_n \to \mathscr{L}_{m+n}, \qquad (uG_{m+1}, vG_{n+1}) \mapsto [u,v]G_{m+n+1}.$$

A typical example is obtained by letting (G_n) be the lower central series $(\gamma_n(G))$ of G, defined by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. The subgroups A_n described in the introduction yield an interesting (sometimes different) series.

We have an action of G on G_n by conjugation, which factors to an action of G/G_1 on $\mathscr{L}$ since G_1 acts trivially on G_n/G_{n+1}.

Conversely, if G_n is characteristic in G for all n, then we may set $H = G \rtimes \mathsf{Aut}\, G$ the *holomorph* of G, and consider the sequence $(G_n)_{n\geq 1}$ as sitting inside H. The resulting Lie algebra $\mathscr{L}$ admits, by the above, a linear action of $\mathsf{Aut}(G)$. The *IA-automorphisms* of G — those automorphisms that act trivially on $G/[G,G]$ — act trivially on $\mathscr{L}$ because $[G,G] \subset G_2$, so the linear group $\mathsf{GL}(H_1G) = \mathsf{Aut}(G)/\mathsf{IA}(G)$ acts on $\mathscr{L}$.

Lubotzky originally suggested to me that the structure of the groups $A = \mathsf{Aut}(F)$ and $B = \mathsf{Aut}(\widehat{F})$ could be understood by considering their Lie algebras with $\mathsf{GL}_r(\mathbb{Z})$-action; see [2]. For fruitful developments of this idea see [6].

3.1. Pronilpotent groups. A pronilpotent group is a limit of nilpotent groups. We recall some useful facts gleaned from [1]. Let F denote a (usual) free group of rank r. Give F a topology by choosing as basis of open neighbourhoods of the identity the collection of subgroups F_n in F's lower central series, and let $\widehat{F}$ be the completion of F in this topology. We naturally view F as a dense subgroup of $\widehat{F}$.

In considering series $(\widehat{F}_n)$ of subgroups of $\widehat{F}$, we further require that the $\widehat{F}_n$ be closed in $\widehat{F}$. Let $(\widehat{F}_n)$ be the (closed) lower central series of $\widehat{F}$, defined by $\widehat{F}_{n+1} = \overline{[\widehat{F}_n, \widehat{F}]}$. We have $F_n = F \cap \widehat{F}_n$, and F_n is dense in $\widehat{F}_n$. Therefore $\mathscr{L}_n(\widehat{F}) = \mathscr{L}_n(F)$, and by [22, Chapter 5] the module $\mathscr{L}_n(\widehat{F})$ is $\mathbb{Z}$-free, of rank r_n given by Witt's formula

$$r_n = \frac{1}{n}\sum_{d|n} \mu(d) r^{n/d},$$

where μ denotes the Moebius function.

From now on, we reserve the symbol $\mathscr{J}$ for this Lie algebra $\mathscr{L}(\widehat{F})$. It is naturally equipped with a $\mathsf{GL}_r(\mathbb{Z})$-action.

3.2. Automorphisms. We next turn to the group B of continuous automorphisms of $\widehat{F}$, in the usual compact-open topology. Since F is dense in $\widehat{F}$, every automorphism $\phi \in B$ is determined by the images $x_1^\phi, \dots, x_r^\phi \in \widehat{F}$ of a basis of F. The images of these in $\widehat{F}/\widehat{F}_2 = \mathbb{Z}^r$ determine a homomorphism $B \to \mathsf{GL}_r(\mathbb{Z})$, which is onto just as for the homomorphism $A \to \mathsf{GL}_r(\mathbb{Z})$.

The embedding $F \mapsto \widehat{F}$ induces an embedding $A \to B$ which, as we shall see, does not have dense image.

Let B_1 denote the kernel of the map $B \twoheadrightarrow \mathsf{GL}_r(\mathbb{Z})$; more generally, denote by π_n the natural map $\pi_n : B \to \mathsf{Aut}(F/F_{n+1})$ for all $n \geq 0$, and set $B_n = \ker \pi_n$. this defines a series of normal subgroups $B = B_0 > B_1 > \cdots$. Furthermore, by a straightforward adaptation of [1, Theorem 1.1], we have $[B_m, B_n] \subset B_{m+n}$, and therefore a Lie algebra

$$\mathscr{M} = \bigoplus_{n\geq 1} \mathscr{M}_n = \bigoplus_{n\geq 1} B_n/B_{n+1}.$$

Every element $\phi \in B_1$ is determined by the elements $x_1^{-1}x_1^\phi, \dots, x_r^{-1}x_r^\phi \in \widehat{F}_2$; and, conversely, every choice of $f_1, \dots, f_r \in \widehat{F}_2$ determines a homomorphism $\phi_0 : F \to \widehat{F}$ defined on the basis by $x_i^{\phi_0} = x_i f_i$, and extended multiplicatively; the so-defined map ϕ_0 extends to a continuous map $\phi : \widehat{F} \to \widehat{F}$, since $\phi^{-1}(\widehat{F}_n)$ contains $\widehat{F}_n$ and the $(\widehat{F}_n)$ are a basis for the topology on $\widehat{F}$. Finally, ϕ is onto, because $\phi\pi_n$ is onto for all $n \geq 0$; indeed, a family of elements $\{x_1^{\phi\pi_n}, \dots, x_r^{\phi\pi_n}\}$ generates the nilpotent group F/F_n if and only if it generates its abelianization F/F_2.

By [1, §4] or [19, Theorem 5.8], the $\mathbb{Z}$-module $\mathscr{M}_n$ is free of rank $r \cdot r_{n+1}$. Furthermore, $B_0/B_1 \cong \mathsf{GL}_r(\mathbb{Z})$, so $\mathscr{M}$ is naturally equipped with a $\mathsf{GL}_r(\mathbb{Z})$-action.

3.3. Structure of $\mathscr{J}$. In this § we set $V = H_1(\widehat{F}, \mathbb{Z}) \cong \mathbb{Z}^r$, and study the structure of $\mathscr{J}$ as a $\mathsf{GL}_r(\mathbb{Z})$-module. It is well-known [20] that, as a Lie algebra, $\mathscr{J}$ is the free $\mathbb{Z}$-algebra generated by V.

$\widehat{F}_n$ is topologically spanned by n-fold commutators of elements of $\widehat{F}$, which can be written as functions $f = f(v_1, \dots, v_n)$, where f is now also seen as a commutator expression, evaluated at elements $v_i \in \widehat{F}$. Of the other hand, if we consider f as an element of $\mathscr{J}_n = \widehat{F}_n/\widehat{F}_{n+1}$, these v_i should actually be seen as elements of $V = \widehat{F}/\widehat{F}_2$, since $f(v_1, \dots, v_n) \in \widehat{F}_{n+1}$ as soon as one of the v_i belongs to $\widehat{F}_2$.

The action of $\mathsf{GL}_r(\mathbb{Z})$ on commutator expressions $f = f(v_1, \dots, v_n)$ is diagonal: $f^\rho = f(v_1^\rho, \dots, v_n^\rho)$. This yields immediately

Lemma 3.2. *The representation $\mathscr{J}_n$ of $\mathsf{GL}_r(\mathbb{Z})$ has degree n.*

The following result dates back to the origins of the study of free Lie rings [4], and is even implicit in Witt's work; it appears in the language of operads in [12, Proposition 5.3].

Theorem 3.3. *The decomposition of $\mathscr{J}_n$ as a $\mathsf{GL}_r(\mathbb{Z})$-module is given by inclusion-exclusion as follows:*

$$\mathscr{J}_n = \frac{1}{n} \bigoplus_{d|n} \mu(d)(\psi_d V)^{\otimes n/d},$$

where ψ_d is the d-th Adams operation (keeping the underlying vector space, raising eigenvalues to the d-th power).

Although this formula is explicit and allows fast computation of character values, it is not quite sufficient to write down $\mathscr{J}_n$ conveniently — it would be better to express $\mathscr{J}_n$ as $V^{\otimes n} \otimes_{\mathbb{Z}\mathfrak{S}_n} S_n$ for an appropriate $\mathfrak{S}_n$-representation S_n.

Let us assume for a moment that K is a ring containing a primitive n-th root of unity ε, and that V is a free K-module of rank r. Then by [17] we have

$$S_n = \operatorname{Ind}_{\mathbb{Z}/n}^{\mathfrak{S}_n} K_\varepsilon, \tag{3}$$

where $\mathbb{Z}/n$ acts on $K_\varepsilon \cong K$ by multiplication by ε. Furthermore, if K contains $\frac{1}{n}$, Klyashko gives an isomorphism between the functors $V \mapsto \mathscr{J}_n(V)$ and

$$V \mapsto C_n(V) = \operatorname{Hom}_{K[\mathbb{Z}/n]}(K_\varepsilon, V^{\otimes n}) \simeq \left\{ v \in V^{\otimes n} \,|\, v\gamma = \varepsilon v \right\},$$

where $\mathbb{Z}/n = \langle \gamma \rangle$ acts on $V^{\otimes n}$ by permutation of the factors.

We may not assume that $\mathbb{Z}$ contains n-th roots of unity — it does not; however, S_n is defined over $\mathbb{Z}$ and may be constructed without reference to any ε. Numerous authors [16, 18] have studied the decomposition in irreducibles of the induction from a cyclic subgroup of a one-dimensional representation. We reproduce it here in our notation.

Proposition 3.4 ([18])**.** *The multiplicity of the irreducible representation U_λ in S_n is the number of standard Young tableaux of shape λ and major index congruent to* 1 *modulo n.*

The following result seems new, and constructs efficiently the representation S_n without appealing to n-th roots of unity:

Proposition 3.5. *Inside $\mathfrak{S}_n$, consider the following subgroups: a cyclic subgroup $\mathbb{Z}/n$ generated by a cycle γ of length n; its automorphism group $(\mathbb{Z}/n)^*$; and its subgroup $(n/d)\mathbb{Z}/n$ generated by $\gamma^{n/d}$, isomorphic to a cyclic group of order d. Then*

$$S_n = \bigoplus_{d|n} \mu(d) \operatorname{Ind}_{((n/d)\mathbb{Z}/n)\rtimes(\mathbb{Z}/n)^*}^{\mathfrak{S}_n} \mathbf{1}, \tag{4}$$

where $\mathbf{1}$ *denotes the trivial representation.*

For concreteness, we may identify $\mathfrak{S}_n$ with the symmetric group of $\mathbb{Z}/n$. Then γ is the permutation $i \mapsto i+1 \pmod n$, and $(\mathbb{Z}/n)^*$ is the group of permutations of the form $i \mapsto ki \pmod n$ for all k coprime to n.

Proof. The proof proceeds by direct computation of the characters of the left- and right-hand side of (4), using the expression (4).

To simplify notation, we will write $C = \mathbb{Z}/n$. We write elements of $C \rtimes C^*$ as (m, u). For $m \in C$ we write $m^* = n/\gcd(m, n)$ its order in C. We enumerate $C^* = \{u_1, \dots, u_{\phi(n)}\}$.

It suffices actually to prove that the inductions of K_ε and $\mathbf{1}$ to $C \rtimes C^*$ are isomorphic. Let α denote the character of $\operatorname{Ind}_C^{C\rtimes C^*} K_\varepsilon$; then

$$\alpha(m,u) = \begin{cases} \mu(m^*)\frac{\phi(n)}{\phi(m^*)} & \text{if } u = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ is Euler's totient function. Indeed $\operatorname{Ind}_C^{C\rtimes C^*} \alpha(m,u)$ is a $\phi(n) \times \phi(n)$-monomial matrix; it is the product of a diagonal matrix with entries ε^{μ_1}, $\dots, \varepsilon^{\mu_{\phi(n)}}$ and the permutational matrix given by u's natural action on C^*. This matrix has trace 0 unless $u = 1$, in which case its trace is $\phi(n)/\phi(m^*)$ the sum of all primitive m^*-th roots of unity.

Let β_d denote the character of $\operatorname{Ind}_{C^{n/d}\rtimes C^*}^{C\rtimes C^*} \mathbf{1}$. Then by similar reasoning

$$\beta(m,u) = \begin{cases} \gcd(\frac{n}{d}, u-1) & \text{if } \gcd(\frac{n}{d}, u-1)|m, \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from the next, elementary lemma, whose proof is immediate, by noting that the left- and right-hand sides are multiplicative, and agree when n is a prime power. □

Lemma 3.6. *For any $\ell|n$, we have*

$$\sum_{\ell|d|n}\mu(d)\frac{n}{d}=\mu(\ell)\frac{\phi(n)}{\phi(\ell)}.$$

If $1<u<n$, we also have

$$\sum_{\ell|d|n}\mu(d)\gcd\left(\tfrac{n}{d},u-1\right)=0.$$

4. Free differential calculus

We recall the basic notions from [9]. Let again F denote a free group of rank r, with basis $\{x_1,\dots,x_r\}$. Define derivations

$$\frac{\partial}{\partial x_i}:\mathbb{Z}F\to\mathbb{Z}F$$

by the rules $\frac{\partial}{\partial x_i}x_i=1$, $\frac{\partial}{\partial x_i}(x_i^{-1})=x_i^{-1}$, and $\frac{\partial}{\partial x_i}(x_j^{\pm1})=0$ if $i\neq j$, extended to $\mathbb{Z}F$ linearity and by the Leibniz rule $\frac{\partial}{\partial x_i}(uv)=\frac{\partial u}{\partial x_i}v^o+u\frac{\partial v}{\partial x_i}$, where $o:\mathbb{Z}F\to\mathbb{Z}$ denotes the augmentation map.

A simple calculation proves the formula

$$\frac{\partial}{\partial x_i}[u,v]=u^{-1}v^{-1}\Big((u-1)\frac{\partial v}{\partial x_i}-(v-1)\frac{\partial u}{\partial x_i}\Big). \tag{5}$$

In particular, if $u\in\gamma_n(F)$, then modulo $\gamma_{n+2}(F)$ we have

$$\frac{\partial}{\partial x_i}[u,x_i]\equiv(u-1)-(x_i-1)\frac{\partial u}{\partial x_i},\qquad \frac{\partial}{\partial x_i}[u,x_j]\equiv-(x_j-1)\frac{\partial u}{\partial x_i}\ \text{if } j\neq i.$$

Denote by $\varpi\leq\mathbb{Z}F$ the kernel of o; then for all $u\in\mathbb{Z}F$ we have the "fundamental relation" [9, (2.3)]

$$u-u^o=\sum_{i=1}^{r}\frac{\partial u}{\partial x_i}(x_i-1).$$

Write $X_i=x_i-1$ for $i\in\{1,\dots,r\}$, and consider the ring

$$\mathcal{R}=\mathbb{Z}\langle\langle X_1,\dots,X_r\rangle\rangle$$

of noncommutative formal power series. The map $\tau:x_i\mapsto X_i+1$ defines an embedding of F in $\mathcal{R}$, which extends to an embedding $\tau:\mathbb{Z}F\to\mathcal{R}$. Let ϖ denote also the fundamental ideal $\langle X_1,\dots,X_r\rangle$ of $\mathcal{R}$; this should be no cause of confusion, since $\varpi^\tau=\varpi\cap(\mathbb{Z}F)^\tau$. The ring $\mathcal{R}$ is graded, with homogeneous component $\mathcal{R}_n$ of rank r^n, spanned by words of length n in $X_1,\dots,X_r$.

The dense subalgebra of $\mathcal{R}$ generated by the X_i is free of rank r; it is therefore a Hopf algebra, isomorphic to the enveloping algebra of the free Lie algebra $\mathscr{J}$. From now on, we consider $\mathscr{J}$ as a Lie subalgebra of $\mathcal{R}$ in this manner.

5. Structure of $\mathscr{M}$

We are ready to understand the Lie algebra $\mathscr{M}$ associated with the automorphism group B of $\widehat{F}$.

The module $V = H_1(\widehat{F};\mathbb{Z})$ naturally identifies with $\widehat{F}/\widehat{F}_2$. Its dual, V^*, identifies with homomorphisms $\widehat{F} \to \mathbb{Z}$.

Theorem 5.1. *The $\mathsf{GL}_r(\mathbb{Z})$-module $\mathscr{M}_n$ is isomorphic to $V^* \otimes \mathscr{J}_{n+1}$. The isomorphism $\rho : V^* \otimes \mathscr{J}_{n+1} \to \mathscr{M}_n$ is defined on elementary tensors $\alpha \otimes f$ by*

$$\alpha \otimes f \mapsto \{x_i \mapsto x_i f^{\alpha(x_i)}\},$$

and extended by linearity.

The proof is inspired from [24]; see also [19, Lemma 5.7].

Proof. Consider an elementary tensor $\alpha \otimes f$. There is a unique endomorphism $\phi : \widehat{F} \to \widehat{F}$ satisfying $x_i^\phi = x_i f^{\alpha(x_i)}$, so ρ's image in contained in B. Next, $\{x_i f^{\alpha(x_i)}\}$ is a basis of $\widehat{F}$, since it spans $\widehat{F}/\widehat{F}_2$, so ϕ is invertible.

The map ρ is well-defined: if $f \in F_{n+2}$, then the automorphism

$$x_i \mapsto x_i f^{\alpha(x_i)}$$

of $\widehat{F}$ belongs to B_{n+1}, so the automorphism $\alpha \otimes f$ is may be defined indifferently for an element $f \in \mathscr{J}_{n+1}$ or its representative $f \in F_{n+1}$.

Let us denote temporarily by $x_1^*, \dots, x_r^*$ the dual basis of V^*, defined by $x_i^*(x_j) = \mathbf{1}_{ij}$.

We construct a map $\sigma : B_n \to V^* \otimes \widehat{F}_{n+1}$. Let $\phi \in B_n$ be given. Then $x_i^\phi \equiv x_i \mod \widehat{F}_{n+1}$, so $x_i^\phi = x_i f_i$ for some $f_i \in \widehat{F}_{n+1}$. We set

$$\phi^\sigma = \sum_{i=1}^r x_i^* \otimes f_i.$$

If $\phi \in B_{n+1}$, then $f_i \in \widehat{F}_{n+2}$, so $\phi^\sigma \in V^* \otimes \widehat{F}_{n+2}$. It follows that σ induces a well-defined map $\mathscr{M}_n \to V^* \otimes \mathscr{J}_{n+1}$. Furthermore the maps ρ and σ are inverses of each other.

Next, we check that ρ is linear. Consider $(\alpha \otimes f)^\rho = \phi$ and $(\beta \otimes g)^\rho = \chi$. Then

$$x^{\phi\chi} = (x f^{\alpha(x)})^\chi = (x f^{\alpha(x)}) g^{\beta(x f^{\alpha(x)})} = x f^{\alpha(x)} g^{\beta(x)} = x^{\rho(\alpha\otimes f + \beta\otimes g)}.$$

Finally, we show that the $\mathsf{GL}_r(\mathbb{Z})$-actions are compatible. Choose an element of $\mathsf{GL}_r(\mathbb{Z})$, and lift it to some $\mu \in B$. Consider $(\alpha \otimes f)^\rho = \phi$. Then $(\alpha \otimes f)^\mu = \alpha' \otimes f^\mu$, where $\alpha' \in V^*$ is defined by $\alpha'(x) = \alpha(x^{\mu^{-1}})$, so

$$x^{\phi^\mu} = x^{\mu^{-1}\phi\mu} = \left(x^{\mu^{-1}} f^{\alpha(x^{\mu^{-1}})}\right)^\mu = x(f^\mu)^{\alpha(x^{\mu^{-1}})} = x^{(\alpha'\otimes f^\mu)^\rho}. \qquad \square$$

The Lie bracket on $\mathscr{M}$ may be expressed via the identification ρ of Theorem 5.1.

Theorem 5.2. *Consider* $\alpha, \beta \in V^*$ *and* $f = f(v_0, \dots, v_m) \in \mathscr{J}_{m+1}$ *and* $g = g(w_0, \dots w_n) \in \mathscr{J}_{n+1}$. *Then the bracket* $\mathscr{M}_m \times \mathscr{M}_n \to \mathscr{M}_{m+n}$ *is given by*

$$[\alpha \otimes f, \beta \otimes g] = \alpha \otimes \sum_{i=0}^{m} \beta(v_i) f(v_0, \dots, v_{i-1}, g, v_{i+1}, \dots, v_m) \\ - \beta \otimes \sum_{i=0}^{n} \alpha(w_i) g(w_0, \dots, w_{i-1}, f, w_{i+1}, \dots, w_n).$$

Proof. Write $(\alpha \otimes f)^\rho = \phi$ and $(\beta \otimes g)^\rho = \chi$. Then $\phi^{-1} = (\alpha \otimes f^{-\phi^{-1}})^\rho$ and $\chi^{-1} = (\beta \otimes f^{-\chi^{-1}})^\rho$; indeed

$$(x^\phi)^{\phi^{-1}} = (x f^{\alpha(x)})^{\phi^{-1}} = x f^{-\phi^{-1}\alpha(x)} f^{\alpha(x)\phi^{-1}} = x.$$

We then compute

$$\begin{aligned} x^{[\phi,\chi]} &= x^{\phi^{-1}\chi^{-1}\phi\chi} = (x f^{-\phi^{-1}\alpha(x)})^{\chi^{-1}\phi\chi} \\ &= (x g^{-\chi^{-1}\beta(x)})^{\phi\chi} f^{-\alpha(x)\phi^{-1}\chi^{-1}\phi\chi} \\ &= (x f^{\alpha(x)})^{\chi} g^{-\beta(x)\chi^{-1}\phi\chi} f^{-\alpha(x)\phi^{-1}\chi^{-1}\phi\chi} \\ &= x g^{\beta(x)} f^{\alpha(x)\chi} g^{-\beta(x)\chi^{-1}\phi\chi} f^{-\alpha(x)\phi^{-1}\chi^{-1}\phi\chi} \\ &\equiv x \left(f^{\alpha(x)\chi} f^{-\alpha(x)\phi^{-1}\chi^{-1}\phi\chi}\right)\left(g^{\beta(x)} g^{-\beta(x)\chi^{-1}\phi\chi}\right) \quad \bmod \widehat{F}_{n+m+2} \\ &\equiv x f^{\alpha(x)(\chi-1)} g^{\beta(x)(1-\phi)} = x^{(\alpha\otimes f^{\chi-1} - \beta\otimes g^{\phi-1})^\rho} \quad \bmod \widehat{F}_{n+m+2}. \end{aligned}$$

Now, again computing modulo $\widehat{F}_{n+m+2}$, we have

$$f^\chi = f(v_0 g^{\beta(v_0)}, \dots, v_m g^{\beta(v_m)}) \equiv f \prod_{i=0}^{m} f(v_0, \dots, v_{i-1}, g, v_{i+1}, \dots, v_m)^{\beta(v_i)},$$

and similarly for g, so the proof is finished. □

In fact, the dual basis $\{x_i^*\}$ of V^* is naturally written $\{\frac{\partial}{\partial x_i}\}$; in that language, Theorem 5.1 can be rephrased in an isomorphism

$$\rho : \sum_{i=1}^{r} f_i \frac{\partial}{\partial x_i} \mapsto \left(\phi : x_j \mapsto x_j \prod_{i=1}^{r} f_i^{\frac{\partial x_j}{\partial x_i}} = x_j f_j\right)$$

between $\mathscr{M}$ and order-1 differential operators on $\mathcal{R}$. Moreover, if $\sum f_i \frac{\partial}{\partial x_i} \in \mathscr{M}_n$ then $f_i \in \widehat{F}_{n+1}$, so $f_i - 1 \in \mathcal{R}_{n+1}$. Theorem 5.2 then expresses the Lie bracket on $\mathscr{M}$ as a kind of "Poisson bracket": for $Y \in \mathcal{R}_{n+1}$ and $Z \in \mathcal{R}_{m+1}$, we have

$$\left[Y \frac{\partial}{\partial x_i}, Z \frac{\partial}{\partial x_j}\right] = \frac{\partial Y}{\partial x_j} Z \frac{\partial}{\partial x_i} - \frac{\partial Z}{\partial x_i} Y \frac{\partial}{\partial x_j} \in V^* \otimes \mathcal{R}_{n+m+1}.$$

The representation $\mathscr{M}_n$ can also be written in terms of representations of the symmetric group, as follows. The representation $\mathscr{M}_n \otimes \det$ has degree

$n+r$, and therefore may be written as $V^{\otimes(n+r)} \otimes_{\mathbb{Z}\mathfrak{S}_{n+r}} T_n$, for some representation T_n of $\mathfrak{S}_{n+r}$. Recall that S_n denotes the representation of $\mathfrak{S}_n$ corresponding to the Lie submodule $\mathscr{J}_n \subset \mathcal{R}_n$.

Proposition 5.3. *Let (-1) denote the sign representation of $\mathfrak{S}_{r-1}$. Then*

$$T_n = \operatorname{Ind}_{\mathfrak{S}_{n+1}\times\mathfrak{S}_{r-1}}^{\mathfrak{S}_{n+r}} S_{n+1} \otimes (-1).$$

Proof. Let W, W' be two $\mathsf{GL}_r(\mathbb{Z})$-representations, of degrees m, m' respectively. Then they may be written $W = V^{\otimes m} \otimes_{\mathbb{Z}\mathfrak{S}_m} T$ and $W' = V^{\otimes m'} \otimes_{\mathbb{Z}\mathfrak{S}_{m'}} T'$ for representations T, T' of $\mathfrak{S}_m, \mathfrak{S}_{m'}$ respectively. Their tensor product $W \otimes W'$ then satisfies

$$W \otimes W' \cong V^{\otimes(m+m')} \otimes_{\mathbb{Z}\mathfrak{S}_{m+m'}} \operatorname{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_{m'}}^{\mathfrak{S}_{m+m'}} T \otimes T'.$$

The proposition then follows from Theorem 5.1, with $W = V^*$ and $W' = \mathscr{J}_{n+1}$, since $V^* \otimes \det = V^{\otimes r-1} \otimes_{\mathbb{Z}\mathfrak{S}_{r-1}} (-1)$. □

5.1. Decomposition in $\mathsf{GL}_r(\mathbb{Q})$-modules. We now turn to the fundamental decomposition of the module $\mathscr{M}_n$. Its $\mathsf{GL}_r(\mathbb{Z})$-module structure seems quite complicated; so we content ourselves with a study of the $\mathsf{GL}_r(\mathbb{Q})$-module $\mathscr{M}_n \otimes \mathbb{Q}$. We define the following two submodules of $\mathscr{M}_n \otimes \mathbb{Q}$. The first, $\mathscr{T}_n$, is spanned by the $\mathsf{GL}_r(\mathbb{Q})$-orbit of the automorphisms

$$T_w : x_i \mapsto x_i \text{ for all } i < r, \quad x_r \mapsto x_r w$$

for all choices of $w \in F_{n+1} \cap \langle x_1, \dots, x_{r-1}\rangle$. The second subspace, $\mathscr{A}_n$, is spanned by the automorphisms

$$A_{a_1\dots a_n} : x_i \mapsto x_i[x_i, a_1, \dots, a_n] \text{ for all } i,$$

for all choices of $a_1, \dots, a_n \in F$.

Lemma 5.4. *We have $\mathscr{M}_n \otimes \mathbb{Q} = \mathscr{T}_n \oplus \mathscr{A}_n$ qua $\mathsf{GL}_r(\mathbb{Q})$-modules.*

Proof. We may view $\mathscr{T}_n$ and $\mathscr{A}_n$ as submodules of $V^* \otimes \mathscr{J}_{n+1}$ via Theorem 5.1. Here, $\mathscr{T}_n$ is spanned by all those $\alpha \otimes f(v_0, \dots, v_n)$ such that $\alpha(v_i) = 0$ for all $i \in \{0, \dots, n\}$, when f ranges over n-fold commutators. On the other hand, $\mathscr{A}_n$ is spanned by the $\sum_i x_i^* \otimes f(x_i, v_1, \dots, v_n)$ when f ranges over n-fold commutators. We conclude that $\mathscr{T}_n \cap \mathscr{A}_n = 0$, and it remains to check, by dimension counting, that $\mathscr{T}_n + \mathscr{A}_n = \mathscr{M}_n \otimes \mathbb{Q}$.

By the Littlewood–Richardson rule, the module T_n from Proposition 5.3 is a sum of irreducible representations of $\mathfrak{S}_{n+r}$ of all possible skew shapes λ obtained by playing the "jeu du taquin" on a column of height $r-1$ (the Young diagram of the sign representation) and shapes μ appearing in S_{n+1}.

In the diagram λ, the column of height $r-1$ occupies either the places $(1,1), \dots, (r-1,1)$, or the places $(2,1), \dots, (r,1)$. We shall see that the first case corresponds to summands of $\mathscr{T}_n$, and the second case corresponds to summands of $\mathscr{A}_n$.

In the first case, the original representation μ of $\mathfrak{S}_{n+1}$ subsists, on the condition that it contains at most $r-1$ lines. These summands therefore precisely describe those representations of $\mathfrak{S}_{n+1}$ on $\mathscr{L}_{n+1}$ that come from $F_{n+1} \cap \langle x_1, \dots, x_{r-1} \rangle$.

In the second case, the "jeu du taquin" procedure asks us to remove box $(1,2)$ from μ to fill position $(1,1)$, and to propagate this hole in μ. This amounts to restricting S_{n+1} to the natural subgroup $\mathfrak{S}_n$ of $\mathfrak{S}_{n+1}$. Recall that $S_{n+1} = \mathrm{Ind}_C^{\mathfrak{S}_{n+1}} \chi$, for a primitive character χ of the cyclic group C generated by a cycle of length $n+1$. By Mackey's theorem,

$$\mathrm{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S_{n+1} = \mathrm{Ind}_{C\cap\mathfrak{S}_n}^{\mathfrak{S}_n} \mathrm{Res}_{C\cap\mathfrak{S}_n}^{C} \chi = \mathrm{Ind}_1^{\mathfrak{S}_n} \mathbf{1},$$

since $C\mathfrak{S}_n = \mathfrak{S}_{n+1}$ and $C \cap \mathfrak{S}_n = 1$. Now the $\mathsf{GL}_r(\mathbb{Q})$-representation associated with the $\mathfrak{S}_n$-representation $\mathbb{Q}\mathfrak{S}_n$ is the full space $V^{\otimes n}$, which spans $\mathscr{A}_n$ naturally. □

The correspondence $X_{i_1} \dots X_{i_n} \mapsto A_{x_{i_1} \dots x_{i_n}}$ defines a linear map $\theta'_n : \mathcal{R}_n \to \mathscr{A}_n$.

Lemma 5.5. *If $r \geq n$ then θ'_n is bijective, and makes $\mathscr{A}_n$ isomorphic to $V^{\otimes n}$ qua $\mathsf{GL}_r(\mathbb{Q})$-module.*

Proof. It is clear that θ'_n is onto, and is compatible with the $\mathsf{GL}_r(\mathbb{Q})$-action.

Continuing with the argument of the previous lemma, the Young diagrams λ and μ automatically have at most r rows because $r \geq n$; so, since $\mathrm{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S_{n+1}$ is the regular representation, $\mathscr{A}_n \cong V^{\otimes n}$ has the same dimension as $\mathcal{R}_n$, so θ'_n is injective. □

Note, however, that θ'_n is not injective for $n > r$, and that the θ'_n do not assemble into an algebra homomorphism $\mathcal{R} \to \mathscr{M}$. There does exist, however, an algebra homomorphism $\theta : \mathcal{R} \to \mathscr{M}$, defined as θ'_1 on V and extended multiplicatively to $\mathcal{R}$. It gives $\mathscr{M}$ the "matrix-like" algebra structure (compare with (6))

$$\Big(Y\frac{\partial}{\partial x_i}\Big) \cdot \Big(Z\frac{\partial}{\partial x_j}\Big) = \frac{\partial Y}{\partial x_j} Z \frac{\partial}{\partial x_i}.$$

There does not seem to be any simple formula for the components θ_n of θ, which are "deformations" of θ'_n.

Lemma 5.6. *θ is an algebra homomorphism $\mathcal{R} \to \mathscr{M}$, that is injective up to degree r. Its image is $\bigoplus_{n\geq 0} \mathscr{A}_n$. Its restriction $\theta \downharpoonleft \mathscr{J}$ is injective, and has as image the inner automorphisms of $\mathscr{M}$.*

Proof. Consider the following filtration of $\mathcal{R}_n$:

$$\mathcal{R}_n^i = \langle \text{products of elements of } \mathcal{R}_1 \text{ involving} \geq i \text{ Lie brackets} \rangle.$$

Then $\mathcal{R}_n^0 = \mathcal{R}_n$, and $\mathcal{R}_n^{n-1} = \mathscr{J}_n$. Let $\overline{\mathcal{R}}_n = \bigoplus_{i=0}^{n-2} \mathcal{R}_n^i / R_n^{i+1}$ be the associated graded. A direct calculation gives

$$A_{a_1 \dots a_m} \cdot A_{b_1 \dots b_n} = A_{a_1 \dots a_m b_1 \dots b_n} + \sum_{j=1}^{m} A_{a_1, \dots, [a_j, b_1, \dots, b_n], \dots, a_m}.$$

Therefore $\theta_n = \theta'_n$ on $\mathscr{J}_n$, and the associated graded maps $\overline{\theta'}_n$ and $\overline{\theta}_n$ coincide; therefore, by Lemma 5.5, the map θ_n is injective if $n \leq r$.

This also shows that $\mathscr{A} = \bigoplus_{n\geq 0} \mathscr{A}_n$ is closed under multiplication. The Lie subalgebra $\mathscr{J}$ of $\mathcal{R}$ then naturally corresponds under θ to the span of the $A_{[a_1,\dots,a_n]}$, namely to inner automorphisms, acting by conjugation by $[a_1, \dots, a_n]$. □

The following description is clear from the Young diagram decomposition of $\mathscr{T}_n$ given in Lemma 5.4:

Lemma 5.7. *The module $\mathscr{T}_n$ is isomorphic to the inflation of the module $\mathscr{J}_{n+1}(\langle x_1, \dots, x_{r-1}\rangle)$ from $\mathsf{GL}_{r-1}(\mathbb{Q})$ to $\mathsf{GL}_r(\mathbb{Q})$.*

6. Structure of $\mathscr{L}$

We now turn to the Lie subalgebra $\mathscr{L}$ of $\mathscr{M}$, associated with the automorphism group of F. The main tool in identifying, within $\mathscr{M}_n$, those automorphisms of $\widehat{F}$ which "restrict" to automorphisms of F is provided by Birman's theorem. For an endomorphism $\phi : F \to F$, we define its *Jacobian matrix* and *reduced Jacobian matrix*

$$(6) \qquad D\phi = \left(\frac{\partial(x_i^\phi)}{\partial x_j}\right)_{i,j=1}^{r} \in M_r(\mathbb{Z}F), \qquad \overline{D}\phi = D\phi - \mathbf{1}.$$

Theorem 6.1 ([3])**.** *The map $\phi : F \to F$ is invertible if and only if its Jacobian matrix $D\phi$ is invertible over $\mathbb{Z}F$.*

If $\phi \in \mathscr{M}_n$, then ϕ may be written in the form $\sum f_i \otimes \frac{\partial}{\partial x_i}$ with $f_i \in F_{n+1}$. Then $\frac{\partial f_i}{\partial x_j} \in \varpi^n$, so $\overline{D}\phi \in M_r(\varpi^n)$. By the chain rule, the Jacobian matrix of a product of automorphisms is the product of their Jacobian matrices. Consider automorphisms $\phi \in \mathscr{M}_m, \psi \in \mathscr{M}_n$, so that $\overline{D}\phi \in M_r(\varpi^m)$ and $\overline{D}\psi \in M_r(\varpi^n)$. Then $\overline{D}[\phi, \psi] \in M_r(\varpi^{m+n})$, and

$$\overline{D}[\phi, \psi] \equiv [\overline{D}\phi, \overline{D}\psi] \pmod{\varpi^{m+n+1}}.$$

The following result by Bryant, Gupta, Levin and Mochizuki gives a necessary condition for invertibility, which we will show is sufficient in many cases. Recall that $\mathbb{Z}F$ has an augmentation ideal ϖ, and that ϖ^n/ϖ^{n+1} can be naturally mapped into $\mathcal{R}_n$ via $\tau : (x_{i_1} - 1) \cdots (x_{i_n} - 1) \mapsto X_{i_1} \cdots X_{i_n}$. The cyclic group $\mathbb{Z}/n = \langle\gamma\rangle$ naturally acts on $\mathcal{R}_n$ by cyclic permutation of the variables:

$$(X_{i_1} \cdots X_{i_n})^\gamma = X_{i_2} \cdots X_{i_n} X_{i_1}.$$

Let $\mathcal{R}_n^+$ denote the subspace of "cyclically balanced" elements

$$\mathcal{R}_n^+ = \mathcal{R}_n \cdot (1 - \gamma) = \{u \in \mathcal{R}_n : u \cdot (1 + \gamma + \cdots + \gamma^{n-1}) = 0\}.$$

Theorem 6.2 ([5])**.** *Let $J \in M_r(\mathbb{Z}F)$ be such that $J - \mathbf{1} \in M_r(\varpi^n)$. If J is invertible and $n \geq 2$, then*

(1) *the trace of $J - \mathbf{1}$ belongs to $(\varpi^n \cap [\varpi, \varpi]) + \varpi^{n+1}$;*

(2) $\operatorname{tr}(J-\mathbf{1})^\tau \in \mathcal{R}_n^+$.

Returning to our description of automorphisms $\phi \in B$ as $\sum f_i \frac{\partial}{\partial x_i}$, we get the

Corollary 6.3. *If $n \geq 2$ and $\phi = \sum f_i \frac{\partial}{\partial x_i} \in \mathscr{M}_n$ is in the closure of $\mathscr{L}_n$, then $\sum \frac{\partial f_i}{\partial x_i} \in \mathcal{R}_n^+$.*

Proof. We have $x_i^\phi = x_i f_i$, so $(D\phi)_{i,j} = \mathbf{1}_{i,j} + x_i \frac{\partial f_i}{\partial x_j}$, and $x_i X_{i_1} \dots X_{i_n} \equiv X_{i_1} \dots X_{i_n} \pmod{\varpi^{n+1}}$ for all $i, i_1, \dots, i_n \in \{1, \dots, r\}$. We get $\operatorname{tr}(D\phi - \mathbf{1}) = \sum_i \frac{\partial f_i}{\partial x_i}$, and we apply Theorem 6.2. □

The authors of [5] ask whether the condition in Theorem 6.2 could be sufficient for J to be invertible and therefore for an endomorphism $\phi : F \to F$ to be an automorphism.[1] This is not so; for example, consider $r = 2$ and $n = 4$, in which case all automorphisms in A_4 are interior. The map

$$\phi : x \mapsto x[[x,y],[[x,y],y]], \quad y \mapsto y$$

is an element of $B_4 \setminus A_4$. However,

$$\Big(\frac{\partial}{\partial x}[[x,y],[[x,y],y]]\Big)^\tau = \frac{\partial}{\partial X}[[X,Y],[[X,Y],Y]] = YXY^2 - Y^2XY$$

is in $\mathcal{R}_4^+$.

On the other hand, we shall see below that the condition $r \leq n$ implies the sufficiency of Theorem 6.2's condition.

6.1. Generators of A. Generators of A_1, and therefore of $\mathscr{L}$, have been identified by Magnus. He showed in [21] that the following automorphisms generate A_1:

$$K_{i,j,k} : \begin{cases} x_i \mapsto x_i[x_j, x_k] & \\ x_\ell \mapsto x_\ell & \text{for all } \ell \neq i. \end{cases}$$

In particular $K_{i,j} := K_{i,i,j}$ conjugates x_i by x_j, leaving all other generators fixed. If we let $e_{i,j}$ denote the elementary matrix with a '1' in position (i,j) and zeros elsewhere, then the Jacobian matrix of $K_{i,j}$ is readily computed:

Lemma 6.4.

$$\overline{D}K_{i,j,k} \equiv X_j e_{i,k} - X_k e_{i,j} \pmod{\varpi^2}.$$

Proof. This follows directly from (5). □

Lemma 6.5. *For every $\phi \in \mathscr{M}$ with $\overline{D}\phi = (u_{i,j})_{i,j}$ we have*

$$\operatorname{tr}[\overline{D}\phi, \overline{D}K_{i,j,k}] = [u_{k,i}, X_j] - [u_{j,i}, X_k].$$

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[1] Added in proof: examples of the insufficiency of the condition in Theorem 6.2 for the invertibility of J have also been given in [25] and [13, Example 4.4].

We now write $\Omega = \{1,\dots,r\}^*$ as index set of a basis of $\mathcal{R}$, and for $\omega = \omega_1\dots\omega_n \in \Omega$ we write $X_\omega = X_{\omega_1}\cdots X_{\omega_n}$. We also write '$i \in \omega$' to mean there is an index j such that $\omega_j = i$. We also write '$*$' for an element of $\mathcal{R}$ that we don't want to specify, because its value will not affect further calculations.

For $i,j \in \{1,\dots,r\}$ and $\omega = \omega_1\dots\omega_n \in \Omega$ with $n \geq 2$ such that $i \neq j$ and $i \notin \omega$, choose $k \neq i, \omega_{n-1}$ and define inductively

$$K_{i,\omega,j} = [K_{i,\omega_1\dots\omega_{n-1},k}, K_{k,\omega_n,j}].$$

Lemma 6.6. *For $j \neq i \notin \omega$ we have*

$$\overline{D}K_{i,\omega,j} = X_\omega e_{i,j} - X_{\omega_1\dots\omega_{n-1}j}e_{i,\omega_n}.$$

Proof. The induction starts with $n = 1$, and follows from Lemma 6.4. Then, for $n \geq 2$, choose k as above and compute:

$$\begin{aligned}\overline{D}K_{i,\omega,j} &= [\overline{D}K_{i,\omega_1\dots\omega_{n-1},k}, \overline{D}K_{k,\omega_n,j}]\\ &= [X_{\omega_1\dots\omega_{n-1}}e_{i,k} - *e_{i,\omega_{n-1}}, X_{\omega_n}e_{k,j} - X_j e_{k,\omega_n}]\\ &= X_\omega e_{i,j} - X_{\omega_1\dots\omega_{n-1}j}e_{i,\omega_n},\end{aligned}$$

since for $s \in \{j,\omega_n\}$ and $t \in \{k,\omega_{n-1}\}$ the two terms $e_{i,\omega_{n-1}}e_{k,s}$ and the four terms $e_{k,s}e_{i,t}$ vanish. □

Define next, for $i \neq j \neq k \neq i$ and $i \notin \omega$,

$$L_{i,\omega,j,k} = [K_{i,\omega_2\dots\omega_n k,j}, K_{j,\omega_1,i}].$$

Lemma 6.7. *For $i \neq j \neq k \neq i \notin \omega$ we have*

$$\overline{D}L_{i,\omega,j,k} - 1 = X_{\omega_2\dots\omega_n k\omega_1}e_{i,i} - X_{\omega k}e_{j,j} + X_{\omega j}e_{j,k} - *e_{i,\omega_1}.$$

Proof. Again this is a direct calculation, using Lemma 6.6:

$$\begin{aligned}\overline{D}L_{i,\omega,j,k} &= [\overline{D}K_{i,\omega_2\dots\omega_n k,j}, \overline{D}K_{j,\omega_1,i}]\\ &= [X_{\omega_2\dots\omega_n k}e_{i,j} - X_{\omega_2\dots\omega_n j}e_{i,k}, X_{\omega_1}e_{j,i} - X_i e_{j,\omega_1}]\\ &= X_{\omega_2\dots\omega_n k\omega_1}e_{i,i} - X_{\omega k}e_{j,j} + X_{\omega j}e_{j,k} - X_{\omega_2\dots\omega_n ki}e_{i,\omega_1},\end{aligned}$$

since for $s \in \{j,k\}$ the two terms $e_{j,\omega_1}e_{i,s}$, and for $t \in \{i,\omega_1\}$ the two terms $e_{i,k}e_{j,t}$, vanish. □

7. Proofs of the main theorems

7.1. Theorem A. We start by Theorem A from the Introduction. Recall that for a subgroup $H \leq G$ we write

$$\sqrt{H} = \{g \in G \colon g^k \in H \text{ for some } k \neq 0\}.$$

It is clear that $\gamma_n(A_1) \leq A_n$, since (A_n) is a central series. Moreover, consider $\phi \in A$, written $x_i^\phi = x_i f_i$ for all i. Assume $\phi^k \in A_n$ for some n. Then $x_i^{\phi^k} \in x_i F_{n+1}$, so $f_i^k \in F_{n+1}$. Now $\sqrt{F_{n+1}} = F_{n+1}$, so $f_i \in F_{n+1}$, and therefore

$$\sqrt{A_n} = A_n.$$

We now turn to prove $\sqrt{A_n} = \sqrt{\gamma_n(A_1)}$. Consider the group ring $\mathbb{Q}A_1$, let ϖ denote its augmentation ideal, and set $A'_n = A_1 \cap (1 + \varpi^n)$. It is well-known (see [26, Theorem 11.1.10]) that $\sqrt{\gamma_n(A_1)} = A'_n$. Furthermore, A'_n/A'_{n+1} are free $\mathbb{Z}$-modules, and $[A'_m, A'_n] \leq A'_{m+n}$ for all $m, n \geq 1$, so $\mathscr{L}' = \bigoplus_{n\geq 1} A'_n/A'_{n+1}$ is a torsion-free Lie algebra. Also, $A = A_1 = A'_1$ and $A_2 = A'_2$, and both $\mathscr{L}\otimes\mathbb{Q}$ and $\mathscr{L}'\otimes\mathbb{Q}$ are $\mathsf{GL}_r(\mathbb{Q})$-modules. Furthermore, all three of (A_n), (A'_n) and $(\gamma_n(A_1))$ are filtrations of A with trivial intersection, so the *nongraded* $\mathsf{GL}_r(\mathbb{Q})$-modules $\mathscr{L}\otimes\mathbb{Q}$ and $\mathscr{L}'\otimes\mathbb{Q}$ are isomorphic, because they are all sums of the same irreducible components.

Now the modules $\mathscr{L}_n \otimes \mathbb{Q}$ and $\mathscr{L}'_n \otimes \mathbb{Q}$ are both characterised, within $\mathscr{L} \otimes \mathbb{Q}$ and $\mathscr{L}' \otimes \mathbb{Q}$ respectively, as the degree-n homogeneous part (that on which the scalar matrix $\lambda\mathbf{1} \in \mathsf{GL}_r(\mathbb{Q})$ acts by λ^n). It follows that $\mathscr{L} \otimes \mathbb{Q}$ and $\mathscr{L}' \otimes \mathbb{Q}$ are isomorphic qua graded algebras.

The filtrations (A'_n) and (A_n) have trivial intersections, and are such that their successive quotients are free abelian. Clearly A'_n is contained in A_n, because A'_n is characterized as the fastest-descending normal series with torsion-free successive quotients. Furthermore, the free abelian quotient A_n/A_{n+1} is characterized as mapping into the degree-n part $\mathscr{L}_n \otimes \mathbb{Q}$ of the $\mathsf{GL}_r(\mathbb{Q})$-module $\mathscr{L}\otimes\mathbb{Q}$, and so is A'_n/A'_{n+1}. For the former this follows from Theorem 5.1, and for the latter this follows from its description as n-fold iterated commutators. The claim $A'_n = A_n$ then follows by induction on n.

Next, $A_1/\gamma_n(A_1)$ is a finitely generated nilpotent subgroup, so its torsion subgroup is finite; therefore, $\gamma_n(A_1)$ has finite index in A_n.

The last statement of Theorem A is purely computational. Using the computer system GAP [11], I have:

(1) defined for a large prime p (I chose $p = 61$) a group
$$\tilde{G} = \langle x, y, z \mid x^p, y^p, z^p \rangle;$$
(2) constructed its maximal class-7 nilpotent quotient G, a finite group of order $p^{3+3+8+18+48+116+312}$;
(3) (in 30 minutes) constructed the set S_1 of Magnus generators x_{ij} and x_{ijk} of the group of IA automorphisms of G;
(4) (in 30 minutes) constructed the set S_2 of commutators of elements of S_1;
(5) (in 5 hours) constructed the set S_3 comprised of all commutators $[S_1, S_2]$ and all those quotients of elements of S_2 that belong to A_3;
(6) (in 50 hours) constructed the set S_4 comprised of all commutators $[S_1, S_3]$ and quotients of elements of S_3 that belong to A_4;
(7) (in 2 hours) identified elements of S_4 with their image T_4 in the $\mathbb{F}_p$-vector space G_4^3, via the map $\phi \mapsto (x^{-1}x^\phi, y^{-1}y^\phi, z^{-1}z^\phi)$;
(8) let T_5 denote those elements of T_4 that act trivially on G_4/G_5;
(9) let T_6 denote those elements of T_5 that act trivially on G_5/G_6.

The resulting vector space T_6 has dimension 806, which is therefore the dimension of $\mathscr{L}_6$ when $r = 3$. The running times are approximative, and the computation was performed twice on a standard (2004) PC computer.

On the other hand, I have also computed, for all primes $p \leq 11$ and all $1 < n < 7$, an independent set S'_n in $\mathscr{M}_n \otimes \mathbb{F}_p$ among the set of commutators $[S_1, S_{n-1}]$; and have let T'_6 denote the span of S'_6. It turned out to be a vector space of dimension 805 for $p = 3$, and 806 in all other cases.

I have then lifted a generator of $T_6 \otimes \mathbb{F}_3 / T'_6 \otimes \mathbb{F}_3$ to $\mathscr{M}_6$, as follows:

$$\begin{aligned}
x &\mapsto x \cdot [x, [[x, y], [x, [[y, z], z]]]] \cdot [x, [[x, y], [[x, z], [y, z]]]]^{-1},\\
y &\mapsto y \cdot [[x, [y, [y, z]]], [x, [y, z]]]^{-1} \cdot [[x, [y, [y, [y, z]]]], [x, z]]^{-1},\\
z &\mapsto z \cdot [[[x, z], [[x, z], [y, z]]], y] \cdot [[x, [y, [y, z]]], [[x, z], z]]\\
&\qquad \cdot [[x, [y, [[y, z], z]]], [x, z]] \cdot [[[x, [y, z]], [y, z]], [x, z]]^{-1}.
\end{aligned}$$

This is the image in $\mathscr{M}_6$ of an automorphism of F; it does not belong to $\gamma_6(A_1)$, but its cube does, as another lengthy calculation shows.

7.2. Theorem C. First, we show that $\mathscr{T}_n$ is contained in $\mathscr{L}_n$. For every $w \in \langle x_1, \dots, x_{r-1} \rangle$, the endomorphism T_w of F is invertible (either directly, noting its inverse is $T_{w^{-1}}$, or because its Jacobian is unipotent); if furthermore $w \in F_{n+1}$, then T_w defines an element of $\mathscr{T}_n \cap \mathscr{L}_n$. Since $\mathscr{T}_n$ is generated by the T_w qua $\mathsf{GL}_r(\mathbb{Q})$-module, we are done.

By Corollary 6.3, the image of $\mathscr{L}_n$ under the trace map belongs to the space $\mathcal{R}_n^+$ of cyclically balanced elements as soon as $n \geq 2$. We now claim that, if furthermore $n \leq r$ and $W \in \mathcal{R}_n^+$ is cyclically balanced, then there exists an automorphism $\phi \in \mathscr{L}_n$ with trace W.

Using the $\mathsf{GL}_r(\mathbb{Q})$-action, it suffices to check this for an elementary W, of the form $W = x_r U - U x_r$ with $U \in \mathcal{R}_{n-1}$. In fact, by linearity, we may even reduce ourselves to considering $U = X_\omega$ a word in $\{X_1, \dots, X_r\}$ of length $n - 1$, see the notation of §6.1.

We first note that, if there is a letter x_i which does not occur in W, then by Lemmata 6.6 and 6.5 the automorphism $K_{i,\omega,j}$ belongs to $\mathscr{L}$, and then

$$\operatorname{tr}[K_{i,\omega,j}, K_{j,r,i}] = [X_\omega, X_r] = -W.$$

This shows that all cyclically balanced elements of degree $< r$ appear as traces of automorphisms in $\mathscr{L}$. To complete the argument, it suffices to consider the case $r = n$. A cyclically balanced word is either covered by the previous case, or contains a single instance of each letter. Using Lemma 6.7, consider the expression $\operatorname{tr}[L_{i,\omega,j,k}, K_{k,\ell,j}] = [X_{\omega j}, X_\ell]$. The restrictions on it are that there exists $i, k \in \{1, \dots, r\}$ with $\ell \neq j \neq k \neq i$ and $i \notin \omega j$; all words with a single instance of each letter are covered by these conditions.

7.3. Theorems B and D. Let $\mathcal{R}_n^i$ denote the subspace of $\mathcal{R}_n$ spanned by Young tableaux with major index $\equiv i \pmod{n}$. Then, by Theorem D, the rank of $\mathscr{L}_n$ is $\operatorname{rank} \mathscr{M}_n - \operatorname{rank} \mathcal{R}_n^0$. Imitating Klyashko's argument [17], let ε denote a primitive nth root of unity. Then $\mathcal{R}_n^i$ may be written, qua $\mathsf{GL}_r(\mathbb{Q})$-module, as

$$\mathcal{R}_n^i \cong \mathcal{R}_n \otimes_{\mathbb{Q}\mathfrak{S}_n} \mathbb{Q}\mathfrak{S}_n\kappa_n \cong \mathcal{R}_n \otimes_{\mathbb{Q}\mathfrak{S}_n} \mathbb{Q}\mathfrak{S}_n\theta_n,$$

where the idempotents $\kappa_n, \theta_n \in \mathbb{Q}\mathfrak{S}_n$ are given by

$$\kappa_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \varepsilon^{i\,\mathrm{maj}(\sigma)}\sigma, \qquad \theta_n = \frac{1}{n}\sum_{j=1}^{n} \varepsilon^{ij}(1,2,\dots,n)^i.$$

We therefore get

$$\dim \mathcal{R}_n^i = \frac{1}{n}\sum_{j=1}^{n} \varepsilon^{ij} r^{(n,j)},$$

so in particular

$$\dim \mathcal{R}_n^1 = \frac{1}{n}\sum_{d|n} \mu(d) r^{n/d}, \qquad \dim \mathcal{R}_n^0 = \frac{1}{n}\sum \phi(d) r^{n/d} \tag{7}$$

using the identities $\sum_{\gcd(n,j)=n/d} \varepsilon^j = \mu(d)$ and $\sum_{\gcd(n,j)=n/d} 1 = \phi(d)$. Inserting (7) in $\operatorname{rank} \mathscr{L}_n = r \operatorname{rank} \mathcal{R}_{n+1}^1 - \operatorname{rank} \mathcal{R}_n^0$ yields (1).

8. Examples and illustrations

Recall from Theorem 3.4 that the $\mathsf{GL}_r(\mathbb{Q})$-module $\mathscr{J}_n$ decomposes as a direct sum of irreducible representations, indexed by tableaux with n boxes and major index $\equiv 1 \mod n$. Here are the small-dimensional cases; recall that the $\mathsf{GL}_r(\mathbb{Q})$-decomposition of $\mathscr{J}_n$ consists of those representations indexed by diagrams with at most r lines:

n	1	2	3	4	5	6
$\dim S_n$	1	1	2	6	24	120
S_n	[1]	[1 / 2]	[1 3 / 2]	[1 3 4 / 2]; [1 2 / 3 / 4]	[1 3 4 5 / 2]; [1 2 4 / 3 5]; [1 2 4 / 3 / 5]; [1 2 / 3 4 / 5]; [1 5 / 2 / 3 / 4]	[1 3 4 5 6 / 2]; [1 2 4 5 / 3 6]; [1 2 5 / 3 4 6]; [1 2 3 / 4 6 / 5]; [1 2 5 / 3 4 / 6]; [1 4 6 / 2 5 / 3]; [1 2 3 6 / 4 / 5]; [1 2 4 5 / 3 / 6]; [1 4 6 / 2 / 3 / 5]; [1 3 / 2 4 / 5 / 6]; [1 4 / 2 6 / 3 / 5]; [1 3 / 2 / 4 / 5 / 6]
$\dim \mathscr{J}_n, r=2$	2	1	2	3	6	9
$\dim \mathscr{J}_n, r=3$	3	3	8	18	48	116
$\dim \mathscr{J}_n, r=4$	4	6	20	60	204	670

We next describe the decomposition of $\mathscr{M}_n$. Using Theorem 5.1, it may be computed by the Littlewood–Richardson rule [30]. We concentrate on small values of r:

8.1. $r = 2$. Representations of $\mathsf{GL}_r(\mathbb{Q})$ are classified by Young diagrams with at most two lines. We may therefore express $\mathscr{M}_n$ as a combination of submodules of shape (a, b) for some $a + b = n$ with $a \geq b$. Theorem 3.3 gives a simple answer for $\mathscr{J}_n$; the decomposition of $\mathscr{M}_n$ is then obtained via Theorem 5.1:

Proposition 8.1. *Define the function*

$$\theta(a,b) = \frac{1}{a+b} \sum_{d|(a,b)} \mu(d) \binom{(a+b)/d}{a/d}.$$

Then the multiplicity of (a, b) *in* $\mathscr{J}_n$ *is*

$$\chi^{\mathscr{J}_n}_{(a,b)} = \theta(a,b) - \theta(a-1, b+1).$$

The total number of irreducible representations in $\mathscr{J}_n$ *is*

$$i_n = \begin{cases} \theta(n/2, n/2) & \text{if } n \equiv 0[2], \\ \theta((n+1)/2, (n-1)/2) = \frac{1}{n+1}\binom{n}{(n-1)/2} & \text{if } n \equiv 1[2]. \end{cases}$$

The multiplicity of (a, b) *in* $\mathscr{M}_n$ *is*

$$\chi^{\mathscr{M}_n}_{(a,b)} = \begin{cases} \theta(a+1, b) - \theta(a-1, b+2) & \text{if } a > b, \\ \theta(a+1, b) - \theta(a, b+1) & \text{if } a = b, \end{cases}$$

while the total number of irreducible representations in $\mathscr{M}_n$ *is*

$$a_n = \begin{cases} 2\theta(n/2+1, n/2) & \text{if } n \equiv 0[2], \\ \theta((n+1)/2, (n+1)/2) + \theta((n+3)/2, (n-1)/2) & \text{if } n \equiv 1[2]. \end{cases}$$

Proof. These are special cases of classical formulas, see for instance [8]. We have

$$\chi^{\mathscr{J}_n}_{(a,b)} = \frac{1}{n} \sum_{d|n} \mu(d) \chi_{(a,b)}((d, \dots, d)),$$

where $\chi_{(a,b)}(\mu)$ denotes the value of the character $\chi_{(a,b)}$ on a permutation of cycle type μ. Using the Jacobi-Trudi identity, Foulkes expresses $\chi_{(a,b)}((d, \dots, d))$ as the determinant

$$(n/d)! \begin{vmatrix} [a/d]!^{-1} & [(a+1)/d]!^{-1} \\ [(b-1)/d]!^{-1} & [b/d]!^{-1} \end{vmatrix},$$

where $[x]!^{-1}$ is $x!^{-1}$ if x is an integer, and 0 otherwise. All formulas for $\mathscr{J}_n$ follow.

Since $\mathscr{M}_n = V^* \otimes \mathscr{J}_{n+1}$ by 5.1, the multiplicity of (a, b) in $\mathscr{M}_n$ is obtained by the "Jeu du taquin" procedure, as

$$\chi^{\mathscr{M}_n}_{(a,b)} = \chi^{\mathscr{J}_{n+1}}_{(a+1,b)} + \chi^{\mathscr{J}_{n+1}}_{(a,b+1)},$$

where the last summand is understood as 0 if $a = b$. Again all formulas for $\mathscr{M}_n$ follow. □

The multiplicities of irreducibles of $\mathsf{GL}_r(\mathbb{Q})$ in $\mathscr{J}_n$ and $\mathscr{M}_n$ are listed in the following table, for $n \le 12$:

	n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathscr{J}_n$:	$(n,0)$	1											
	$(n-1,1)$		1	1	1	1	1	1	1	1	1	1	1
	$(n-2,2)$					1	1	2	2	3	3	4	4
	$(n-3,3)$						1	2	4	5	8	10	13
	$(n-4,4)$								1	5	8	15	22
	$(n-5,5)$										5	12	26
	$(n-6,6)$												9
	total	1	1	1	1	2	3	5	8	14	25	42	75
$\mathscr{M}_n$:	$(n,0)$	1	1	1	1	1	1	1	1	1	1	1	1
	$(n-1,1)$		1	1	2	2	3	3	4	4	5	5	6
	$(n-2,2)$				1	2	4	6	8	11	14	17	21
	$(n-3,3)$						2	5	10	16	25	35	49
	$(n-4,4)$								5	13	37	48	77
	$(n-5,5)$										12	35	77
	$(n-6,6)$												33
	total	1	2	2	4	5	10	15	28	45	84	141	264

8.2. $r = 3$ and $r = 4$. For $r > 2$, some degree-n irreducibles for $\mathsf{GL}_r(\mathbb{Q})$ in $\mathscr{M}_n$ are represented by a Young diagram with n boxes, and others are represented as a formal quotient of a Young diagram with $n + r$ boxes by the degree-r determinant representation, following Lemma 5.7. We indicate the latter representations by putting $*$'s in the first column (of height $r-1$). Table 1 describes the irreducibles of $\mathscr{M}_n \otimes \mathbb{Q}$ for $1 \le n \le 5$ and $r = 3$, while Table 2 describes them for $r = 4$. Whenever possible, the Young diagrams are filled in so as to separate them according to their major index.

Some of the Young diagrams with no $*$'s are not filled in; Theorem D does not apply since $n > r$ there, and the decomposition of $\mathscr{L}_n$ is only partly understood. In particular, note for $r = 3, n = 5$ that there are six tableaux with shape $3+1+1$, but only five appear in the decomposition of $\mathscr{M}_n$, and there are five tableaux with shape $2+2+1$, and only three appear in $\mathscr{M}_n$. For $r = 4, n = 5$, there are four tableaux with shape $2+1+1+1$ but only three appear in $\mathscr{M}_n$. It seems that the tableaux with the largest number of rows are more likely to disappear first, as n increases.

n	1	2	3	4	5	6
$\dim \mathscr{M}_n$	9	24	54	144	348	936
$\dim \mathscr{L}_n$	9	18	43	120	297	806
$\dim \mathscr{T}_n$	6	15	27	66	117	279
$\dim \mathscr{I}_n$	3	3	8	18	48	116

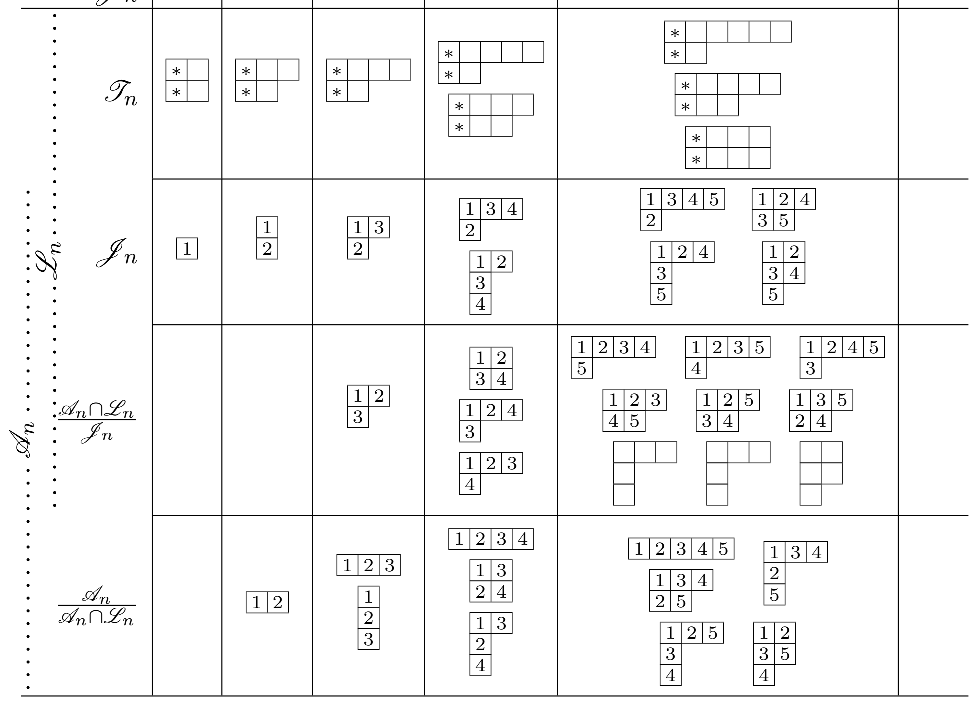

TABLE 1. The decomposition of $\mathscr{L} \otimes \mathbb{Q}$ and $\mathscr{M} \otimes \mathbb{Q}$ for $r = 3$

n	1	2	3	4	5	6
$\dim \mathscr{M}_n$	24	80	240	816	2680	9360
$\dim \mathscr{L}_n$	24	70	216	746	2472	8660
$\dim \mathscr{T}_n$	20	64	176	560	1660	5296
$\dim \mathscr{J}_n$	4	6	20	60	204	670

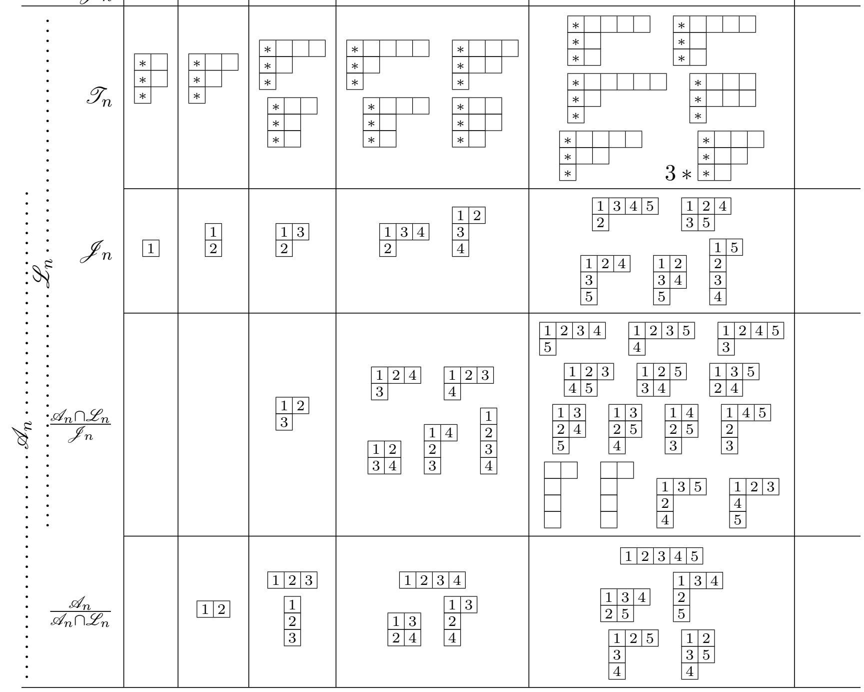

TABLE 2. The decomposition of $\mathscr{L} \otimes \mathbb{Q}$ and $\mathscr{M} \otimes \mathbb{Q}$ for $r = 4$

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