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An equivalence theorem for reduced Fell bundle C^*-algebras

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Abstract. We show that if $\mathscr{E}$ is an equivalence of upper semicontinuous Fell bundles $\mathscr{B}$ and $\mathscr{C}$ over groupoids, then there is a linking bundle $L(\mathscr{E})$ over the linking groupoid $\mathcal{L}$ such that the full cross-sectional algebra of $L(\mathscr{E})$ contains those of $\mathscr{B}$ and $\mathscr{C}$ as complementary full corners, and likewise for reduced cross-sectional algebras. We show how our results generalise to groupoid crossed-products the fact, proved by Quigg and Spielberg, that Raeburn's symmetric imprimitivity theorem passes through the quotient map to reduced crossed products.

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1. Introduction

The purpose of this paper is to prove a reduced equivalence theorem for cross-sectional algebras of Fell bundles over groupoids, and to prove that the imprimitivity bimodule which implements the equivalence between the reduced C^*-algebras is a quotient of the Muhly–Williams equivalence bimodule between the full C^*-algebras [16].

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An increasingly influential interpretation of Hilbert bimodules (or C^*-correspondences) is to regard them as generalized endomorphisms of C^*-algebras. Imprimitivity bimodules represent isomorphisms, and a Fell bundle over a groupoid G is then the counterpart of an action of G on a $C_0(G^0)$-algebra A. The cross-sectional algebras of the bundle are analogues of groupoid crossed products. For example, if G is a group and each imprimitivity module is of the form ${}_\alpha A$ for an automorphism α of A (see, for example [18]), then the cross-sectional algebras of such bundles are precisely those arising from group crossed products; Fell and Doran called these semidirect products in their magnum opus [5, §VIII.4.2]. In particular, if $A = C_0(G^{(0)})$ and each fibre of the Fell-bundle is 1-dimensional, then the cross-sectional algebras are the usual groupoid C^*-algebras.

The classical result which motivates this paper is that if groups G and H act freely, properly and transitively on the same locally compact Hausdorff space P and the actions commute, then the groups are the same. To see why, fix $x \in P$. Then for each $g \in G$, there is a unique $h \in H$ such that $g \cdot x = x \cdot h$, and since the actions commute, $g \mapsto h$ is an isomorphism of G with H. Hence $C^*(G) \cong C^*(H)$ and $C^*_r(G) \cong C^*_r(H)$. A particularly powerful viewpoint on this is the following. If P^{op} is a copy of the space P, but with the actions reversed so that G acts on the right and H on the left, then $L = G \sqcup P \sqcup P^{\mathrm{op}} \sqcup H$ is a groupoid, called the linking groupoid, with two units. The isotropy at one unit is G and the isotropy at the other is H, and conjugation in L by any element of P determines an isomorphism from G to H. At the level of C^*-algebras, we obtain the following very nice picture: the actions of G and H on P induce convolution-like products $C_c(G) \times C_c(P) \to C_c(P)$ and $C_c(P) \times C_c(H) \to C_c(P)$, and $C_c(L)$ decomposes as a block 2×2 matrix algebra

$$C_c(L) \cong \begin{pmatrix} C_c(G) & C_c(P) \\ C_c(P^{\mathrm{op}}) & C_c(H) \end{pmatrix}.$$

Moreover, the universal norm on $C_c(L)$ restricts to the universal norm on each of $C_c(G)$ and $C_c(H)$, and likewise for reduced norms. The characteristic function 1_P of P is a partial isometry in the multiplier algebra of each of $C^*(L)$ and $C^*_r(L)$ and conjugation by 1_P implements the isomorphisms $C^*(G) \cong C^*(H)$ and $C^*_r(G) \cong C^*_r(H)$.

When G and H do not act transitively, the actions of G and H on P induce actions of G on P/H and of H on $G\backslash P$. The picture at the level of groups is now somewhat more complicated, but the C^*-algebraic picture carries over nicely: replacing $C_c(G)$ with $C_c(G, C_c(P/H))$ and $C_c(H)$ with $C_c(H, C_c(G\backslash P))$ in the matrix above, we obtain a *-algebra $L(P)$. The C^*-identity allows us to extend the norm on $C_0(P/H) \rtimes G$ to a norm on $L(P)$. Moreover, this norm is consistent with the norm on $C_0(G\backslash P) \rtimes H$, and the completion of $L(P)$ in this norm contains $C_0(P/H) \rtimes G$ and $C_0(G\backslash P) \rtimes H$ as complementary full corners. Further, this whole apparatus descends under quotient maps to reduced crossed products.

To prove an analogue of this equivalence theorem in the context of Fell bundles, one uses the notion of an *equivalence* of Fell bundles specified in [16]. The concept is closely modeled on the situation of groups; but the natural objects on which Fell bundles act are Banach bundles in which the fibres are equivalence bimodules. That is, given Fell bundles $\mathscr{B}$ and $\mathscr{C}$ over groupoids G and H, an equivalence between the two is, roughly speaking, an upper semicontinuous Banach bundle $\mathscr{E}$ over a space Z such that Z admits actions of G and H making it into an equivalence of groupoids in the sense of Renault, each fibre E_z of $\mathscr{E}$ is an imprimitivity bimodule from the fibre $B_{r(z)}$ of $\mathscr{B}$ over $r(z)$ to $C_{s(z)}$, and there are fibred multiplication operations $\mathscr{B}*\mathscr{E} \to \mathscr{E}$ and $\mathscr{E}*\mathscr{C} \to \mathscr{E}$ which are compatible with the bundle maps, and which implement isomorphisms $B_x \otimes_{B_u} E_z \cong E_{x\cdot z}$. Muhly and Williams show in [16, Theorem 6.4] that given such an equivalence, the full cross-sectional algebras $C^*(G,\mathscr{B})$ and $C^*(H,\mathscr{C})$ are Morita equivalent (Kumjian proves the corresponding statement for reduced C^*-algebras in the r-discrete situation in [13]).

In this paper, we show that Muhly and Williams's Morita equivalence passes to reduced algebras. We do so by constructing a linking bundle $L(\mathscr{E}) = \mathscr{B} \sqcup \mathscr{E} \sqcup \mathscr{E}^{\mathrm{op}} \sqcup \mathscr{C}$ and showing that $\Gamma_c(L; L(\mathscr{E}))$ has a matrix decomposition as above. We then prove that the completion of $L(\mathscr{E})$ in the universal norm is a linking algebra for a Morita equivalence between $C^*(G,\mathscr{B})$ and $C^*(H,\mathscr{C})$, and likewise for reduced C^*-algebras.

We conclude by showing how to recover a generalisation of Quigg and Spielberg's theorem [19] which says that the symmetric imprimitivity bimodule arising in Raeburn's symmetric imprimitivity theorem [20] passes under the quotient map to an imprimitivity bimodule for reduced crossed products.

Our reduced equivalence theorem itself is not new: late in the development of this paper, we learned that Moutuou and Tu also prove that equivalent Fell bundles have Morita equivalent reduced cross-sectional algebras [14]. It appears that Moutuou and Tu deal only with Fell bundles whose underlying Banach bundles are required to be continuous rather than just upper semicontinuous. (Upper semicontinuous bundles turn out to be the more natural object in the context of C^*-algebras — see [16] and especially [23, Appendix C]). Moreover Moutuou and Tu restrict attention to *principle* G-spaces for their groupoid equivalences. But these are minor points and the arguments of [14] would surely go through unchanged to our setting. The main new contribution in this article is that we develop the linking bundle technology to show explicitly that the full cross-sectional algebra of the linking bundle is a linking algebra for the full cross sectional algebras of $\mathscr{B}$ and $\mathscr{C}$, and that the quotient map from the full to the reduced cross-sectional algebra of the linking bundle implements the quotients $C^*(G,\mathscr{B}) \to C^*_r(G,\mathscr{B})$ and $C^*(H,\mathscr{C}) \to C^*_r(H,\mathscr{C})$. In particular, if $I_r^{\mathscr{C}}$ is the ideal of $C^*(H,\mathscr{C})$

consisting of elements whose reduced norm is zero, then the equivalence bimodule X_r which we obtain between reduced cross-sectional algebras is the quotient of the equivalence bimodule X between full algebras by $X \cdot I_r^{\mathscr{C}}$. Consequently, induction over X carries $I_r^{\mathscr{C}}$ to the corresponding ideal $I_r^{\mathscr{B}}$ of $C^*(G, \mathscr{B})$.

2. Background

Recall that for second-countable locally compact Hausdorff groupoids G and H, a G–H equivalence is a locally compact Hausdorff space Z which is simultaneously a free and proper left G-space and a free and proper right H-space (with continuous *open* fibre maps) such that the actions of G and H on Z commute, the map r_Z induces a homeomorphism of Z/H with $G^{(0)}$ and the map s_Z induces a homeomorphism of $G\backslash Z$ with $H^{(0)}$. Then G acts on $Z *_r Z$ by $g \cdot (y, z) = (g \cdot y, g \cdot z)$, and the formula $h \cdot [g, h]_H = g$ defines a homeomorphism $[\cdot,\cdot]_H : G\backslash(Z *_r Z) \to H$; and ${}_G[\cdot,\cdot] : Z *_s Z \to G$ is defined similarly (see [15, Definition 2.1] for details).

Recall that an upper semicontinuous Banach bundle over a locally compact Hausdorff space Z is a topological space $\mathscr{B}$ together with a continuous open surjection $q : \mathscr{B} \to Z$ such that each $\mathscr{B}_z := q^{-1}(z)$ is a Banach space and: $b \mapsto \|b\|$ is upper semicontinuous; addition is continuous from $\mathscr{B} *_q \mathscr{B} \to \mathscr{B}$; scalar multiplication is continuous on $\mathscr{B}$; and $\|b_i\| \to 0$ and $q(b_i) \to z$ implies $b_i \to 0_z \in q^{-1}(z)$. The concept of an upper semicontinuous Banach bundle goes back to [3], where they were called (H)-bundles, and the work of Hofmann [2, 6–8]. Fell calls such bundles loose in [4, Remark C.1]. Further details and comments concerning upper semicontinuous Banach bundles are given in [16, Appendix A] and in the C^*-case in [23, Appendix C]. As in [16], a *Fell bundle* over a locally compact Hausdorff groupoid G is an upper semicontinuous Banach bundle $q : \mathscr{B} \to G$ endowed with a continuous bilinear associative map $(a, b) \mapsto ab$ from $\mathscr{B}^{(2)} := \{\, (a, b) \in \mathscr{B} \times \mathscr{B} : s(q(a)) = r(q(b)) \,\}$ to $\mathscr{B}$ such that:

(a) $q(ab) = q(a)q(b)$ for all $(a, b) \in \mathscr{B}^{(2)}$.
(b) $q(a^*) = q(a)^{-1}$ for all $a \in \mathscr{B}$.
(c) $(ab)^* = b^*a^*$ for all $(a, b) \in \mathscr{B}^{(2)}$.
(d) For each $u \in G^{(0)}$, the fibre $A_u := q^{-1}(u)$ is a C^*-algebra under these operations.
(e) For each $g \in G \setminus G^{(0)}$, the fibre $B_g := q^{-1}(g)$ is an $A_{r(g)}$–$A_{s(g)}$-imprimitivity bimodule with actions determined by multiplication in $\mathscr{B}$ and inner products $\langle a\ ,\ b\rangle = a^*b$ and $\langle a\ ,\ b\rangle = ab^*$.

As a notational convenience, we define $r, s : \mathscr{B} \to G^{(0)}$ by $r(a) := r_G(q(a))$ and $s(a) := s_G(q(a))$. See [16] for more details regarding Fell bundles over groupoids.

Remark 1. In the context of bundles over groups, the fibres in a Fell bundle are not always assumed to be imprimitivity bimodules (they are not assumed

to be full — see [13, 2.4]). Bundles in which all the fibres are indeed imprimitivity bimodules are then called *saturated*. We take this condition as part of our definition. It should also be observed that the underlying Banach bundle of a Fell bundle over a *group* is always continuous [1, Lemma 3.30].

Remark 2. In our notation the fibre of $\mathscr{B}$ over a unit u can be denoted either A_u or B_u. The dual notation allows us to emphasise its dual roles. We write A_u to emphasise its role as a C^*-algebra, and B_u to emphasise its role as an imprimitivity bimodule. The C^*-algebra $A := \Gamma_0(G^{(0)}; \mathscr{B})$ is called the *C^*-algebra of the Fell bundle $\mathscr{B}$ over $G^{(0)}$*.

We recall from [16] the definition of an equivalence of Fell bundles. First, fix a second-countable locally compact Hausdorff groupoid G, a left G-space Z, a Fell bundle $q_G : \mathscr{B} \to G$, and a Hausdorff space $\mathscr{E}$ together with a continuous open surjection $q : \mathscr{E} \to Z$. Again, as a notational convenience, we shall write r for the composition $r_Z \circ q : \mathscr{E} \to G^{(0)}$. we say that $\mathscr{B}$ *acts on the left of* $\mathscr{E}$ if there is a pairing $(b, e) \mapsto b \cdot e$ from

$$\mathscr{B} * \mathscr{E} = \{\, (b, e) \in \mathscr{B} \times \mathscr{E} : s(b) = r(e) \,\}$$

to $\mathscr{E}$ such that:

(a) $q(b \cdot e) = q_G(b)q(e)$ for $(b, e) \in \mathscr{B} * \mathscr{E}$.
(b) $a \cdot (b \cdot e) = (ab) \cdot e$ whenever $(a, b) \in \mathscr{B}^{(2)}$ and $(b, e) \in \mathscr{B} * \mathscr{E}$.
(c) $\|b \cdot e\| \le \|b\|\|e\|$ for $(b, e) \in \mathscr{B} * \mathscr{E}$.[1]

If $\mathscr{E}$ is a right H-space, and $q_H : \mathscr{C} \to H$ is a Fell bundle, then a right action of $\mathscr{C}$ on $\mathscr{E}$ is defined similarly.

Now fix second-countable locally compact Hausdorff groupoids G and H and a G–H equivalence Z. Suppose that $q_G : \mathscr{B} \to G$ and $q_H : \mathscr{C} \to H$ are Fell bundles. Fix a Banach bundle $q : \mathscr{E} \to Z$. We write

$$\mathscr{E} *_s \mathscr{E} = \{\, (e, g) \in \mathscr{E} \times \mathscr{E} : s(e) = s(g) \,\}$$

and we define $\mathscr{E} *_r \mathscr{E}$ similarly. We call $\mathscr{E}$ a $\mathscr{B}$–$\mathscr{C}$ equivalence if:

(a) There are a left action of $\mathscr{B}$ on $\mathscr{E}$ and a right action of $\mathscr{C}$ on $\mathscr{E}$ which commute.
(b) There are sesquilinear maps

$$ {}_{\mathscr{B}}\langle \cdot\ , \cdot \rangle : \mathscr{E} *_s \mathscr{E} \to \mathscr{B} \quad \text{and} \quad \langle \cdot\ , \cdot \rangle_{\mathscr{C}} : \mathscr{E} *_r \mathscr{E} \to \mathscr{C} $$

such that the relations
  (i) $q_G({}_{\mathscr{B}}\langle e\ , f\rangle) = {}_G[q(e), q(f)]$ and $q_H(\langle e\ , f\rangle_{\mathscr{C}}) = [q(e), q(f)]_H$,
  (ii) ${}_{\mathscr{B}}\langle e\ , f\rangle^* = {}_{\mathscr{B}}\langle f\ , e\rangle$ and $\langle e\ , f\rangle_{\mathscr{C}}^* = \langle f\ , e\rangle_{\mathscr{C}}$,
  (iii) $b\,{}_{\mathscr{B}}\langle e\ , f\rangle = {}_{\mathscr{B}}\langle b \cdot e\ , f\rangle$ and $\langle e\ , f\rangle_{\mathscr{C}} c = \langle e\ , f \cdot c\rangle_{\mathscr{C}}$ and
  (iv) ${}_{\mathscr{B}}\langle e\ , f\rangle \cdot g = e \cdot \langle f\ , g\rangle_{\mathscr{C}}$
are satisfied whenever they make sense.
(c) Under the actions described in (a) and the inner-products defined in (b), each $\mathscr{E}_z := q^{-1}(z)$ is an $A_{r(z)}$–$D_{s(z)}$-imprimitivity bimodule.

[1] The equality appearing in the corresponding item in [16] is a typographical error.

As in [22], if G, H are second-countable locally compact Hausdorff groupoids with Haar systems λ, β and Z is a G–H equivalence, we write Z^{op} for the "opposite equivalence" $Z^{\mathrm{op}} = \{\, \bar{z} : z \in Z \,\}$ with $r(\bar{z}) = s(z)$, $s(\bar{z}) = r(z)$, $h \cdot \bar{z} := \overline{z \cdot h^{-1}}$ and $\bar{z} \cdot g := \overline{g^{-1} \cdot z}$. Then $L := G \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup H$ with

$$L^{(0)} := G^{(0)} \sqcup H^{(0)} \subseteq L$$

is a groupoid containing G and H as subgroupoids: we extend the inverse map to Z and Z^{op} by setting $z^{-1} := \bar{z}$; and multiplication between Z and G, H is implemented by the left and right actions, while multiplication between Z and Z^{op} is implemented by ${}_G[\cdot,\cdot]$ and $[\cdot,\cdot]_H$. See [22, Lemma 5] for details. There is a Haar system on L determined by

$$\kappa^w(F) := \begin{cases} \int_G F(g)\, d\lambda^w(g) + \int_H F(z \cdot h)\, d\beta^{s(z)}(h) & \text{if } w \in G^{(0)} \\ \int_G F(\bar{y} \cdot g)\, d\lambda^{s(\bar{y})}(g) + \int_H F(h)\, d\beta^w(h) & \text{if } w \in H^{(0)} \end{cases}$$

for $F \in C_c(L)$ and $w \in L^{(0)}$ (see [22, Lemma 6]). For $u \in G^{(0)}$ and $v \in H^{(0)}$, we write σ^u_Z and $\sigma^v_{Z^{\mathrm{op}}}$ for the restrictions of κ^u to Z and of κ^v to Z^{op}. The main results of [22] say that $C^*(L, \kappa)$ contains $C^*(G, \lambda)$ and $C^*(H, \beta)$ as the complementary full corners determined by the multiplier projections $1_{G^{(0)}}$ and $1_{H^{(0)}}$, and that this Morita equivalence passes under the quotient map $C^*(L, \kappa) \to C^*_r(L, \kappa)$ to reduced groupoid C^*-algebras. Our goal in this article is to establish the corresponding statement for Fell bundles. As a first step, we show in the next section how to construct from an equivalence of Fell bundles a linking bundle over the linking groupoid.

3. Linking bundles

Let G and H be locally compact Hausdorff groupoids, let Z be a G–H equivalence, and let L be the linking groupoid as above. Suppose that $p_G : \mathscr{B} \to G$ and $p_H : \mathscr{C} \to H$ are upper-semicontinuous Fell bundles, and that $q : \mathscr{E} \to Z$ is a bundle equivalence. We denote by A the C^*-algebra $\Gamma_0(G^{(0)}; \mathscr{B})$ of the bundle $\mathscr{B}$, and by D the C^*-algebra $\Gamma_0(H^{(0)}; \mathscr{C})$ of $\mathscr{C}$; so the fibre over $u \in G^{(0)}$ is A_u, the fibre over $v \in H^{(0)}$ is D_v, and each $\mathscr{E}_z$ is an $A_{r(z)}$–$D_{s(z)}$-imprimitivity bimodule.

Let $\mathscr{E}^{\mathrm{op}} = \{\, \bar{e} : e \in \mathscr{E} \,\}$ be a copy of the topological space $\mathscr{E}$ endowed with the conjugate Banach space structure $\alpha\bar{e} + \bar{f} = \overline{(\overline{\alpha} e + f)}$ on each fibre. Then $q^{\mathrm{op}} : \mathscr{E}^{\mathrm{op}} \to Z^{\mathrm{op}}$ is an upper-semicontinuous Banach bundle with $q^{\mathrm{op}}(\bar{e}) = \overline{q(e)}$. We have $s(\bar{e}) = s(q^{\mathrm{op}}(\bar{e})) = r(e)$ and likewise $r(\bar{e}) = s(e)$, so we obtain a right $\mathscr{B}$-action and a left $\mathscr{C}$-action on $\mathscr{E}^{\mathrm{op}}$ by

$$\bar{e} \cdot b = \overline{b^* \cdot e} \quad \text{and} \quad c \cdot \bar{e} = \overline{e \cdot c^*}. \tag{1}$$

The inner products on $\mathscr{E}^{\mathrm{op}} *_r \mathscr{E}^{\mathrm{op}}$ and $\mathscr{E}^{\mathrm{op}} *_s \mathscr{E}^{\mathrm{op}}$ are given by

$$\langle \bar{e}\,,\, \bar{f} \rangle_{\mathscr{B}} = {}_{\mathscr{B}}\langle e\,,\, f \rangle \quad \text{and} \quad {}_{\mathscr{C}}\langle \bar{e}\,,\, \bar{f} \rangle = \langle e\,,\, f \rangle_{\mathscr{C}}.$$

Routine calculations show that each $E^{\mathrm{op}}(\bar{z})$ is the dual imprimitivity bimodule $E(z)^\sim$ of $E(z)$. Since $s(z) = r(\bar{z})$ and $r(z) = s(\bar{z})$, axioms (a), (b) and (c) of [16, Definition 6.1] hold, so $\mathscr{E}^{\,\mathrm{op}}$ is a $\mathscr{C}$–$\mathscr{B}$-equivalence.

Let $L(\mathscr{E}) = \mathscr{B} \sqcup \mathscr{E} \sqcup \mathscr{E}^{\,\mathrm{op}} \sqcup \mathscr{C}$ and define $L(q) : L(\mathscr{E}) \to L$ by

$$L(q)|_{\mathscr{B}} = p_G, \quad L(q)|_{\mathscr{C}} = p_H, \quad L(q)|_{\mathscr{E}} = q \quad \text{and} \quad L(q)|_{\mathscr{E}^{\,\mathrm{op}}} = q^{\mathrm{op}}.$$

Since $e \mapsto \bar{e}$ is a fiberwise-isometric homeomorphism from $\mathscr{E}$ to $\mathscr{E}^{\,\mathrm{op}}$ and since $z \mapsto \bar{z}$ is a homeomorphism from Z to Z^{op}, the bundle $L(\mathscr{E})$ is an upper semicontinuous Banach bundle. Let

$$L(\mathscr{E})^{(2)} = \{\, (\mathfrak{a}, \mathfrak{b}) \in L(\mathscr{E}) \times L(\mathscr{E}) : s(L(q)(\mathfrak{a})) = r(L(q)(\mathfrak{b})) \,\}.$$

Define $m : L(\mathscr{E})^{(2)} \to L(\mathscr{E})$ to coincide with the given multiplications on $\mathscr{B}$ and $\mathscr{C}$ and with the actions of $\mathscr{B}$ and $\mathscr{C}$ on $\mathscr{E}$ and $\mathscr{E}^{\,\mathrm{op}}$, and to satisfy

$$m(e, \bar{f}) = {}_{\mathscr{B}}\langle e \,,\, f\rangle \text{ for } (e, f) \in \mathscr{E} *_s \mathscr{E},$$
$$m(\bar{e}, f) = \langle e \,,\, f\rangle_{\mathscr{C}} \text{ for } (e, f) \in \mathscr{E} *_r \mathscr{E}.$$

We define $\mathfrak{a} \mapsto \mathfrak{a}^*$ on $L(\mathscr{E})$ to extend the given involutions on $\mathscr{B}$ and $\mathscr{C}$ by setting $e^* = \bar{e}$ on $\mathscr{E}$ and $\bar{e}^* = e$ on $\mathscr{E}^{\,\mathrm{op}}$.

Lemma 3. *With notation as above, the bundle $L(\mathscr{E})$ is a Fell bundle over L. Moreover, the C^*-algebra $\Gamma_0(L^{(0)}; L(\mathscr{E}))$ is isomorphic to $A \oplus D$.*

Proof. We already know $L(\mathscr{E})$ is an upper-semicontinuous Banach bundle, that each $L(\mathscr{E})_u$ is a C^*-algebra and each $L(\mathscr{E})_x$ is a $L(\mathscr{E})_{r(x)}$–$L(\mathscr{E})_{s(x)}$-imprimitivity bimodule. The fibre map q preserves multiplication and involution by definition of these operations. The operations are continuous because they are continuous on each component of $L(\mathscr{E})$ and of $L(\mathscr{E}) * L(\mathscr{E})$, and the components are topologically disjoint. That $(\mathfrak{a}\mathfrak{b})^* = \mathfrak{b}^*\mathfrak{a}^*$ is clear on $\mathscr{B} * \mathscr{B}$ and $\mathscr{C} * \mathscr{C}$, follows from the inner-product axioms on $\mathscr{E} * \mathscr{E}^{\,\mathrm{op}}$ and $\mathscr{E}^{\,\mathrm{op}} * \mathscr{E}$, and follows from (1) for the remaining pairings. Associativity for triples from $\mathscr{E} * \mathscr{E}^{\,\mathrm{op}} * \mathscr{E}$ and $\mathscr{E}^{\,\mathrm{op}} * \mathscr{E} * \mathscr{E}^{\,\mathrm{op}}$ follows from the imprimitivity bimodule axiom ${}_{\star}\langle e \,,\, f\rangle \cdot g = e \cdot \langle f \,,\, g\rangle_{\star}$, and is clear for all other triples.

The map $f \mapsto (f|_{G^{(0)}}, f|_{H^{(0)}})$ is a surjection

$$\Gamma_0(L^{(0)}; L(\mathscr{E})) \to A \oplus D,$$

and the inverse makes sense because $G^{(0)}$ and $H^{(0)}$ are topologically disjoint. Hence $\Gamma_0(L^{(0)}; L(\mathscr{E})) \cong A \oplus D$. □

Resume the hypotheses of Lemma 3. It is routine to check that

$$(\mathfrak{p}_G\varphi)(g) := \chi_{G^{(0)}}(r(g))\varphi(g)$$

determines a bounded self-adjoint map on $\Gamma_c(G; \mathscr{B})$ under the inner-product $(\varphi, \psi) \mapsto \varphi^*\psi$, and hence extends to a multiplier projection, also denoted $\mathfrak{p}_G$, of $C^*(G, \mathscr{B})$. Taking adjoints, $(\varphi\mathfrak{p}_G)(\mathfrak{a}) = \chi_{G^{(0)}}(s(\mathfrak{a}))\varphi(\mathfrak{a})$. The corresponding projection $\mathfrak{p}_H$ for H is defined similarly.

Remark 4. As in [22], we think of $\varphi \in \Gamma_c(L; L(\mathscr{E}))$ as a matrix

$$\begin{pmatrix} \varphi_G & \varphi_Z \\ \varphi_{Z^{\mathrm{op}}} & \varphi_H \end{pmatrix}$$

where φ_G is the restriction of φ to $G \subseteq L$ and similarly for the other terms. With respect to this decomposition, we have

$$\varphi\psi = \begin{pmatrix} \varphi_G\psi_G + \varphi_Z\psi_{Z^{\mathrm{op}}} & \varphi_G\psi_Z + \varphi_Z\psi_H \\ \varphi_{Z^{\mathrm{op}}}\psi_G + \varphi_H\psi_Z & \varphi_{Z^{\mathrm{op}}}\psi_Z + \varphi_H\psi_H \end{pmatrix},$$

where we have used juxtaposition for the convolution product restricted to the various corners.[2] Moreover $\varphi_G = \mathfrak{p}_G\varphi\mathfrak{p}_G$, $\varphi_Z = \mathfrak{p}_G\varphi\mathfrak{p}_H$, $\varphi_{Z^{\mathrm{op}}} = \mathfrak{p}_H\varphi\mathfrak{p}_G$, and $\varphi_H = \mathfrak{p}_H\varphi\mathfrak{p}_H$.

Lemma 5. *Resume the hypotheses of Lemma* 3. *Then* $\mathfrak{p}_G$ *and* $\mathfrak{p}_H$ *are full multiplier projections of* $C^*(L, L(\mathscr{E}))$.

Proof. We just show that $\mathfrak{p}_G$ is full; the corresponding statement for $\mathfrak{p}_H$ follows by symmetry. Fix $\varphi, \psi \in \Gamma_c(L; L(\mathscr{E}))$. Using the matrix notation established above, we have

$$\varphi\mathfrak{p}_G\psi = \begin{pmatrix} \varphi_G\psi_G & \varphi_G\psi_Z \\ \varphi_{Z^{\mathrm{op}}}\psi_G & \varphi_{Z^{\mathrm{op}}}\psi_Z \end{pmatrix}.$$

That elements of the form $\varphi_G\psi_G$ span a dense subalgebra of $\Gamma_c(G; \mathscr{B})$ is clear. That elements of the form $\varphi_G\psi_Z$ span a dense subspace of $\Gamma_c(Z; \mathscr{E})$ and likewise that elements of the form $\varphi_{Z^{\mathrm{op}}}\psi_G$ span a dense subspace of $\Gamma_c(Z^{\mathrm{op}}; \mathscr{E}^{\mathrm{op}})$ follows from [16, Proposition 6.10]. That elements of the form $\varphi_{Z^{\mathrm{op}}}\psi_Z$ span a dense subspace of $\Gamma_c(H; \mathscr{C})$ follows from the argument which establishes axiom (IB2) in [16, Section 7]. □

Recall that the inductive-limit topology on $C_c(X)$ for a locally compact Hausdorff space X is the unique finest locally convex topology such that for each compact $K \subseteq X$, the inclusion of

$$C_c(X)^K = \{f \in C_c(X) : \operatorname{supp}(f) \subseteq K\}$$

into $C_c(X)$ is continuous (see for example [4, II.14.3] or [21, §D.2]). In particular, [21, Lemma D.10] says that to check that a linear map L from $C_c(X)$ into any locally convex space M is continuous, it suffices to see that if $f_n \to f$ uniformly and if all the supports of the f_n are contained in the same compact set K, then $L(f_n) \to L(f)$.

Remark 6. We are now in a situation analogous to that of [22, Remark 8]. By the Disintegration Theorem for Fell bundles, [16, Theorem 4.13], any pre-C^*-norm $\|\cdot\|_\alpha$ on $\Gamma_c(L; L(\mathscr{E}))$ which is continuous in the inductive-limit topology is dominated by the universal norm. Hence the argument

[2] In fact, the products in the matrix can be expressed in terms of the inner-products and module actions from [16, Theorem 6.4].

of [22, Remark 8] shows that $\mathfrak{p}_G C^*_\alpha(L, L(\mathscr{E}))\mathfrak{p}_H$ is a $C^*_\alpha(G,\mathscr{B})$–$C^*_\alpha(H,\mathscr{C})$-imprimitivity bimodule. So to prove that $C^*(G,\mathscr{B})$ is Morita equivalent to $C^*(H,\mathscr{C})$ we just need to show that for $F \in \mathfrak{p}_G\Gamma_c(L; L(\mathscr{E}))\mathfrak{p}_G$, the universal norms $\|F\|_{C^*(L,L(\mathscr{E}))}$ and $\|F|_G\|_{C^*(G,\mathscr{B})}$ coincide, and similarly for the reduced algebras (the corresponding statements for H hold by symmetry).

4. The reduced norm

In this section we recall the construction of the reduced cross-sectional algebra of a Fell bundle. We first discuss how to induce representations from the C^*-algebra of the restriction of a Fell bundle to a closed subgroupoid up to representations of the C^*-algebra of the whole bundle. We then apply this construction to the closed subgroupoid $G^{(0)}$ of G to induce representations of the C^*-algebra $A = \Gamma_0(G^{(0)};\mathscr{B})$ up to representations of $C^*(G,\mathscr{B})$. These are, by definition, the regular representations whose supremum determines the reduced norm.

4.1. Induced representations. Let G be a second countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Let $q : \mathscr{B} \to G$ be a separable Fell bundle as described in [11, §1.3]. Assume that H is a closed subgroupoid of G with Haar system $\{\alpha^u\}_{u\in H^{(0)}}$. We write $q_H : \mathscr{B}|_H \to H$ for the Fell bundle obtained by restriction to H. We want to induce representations of $C^*(H,\mathscr{B})$ to $C^*(G,\mathscr{B})$ using the Equivalence Theorem [16, Theorem 6.4] for Fell bundles. We will use the set-up and notation from [10, §2]. In particular, we recall that $G_{H^{(0)}} = s^{-1}(H^{(0)})$ is a (H^G, H)-equivalence where H^G is the *imprimitivity groupoid* $(G_{H^{(0)}} *_s G_{H^{(0)}})/H$. Let $\sigma : H^G \to G$ be the continuous map given by $\sigma\big([x,y]\big) = xy^{-1}$. The pull-back Fell bundle $\sigma^*q : \sigma^*\mathscr{B} \to H^G$ is the Fell bundle

$$\sigma^*\mathscr{B} = \{([x,y],b) : [x,y] \in H^G, b \in \mathscr{B}, \sigma([x,y]) = q(b)\}$$

with bundle map $\sigma^*q([x,y],b) = [x,y]$ over H^G.

Let $\mathscr{E} = q^{-1}(G_{H^{(0)}})$; then q restricts to a map $q : \mathscr{E} \to G_{H^{(0)}}$. We wish to make this Banach bundle into a $\sigma^*\mathscr{B}$–$\mathscr{B}|_H$-equivalence (see [16, Definition 6.1]). It is clear how $\mathscr{B}|_H$ acts on the right of $\mathscr{E}$, and we get a left action of $\sigma^*\mathscr{B}$ via

$$\big([x,y],b\big)\cdot e := be \quad \text{for } q(e) = yh.$$

(Since $q(b) = xy^{-1}$, $q(be) = xh$ as required.) The "inner products" on $\mathscr{E} *_r \mathscr{E}$ and $\mathscr{E} *_s \mathscr{E}$ are given by

$$\langle e\ ,\ f\rangle_{\mathscr{B}|_H} = e^*f \quad \text{and} \quad {}_{\sigma^*\mathscr{B}}\langle e\ ,\ f\rangle = \big([q(e),q(f)], ef^*\big),$$

respectively. It now straightforward to check that $\mathscr{E}$ is a $\sigma^*\mathscr{B}$–$\mathscr{B}|_H$-equivalence. By [16, Theorem 6.4], $\Gamma_c(G_{H^{(0)}};\mathscr{E})$ is a pre-imprimitivity bimodule

with actions and inner products determined by

$$F \cdot \varphi(z) = \int_G F\big([z,y]\big)\varphi(y)\, d\lambda_{s(z)}(y), \tag{2}$$

$$\varphi \cdot g(z) = \int_H \varphi(zh)g(h^{-1})\, d\alpha^{s(z)}(h), \tag{3}$$

$$\langle \varphi\ ,\ \psi\rangle_\star(h) = \int_G \varphi(y)^*\psi(yh)\, d\lambda_{r(h)}(y), \tag{4}$$

$${}_\star\langle \varphi\ ,\ \psi\rangle\big([x,y]\big) = \int_H \varphi(xh)\psi(yh)^*\, d\alpha^{s(x)}(h) \tag{5}$$

for $F \in \Gamma_c(H^G; \sigma^*\mathscr{B})$, $\varphi, \psi \in \Gamma_c(G_{H^{(0)}}; \mathscr{E})$ and $g \in \Gamma_c(H; \mathscr{B}|_H)$. The completion $\mathsf{X} = \mathsf{X}_H^G$ is a $C^*(H^G, \sigma^*\mathscr{B})$–$C^*(H, \mathscr{B}|_H)$-imprimitivity bimodule.

Remark 7. It is pleasing to note that the formalism of Fell bundles is such that equations (2)–(5) are virtually identical to those in the scalar case: see [10, Eq. (1)–(4)].[3] The only difference is that complex conjugates in the scalar case are replaced by adjoints.

To construct induced representations using the machinery of [21, Proposition 2.66], we need a nondegenerate homomorphism $V : C^*(G, \mathscr{B}) \to \mathcal{L}(\mathsf{X})$ which will make X into a right Hilbert $C^*(G, \mathscr{B})$–$C^*(H, \mathscr{B}|_H)$-bimodule (the data needed to induce representations *a la* Rieffel.) Define

$$V : \Gamma_c(G; \mathscr{B}) \to \mathrm{Lin}(\Gamma_c(G_{H^{(0)}}; \mathscr{E}))$$

by

$$V(f)(\varphi)(z) := \int_G f(y)\varphi(y^{-1}z)\, d\lambda^{r(z)}(y). \tag{6}$$

By the Tietze Extension Theorem for upper semicontinuous Banach bundles [16, Proposition A.5], each $\varphi \in \Gamma_c(G_{H^{(0)}}; \mathscr{E})$ is the restriction of an element of $\Gamma_c(G; \mathscr{B})$. So the argument of [10, Remark 1] and the paragraph which follows yields

$$\langle V(f)\varphi\ ,\ \psi\rangle_\star = \langle \varphi\ ,\ V(f^*)\psi\rangle_\star.$$

The map $f \mapsto \langle V(f)\varphi\ ,\ \psi\rangle_\star$ is continuous in the inductive-limit topology, and the existence of approximate units in $\Gamma_c(G; \mathscr{B})$ implies that

$$\{\, V(f)\varphi : f \in \Gamma_c(G; \mathscr{B}) \text{ and } \varphi \in \Gamma_c(G_{H^{(0)}}; \mathscr{E})\,\}$$

spans a dense subspace of $\Gamma_c(G_{H^{(0)}}; \mathscr{E})$. Then [11, Proposition 1.7] implies that V is bounded and extends to a nondegenerate homomorphism as required.

Now if L is a representation of $C^*(H, \mathscr{B}|_H)$, then the induced representation $\mathrm{Ind}_H^G L$ of $C^*(G, \mathscr{B})$ acts on the completion of $\mathsf{X} \odot \mathcal{H}_L$ with respect to

$$(\varphi \otimes h \mid \psi \otimes k) = \big(L(\langle \psi\ ,\ \varphi\rangle_\star)h \mid k\big)_{\mathcal{H}_L}.$$

[3] Well, they would be if it weren't for the typos in equations (1) and (4) in [10].

Fix $\varphi \in \Gamma_c(G_{H^{(0)}}; \mathscr{E})$. Writing $f \cdot \varphi$ for $V(f)\varphi$, we have

$$(\operatorname{Ind}_H^G L)(f)(\varphi \otimes h) = f \cdot \varphi \otimes h,$$

and, as in [10, Remark 1], $f \cdot \varphi = f * \varphi$.

4.2. Regular representations and the reduced C^*-algebra. *Regular representations* are, by definition, those induced from $A = \Gamma_0(G^{(0)}; \mathscr{B})$. Thus, in the notation of Section 4.1, $H = G^{(0)}$, $G_{H^{(0)}} = G$, $\mathscr{E} = \mathscr{B}$, and we write $\mathscr{A}$ in place of $\mathscr{B}|_{G^{(0)}}$ (see Remark 2); in particular each $B(x)$ is a $A(r(x))$–$A(s(x))$-imprimitivity bimodule. We also have $\langle \varphi \, , \, \psi \rangle_\star = \varphi^* \psi|_{G^{(0)}}$, with the product being computed in $\Gamma_c(L; L(\mathscr{E}))$.

Let π be a representation of A on H_π. Let $\tilde{\pi}$ be the extension of π to $M(A)$, and let $i : C_0(G^{(0)}) \to M(A)$ be the map characterised by

$$\big(i(f)a\big)(u) = f(u)a(u)$$

for $u \in G^{(0)}$. Then $\varphi := \tilde{\pi} \circ i$ is a representation of $C_0(G^{(0)})$ on H_π which commutes with π. Example F.25 of [23] shows that there is a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$ and a finite Radon measure μ on $G^{(0)}$ such that π is equivalent to a direct integral $\int_{G^{(0)}}^{\oplus} \pi_u \, d\mu$, and such that if $L : f \to L_f$ is the diagonal inclusion of $C_0(G^{(0)})$ in $B(L^2(G^{(0)} * \mathscr{H}, \mu))$, then $\pi(i(f)a) = L_f \pi(a)$ for $a \in A$. So each π_u factors through A_u. We will usually write $\pi_u\big(a(u)\big)$ in place of $\pi_u(a)$ for $a \in A$. See [17, p. 46] for more details. The regular representation $\operatorname{Ind}\pi = \operatorname{Ind}_{G^{(0)}}^G \pi$ then acts on the completion of $\Gamma_c(G; \mathscr{B}) \odot L^2(G^{(0)} * \mathscr{H}, \mu)$ with respect to $\big(\varphi \otimes h \mid \psi \otimes k\big) = \big(\pi\big(\psi^* * \varphi\big)h \mid k\big)$, and a quick calculation yields

$$\big(\varphi \otimes h \mid \psi \otimes k\big) = \int_{G^{(0)}} \int_G \big(\pi_u\big(\psi(x)^* \varphi(x)\big)h(u) \mid k(u)\big) \, d\lambda_u(x) \, d\mu(u). \tag{7}$$

Then $\operatorname{Ind}\pi$ acts by:

$$(\operatorname{Ind}\pi)(f)(\varphi \otimes h) = V(f)(\varphi) \otimes h = f \cdot \varphi \otimes h. \tag{8}$$

We next define the reduced algebra of a Fell bundle. We define the reduced norm by analogy with the one-dimensional case as the supremum of the norms determined by induced representations of A. We then show that this agrees, via V, with the operator norm on $\mathcal{L}(\mathsf{X})$. This is equivalent to Definition 2.4 and Lemma 2.7 of [14], though the roles of definition and lemma are interchanged.

Definition 8. We define the *reduced norm* on $\Gamma_c(G; \mathscr{B})$ by

$$\|f\|_r := \sup\{\, \|(\operatorname{Ind}\pi)(f)\| : \pi \text{ is a representation of } A \,\}.$$

Since the kernel of $\operatorname{Ind}\pi$ depends only on the kernel of π (see [21, Corollary 2.73]), we have $\|f\|_r = \|(\operatorname{Ind}\pi)(f)\|$ for any faithful representation π of A. We define the *reduced C^*-algebra* $C_r^*(G, \mathscr{B})$ of $\mathscr{B}$ to be the quotient of $C^*(G, \mathscr{B})$ by $I_{C_r^*(G,\mathscr{B})} := \{\, a \in C^*(G, B) : \|a\|_r = 0 \,\}$.

Lemma 9. *Let* $\mathsf{X} = \mathsf{X}^G_{G^{(0)}}$ *and* $V : C^*(G,\mathscr{B}) \to \mathcal{L}(\mathsf{X})$ *the homomorphism determined by* (6)*. Then* $\ker V = I_{C^*_r(G,\mathscr{B})}$ *and* V *factors through an injection of* $C^*_r(G,\mathscr{B})$ *into* $\mathcal{L}(\mathsf{X})$*. In particular,* $\|V(f)\| = \|f\|_r$*.*

Proof. Let π be a faithful representation of A. Then for any $x \in \mathsf{X}$, $h \in \mathcal{H}_\pi$ and $f \in C^*(G,\mathscr{B})$, we have

$$\begin{aligned}\big\|(\operatorname{Ind}\pi)(f)(x\otimes h)\big\|^2 &= \big\|V(f)(x)\otimes h\big\|^2 \\ &= \big(\pi\big(\langle V(f)(x)\,,\,V(f)(x)\rangle_\star\big)h \mid h\big).\end{aligned} \tag{9}$$

Thus if $V(f) = 0$, then $(\operatorname{Ind}\pi)(f) = 0$. On the other hand, given x and f, we can find a unit vector h such that the right-hand side of (9) is at least

$$\frac{1}{2}\big\|\pi\big(\langle V(f)(x)\,,\,V(f)(x)\rangle_\star\big)\big\| = \frac{1}{2}\big\|V(f)(x)\big\|^2.$$

Thus $\operatorname{Ind}\pi(f) = 0$ implies $V(f) = 0$. We have shown $\ker V = \ker(\operatorname{Ind}\pi)$, and hence V factors through an injection of $C^*_r(G,\mathscr{B})$ into $\mathcal{L}(X)$ as claimed. □

We digress briefly to check that the definition of the reduced C^*-algebra which we have given is compatible with existing definitions on some special cases.

Example 10 (The Scalar Case: Groupoid C^*-Algebras)**.** Let $\mathscr{B} = G\times\mathbf{C}$ so that $C^*(G,\mathscr{B}) = C^*(G)$. So $A = C_0(G^{(0)})$, and π defined by multiplication on $L^2(G^{(0)},\mu)$ is a faithful representation of A. Then $\operatorname{Ind}\pi$ acts on the completion H of $C_c(G)\odot L^2(G^{(0)})$ and, if we let $\nu = \mu\circ\lambda$, then (7) becomes

$$\big(\varphi\otimes h \mid \psi\otimes k\big) = \int_G \big(\varphi(x)h(s(x)) \mid \psi(x)k(s(x))\big)\,d\nu^{-1}(x).$$

Hence there is a unitary U from H onto $L^2(G,\nu^{-1})$ defined by

$$U(\varphi\otimes h)(x) = \varphi(x)h(s(x)),$$

and U intertwines $\operatorname{Ind}\pi$ with the representation

$$(\operatorname{Ind}\mu)(f)\xi(x) = \int_G f(y)\xi(y^{-1}x)\,d\lambda^{r(x)}(y).$$

Hence our definition of the reduced norm agrees with the usual definition (see [22, §3], for example), and $C^*_r(G, G\times\mathbf{C}) = C^*_r(G)$.

Example 11 (Groupoid Crossed Products)**.** Suppose that $(\mathscr{A},G,\alpha)$ is a dynamical system and form the associated semidirect product Fell bundle $\mathscr{B} = r^*\mathscr{A}$ as in [16, Example 2.1]. Working with the appropriate $\mathscr{A}$-valued functions, as in [16, Example 2.8], a quick calculation starting from (7) gives

$$\begin{aligned}&\big(f\otimes h \mid g\otimes k\big) = \\ &\qquad\int_{G^{(0)}}\int_G \big(\pi_u\big(\alpha_x^{-1}\big(f(x)\big)\big)h(u) \mid \pi_u\big(\alpha_x^{-1}\big(g(x)\big)\big)k(u)\big)\,d\lambda_u(x)\,d\mu(u).\end{aligned} \tag{10}$$

(Since λ_u is supported on G_u, each $\alpha_x^{-1}(g(x)) \in A_u$, so the integrand makes sense.) Let $G*\mathscr{H}_s$ be the pull back of $G^{(0)}*\mathscr{H}$ via s. Given a representation π of A, there is a unitary U from the space of $\operatorname{Ind}\pi$ to $L^2(G*\mathscr{H}_s, \nu^{-1})$ defined by $U(f\otimes h) = \pi_{s(x)}\big(\alpha_x^{-1}\big(f(x)\big)\big)h\big(s(x)\big)$. This U intertwines $\operatorname{Ind}\pi$ with the representation L^π given by

$$L^\pi(f)\xi(x) = \int_G \pi_{s(x)}\big(\alpha_x^{-1}\big(f(y)\big)\big)\xi(y^{-1}x)\,d\lambda^{r(x)}(y). \tag{11}$$

Applying this with a faithful representation π of A, we deduce that

$$\mathscr{A} \rtimes_{\alpha,r} G \cong C_r^*(G, r^*\mathscr{A}).$$

Remark 12. In Examples 10 and 11, the essential step in finding a concrete realization of the space of $\operatorname{Ind}\pi$ is to "distribute the π_u" in the integrand in (7) to both sides of the inner product. But for general Fell bundles, $\pi_u(\varphi(x))$ makes no sense for general $x \in G_u$. This often makes analyzing regular representations of Fell bundle C^*-algebras considerably more challenging.

Example 13. Any representation π_u of A_u determines a representation $\pi_u \circ \epsilon_u$ of A by composition with evaluation at u (in the direct-integral picture, $\pi = \pi_u \circ \epsilon_u$ is a direct integral with respect to the point-mass δ_u). We abuse notation slightly and write $\operatorname{Ind}\pi_u$ for $\operatorname{Ind}(\pi_u \circ \epsilon_u)$ which acts on the completion of $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_{\pi_u}$ under

$$(\varphi\otimes h \mid \psi\otimes k) = \int_G \big(\pi_u\big(\psi(x)^*\varphi(x)\big)h(u) \mid k(u)\big)\,d\lambda_u(x). \tag{12}$$

Equation (12) depends only on $\varphi|_{G_u}$ and $\psi|_{G_u}$; and conversely each element of $\Gamma_c(G_u;\mathscr{B})$ is the restriction of some $\varphi \in \Gamma_c(G;\mathscr{B})$ by the Tietze Extension Theorem for upper semicontinuous Fell bundles [16, Proposition A.5]. So we can view the space of $\operatorname{Ind}\pi_u$ as the completion of $\Gamma_c(G_u;\mathscr{B}) \odot \mathcal{H}_{\pi_u}$ with respect to (12).

5. The equivalence theorem

Fix for this section second-countable locally compact Hausdorff groupoids G and H with Haar systems λ and β, a G–H equivalence Z, Fell bundles $p_G : \mathscr{B} \to G$ and $p_H : \mathscr{C} \to H$ and a $\mathscr{B}$–$\mathscr{C}$ equivalence $q : \mathscr{E} \to Z$. Let κ denote the Haar system on L obtained from [22, Remark 11], and let $L(q) : L(\mathscr{E}) \to L$ be the linking bundle of Section 3.

Theorem 14. *Suppose that $F \in \Gamma_c(L; L(\mathscr{E}))$ satisfies $f(\zeta) = 0$ for all $\zeta \in L \setminus G$. Let $f := F|_G \in \Gamma_c(G;\mathscr{B})$. Then $\|F\|_{C^*(L,L(\mathscr{E}))} = \|f\|_{C^*(G,\mathscr{B})}$ and $\|F\|_{C_r^*(L,L(\mathscr{E}))} = \|f\|_{C_r^*(G,\mathscr{B})}$.*

Moreover, $\mathfrak{p}_G C^*(L, L(\mathscr{E}))\mathfrak{p}_H$ is a $C^*(G,\mathscr{B})$–$C^*(H,\mathscr{C})$-imprimitivity bimodule, and $\mathfrak{p}_G C_r^*(L, L(\mathscr{E}))\mathfrak{p}_H$ is a $C_r^*(G,\mathscr{B})$–$C_r^*(H,\mathscr{C})$-imprimitivity bimodule which is the quotient module of $\mathfrak{p}_G C^*(L, L(\mathscr{E}))\mathfrak{p}_H$ by the kernel I_r of the canonical homomorphism of $C^*(H,\mathscr{C})$ onto $C_r^*(H,\mathscr{C})$.

Remark 15. Recall the set-up of Example 10. It is not difficult to see that if G and H are groupoids and Z is a G–H equivalence, then the trivial bundle $Z \times \mathbf{C}$ is a $(G \times \mathbf{C})$–$(H \times \mathbf{C})$ equivalence. Hence we recover Theorem 13, Proposition 15 and Theorem 17 of [22] from Theorem 14.

To prove Theorem 14, we first establish some preliminary results. Our key technical result is a norm-estimate for the representations of

$$\Gamma_c(G; \mathscr{B}) \subseteq \Gamma_c(L; L(\mathscr{E}))$$

coming from elements of H^0.

Let $\{\rho_Z^v\}_{v \in H^{(0)}}$ be the Radon measures on Z introduced in [22, Theorem 13]. For each $v \in H^{(0)}$, fix $\zeta \in Z$ with $s(\zeta) = v$, and define a $D(v)$-valued form on $\mathsf{Y}_0 = \Gamma_c(Z; \mathscr{E})$ by

$$\langle \varphi \, , \, \psi \rangle_D(v) = \int_G \langle \varphi(x^{-1} \cdot \zeta) \, , \, \psi(x^{-1} \cdot \zeta) \rangle_{\mathscr{E}} \, d\lambda^{r(\zeta)}(x). \tag{13}$$

Left-invariance of ρ implies that this formula does not depend on the choice of $\zeta \in Z$ such that $s(\zeta) = v$. The map $(\varphi, \psi) \mapsto \langle \varphi \, , \, \psi \rangle_D(v)$ is the restriction to $L^{(0)}$ of the product $\varphi^* \psi$ computed in $\Gamma_c(L; L(\mathscr{E}))$.

The following lemma constructs what is essentially an "integrated form" of the modules used in [14, Proposition 4.3].

Lemma 16. *With respect to the pre-inner product* (13), Y_0 *is a pre-Hilbert* D*-module whose completion,* Y *is a full right Hilbert* D*-module.*

Proof. That (13) takes values in D follows from the observation above that $\langle \varphi \, , \, \psi \rangle_D = (\varphi^* \psi)|_{L^{(0)}}$. Since each E_z is an imprimitivity bimodule, the range of $\langle \cdot \, , \, \cdot \rangle_D(v)$ is all of D_v. Since D is a $C_0(H^{(0)})$-algebra, to see that $X := \overline{\operatorname{span}}\{ \langle \varphi \, , \, \psi \rangle_D : \varphi, \psi \in \mathsf{Y}_0 \} = D$, it therefore suffices to show that X is a $C_0(H^{(0)})$-module (see, for example [23, Proposition C.24]), for which one uses the right action of $C_0(H^{(0)})$ on Y_0 to check that

$$(\langle \varphi \, , \, \psi \rangle_D \cdot f)(v) := f(v) \langle \varphi \, , \, \psi \rangle_D(v)$$

is bilinear from $X \times C_0(H^{(0)})$ to X. An argument like the one immediately after the proof of Lemma 3 shows that for $f \in \Gamma_c(L^{(0)}; L(q)^{-1}(L^{(0)}))$, $M_f(\psi)(g) := f(r(g))\psi(g)$ determines a multiplier of $C^*(L, L(\mathscr{E}))$, so Y_0 is a pre-Hilbert D-module which is full since $X = D$. □

Remark 17. Since D is a $C_0(H^{(0)})$-algebra, to each $v \in H^{(0)}$ there corresponds a quotient module

$$\mathsf{Y}(v) := Y / Y \cdot I_v,$$

where $I_v = \{ d \in D : d(v) = 0 \}$. As in [21, Proposition 3.25], $\mathsf{Y}(v)$ is a right Hilbert $D(v)$-module: if we denote by $x(v)$ the image of x in $\mathsf{Y}(v)$, then we have $\big\langle x(v) \, , \, y(v) \big\rangle_{D(v)} = \langle x \, , \, y \rangle_D(v)$. Since $\|y\|^2 = \|\langle y \, , \, y \rangle_D\|$, we obtain $\|y\| = \sup_{v \in H^{(0)}} \|y(v)\|$. Indeed, the $\mathsf{Y}(v)$ are isomorphic to the modules used in [14, Proposition 4.3].

Fix $T \in \mathcal{L}(\mathsf{Y})$. Since T is D-linear, for each $v \in H^{(0)}$ there is an adjointable operator T_v on $\mathsf{Y}(v)$ satisfying $T_v(x(v)) = (Tx)(v)$ for all $x \in \mathsf{Y}$. Since $\langle T_v(x(v)) \, , \, y(v)\rangle_{D(v)} = \langle Tx \, , \, y\rangle_D(v)$, we have $\|T\| = \sup_{v\in H^{(0)}} \|T_v\|$.

Proposition 18 ([14, Proposition 4.3])**.** *There is a homomorphism M from $C^*(G,\mathscr{B})$ to $\mathcal{L}(\mathsf{Y})$ such that if $f \in \Gamma_c(G;\mathscr{B})$ and $\varphi \in \mathsf{Y}_0$, then*

$$M(f)\varphi(\zeta) = \int_G f(x)\varphi(x^{-1}\cdot\zeta)\, d\lambda^{r(\zeta)}(x). \tag{14}$$

We have $I_{C_r^*(G,\mathscr{B})} \subset \ker M$, and M factors through $C_r^*(G,\mathscr{B})$. In particular, $\|M(f)\| \le \|f\|_r$.

Proof. Direct computation shows that

$$\langle M(f)\varphi \, , \, \psi\rangle_D = \langle \varphi \, , \, M(f^*)\psi\rangle_D \quad \text{for } f \in \Gamma_c(G;\mathscr{B}) \text{ and } \varphi,\psi \in \Gamma_c(Z;\mathscr{E}).$$

A calculation using Remark 17, the Cauchy–Schwarz inequality for Hilbert modules ([21, Lemma 2.5]) and the characterization of inductive-limit topology continuous maps out of $C_c(G)$ in terms of eventually compactly supported uniform convergence shows that $f \mapsto \langle M(f)\varphi \, , \, \psi\rangle_D$ is continuous in the inductive-limit topology. The existence of approximate identities as in [16, Proposition 6.10] then implies that

$$\operatorname{span}\{\, M(f)\varphi : f \in \Gamma_c(G;\mathscr{B}) \text{ and } \varphi \in \mathsf{Y}_0 \,\}$$

is dense in Y in the inductive limit topology, so [11, Proposition 1.7] implies that M is bounded and extends to $C^*(G,\mathscr{B})$.

Since $\|M(f)\| = \sup_{v\in H^{(0)}} \|M_v(f)\|$, it now suffices to show that

$$\|M_v(f)\| \le \|f\|_r \quad \text{for all } v \in H^{(0)}. \tag{15}$$

Fix $v \in H^{(0)}$ and choose $\zeta \in Z$ such that $s(\zeta) = v$. Let $u = r(\zeta)$. For any $e \in q^{-1}(\zeta)$ we have ${}_{\mathscr{B}}\langle \varphi(x\cdot\zeta) \, , \, e\rangle \in B(_G[x\cdot\zeta,\zeta]) = B(x)$. Thus we can define

$$U_e : \Gamma_c(Z;\mathscr{E}) \to \Gamma_c(G_u;\mathscr{B}) \quad \text{by} \quad U(\varphi)(x) = {}_{\mathscr{B}}\langle \varphi(x\cdot\zeta) \, , \, e\rangle.$$

Just as in Remark 17, we can form the quotient module $\mathsf{X}(u)$, and the map $V : C^*(G,\mathscr{B}) \to \mathcal{L}(\mathsf{X})$ from Lemma 9 gives operators $V_u(f) \in \mathcal{L}(\mathsf{X}(u))$ such that $\|V_u(f)\| \le \|f\|_r$. The inner product $\langle \varphi \, , \, \psi\rangle_\star(u)$ depends only on the $\varphi|_{G_u}$ and $\psi|_{G_u}$, and every element of $\Gamma_c(G_u;\mathscr{B})$ extends to an element of $\Gamma_c(G;\mathscr{B})$ by the Tietze Extension Theorem for upper semicontinuous Banach bundles [16, Proposition A.5]. So we can view U_e as a map from $\Gamma_c(Z;\mathscr{E})$ to $\mathsf{X}(u)$.

Using that

$${}_{\mathscr{B}}\langle \varphi(x^{-1}\zeta) \, , \, e\rangle^* \, {}_{\mathscr{B}}\langle \varphi(x^{-1}\cdot\zeta) \, , \, e\rangle = {}_{\mathscr{B}}\langle e\cdot\langle \varphi(x^{-1}\cdot\zeta) \, , \, \varphi(x^{-1}\cdot\zeta)\rangle_{\mathscr{C}} \, , \, e\rangle,$$

one computes to see that

$$\langle U_e(\varphi) \, , \, U_e(\varphi)\rangle_\star(u) = \langle e\cdot\langle \varphi \, , \, \varphi\rangle_{\mathscr{B}}(v) \, , \, e\rangle_D. \tag{16}$$

So if $\|e\| \leq 1$, the Cauchy–Schwarz inequality for Hilbert modules ([21, Lemma 2.5]) implies that $\|U_e(\varphi)\|_{\mathsf{X}(u)} \leq \|\varphi\|_{\mathsf{Y}(v)}$.

For $x \in G_u$, that the pairing ${}_{\mathscr{B}}\langle \cdot\,,\cdot \rangle$ is A-linear in the first variable gives

$$U_e(M(f)\varphi)(x) = \int_G {}_{\mathscr{B}}\langle f(y)\varphi(y^{-1}x\cdot\zeta)\,,\,e\rangle\, d\lambda^{r(x)}(y) = V_u(f)U(\varphi)(x).$$

Fix $f \in \Gamma_c(G;\mathscr{B})$ and $\epsilon > 0$. Fix $\varphi \in \Gamma_c(Z;\mathscr{E})$ such that $\|\varphi\|_{\mathsf{Y}(v)} = 1$ and such that $\|M(f)\varphi\|_{\mathsf{Y}(v)} > \|M_v(f)\| - \epsilon$. By (16), there exists $e \in q^{-1}(\zeta)$ with $\|e\| = 1$ such that

$$\|U_e(M(f)\varphi)\|_{\mathsf{X}(u)} > \|M(f)\varphi\|_{\mathsf{Y}(v)} - \epsilon.$$

Hence

$$\begin{aligned}\|M_v(f)\| - 2\epsilon < \|M(f)\varphi\|_{\mathsf{Y}(v)} - \epsilon &< \|U_e(M(f)\varphi)\|_{\mathsf{X}(u)}\\ &= \|V_u(f)U_e(\varphi)\|_{\mathsf{X}(u)} \leq \|f\|_r.\end{aligned}$$

Letting $\varepsilon \to 0$ gives (15). □

Proof of Theorem 14. Since every representation of $C^*(L, L(\mathscr{E}))$ restricts to a representation of $C^*(G,\mathscr{B})$, we have $\|F\|_{C^*(L,L(\mathscr{E}))} \leq \|f\|_{C^*(G,\mathscr{B})}$, so we just have to establish the reverse inequality. The argument for this is nearly identical to that of [22, Proposition 15]. The key differences are that: [16, Theorem 6.4] is used in place of [17, Theorem 5.5]; and [16, Proposition 6.10] is used to obtain an approximate identity for both $\Gamma_c(G;\mathscr{B})$ and $\Gamma_c(Z;\mathscr{E})$ which can be used in place of the approximate identity in $\mathscr{C}(L)$ to establish the analogue of [22, Equation (10)] and to complete the norm approximation at the end of the proof.

We now turn to the proof that the reduced norms agree. Fix faithful representations π_u of the $A(u)$ and τ_v of the $D(v)$. Then

$$\|F\|_{C_r^*(L,L(\mathscr{E}))} = \max\left\{\sup_{u\in G^{(0)}} \|(\operatorname{Ind}^L \pi_u)(F)\|,\ \sup_{v\in H^{(0)}} \|(\operatorname{Ind}\tau_v)(F)\|\right\}.$$

Fix $u \in G^{(0)}$. Let $\mathcal{H}_1$ be the space of $\operatorname{Ind}^G \pi_u$; that is, the completion of $\Gamma_c(G_u;\mathscr{B}) \odot \mathcal{H}_{\pi_u}$ as in Example 13. The representation $\operatorname{Ind}^L \pi_u$ acts on the completion of $\Gamma_c(L_u; L(\mathscr{E})) \odot \mathcal{H}_{\pi_u}$ which decomposes as $\mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 = \overline{\Gamma_c(G_u; L(\mathscr{E})) \odot \mathcal{H}_{\pi_u}}$ and $\mathcal{H}_2 = \overline{\Gamma_c(Z_u; L(\mathscr{E})) \odot \mathcal{H}_{\pi_u}}$. Moreover, the restriction of $\operatorname{Ind}^L \pi_u$ to H_2 is the zero representation. Hence

$$\|F\|_{C_r^*(L,L(\mathscr{E}))} \geq \sup_{u\in G^{(0)}} \|\operatorname{Ind}^L \pi_u(f)\| = \|f\|_{C_r^*(G,\mathscr{B})},$$

and it suffices now to establish that $\|\operatorname{Ind}^L \tau_v(F)\| \leq \sup_u \|\operatorname{Ind}^L \pi_u(F)\|$ for all $F \in \Gamma_c(L; L(\mathscr{E}))$ and $v \in H^{(0)}$.

Fix $v \in H^{(0)}$. Then $\operatorname{Ind}^L \tau_v$ acts on the completion of $\Gamma_c(L_v; L(\mathscr{E})) \odot \mathcal{H}_{\tau_v}$ which again decomposes as a direct sum $\mathcal{H}_3 \oplus \mathcal{H}_4$, where

$$\mathcal{H}_3 = \overline{\Gamma_c(Z_v^{\mathrm{op}}; L(\mathscr{E})) \odot \mathcal{H}_{\tau_v}} \quad \text{and} \quad \mathcal{H}_4 = \overline{\Gamma_c(H_v; L(\mathscr{E})) \odot \mathcal{H}_{\tau_v}}.$$

The restriction to $\mathcal{H}_4$ is zero, and $\mathcal{H}_3$ is the completion of $\Gamma_c(Z;\mathscr{E})$ under

$$(\varphi\otimes h \mid \psi\otimes k) = \int_Z \big(\tau_v\big(\langle\psi(\zeta)\ ,\ \varphi(\zeta)\rangle_{\mathscr{C}}\big)h \mid k\big)\,d\rho^v(\zeta).$$

An inner-product computation shows that if Y is the Hilbert D-module of Lemma 16, then $\mathcal{H}_3$ is isomorphic to the completion of $\mathsf{Y}\odot\mathcal{H}_{\tau_v}$ under

$$(\varphi\otimes h \mid \psi\otimes k) = \big(\tau_v\big(\langle\psi\ ,\ \varphi\rangle_{D}(v)\big)h \mid k\big),$$

and then the restriction of $(\operatorname{Ind}^L\tau_v)(F)$ to $\mathcal{H}_3$ is $\mathsf{Y}\text{-}\operatorname{Ind}\tau_v$. Hence

$$\|\operatorname{Ind}^L\tau_v(F)\| = \|\mathsf{Y}\text{-}\operatorname{Ind}\tau_v(f)\|.$$

Since $\mathsf{Y}\text{-}\operatorname{Ind}\tau_v[x\otimes h] = [M(f)x\otimes h]$ for all x, we have $\ker M\subset\ker\mathsf{Y}\text{-}\operatorname{Ind}\tau_v$. Hence Proposition 18, implies that $\|\mathsf{Y}\text{-}\operatorname{Ind}\tau_v(f)\|\le\|f\|_{C^*_r(G,\mathscr{B})}$ as required.

The final statement follows from [21, Theorem 3.22]. □

6. The reduced symmetric imprimitivity theorem

Suppose that K and H are locally compact groups acting freely and properly on the left and right, respectively, of a locally compact space P. Suppose also that we have commuting actions α and β of K and H, respectively, on a C^*-algebra D. Then we can form the induced algebras $\operatorname{Ind}_H^P(D,\beta)$ and $\operatorname{Ind}_K^P(D,\alpha)$ and get dynamical systems

$$\check\sigma : K\to\operatorname{Aut}\big(\operatorname{Ind}_H^P(D,\beta)\big)\quad\text{and}\quad\check\tau : H\to\operatorname{Aut}\big(\operatorname{Ind}_K^P(D,\alpha)\big)$$

for the diagonal actions as in [23, Lemma 3.54]. Then Raeburn's Symmetric Imprimitivity Theorem says that the crossed products

$$\operatorname{Ind}_H^P(D,\beta)\rtimes_{\check\sigma}K\quad\text{and}\quad\operatorname{Ind}_K^P(D,\alpha)\rtimes_{\check\tau}H$$

are Morita equivalent. In [19], Quigg and Spielberg proved that Raeburn's Morita equivalence passed to the reduced crossed products. (Kasparov had a different proof in [12, Theorem 3.15] and an Huef and Raeburn gave a different proof of the Quigg and Spielberg result in [9, Corollary 3].)

We consider the corresponding statements for groupoid dynamical systems. Let $(\mathscr{A},G,\alpha)$ be a groupoid dynamical system as in [17, §4]. Recall that the associated crossed product $\mathscr{A}\rtimes_\alpha G$ is a completion of $\Gamma_c(G;r^*\mathscr{A})$. If π is a representation of $A:=\Gamma_0(G^{(0)};\mathscr{A})$, then the associated regular representation of $\mathscr{A}\rtimes_\alpha G$ is the representation $L^\pi := \operatorname{Ind}^{\mathscr{A}}\pi$ acting on $L^2(G*\mathcal{H}_s,\nu^{-1})$ as in (11). The reduced crossed product, $\mathscr{A}\rtimes_{\alpha,r}G$ is the quotient of $\mathscr{A}\rtimes_\alpha G$ by the common kernel of the L^π with π faithful. Let $\mathscr{B}:=r^*\mathscr{A}$ with the semidirect product Fell bundle structure so that $C^*(G,\mathscr{B})$ is isomorphic to $\mathscr{A}\rtimes_\alpha G$, then it follows from Example 11 that $C^*_r(G,\mathscr{B})$ is isomorphic to the reduced crossed product $\mathscr{A}\rtimes_{\alpha,r}G$.

Now let H be a locally compact group and let X be a locally compact Hausdorff H-space. Let G be the transformation groupoid $G=H\times X$. As in [17, Example 4.8], suppose that $A=\Gamma_0(X;\mathscr{A})$ is a $C_0(X)$-algebra,

and define $\mathrm{lt} : H \to \mathrm{Aut}(C_0(X))$ by $\mathrm{lt}_h(\varphi)(x) := \varphi(h^{-1} \cdot x)$. Suppose that $\beta : H \to \mathrm{Aut}\, A$ is a C^*-dynamical system such that

$$\beta_h(\varphi \cdot a) = \mathrm{lt}_h(\varphi) \cdot \beta_h(a) \quad \text{for } h \in H,\ \varphi \in C_0(X) \text{ and } a \in A.$$

Then, following [17, Example 4.8], we obtain a groupoid dynamical system $(\mathscr{A}, G, \alpha)$ where

$$\alpha_{(h,x)}\big(a(h^{-1} \cdot x)\big) = \beta_h(a)(x).$$

Let Δ be the modular function on H. Then the map

$$\Phi : C_c(H, A) \to \Gamma_c(G; r^*\mathscr{A})$$

given by

$$\Phi(f)(h, x) = \Delta(h)^{\frac{1}{2}} f(h)(x)$$

extends to an isomorphism of $A \rtimes_\beta H$ with $\mathscr{A} \rtimes_\alpha G$.

Fix a representation π of A. By decomposing π as a direct integral over X one checks that $(f \otimes_\pi \xi | g \otimes_\pi \eta) = (\Phi(f) \otimes_\pi \xi | \Phi(g) \otimes_\pi \eta)$ for $f, g \in C_c(G, A)$ and $\xi, \eta \in \mathcal{H}_\pi$. We use this to show that $U(f \otimes h) = \Phi(f) \otimes h$ determines a unitary from the space of the regular representation $\mathrm{Ind}^A \pi$ of $A \rtimes_\beta H$ to the space of the regular representation $\mathrm{Ind}^{\mathscr{A}} \pi$ of $\mathscr{A} \rtimes_\alpha G$ which intertwines $\mathrm{Ind}^A \pi(f)$ and $\mathrm{Ind}^{\mathscr{A}} \pi(\Phi(f))$ for all f. Therefore Φ factors through an isomorphism $A \rtimes_{\beta,r} H \cong \mathscr{A} \rtimes_{\alpha,r} G$.

Now, back to the set-up of Raeburn's Symmetric Imprimitivity Theorem. Since $\mathrm{Ind}_H^P(D, \beta)$ is a $C_0(P/H)$-algebra, it is the section algebra of a bundle $\mathscr{B}$ over P/H. It is shown in [17, Example 5.12] that there is a groupoid action σ of the transformation groupoid $K \times P/H$ on $\mathscr{B}$ such that

$$\mathrm{Ind}_H^P(D, \beta) \rtimes_{\check{\sigma}} K \cong \mathscr{B} \rtimes_\sigma (K \times P/H).$$

Similarly,

$$\mathrm{Ind}_K^P(D, \alpha) \rtimes_{\check{\tau}} H \cong \mathscr{A} \rtimes_\tau (K\backslash P \rtimes H)$$

for an appropriate bundle $\mathscr{A}$ over $K\backslash P$ and action τ. Furthermore, the trivial bundle $\mathscr{E} := P \times A$ is an equivalence between $(\mathscr{B}, K \times P/H, \sigma)$ and $(\mathscr{A}, K\backslash P \times H, \tau)$ in the sense of [17, Definition 5.1]. (Thus Raeburn's Symmetric Imprimitivity Theorem is a special case of [17, Theorem 5.5].) Therefore the Quigg-Spielberg result follows from following corollary of our main theorem.

Corollary 19. *Suppose that $q : \mathscr{E} \to Z$ is an equivalence between the groupoid dynamical systems $(\mathscr{B}, H, \beta)$ and $(\mathscr{A}, G, \alpha)$. Then the Morita equivalence of* [17, Theorem 5.5] *factors through a Morita equivalence of the reduced crossed products $\mathscr{B} \rtimes_{\beta,r} H$ and $\mathscr{A} \rtimes_{\alpha,r} G$.*

Proof. Recall that $r^*\mathscr{A} := \{(a, x) \in \mathscr{A} \times G : r(a) = r(x)\}$ is a Fell bundle over G with bundle map $(a, x) \mapsto x$, multiplication $(a, x)(b, y) = (a\alpha_x(b), xy)$ and involution $(a, x)^* = (\alpha_{x^{-1}}(a), x^{-1})$ (see [16, Example 2.1]), and similarly for $r * \mathscr{B}$. Define maps $r^*\mathscr{A} * \mathscr{E} \to \mathscr{E}$ and $\mathscr{E} * r^*\mathscr{B} \to \mathscr{E}$ by

$$(a, g) \cdot \mathfrak{a} := a \cdot \alpha_g(\mathfrak{a}) \quad \text{and} \quad \mathfrak{a} \cdot (b, h) := \beta_h(\mathfrak{a}) \cdot b.$$

Define pairings ${}_{r^*\mathscr{A}}\langle\cdot\,,\cdot\rangle : \mathscr{E} *_s \mathscr{E} \to \mathscr{A}$ and $\langle\cdot\,,\cdot\rangle_{r^*\mathscr{B}} : \mathscr{E} *_r \mathscr{E} \to \mathscr{B}$ by

$$
\begin{aligned}
{}_{r^*\mathscr{A}}\langle \mathfrak{a}\,,\,\mathfrak{b}\rangle &= \big(\langle \mathfrak{a}\,,\,\alpha_{G[q(\mathfrak{a}),q(\mathfrak{b})]}(\mathfrak{b})\rangle_{A_{r(\mathfrak{a})}}, G[q(\mathfrak{a}),q(\mathfrak{b})]\big) \quad \text{and}\\
\langle \mathfrak{a}\,,\,\mathfrak{b}\rangle_{r^*\mathscr{B}} &= \big(\langle \mathfrak{a}\,,\,\beta_{[q(\mathfrak{a}),q(\mathfrak{b})]_H}(\mathfrak{b})\rangle_{A_{r(\mathfrak{a})}}, [q(\mathfrak{a}),q(\mathfrak{b})]_H\big).
\end{aligned}
$$

It is routine though tedious to show that $\mathscr{E}$ is an $r^*\mathscr{A}$–$r^*\mathscr{B}$ equivalence.

The Morita equivalence X of [17, Theorem 5.5] and the Morita equivalence $\mathfrak{p}_G C^*(L, L(\mathscr{E}))\mathfrak{p}_H$ of Theorem 14 are both completions of $\Gamma_c(Z;\mathscr{E})$. From the formulae for the actions of $r^*\mathscr{A}$ on $\mathscr{E}$, we see that the identity map on $\Gamma_c(Z;E)$ determines a left-module map from X to $\mathfrak{p}_G C^*(L, L(\mathscr{E}))\mathfrak{p}_H$, and similarly on the right. So it suffices to show that the norms on X and on $\mathfrak{p}_G C^*(L, L(\mathscr{E}))\mathfrak{p}_H$ coincide. For this, observe that the formula [17, Equation (5.1)] for the $\mathscr{A}\times_\alpha G$-valued inner-product on $\Gamma_c(Z;\mathscr{E})$ is precisely the convolution formula for multiplication of the corresponding elements of $\Gamma_c(L; L(\mathscr{E}))$ with respect to the Haar system κ described in [22]. □

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