New York Journal of Mathematics

New York J. Math. **17** (2011) 383–435.

# On small geometric invariants of 3-manifolds

**Paul J. Kapitza**

Abstract. A small geometric invariant is a nonnegative integer invariant associated with a 3-manifold whose value is bounded above by the Heegaard genus of the manifold.

Craggs has studied techniques to detect for a given 3-manifold $M^3$, whether the double $2M = \mathrm{Bd}(M_\star \times [-1,1])$ bounds a 4-manifold $N$ that has the same 3-deformation type as the complement of the interior of a 3-ball in $M$ and has a handle presentation with, in some sense, a minimal number of 1-handles. Here, $M_\star$ is obtained from $M$ by removing an open ball. He exhibits a pair of surgery obstructions, whose vanishing is sufficient for the existence of this type of 4-manifold $N$ and minimal handle presentation.

We show that for the double of one of the Boileau–Zieschang manifolds, there is a certain handle presentation which, in the absence of the obstructions studied by Craggs, is reducible to this minimal number of 1-handles and we provide an explicit construction. For this case, the question of the existence of a minimal handle presentation is reduced to a study of the obstructions defined by Craggs.

Contents



Received June 15, 2010; revised March 29, 2011.

2000 *Mathematics Subject Classification.* Primary 57M20; Secondary 57M50, 57R65.

*Key words and phrases.* Heegaard genus, extended Nielsen genus, Boileau–Zieschang manifolds.

ISSN 1076-9803/2011

## 1. Introduction

A geometric invariant of a 3-manifold is a geometrically defined measure which remains the same across the homeomorphism class of a manifold. The Heegaard genus of a manifold is one such example.

Craggs [3] studies a geometric invariant for 3-manifolds $M$ defined by considering certain 4-manifolds $N$ bounded by the double $2M$ of $M$. He looks at handle presentations of $N$ with handles of index at most 2, and takes the minimum number of 1-handles over all such presentations. This minimum number is bounded above by the Heegaard genus of $M$ and so is a small geometric invariant.

It is known that the rank of the fundamental group of an arbitrary 3-manifold $M^3$ and its associated Heegaard genus do not always agree. In particular, the manifolds of Boileau and Zieschang [2] make up a collection of 3-manifolds for which the Heegaard genus is 3, but the rank of the fundamental group is 2. See also Schultens and Weideman [11].

There have been efforts to show that for a given 3-manifold $M$, the 4 manifold $N = M_\star \times [-1, 1]$ has a minimal 2-handle presentation, where the number of 1-handles is determined by the formal 3-deformation properties of $M_\star$. Here $M_\star$ is the result of removing the interior of a 3-ball from $M$.

Craggs [3] uses the extended Nielsen genus $\mathrm{en}(M)$ of the base manifold $M$ as a measure of the potential minimum number of 1-handles in any handle presentation associated with an appropriate 4-manifold $N$ bounded by $2M$. The extended Nielsen genus of a 3-manifold is bounded above by the Heegaard genus of the 3-manifold, and in the case of the Boileau–Zieschang manifolds it is less than the Heegaard genus. Thus, the extended Nielsen genus is a small geometric invariant that is sometimes less than Heegaard genus.

Craggs defines handle presentations for certain 4-manifolds bounded by the double $2M$ of $M$ to be minimal if the number of 1-handles is equal to the extended Nielsen genus $\mathrm{en}(M)$ of $M$. The geometric realization of $\mathrm{en}(M)$ as the number of 1-handles in a minimal handle presentation for $M$ associates in a natural way a pair of framed surgery obstructions $\{\mathcal{L}, \mathcal{T}\}$ in a cube with handles. If these obstructions are always trivial, then minimal handle presentations always exist, and they provide a new small geometric invariant that is generally not equal to Heegaard genus.

We examine the thesis that one member of the Boileau–Zieschang family bounds a 4-manifold with a minimal handle presentation. In this paper, we construct a 4-manifold of the form $M_\star \times [-1, 1]$ whose associated handle presentation exhibits an algebraic simplicity which agrees with the extended Nielsen genus of $M$.

Some of the material here is taken from the author's Ph.D. thesis at the University of Illinois, Urbana-Champaign. The author wishes to thank Professor Robert Craggs for his help in directing the thesis and to a referee for numerous helpful comments on a previous version of this paper.

**1.1. Notation and conventions.** Let $K = \bigcup_\alpha e_\alpha$ be a finite connected CW complex with characteristic maps $\phi_\alpha : D^n \to K$. Here $D^n$ is a topological ball of dimension n such that $\phi_\alpha \mid \mathrm{Int}(D^n)$ is a homeomorphism onto $e_\alpha$, with $\phi_\alpha(\mathrm{Bd}(D^n)) \subset K^{n-1}$, where $K^n = \bigcup\{e_\alpha \mid \dim e_\alpha \leq n\}$ denotes the *n-skeleton of* $K$.

An *elementary n-expansion* $K \nearrow L$ is defined for $L = K \cup_f D^n$, where $f$ attaches to $K$ all of the boundary of $D^n$, except one open $(n-1)$-cell. An *elementary n-collapse* is the inverse of an elementary n-expansion, denoted as $K \searrow L$.

We work in the PL category. For a piecewise linear 3-manifold $M^3$, a *Heegaard decomposition of* $M^3$ *of genus* $n$ is a triple, $(M; H, J)$, where $M = H \cup J$ and $H$ and $J$ are handlebodies of genus $n$ with $H \cap J = \mathrm{Bd}(H) = \mathrm{Bd}(J)$. The genus of the decomposition is the genus of the handlebody $J$. The *Heegaard genus* of $M^3$, $\mathrm{hg}(M^3)$, is the minimum value of $n$ obtained over all Heegaard decompositions of $M^3$.

The manifold $M^3$ exhibits the structure of a CW complex with cells identified with the piecewise linear cells of $M$ in a piecewise linear cell decomposition of $M$. Every cellular decomposition of $M^3$ with one 0-cell and one 3-cell defines a Heegaard decomposition of $M^3$. Taking $\overline{M \setminus N}$ where $N$ is a regular neighborhood of the 1-skeleton of $M$ results in a handlebody whose genus is the number of 1-cells in the decomposition.

The cell complex obtained from a Heegaard decomposition of genus $n$ provides a handle decomposition of the form

$$M^3 = h^0 \cup \left[\bigcup_{i=1}^{n} h_i^1\right] \cup \left[\bigcup_{j=1}^{n} h_j^2\right] \cup h^3$$

where $h_m^l$ is a three dimensional handle of index $l$, $l = 0, \ldots, 3$ and $n$ is the genus of $J$.

If $K$ is a 2-complex, a *formal* 3*-deformation of* $K$, denoted $K \overset{3}{\nearrow\!\searrow} L$, is a sequence of polyhedra $K = K_0 \to K_1 \to \cdots \to K_n = L$, where $K_i \to K_{i+1}$ results from an expansion ($\nearrow$) or collapse ($\searrow$) of a piecewise linear cell of dimension at most three. A complex $K$ is *collapsible* if $K \searrow \{\star\}$ where $\{\star\}$ denotes a 0-cell of $K$.

**1.2. Presentations and extended Nielsen equivalence.** Let $X$ be a finite set and $R$ a set of words on $X$. A group $G$ is defined by the sets $X$ and $R$ if $G \cong F/N$, where $F$ is the free group on $X$ and $N$ is the normal subgroup of $F$ normally generated by $R$. A *presentation* $P = \langle X \mid R \rangle$ for $G$ consists of the ordered sets $X = \{x_i^{\pm 1} \mid i = 1, \ldots, m\}$, the *generators* of $P$, and $R = \{r_1, r_2, \ldots, r_n\}$, the defining *relators* of $P$. A presentation $P = \langle X \mid R \rangle$ *presents a group* $G$ if $G$ is isomorphic to the quotient group, $F/N$ where $F$ is the free group on the generators $x_1, x_2, \ldots, x_m$, and $N \subset F$ is the smallest normal subgroup containing $R$. The group presented by $P$ is said to be *finitely presented* if there exists a presentation in which both $X$ and $R$ are finite sets. The *rank* of the group $G$ presented by $P$, $\mathrm{rk}(G)$, is the minimum number of generators $m$ for $X = \{x_1, x_2, \ldots, x_m\}$ required to present $G$.

For a 3-manifold $M^3$, let $K = K^2$ in $M^3 \setminus B^3$ be defined by

$$K = e^0 \cup \left[\bigcup_{i=1}^{n} e_i^1\right] \cup \left[\bigcup_{j=1}^{p} e_j^2\right]$$

in which $e_\alpha^k$ is a $k$-cell for $k = 0, 1, 2$ with characteristic maps $\phi_\alpha^k : D^k \to K$. Associated with $K$ is a group presentation of the form

$$P_K = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_p \rangle$$

where each generator $x_i$ is obtained from a 1-cell of $K$ in $K^1 \setminus T(K)$ for some chosen maximal tree $T(K)$. For each relator $r_j$ there is a 2-cell $e_j^2$ and a characteristic map $\phi_j^2 : D^2 \to K$ where $\phi_j^2 \mid \partial D^2$ reads a word $r_j$ in the symbols $x_1, x_2, \ldots, x_n$.

A *geometric presentation associated with* $K$ is a presentation

$$P_K = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_p \rangle$$

where $r_i$ reads the attaching map $\phi_i(\mathrm{Bd}(e_i^2)) \subset K$. The *reduced presentation for* $P = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_p \rangle$ is defined to be

$$|P| = \langle x_1, x_2, \ldots, x_n \mid s_1, s_2, \ldots, s_p \rangle$$

where $P$ is a presentation and $s_i \equiv |r_i|$, where $|r_i|$ denotes the freely reduced form of $r_i$. The *reduced presentation associated with* $K$ is defined to be

$$|P_K| = \langle x_1, x_2, \ldots, x_n \mid s_1, s_2, \ldots, s_p \rangle$$

where $P_K$ is the geometric presentation associated with $K$ and $s_i = |r_i|$, where $|r_i|$ denotes the freely reduced form of $r_i$. In this case, we will also write $|P_K| = \langle x_1, x_2, \ldots, x_n \mid \ |r_1|, |r_2|, \cdots, |r_p| \rangle$ to denote the corresponding abstract presentation with freely reduced relators.

Given a presentation $P = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_p \rangle$, one may construct a presentation $P'$ obtained from $P$ by a finite sequence of *elementary extended Nielsen* operations:

(1) For some $1 \le j \le p$, add or delete the trivial relator $xx^{-1}$ or $x^{-1}x$ in $r_j$, leaving $r_k$ unchanged for $k \neq j$.
(2) For some $1 \le j \le p$, replace $r_j$ with $r_j^{-1}$, leaving $r_k$ unchanged for $k \neq j$.
(3) For some $1 \le j \le p$ and some $1 \le k \le p$, replace $r_j$ with $r_j r_k$, where $k \neq j$, leaving $r_i$ unchanged for $i \neq j$.

(4) For some $1 \leq j \leq p$, replace $r_j$ with $w^{-1}r_jw$, where $w$ is an element in $F(x_1, x_2, \ldots, x_n)$, leaving $r_k$ unchanged for $k \neq j$.
(5) For an automorphism, $\alpha : F(x_1, x_2, \ldots, x_n) \to F(x_1, x_2, \ldots, x_n)$, replace $r_j$ with $\alpha(r_j)$ for $j = 1, \ldots, p$.
(6) Add $x_{n+1}$ to the set of generators and $r_{p+1} = x_{n+1}$ to the set of defining relators.
(7) Remove $x_n$ from the set of generators and the relator $r_k = x_n$ from the set of defining relators, when both occur and $x_n$ appears exactly once among the relators.

Two presentations $P$ and $P'$ which are related by a finite sequence of extended Nielsen operations are said to be *extended Nielsen equivalent*, $P \overset{\text{en}}{\sim} P'$. It is well known that if $P \overset{\text{en}}{\sim} P'$ then $P$ and $P'$ present the same group. The extended Nielsen operations can be shown to generate a subset of the Tietze transformations for groups that accounts for all Tietze II operations. The *extended Nielsen genus of a presentation* $P$, denoted $\text{en}(P)$, is defined to be the minimum number of generators in any presentation which is extended Nielsen equivalent to $P$.

If $K$ is a 2-complex, the *extended Nielsen genus of* $K$, $\text{en}(K)$, is defined to be $\text{en}(P_K)$, where $P_K$ is the standard reading of a presentation from a 2-complex $K$. For a 3-manifold $M^3$, the *extended Nielsen genus of* $M^3$, $\text{en}(M^3)$is the extended Nielsen genus of any 2-spine of $M^3$. See Brown [1], Kreher and Metzler [8], Young [15] and Wright [14] on the equivalence of extended Nielsen equivalence and formal 3-deformation in both the polyhedral and the CW categories. These results imply that $\text{en}(M^3)$ is well-defined.

**1.3. Historical remarks.** Suppose that $M^3$ is a 3-manifold with 2-complex spine $K$, having a geometric presentation $P_K$. It is known that

$$\text{rk}(M) \leq \text{en}(M) \leq \text{hg}(M).$$

Haken [6] and Waldhausen [12] conjecture that $\text{rk}(M) = \text{hg}(M)$ for all 3-manifolds $M$. M. Boileau and H. Zieschang [2] exhibit a collection of Seifert 3-manifolds for which $2 = \text{rk}(\pi_1(M)) < \text{hg}(M) = 3$, providing a counterexample to the conjectures of Waldhausen and Haken.

In an explicit calculation, Montesinos [9] exhibits an extended Nielsen equivalence between a geometric presentation for $\pi_1(M_1, \star)$ and a presentation $P_1'$ that has 2 generators and 2 relators establishing that $\text{en}(M_1) \leq 2$. Here, $M_1$ is one of the family of manifolds exhibited by Boileau and Zieschang. That $\text{en}(M_1) > 1$ follows from the the fact that $\text{rk}(\pi_1(M_1)) \leq \text{en}(M_1)$ . Therefore, when combined with the previous results we have that

$$2 = \text{en}(M_1) = \text{rk}(M_1) < \text{hg}(M_1) = 3.$$

The following definitions come from Craggs [4]. Given a sequence of polyhedra $K = K_0 \to K_1 \to \cdots \to K_n = L$, where $K_i \to K_{i+1}$ is an expansion or collapse of a piecewise linear cell, if there is some polyhedron $X$ (usually a manifold) such that $K(i) \subset X$ for each $i$, then one says $K$

*deforms to $L$ in $X$.* If $M$ is a manifold and $K$ and $L$ are in the interior of $M$, then $K$ deforms to $L$ in $M$ means that $K$ and $L$ have isotopically embedded regular neighborhoods.

Craggs [4] studies the question, for which 2-complexes $K$ in a 3-manifold $M$ do the corresponding 2-complexes $K \times \{0\} \subset M \times [0,1]$ 3-deform in $M \times [0,1]$, keeping 1-skeletons fixed, to a 2-complex $L \subset M \times [0,1]$ so that the associated presentation $P_L$ is obtained from the presentation $P_K$ by freely reducing relator words? He addresses the following: If $M_\star$ 3-deforms in $M_\star \times [-1,1]$ to a 2-spine complex $L$ such that $|P_L|$ has $m$ 1-cells and $k$ 2-cells reading generators, does $L$ 3-deform in $M \times [-1,1]$ to a 2-complex with $m-k$ 1-cells?

A related question as to whether the 2-complex $K \times \{0\} \subset M_\star \times [-1,1]$ 3-deforms in $M_\star \times [-1,1]$ to a 2-complex $L$ having at most $\mathrm{en}(K)$ 1-cells has been addressed by Craggs [4] concerning the family of manifolds $\{M_n\}_{n=1}^{\infty}$.

Material necessary for later calculations is contained in the following sections. Section 1.4 describes the basic objects involved, the singular disk systems. Section 1.5 reviews material on singular systems with an admissibility requirement on the collection of singular disks in the system. Admissible systems provide a connection between modifications of singular systems and 3-deformations in $M_\star \times [-1,1]$.

**1.4. Singular disk systems.** The definitions and results which follow concerning singular and admissible disk systems are due to Craggs.

**Definition 1.1.** A *singular disk system in $H$* is a pair $(D,g)$ where

$$D = \bigcup_{i=1}^{n} D_i$$

is a finite disjoint union of disks and $g : D \to H$ is a proper map such that:

(1) $g^{-1}(\mathrm{Bd}((H))) \subset \mathrm{Bd}(D)$.
(2) The singular set of $g$ is a finite collection of proper disjoint arcs $\bigcup\{A_{i_1}, A_{i_2}\}$ such that each pair corresponds to a transverse double arc intersection.

A singular system is said to be *ordinary* if $g$ is nonsingular.

Figure 1 illustrates a singular disk system consisting of the 2-cells $D = D_1 \cup D_2$. The map $g : D \to H$ identifies the arcs $A_1 \subset D_1$ and $A_2 \subset D_2$ in the image.

**1.5. Admissible disk systems.**

**Definition 1.2.** A singular system $(D,g)$ in $H$ is said to be an *admissible system* if there exists a continuous map $\epsilon : D \to \{-1,0,1\}$ such that:

(1) The map $(g,\epsilon) : D \to H \times [-1,1]$ defined by $(g,\epsilon)(x) = (g(x),\epsilon(x))$ is an embedding.
(2) If a given disk $D_i \in D$ contains a singular arc, then $\epsilon(D_i) \neq 0$.

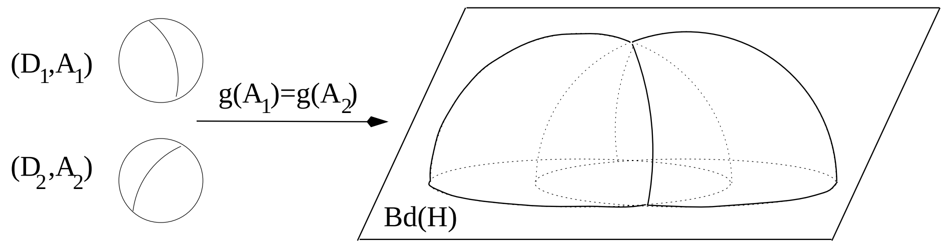


FIGURE 1. Singular disk system.

The quantity $\epsilon(D_i)$ is called the *label* of the disk $D_i$. We will write $\epsilon_i = \epsilon(D_i)$ and place $D_i^{\epsilon_i} = (g(D_i), \epsilon(D_i))$. In particular, if a singular disk system $(D, g)$ becomes an admissible disk system with the addition of some map $\epsilon : D \to \{-1, 0, 1\}$, then the admissible system will be denoted by the triple $(D, g, \epsilon)$.

In an admissible disk system $(D, g, \epsilon)$ in $H$, there is a natural partition of the 2-manifold $D$ into three disjoint submanifolds: $D^+$, $D^-$ and $D^0$, corresponding to those disks in $D$ for which $\epsilon = +1, -1, 0$ respectively.

Suppose $(M; H, J)$ is a decomposition where $(D, g, \epsilon)$ and $(D', g', \epsilon')$ are two admissible singular disk systems on $H$. Then $(D', g', \epsilon')$ *results from* $(D, g, \epsilon)$ *by an admissible sequence of operations* if $(D', g', \epsilon')$ is obtained from $(D, g, \epsilon)$ by a finite sequence of the following operations and their inverses:

(1) (*Bookkeeping*): Replace $(D, g, \epsilon)$ with the system $(D', g', \epsilon')$ where $h : M \to M$ is a homeomorphism that takes $H$ onto itself and $g' = g \circ h$.
(2) (*Level Switch*): For $D_i^\epsilon = (g(D_i), \epsilon(D_i))$ where $g \mid D_i$ is nonsingular, replace $\epsilon(D_i)$ by $\epsilon'(D_i) \in \{-1, 0, 1\}$.
(3) (*Full Isotopy*): Let $h_t : H \times I \to H$ be an isotopy such that $h_0 = 1_H$. Replace $(D, g, \epsilon)$ with the system $(D', g', \epsilon')$ where $g' = h_1 \circ g$.
(4) (*Split Isotopy*): Replace $(D, g, \epsilon)$ with the system $(D', g', \epsilon')$ where for some isotopy $h_t : H \times I \to H$ and $\eta \in \{-1, 1\}$ the following condition holds:
$g' \mid D_i = (h_1 \circ g) \mid D_i$ if $\epsilon(D_i) = \eta$ and $g' \mid D_i = g|D_i$ for $\epsilon(D_i) \neq \eta$.
(5) (*Admissible Disk Slide*): Replace $(D, g, \epsilon)$ with the system $(D', g', \epsilon')$ by sliding $g(D_j)$ over $g(D_k)$ along an arc $\beta$ where, considered as a singular system, $(D', g')$ results from $(D, g)$ by a slide of $g(D_j)$ over $g(D_k)$ and either $\epsilon_j = \epsilon_k$ or at least one of these quantities is 0.
(6) (*Stabilization*): Replace $(D, g, \epsilon)$ with the system $(D'', g'', \epsilon'')$ where $D \subset D''$ and $g = g'' \mid D$. In this operation, $D'' \setminus D = B^2$ is a nonsingular disk containing a properly embedded arc $\beta$. Let $N$ be a regular neighborhood of $\beta$ in $H \subset (M; H, J)$ and delete one of

the two components of $\overline{B^2 \setminus \beta}$ to produce the system $(D'', g'', \epsilon'')$ on $(M; H', J')$, where the genus $(H') = \text{genus}(J') = \text{genus}(J+1)$.

There is a natural association via Craggs [4] between admissible systems and 2-complexes in $M_\star \times [-1,1]$, in which each admissible operation induces an extended Nielsen transformation of the corresponding 2-complex group presentation.

## 2. Algebraic co-k-collapsibility

In this section, a property of the words $\{r_1, r_2, \ldots, r_p\}$ which are associated with a presentation of a collapsible complex $K$ is examined. A form for the relators associated with a collapsible complex is presented in terms of the associated presentation.

The remainder of this section is taken from Whitehead [13]. In what follows, $G = F(x_1, x_2, \ldots, x_n)$ is a free group, $W(x_1, x_2, \ldots, x_n)$ is a word on the symbols $X = \{x_1, x_2, \ldots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$ and $x$ and $y$ are elements of $X$.

**Definition 2.1.** An *elementary transformation* on a word

$$W = W(x_1, x_2, \ldots, x_n)$$

is either an insertion into $W$ or a deletion from $W$ of a pair of successive letters of the form $xx^{-1}$ for $x \in X$.

**Definition 2.2.** A *simple transformation of the first type* on a set of words $\{W_1, \ldots, W_k\}$ in $G$ is a replacement of the form $x \to xy$ and $x^{-1} \to y^{-1}x^{-1}$ for each occurrence of $x$ or $x^{-1}$ in $\{W_1, \ldots, W_k\}$. A *simple transformation of the second type* on a set of words $\{W_1, \ldots, W_k\} \subset G$ is an elementary transformation applied to some word in $\{W_1, \ldots, W_k\}$.

A *simple transformation* on a set of words $\{W_1, \ldots, W_k\} \subset G$ is either a simple transformation of the first or second type.

**Definition 2.3.** A *simple set of words* is a set $\{W_1, \ldots, W_k\}$ of distinct words derived from an independent set of generators $\{x_1, x_2, \ldots, x_n\}$ by a sequence of simple transformations.

The following results on simple sets and simple transformations will be used in later sections.

**Lemma 2.4.** *If $\{W_1, \ldots, W_k\}$ is a simple set of words on $X$, then every subset is also a simple set. Also, if $k < n$, any simple set $\{W_1, \ldots, W_k\}$ may be extended to a simple set $\{W_1, \ldots, W_n\}$.*

Given a simple transformation, say $x_i \to x_i x_j$ of the first type, there is an associated automorphism of $G$. For fixed $i$, $j$, $1 \leq i$, $j \leq n$, $i \neq j$ and for all $k \neq i$ define the map $\alpha_{ij} : G \to G$ by

$$\alpha_{ij}(x_i) = x_i x_j$$
$$\alpha_{ij}(x_k) = x_k, \qquad k \neq i.$$

For a word $W \cong x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \dots x_{i_l}^{\epsilon_l}$ in $G$, extend $\alpha_{ij}$ to $W \in G$ by defining

$$\alpha_{ij}(W) \cong \alpha_{ij}(x_{i_1})^{\epsilon_1} \alpha_{ij}(x_{i_2})^{\epsilon_2} \dots \alpha_{ij}(x_{i_l})^{\epsilon_l}.$$

Associating the simple transformations with automorphisms of $G$ applied to the set of words $\{W_1, \dots, W_k\}$ in this way yields the following result.

**Theorem 2.5.** *The collection $\{W_1, \dots, W_k\}$ is a simple set of words on the generating set $X$ if and only if the elements of $W$ correspond to an independent set of generators in some automorphism of $G$.*

**2.1. Algebraic collapsibility.** Consider the 2-complex $K$

$$K = e^0 \cup \left[ \bigcup_{i=1}^{n} e_i^1 \right] \cup \left[ \bigcup_{j=1}^{p} e_j^2 \right]$$

in which $e_\alpha^k$ is a $k$-cell for $k = 0, 1, 2$ together with the characteristic maps $\phi_\alpha^k : D^k \to K$ where $\phi_\alpha^k \mid \overset{\circ}{D^k}$ is a homeomorphism onto $e_\alpha^k$.

For $i = 1, \dots, n$ let $x_i$ be the generator associated with $e_i^1$. Then for $j = 1, \dots, p$, the attaching map associated with the 2-cell $e_j^2$ yields a word $r_j$ on the symbols $X = \{x_1, x_2, \dots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$.

Let $P_K = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_p \rangle$ be the geometric presentation associated with $K$ and suppose that $K$ collapses to a 2-complex $K(1)$ by an elementary collapse. In particular, suppose that $K \overset{e}{\searrow} K(1)$ by a collapse across the 1-cell $e_1^1$ which removes the 2-cell $e_1^2$ whose associated reading is given by $r_1$. Denote the resulting geometric presentation associated with $K(1)$ in terms of $P_K$ by writing

$$P_{K(1)} = \langle \hat{x}_1, x_2, \dots, x_n \mid \hat{r}_1, r_2, \dots r_p \rangle$$

where $\hat{x}$ indicates the removal of quantity $x$.

Corresponding to the elementary collapse $K \searrow K(1)$ across the 1-cell $e_1^1$, the set of words $\{r_1, r_2, \dots, r_p\}$ in $P_K$ has the following properties:

(1) The symbol $x_1$ or $x_1^{-1}$ occurs exactly once in the relator $r_1$.
(2) For $2 \le j \le p$, no $r_j$ contains an occurrence of $x_1$ or $x_1^{-1}$.

In the case where $K \searrow \{\star\}$ the set of words $\{r_1, r_2, \dots, r_p\}$ in $P_K$ will be called an *algebraically collapsible* set of words. In Section 2.2, the case where $K \searrow L$ for a subcomplex $L \subset K$ is examined.

To formalize this situation, we introduce the following terminology.

Let $W$ be a collection of words on $\{x_1, x_2, \dots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$. For each $1 \le i \le n$, let $\nu_i : W \to \mathbb{Z}$ be the function defined by setting $\nu_i(r)$ equal to the number of occurrences of $\{x_i^{\pm 1}\}$ in $r \in W$.

We will use the following subscript notation for a nonempty set of words $\{r_1, r_2, \dots, r_p\}$ on the set $X$. For $\Delta : \{1, \dots, p\} \to \{1, \dots, p\}$ an element of the symmetric group $S_p$ let $(j)$ denote the image under $\Delta$ of the element $j \in \{1, \dots, p\}$. That is, define $(j) = \Delta(j)$ for $\Delta \in S_p$. For the set of generators $\{x_1, x_2, \ldots, x_n\}$ and for $i \in \{1, \ldots, n\}$ let $[i] = \Gamma(i)$ for $\Gamma \in S_n$ be the image of $i$ under $\Gamma$.

With these conventions, the notation $\nu_{[i]}(r_{(j)})$ refers to the number of occurrences of the generator $x_{[i]}$ in the word $r_{(j)}$ under some pair of permutations $\Delta$ and $\Gamma$ as defined above.

**Definition 2.6.** An ordered collection of words $\{r_1, r_2, \ldots, r_n\}$ on

$$X = \{x_1, x_2, \ldots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$$

is called *algebraically collapsible* if after free reduction, there exist permutations $\Gamma \in S_n$ and $\Delta \in S_n$ such that

$$\nu_{[i]}(r_{(j)}) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i < j. \end{cases}$$

A presentation $P = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$ is called *algebraically collapsible* if $\{r_1, r_2, \ldots, r_n\}$ is an algebraically collapsible collection of words on $X$.

In general, we will assume that when given a collection $\{r_1, r_2, \ldots, r_n\}$ of words, any free reduction is performed prior to testing the collection by Definition 2.6.

**Example 2.7.** The collection of words $\{r_1, r_2, r_3\}$ on generators $\{x_1, x_2, x_3\}$ given by the assignments $r_1 = x_1$, $r_2 = x_1^{-1} x_3 x_2^3$ and $r_3 = x_2 x_1^{-1}$ is algebraically collapsible. Let $\Gamma = (1\ 3) \in S_3$ and $\Delta = (1\ 2\ 3) \in S_3$. Then

$$\begin{aligned} &\nu_{[1]}(r_{(1)}) = 1,\ \nu_{[1]}(r_{(2)}) = \nu_{[1]}(r_{(3)}) = 0 \\ &\nu_{[2]}(r_{(2)}) = 1,\ \nu_{[2]}(r_{(3)}) = 0 \\ &\nu_{[3]}(r_{(3)}) = 1. \end{aligned}$$

**Theorem 2.8.** *Let $\{r_1, r_2, \ldots, r_n\}$ be a collection of words on the alphabet $X$. Then $\{r_1, r_2, \ldots, r_n\}$ is algebraically collapsible if and only if there exist $\Gamma \in S_n$ and $\Delta \in S_n$ where $\Gamma(j) = [j]$, $\Delta(i) = (i)$ such that*

$$r_{(i)} = u_i x_{[i]}^{\pm 1} v_i \qquad 1 \leq i \leq n$$

*where $u_i = u_i(x_{[i+1]}, \ldots, x_{[n]})$ and $v_i = v_i(x_{[i+1]}, \ldots, x_{[n]})$.*

**Proof.** Suppose $\{r_1, r_2, \ldots, r_n\}$ is algebraically collapsible. Then there exist $\Gamma \in S_n$ and $\Delta \in S_n$ so that

$$\nu_{[i]}(r_{(j)}) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \qquad 1 \leq i,\ j \leq n.$$

For $1 \leq i \leq n$, $\nu_{[i]}(r_{(i)}) = 1$ so that $r_{(i)}$ is of the form $r_{(i)} = u_i x_{[i]}{}^{\pm 1} v_i$ where $u_i = u_i(x_{[1]}, \ldots, \hat{x}_{[i]}, \ldots, x_{[n]})$ and $v_i = v_i(x_{[1]}, \ldots, \ldots, \hat{x}_{[i]}, \ldots, x_{[n]})$. Since $\nu_{[m]}(r_{(i)}) = 0$ for $m = 1, \ldots, i-1$ then $u_i = u_i(x_{[i+1]}, \ldots, x_{[n]})$ and $v_i = v_i(x_{[i+1]}, \ldots, x_{[n]})$.

($\Leftarrow$) Suppose $\{r_1, r_2, \ldots, r_n\}$ is given along with $\Delta \in S_n$ and $\Gamma \in S_n$ so that $r_{(i)} = u_i x_{[i]}^{\pm 1} v_i$ for $u_i = u_i(x_{[i+1]}, \ldots, x_{[n]})$ and $v_i = v_i(x_{[i+1]}, \ldots, x_{[n]})$, $1 \leq i \leq n$. Apply the counting function $\nu_{[i]}$ for $i = 1, \ldots, n$ to obtain

$$\nu_{[i]}(r_{(j)}) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \qquad 1 \leq i,\, j \leq n.$$

This implies $\{r_1, \ldots, r_n\}$ is algebraically collapsible on $\{x_1, x_2, \ldots, x_n\}$. $\square$

In general, $\Gamma$ and $\Delta$ are not unique. For example, the collection of words $\{r_1, \ldots, r_n\}$ on the set of generators $\{x_1, x_2, \ldots, x_n\}$ where $r_j \equiv x_j$ for $j = 1, \ldots, n$ is algebraically collapsible for every $\Gamma = \Delta \in S_n$.

Let $F(x_1, x_2, \ldots, x_n)$ be the free group on $\{x_1, x_2, \ldots, x_n\}$. Suppose that $\{r_1, r_2, \ldots, r_n\}$ is a collection of distinct words on $F$. Recall from Theorem 2.5 that $\{r_1, r_2, \ldots, r_n\}$ is a simple set of words if each word $r_j$ corresponds to a generator $x_i$ under some automorphism $\sigma : F \to F$ . In the notation of Definition 2.6 this is equivalent to the statement that $\{r_1, r_2, \ldots, r_n\}$ is a simple set of words if there exist $\Gamma \in S_n$ and $\Delta \in S_n$ so that $\sigma(x_{[i]}) = r_{(j)}$ for $i,\, j = 1, \ldots, n$.

**Lemma 2.9.** *Suppose that $r$ is a word on $X$ of the form $r = u x_i^{\epsilon} v$ where $u$ and $v$ are words on the set of generators $X \setminus \{x_i,\, x_i^{-1}\}$and $\epsilon = \pm 1$. Then there exists an automorphism $\sigma : F \to F$ such that*

$$\sigma(x_j) = \begin{cases} u^{-1} x_i^{\epsilon} v^{-1} & j = i \\ x_j & j \neq i, \end{cases} \qquad j = 1, \ldots, n.$$

**Proof.** Let $r = u x_i v$ where $u$ and $v$ are words on $X \setminus \{x_i \cup x_i^{-1}\}$. Suppose that

$$u = x_{l_s}^{\epsilon_{l_s}} x_{l_{s-1}}^{\epsilon_{l_{s-1}}} \ldots x_{l_2}^{\epsilon_{l_2}} x_{l_1}^{\epsilon_{l_1}}$$
$$v = x_{r_1}^{\epsilon_{r_1}} x_{r_2}^{\epsilon_{r_2}} \ldots x_{r_{t-1}}^{\epsilon_{r_{t-1}}} x_{r_t}^{\epsilon_{r_t}}.$$

For $k = 1, \ldots, s$, consider the simple transformation of the first type defined by $x_i \to x_{l_k}^{-\epsilon_{l_k}} x_i$. By the remarks following Theorem 2.4 there is an associated automorphism $\lambda_{l_k} : F \to F$ where

$$\lambda_{l_k}(x_j) = \begin{cases} x_{l_k}^{-\epsilon_{l_k}} x_i & j = i \\ x_j & j \neq i \end{cases} \quad j = 1, \ldots, n.$$

Define $\lambda_L = \lambda_{l_s} \circ \lambda_{l_{s-1}} \circ \cdots \circ \lambda_{l_2} \circ \lambda_{l_1}$. By construction,

$$\lambda_L(x_j) = \begin{cases} u^{-1} x_i & j = i \\ x_j & j \neq i \end{cases} \quad j = 1, \ldots, n.$$

Similarly, for $k = 1, \ldots, t$, the simple transformation of the first type defined by $x_i \to x_i x_{r_k}^{-\epsilon_{r_k}}$ may be associated with the automorphism

$$\rho_{r_k} : F \to F$$

where

$$\rho_{r_k}(x_j) = \begin{cases} x_i x_{r_k}^{-\epsilon_{r_k}} & j = i \\ x_j & j \neq i \end{cases} \quad j = 1, \dots, n.$$

Define $\rho_R = \rho_{r_1} \circ \rho_{r_2} \circ \cdots \circ \rho_{r_t}$. Then

$$\rho_R(x_j) = \begin{cases} x_i v^{-1} & j = i \\ x_j & j \neq i \end{cases} \quad j = 1, \dots, n.$$

Finally, define $\sigma = \lambda_L \circ \rho_R$, so that

$$\sigma(x_j) = \begin{cases} u^{-1} x_i v^{-1} & j = i \\ x_j & j \neq i \end{cases} \quad j = 1, \dots, n.$$

In the case $r = u x_i^{-1} v$ where $u$ and $v$ are words on the set of generators $X \setminus \{x_i \cup x_i^{-1}\}$, apply the preceding construction to the word $v^{-1} x_i u^{-1}$ to obtain an automorphism $\sigma : F \to F$ such that

$$\sigma(x_j) = \begin{cases} v x_i u & j = i \\ x_j & j \neq i, \end{cases} \quad j = 1, \dots, n.$$

Then $\sigma(x_i^{-1}) = (\sigma(x_i))^{-1} = u^{-1} x_i^{-1} v^{-1}$. □

**Theorem 2.10.** *Suppose that $\{r_1, r_2, \dots, r_n\}$ is an algebraically collapsible set of words on $X$. Then there exists an automorphism $\sigma : F \to F$ such that $\sigma(x_{[i]}) = r_{(i)}$ for $1 \leq i \leq n$.*

*In particular, if $\{r_1, r_2, \dots, r_n\}$ is an algebraically collapsible set then $\{r_1, r_2, \dots, r_n\}$ is a simple set of words.*

**Proof.** Let $\{r_1, r_2, \dots, r_n\}$ be an algebraically collapsible set on $X$. By Theorem 2.8 there exists $\Gamma \in S_n$ and $\Delta \in S_n$, where $(j) = \Delta(j)$ and $[i] = \Gamma(i)$, so that $r_{(j)} = u_j x_{[j]}^{\pm 1} v_j$ for each $j \in \{1, \dots n\}$ where

$$u_j = u_j(x_{[j+1]}, \dots, x_{[n]})$$
$$v_j = v_j(x_{[j+1]}, \dots, x_{[n]}).$$

For each $i = 1, \dots n$, let $\sigma_i : F \to F$ be the automorphism of Lemma 2.9 defined by

$$\sigma_i(x_{[j]}) = \begin{cases} u_i^{-1} x_{[i]}\, v_i^{-1} & j = i \\ x_{[j]} & j \neq i \end{cases} \quad j \in \{1, \dots, n\}.$$

Define $\sigma = \sigma_n \circ \cdots \circ \sigma_1$.

Claim: $\sigma(r_{(j)}) = x_{[j]}$ for $j = 1, \dots, n$.

Let $r_{(j)} = u_j x_{[j]} v_j$, where

$$u_j = u_j(x_{[j+1]}, \dots, x_{[n]}),$$
$$v_j = v_j(x_{[j+1]}, \dots, x_{[n]}).$$

By construction, $\sigma_i(u_j) = u_j$ and also $\sigma_i(v_j) = v_j$ for $i = 1 \dots j$. Therefore,

$$\begin{aligned}\sigma(r_{(j)}) &= \sigma_n \circ \cdots \circ \sigma_{j+1} \circ \sigma_j \circ \cdots \circ \sigma_1(u_j x_{[j]} v_j)\\ &= \sigma_n \circ \cdots \circ \sigma_{j+1} \circ \sigma_j(u_j x_{[j]} v_j)\\ &= \sigma_n \circ \cdots \circ \sigma_{j+1}(u_j(u_j^{-1} x_{[j]} v_j^{-1}) v_j)\\ &= \sigma_n \circ \cdots \circ \sigma_{j+1}(x_{[j]})\\ &= x_{[j]}.\end{aligned}$$

Then the automorphism $\sigma^{-1} : \{x_{[1]}, \dots, x_{[n]}\} \to \{r_{(1)}, \dots, r_{(n)}\}$ exhibits $\{r_{(1)}, \dots, r_{(n)}\}$ as images of the generators $\{x_{[1]}, \dots, x_{[n]}\}$. By Theorem 2.5 $\{r_{(1)}, \dots, r_{(n)}\}$ forms a simple set of words. □

**Corollary 2.11.** *If $P = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n\rangle$ is an algebraically collapsible presentation then $P$ is extended Nielsen equivalent to the empty presentation.*

**Proof.** Let $P = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n\rangle$ be an algebraically collapsible presentation. Then there exists $\Gamma \in S_n$ and $\Delta \in S_n$, with $\nu_{[i]}(r_{(j)}) = 1$ if $i = j$ and $\nu_{[i]}(r_{(j)}) = 0$ where $i < j$ for $1 \le i, j \le n$.

Since $P$ is algebraically collapsible, Lemma 2.10 implies there exists an automorphism $\sigma : F(x_1, x_2, \dots, x_n) \to F(x_1, x_2, \dots, x_n)$ where $\sigma(x_{[i]}) = r_{(i)}$ for $1 \le i \le n$.

Then

$$\begin{aligned}P &= \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n\rangle\\ &\overset{\text{en}}{\sim} \langle x_{[1]}, \dots, x_{[n]} \mid r_{(1)}, \dots, r_{(n)}\rangle\\ &\overset{\text{en}}{\sim} \langle x_{[1]}, \dots, x_{[n]} \mid x_{[1]}, \dots, x_{[n]}\rangle\\ &\overset{\text{en}}{\sim} \langle - \mid - \rangle.\end{aligned}$$ □

**2.2. Algebraic co-k-collapsibility.** As in the previous section, let

$$K = e^0 \cup \left[\bigcup_{i=1}^{n} e_i^1\right] \cup \left[\bigcup_{j=1}^{p} e_j^2\right]$$

and let $P_K = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_p\rangle$ be the geometric presentation associated with $K$. Suppose that $L \subset K$ is a subcomplex of $K$ for which $K \searrow L$. In this case, if the corresponding elementary collapses are across the 1-cells $\{e_1^1, e_2^1, \dots, e_{p-k}^1\}$ for some $k \ge 0$ which remove the 2-cells $\{e_1^2, e_2^2, \dots, e_{p-k}^2\}$, then the corresponding presentation associated with $L$ is

$$P_L = \langle \hat{x}_1, \dots, \hat{x}_{p-k}, \dots, x_n \mid \hat{r}_1, \dots, \hat{r}_{p-k}, r_{p-k+1} \dots, r_p\rangle$$

where $k \le n$.

**Definition 2.12.** A collection of words $\{r_1, r_2, \dots r_q\}$ for $0 \le k \le q$ on the letters $\{x_1, x_2, \dots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$ for $0 < q \le n$ is called *algebraically co-k-collapsible* if after free reduction, there exists $\Gamma \in S_n$ and $\Delta \in S_q$ such that

$$\nu_{[i]}(r_{(j)}) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \qquad 1 \le i,\, j \le q - k.$$

A presentation $P = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_p \rangle$ for $p \le n$ is said to be algebraically co-k-collapsible if $\{r_1, \dots, r_p\}$ is an algebraically co-k-collapsible collection of words on $X$.

**Lemma 2.13.** *If $\{r_1, r_2, \dots, r_p\}$ is algebraically co-k-collapsible, $0 \le k < p$, then there exists a subset of cardinality $p - k$ which forms an algebraically collapsible set.*

**Proof.** Let $\{r_1, r_2, \dots, r_p\}$ be algebraically co-k-collapsible with $0 \le k < p$. Then there exists $\Gamma \in S_n$ and $\Delta \in S_p$ such that

$$\nu_{[i]}(r_{(j)}) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \qquad 1 \le i,\, j \le p - k.$$

From Definition 2.6 it follows directly that $\{r_{(1)}, \dots, r_{(p-k)}\}$ is algebraically collapsible on $X$. □

**Lemma 2.14.** *Let $\{r_1, r_2, \dots, r_p\}$ for $p \le n$ be a collection of words on the generating set $X$. Then $\{r_1, r_2, \dots, r_p\}$ is algebraically co-k-collapsible if and only if there exist $\Gamma \in S_n$ and $\Delta \in S_p$ where $\Gamma(j) = [j]$, $\Delta(i) = (i)$ such that*

$$r_{(i)} = u_i x_{[i]}^{\pm 1} v_i \qquad 1 \le i \le p - k$$

*where $u_i = u_i(x_{[i+1]}, \dots, x_{[n]})$ and $v_i = v_i(x_{[i+1]}, \dots, x_{[n]})$.*

**Proof.** ($\Rightarrow$) Let $\{r_1, r_2, \dots, r_p\}$ be algebraically co-k-collapsible. Then Lemma 2.13 implies there exists $\Gamma \in S_n$ and $\Delta \in S_p$ and an algebraically collapsible subset $\{r_{(1)}, \dots, r_{(p-k)}\}$. Theorem 2.8 applied to this subset implies the result.

($\Leftarrow$) Suppose given $\{r_1, r_2, \dots, r_p\}$, $\Delta \in S_p$ and $\Gamma \in S_n$ so that $r_{(i)} = u_i x_{[i]}^{\pm 1} v_i$ for $u_i = u_i(x_{[i+1]}, \dots, x_{[n]})$ and $v_i = v_i(x_{[i+1]}, \dots, x_{[n]})$, $1 \le i \le p-k$. Apply the counting function $\nu$ to obtain

$$\nu_{[i]}(r_{(j)}) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \qquad 1 \le i,\, j \le p - k.$$

Then $\{r_{(1)}, \dots, r_{(p-k)}\} \subset \{r_1, \dots, r_p\}$ is algebraically collapsible so that $\{r_1, \dots, r_p\}$ is an algebraically co-k-collapsible set on $X$. □

**Lemma 2.15.** *Suppose that $P = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_p \rangle$ for $p \le n$ is algebraically-co-k collapsible. Then $P$ is extended Nielsen equivalent to*

$$P' = \langle x_{[p-k+1]}, \dots, x_{[n]} \mid r'_{(p-k+1)}, \dots, r'_{(p)} \rangle$$

*for $\Gamma \in S_n$, $\Delta \in S_p$, and words $\{r'_{(p-k+1)}, \dots, r'_{(p)}\}$ on $\{x_{[p-k+1]}, \dots, x_{[n]}\}$.*

**Proof.** Let $P = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_p\rangle$ for $p \le n$ be algebraically co-k-collapsible. Then there exists $\Gamma \in S_n$ and $\Delta \in S_p$, with $\nu_{[i]}(r_{(j)}) = 1$ if $i = j$ and $\nu_{[i]}(r_{(j)}) = 0$ where $i < j$ for $1 \le i, j \le p - k$.

By Lemma 2.10 there exists an automorphism $\sigma : F \to F$ where $\sigma(r_{(i)}) = x_{[i]}$ for $1 \le i \le p - k$. For $\Gamma \in S_n$ as above we have

$$\begin{aligned}
(1) \qquad P &= \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_p\rangle \\
&\overset{\text{en}}{\sim} \langle x_{[1]}, \ldots, x_{[p-k]}, x_{[p-k+1]}, \ldots, x_{[n]} \mid r_{(1)}, \ldots, r_{(p-k)}, \ldots r_{(p)}\rangle \\
&\overset{\text{en}}{\sim} \langle x_{[1]}, \ldots, x_{[p-k]}, x_{[p-k+1]}, \ldots, x_{[n]} \\
&\qquad \mid x_{[1]}, \ldots, x_{[p-k]}, \sigma(r_{(p-k+1)}), \ldots, \sigma(r_{(p)})\rangle.
\end{aligned}$$

*Claim*:

$$(x_{[1]}, \ldots, x_{[p-k]}, \sigma(r_{(p-k+1)}), \ldots, \sigma(r_{(p)})) \overset{\text{en}}{\sim} (x_{[1]}, \ldots, x_{[p-k]}, r'_{(p-k+1)}, \ldots, r'_{(p)})$$

where $r'_{(i)} = r'_{(i)}(x_{[p-k+1]}, \ldots, x_{[n]})$ for $p - k + 1 \le i \le p$.

*Proof.* We argue by induction on the number of words $k$ in the set

$$\{\sigma(r_{(p-k+1)}), \ldots, \sigma(r_{(p)})\}.$$

If $k = 0$, then $\{r_1, r_2, \ldots, r_p\}$ is algebraically collapsible on $X$. Corollary 2.11 implies that Equation (1) is extended Nielsen equivalent to the presentation

$$\langle x_{[p+1]}, \ldots, x_{[n]} \mid -\rangle.$$

For $k > 0$, choose $\sigma(r_{(j)}) \in \{\sigma(r_{(p-k+1)}), \ldots, \sigma(r_{(p)})\}$ for some $j$, where $p-k+1 \le j \le p$. For some $\alpha \in \{1, \ldots, p-k\}$, suppose that $x_{[\alpha]}$ is the first occurrence in $\sigma(r_{(j)})$ of a member of $\{x_{[1]}, \ldots, x_{[p-k]}\}$. Then $\sigma(r_{(j)}) = ux_{[\alpha]}v$ where $u = u(x_{[p-k+1]}, \ldots, x_{[n]})$. From Equation (1) we obtain

$$\begin{aligned}
&(x_{[1]}, \ldots, x_{[\alpha]}, \ldots, x_{[p-k]}, \sigma(r_{(p-k+1)}), \ldots, \sigma(r_{(j)}), \ldots, \sigma(r_{(p)})) \\
&\overset{\text{en}}{\sim} (x_{[1]}, \ldots x_{[\alpha]}, \ldots, x_{[p-k]}, \sigma(r_{(p-k+1)}), \ldots, ux_{[\alpha]}v, \ldots, \sigma(r_{(p)})) \\
&\overset{\text{en}}{\sim} (x_{[1]}, \ldots x_{[\alpha]}, \ldots, x_{[p-k]}, \sigma(r_{(p-k+1)}), \ldots, x_{[\alpha]}vu, \ldots, \sigma(r_{(p)})) \\
&\overset{\text{en}}{\sim} (x_{[1]}, \ldots x_{[\alpha]}, \ldots, x_{[p-k]}, \sigma(r_{(p-k+1)}), \ldots, uv, \ldots, \sigma(r_{(p)})).
\end{aligned}$$

Then the number of occurrences of elements of $\{x_{[1]}, \ldots, x_{[p-k]}\}$ has been reduced by one in the word $\sigma(r_{(j)})$. Continuing for a finite number of such occurrences results in a word $r'_j = r'_j(x_{[p-k+1]}, \ldots, x_{[n]})$, which reduces $k$ by 1. The induction hypothesis then implies the existence of the extended Nielsen equivalent presentation,

$$\langle x_{[1]}, \ldots x_{[p-k]}, x_{[p-k+1]}, \ldots, x_{[n]} \mid x_{[1]}, \ldots, x_{[p-k]}, r'_{(p-k+1)}, \ldots, r'_{(p)}\rangle$$

which in turn is equivalent to

$$\overset{\text{en}}{\sim} \langle x_{[p-k+1]}, \ldots, x_{[n]} \mid r'_{(p-k+1)}, \ldots, r'_{(p)}\rangle$$

such that $\{r'_{(p-k+1)}, \dots, r'_{(p)}\}$ are words on $\{x_{[p-k+1]}, \dots, x_{[n]}\}$. □

## 3. Computation of the 2-handle presentation

We adopt the following terminology concerning minimal handle presentations [3].

**Definition 3.1.** Let $M^3$ be a 3-manifold and let $N$ be a 4-manifold with $\mathrm{Bd}(N) = 2M$. A *minimal handle structure for* $N$ (relative to the boundary $2M$) is a handle presentation for $N$ of the form

$$\mathcal{H} = h^0 \cup \bigcup_{i=1}^{\mathrm{en}(M)} h_i^1 \cup \bigcup_{j=1}^{q} h_j^2,$$

where:

(1) $\mathcal{H}$ has one 0-handle and $\mathrm{en}(M)$ 1-handles.
(2) If $K_{\mathcal{H}}$ is a 2-complex associated with $\mathcal{H}$, then $K_{\mathcal{H}}$ formally 3-deforms to $M_\star$.

We establish a partial result in support of the following conjecture:

**Conjecture 3.2** ([3]). *Let $M$ be a 3-manifold. Then there exists a 4-manifold $N$ with boundary $2M$, and there is a minimal handle presentation for $N$.*

Details concerning the manifold $M_1$ are discussed in the following section.

Recall that $\mathrm{en}(K)$ is the minimum number of generators on the presentation $P_K$ which is achievable by formal three deformations on $K$, whereas $\mathrm{en}(M^3)$ is the minimum number of generators in any 2-complex $L$ which 3-deforms to a 2-complex spine $K$ of $M^3$. Here, $\mathrm{en}(P_K) = \mathrm{en}(K) = \mathrm{en}(M^3)$. We consider the problem of reducing the number of 1-handles in $M_\star \times [-1, 1]$, to obtain a handle presentation of $M_\star \times [-1, 1]$ for which the number of 1-handles is strictly less than $\mathrm{hg}(M^3)$for one of a family of manifolds introduced by Boileau–Zieschang.

## 4. The manifolds of Boileau–Zieschang

Recall that a presentation $P$ for a three manifold group $\pi_1(M)$ is said to be *geometric* if there is a 2-spine $K$ of $M_\star$, so that $P$ is the presentation given by

$$P = P_K = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n \rangle$$

where $r_i$ reads the attaching map $\phi_i(\mathrm{Bd}(e_i^2)) \subset K$. The Heegaard genus of a 3-manifold $M^3$ is defined as the minimum number of 1-handles geometrically realizable in any Heegaard decomposition of $M^3$. For each 1-handle in a Heegaard decomposition, there is a free generator in a geometric presentation for $\pi_1(M)$. This implies that a lower bound for the number of 1-handles in $M^3$ in any Heegaard decomposition is given by $\mathrm{rk}(\pi_1(M))$. That is, if $M^3$ is a 3-manifold, then

$$\mathrm{rk}(\pi_1(M^3)) \le \mathrm{hg}((M)). \tag{2}$$

Boileau and Zieschang [2] exhibit a family of manifolds $\{M_i\}_{i=1}^{\infty}$ for which the inequality (2) is strict, that is

**Theorem 4.1** ([2]). *There exists a family of* 3*-manifolds* $\{M_i\}_{i=1}^{\infty}$ *such that for all* $i \ge 1$*,*

$$2 = \mathrm{rk}(\pi_1(M_i)) < \mathrm{hg}(M_i) = 3.$$

The proof of the theorem proceeds by exhibiting particular Heegaard decompositions of genus 3 and reducing the number of generators to 2 by algebraic techniques. A discussion of this occurs in Montesinos [10].

One member of this family will be denoted throughout the rest of this paper as $M_1$. A geometric presentation for $M_1$ is given by Montesinos as

$$P_{M_1} = \langle x_1, x_2, x_3 \mid x_3x_1x_3x_1^{-1}, x_2x_1^{-1}x_2x_1^{-3}, (x_3x_2x_1^{-1})^3(x_2x_1^{-1})^2\rangle. \tag{3}$$

Using the given presentation for $\pi_1(M_1)$, the following theorem [10] verifies that the extended Nielsen genus of $M_1^3$ is 2, so

$$2 = \mathrm{rk}(\pi_1(M_1)) = \mathrm{en}(M_1) < \mathrm{hg}(M_1) = 3. \tag{4}$$

The derivation following the statement of the next theorem is included for reference. It is referenced in Section 5 to calculate a handle presentation for $M_{1\star} \times [-1, 1]$ whose associated presentation is algebraically co-2-collapsible.

**Theorem 4.2** ([10]). *Let* $M_1^3$ *be the manifold of Boileau–Zieschang with presentation* $P_{M_1}$ *as given above. Then the extended Nielsen genus of* $P_{M_1}$ *is* 2.

**Proof.** Let $r_1 = x_3x_1x_3x_1^{-1}$ and $r_2 = x_2x_1^{-1}x_2x_1^{-3}$. Then,

$$\begin{aligned}
&(r_1, r_2, (x_3x_2x_1^{-1})^3(x_2x_1^{-1})^2) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, (x_3x_2x_1^{-1})^3(x_2x_1^{-1})^2(x_2x_1^{-1})^{-2}x_1^2) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_1^{-1}x_3(x_3^{-1}x_1x_3^{-1}x_1^{-1})x_2x_1^{-1}x_3x_2x_1) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_3^{-1}x_1^{-1}x_2(x_2^{-1}x_1x_2^{-1}x_1^3)x_1^{-1}x_3x_2x_1) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_3^{-1}x_2^{-1}x_1^2x_3(x_3^{-1}x_1^{-1}x_3^{-1}x_1)x_2x_1) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_3^{-1}x_2^{-1}x_1x_3^{-1}(x_3x_1^{-1}x_3x_1)x_1x_2x_1) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_3^{-1}x_2^{-1}x_3x_1x_1(x_1x_2^{-1}x_1^3x_2^{-1})x_2x_1) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_3^{-1}x_2^{-1}x_3x_1^3x_2{}^{-1}(x_2x_1^{-3}x_2x_1^{-1})x_1^4) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_3x_2x_3^{-1}x_2^{-1}x_3x_2x_1^2(x_1^{-2}x_2x_1^{-1}x_2x_1^{-1})x_1) \\
&\quad\overset{\text{en}}{\sim} (r_1, r_2, x_1^{-1}(x_2x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2)).
\end{aligned}$$

Substituting into $P_{M_1}$, we obtain

$$\begin{aligned} P_{M_1} &= \langle x_1, x_2, x_3 \mid x_3x_1x_3x_1^{-1}, x_2x_1^{-1}x_2x_1^{-3}, (x_3x_2x_1^{-1})^3(x_2x_1^{-1})^2\rangle \\ &\overset{\text{en}}{\sim} \langle x_1, x_2, x_3 \mid x_3x_1x_3x_1^{-1}, x_2x_1^{-1}x_2x_1^{-3}, x_1^{-1}(x_2x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2)^{-1}\rangle \\ &\overset{\text{en}}{\sim} \langle x_2, x_3 \mid x_3(x_2x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2)^{-1}x_3(x_2x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2), \\ &\qquad x_2(x_2x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2)x_2(x_2x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2)^3\rangle. \end{aligned}$$

So, the extended Nielsen genus of $P_{M_1}$ is at most 2. Since the genus must be at least the rank of the group, it must be equal to 2. □

When inequality (2) is strict for a manifold $M^3$, it follows that no cell decomposition of $M^3$ can result in a 2-spine $K$ having exactly one 0-cell, $\mathrm{rk}(\pi_1(M^3))$ 1-cells and $\mathrm{rk}(\pi_1(M^3))$ 2-cells. From such a spine, a Heegaard decomposition could be constructed with genus $\mathrm{rk}(\pi_1(M^3))$.

Thus, for the Boileau–Zieschang manifold $M_1^3$, Theorem 4.2 yields that

$$\mathrm{en}(M_1^3) = 2 < \mathrm{hg}(M_1^3)$$

so that no handle decomposition consisting of exactly one 0-handle, two 1-handles, two 2-handles and one 3-handle exists.

## 5. A handle presentation for a 4-manifold bounded by $2M_1$

To obtain information about minimal handle structures for 4-manifolds $N$ bounded by $2M$, we examine handle decompositions of $M_{1\star} \times [-1, 1]$ using a handle calculus for handle presentations with no handles of index greater than 2.

A handle presentation $\mathcal{H}$ is *normal* if the attaching spheres for the 2-handles are contained in $(\partial J) \times \{-0.75, 0.75\}$.

See Craggs [3] for a treatment of *normal* handle presentations, and Craggs [4, 5] for material concerning algebraic cancellation, linking obstructions and the free reduction problems.

**Definition 5.1.** A normal handle presentation $\mathcal{H}$ for a 4-manifold $N$ with boundary $2M_*$ is *algebraically minimal* (relative to the boundary $2M_*$) provided:

(1) The handle presentation $\mathcal{H}$ has no handles of index greater than 2.
(2) All but $\mathrm{en}(M_\star)$ of the 1-handles can be canceled algebraically.
(3) If $K_{\mathcal{H}}$ is a 2-complex naturally associated with $\mathcal{H}$, then $M_* \overset{3}{\nearrow\!\!\searrow} K_{\mathcal{H}}$.

Note that if $\mathcal{H}$ is a handle presentation for $M_{1^\star} \times [-1, 1]$ which is algebraically minimal, then in the absence of any linking obstructions, Theorem A, Craggs [5] implies that $\mathcal{H}$ is reducible to a minimal handle structure for $M_1$.

The remainder of this section is devoted to establishing an explicit description of an algebraically minimal handle presentation.

**Theorem 5.2.** *There exists an algebraically minimal normal handle presentation $\mathcal{H}$ for a 4-manifold $N$ with boundary $2M_1$.*

We calculate a handle presentation $\mathcal{H}$ whose associated presentation is algebraically co-2-collapsible. This will imply that $\mathcal{H}$ is an algebraically minimal handle presentation.

Unless stated otherwise, all handle presentations for $M_{1\star} \times [-1, 1]$ are assumed to have handles of index at most two.

We introduce a sequence of admissible operations that will be used extensively in what follows. Suppose that $(D, g)$ is a singular system with members including $D_i$ and $D_j$. If a push is performed on $D_i$ along an arc $\beta$ which encounters $D_j$ the resulting system may be modeled by an appropriately chosen admissible system. Figure 2 illustrates one such possibility. Here, the arc $\beta$, and the relator paths $r_k = \mathrm{Bd}(D_k) \cap \mathrm{Bd}(J)$ for $k = i, j$ are illustrated.

To describe the corresponding operations as an admissible sequence of operations, we introduce the following notation: Let $(r_i, \epsilon_i)$ denote the relator curve within an admissible system $(D, g, \epsilon)$, that is, let

$$(r_i, \epsilon_i) = \mathrm{Bd}(g(D_i, \epsilon_i)) \cap \mathrm{Bd}(J)$$

where $\epsilon_i \in \{-1, 0, 1\}$ is the label associated with $g(D_i)$. Let $(r_i, \epsilon_i) \to (r_i, \epsilon_i')$ denote a change of label corresponding to a level change, and denote an admissible slide of $g(D_i, \epsilon_i)$ over $g(D_j, \epsilon_j)$ by the notation $(r_i, \epsilon_i) \curvearrowright (r_j, \epsilon_j)$.

The next lemma states that the configuration of Figure 2 may be obtained entirely within an admissible context.

**Lemma 5.3.** *Suppose that $A$ is an admissible system having $(r_i, \epsilon_i)$ and $(r_j, \epsilon_j)$ as relators. Let $\beta$ be an arc joining a point of $(r_i, \epsilon_i)$ with a point of $(r_j, \epsilon_j)$ and let $N$ be a regular neighborhood of $\beta$ in $\mathrm{Bd}(J)$ which fails to intersect the other arcs of the system. Then there exists an admissible system $A'$ and a sequence of admissible operations taking $A$ to $A'$ which result in the configuration given by Figure 2.*

**Proof.** The result consists of calculating a suitable sequence of admissible operations. Let $(r_i, \epsilon_i)$, $(r_j, \epsilon_j)$ and $\beta$ be given for some admissible system $A$.

Begin by stabilizing as indicated in Figure 3 to obtain relators $(r_k, 0)$ and $(r_l, 0)$ on the generators $y_1$ and $y_2$ respectively. Then the label changes

$$(r_l, 0) \to (r_l, \epsilon_j), \quad (r_k, 0) \to (r_k, \epsilon_i)$$

allow the nonsingular slides,

$$(r_j, \epsilon_j) \curvearrowright (r_l, \epsilon_j), \quad (r_i, \epsilon_i) \curvearrowright (r_k, \epsilon_i).$$

Since the sliding operations leave their respective targets nonsingular, we may adjust labels again resulting in the pair

$$(r_l, \epsilon_j) \to (r_l, 0),\ (r_k, \epsilon_i) \to (r_k, 0).$$

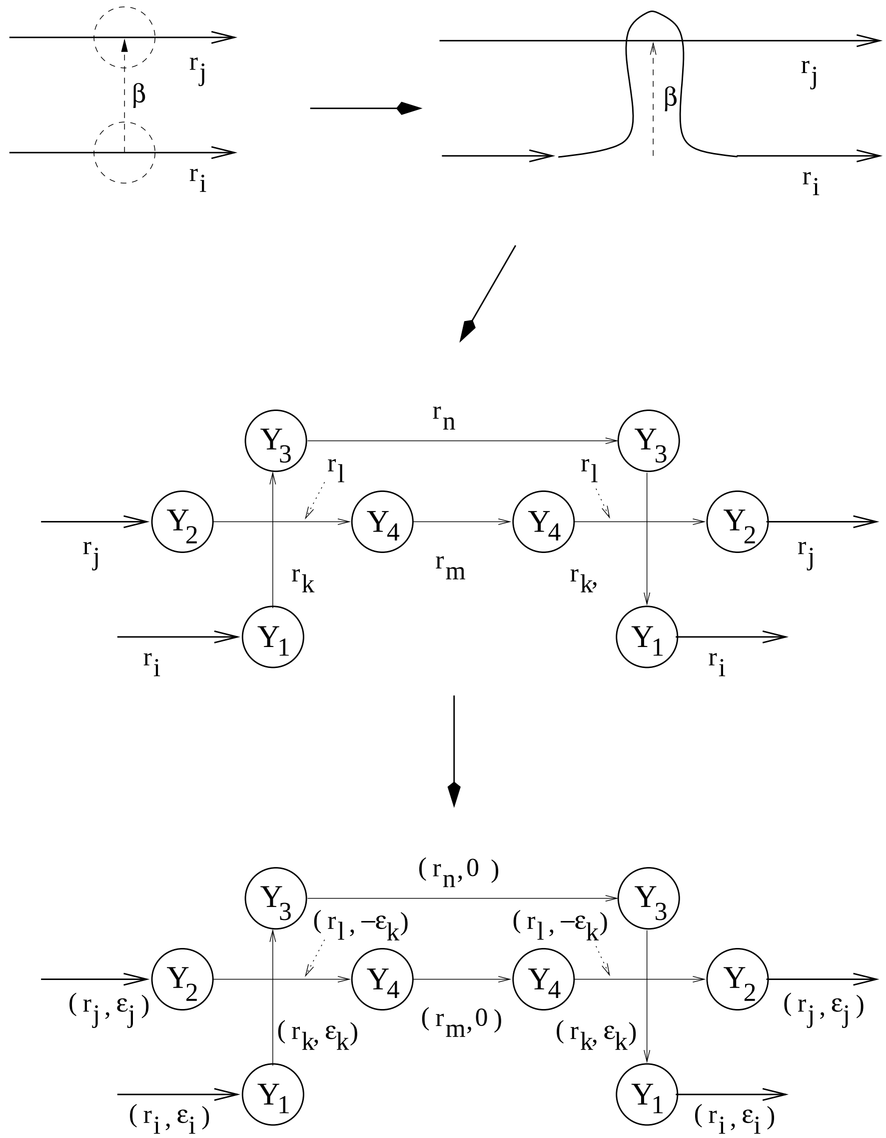


FIGURE 2. A singular push as an admissible system.

This situation forms the basis for Step 2, indicated in the upper right hand corner of Figure 3.

Stabilizing again, we obtain relators $(r_n, 0)$ and $(r_m, 0)$ on the generating symbols $y_3$ and $y_4$. Set $(r_k, 0) \to (r_k, \epsilon_k)$, where $\epsilon_k \neq 0$ is some choice of label, and set $(r_l, 0) \to (r_l, -\epsilon_k)$, performing a feeler push along the arc $\beta$.

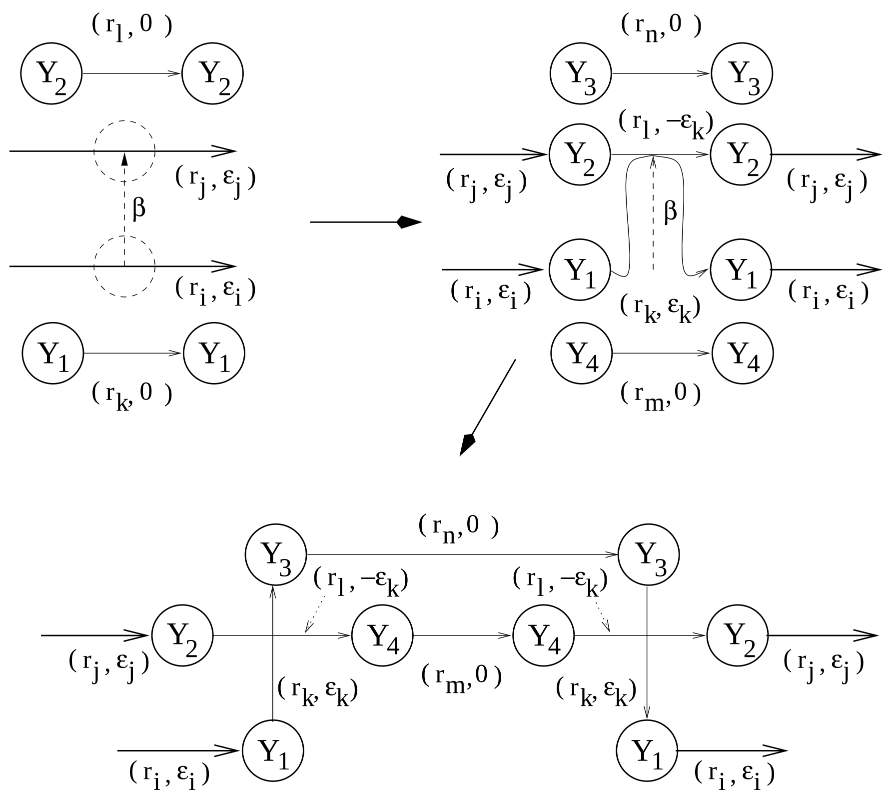


FIGURE 3. Admissible operations to obtain a singular feeler push.

This allows the following sequence of admissible moves:

$$(r_n, 0) \to (r_n, \epsilon_k),\ (r_m, 0) \to (r_m, -\epsilon_k)$$
$$(r_k, \epsilon_k) \curvearrowright (r_n, \epsilon_k), (r_l, -\epsilon_k) \curvearrowright (r_m, -\epsilon_k)$$
$$(r_n, \epsilon_k) \to (r_n, 0),\ (r_m, -\epsilon_k) \to (r_m, 0).$$

The resulting configuration is illustrated in the lower section of Figure 3 which is the desired result. □

**Lemma 5.4.** *There exists an admissible system $A^1$, having the following properties:*

(1) *$A^1$ is admissibly equivalent to the geometric presentation $P_{M_1}$.*
(2) *$P_{A^1} = \langle x_1, x_2, x_3, y_1, \ldots, y_{92} \mid r_1, r_2, \ldots, r_{95}\rangle$, where $r_i$ is given in Table* 1.
(3) *$P_{A^1}$ is extended Nielsen equivalent to $P(M_1)$.*

**Proof.** Section 6 contains a derivation of the admissible system $A_1$. Geometric readings are presented at intermediate stages ending in an explicit list of the relators of the corresponding presentation $P_{A^1}$. Table 1 represents the reduced form of this final entry. Each admissible operation induces an extended Nielsen transformation of the original presentation $P_{M_1}$. Therefore, the third part of the lemma follows immediately. □

Table 1: $P_{A^1}$ with relators $\{r_1, r_2, \ldots, r_{95}\}$.

$$
\begin{aligned}
r_1^0 &= x_3\,x_1\,x_3\,x_{49}\,x_1^{-1}\\
r_2^0 &= x_2\,x_1^{-1}\,x_2\,x_1^{-3}\\
r_3^0 &= y_{10}\,y_8\,y_1\,y_5\,y_4\,y_{17}\\
r_4^+ &= y_{29}\,y_{37}^{-1}\,y_3\,y_{57}\,y_{73}^{-1}\,y_1^{-1}\\
r_5^- &= y_4\,y_2^{-1}\\
r_6^0 &= y_9^{-1}\,y_2\\
r_7^0 &= y_{16}^{-1}\,y_3^{-1}\\
r_8^+ &= y_6^{-1}\,y_{18}^{-1}\,y_8\\
r_9^- &= y_{22}^{-1}\,y_7^{-1}\,y_5\,y_{33}^{-1}\\
r_{10}^0 &= y_6\\
r_9^- &= y_{22}^{-1}\,y_7^{-1}\,y_5\,y_{33}^{-1}\\
r_{10}^0 &= y_6\\
r_{11}^0 &= y_7^{-1}\\
r_{12}^+ &= y_{11}^{-1}\,y_9\\
r_{13}^- &= y_{77}\,y_{53}^{-1}\,y_{12}^{-1}\,x_3^{-1}\,y_{41}\,y_{25}^{-1}\,y_{10}\\
r_{14}^0 &= y_{12}\\
r_{15}^0 &= y_{15}^{-1}\,y_{11}^{-1}\\
r_{16}^+ &= y_{13}^{-1}\,y_{15}\\
r_{17}^- &= y_{45}^{-1}\,x_3^{-1}\,y_{14}^{-1}\,y_{16}\\
r_{18}^0 &= y_{14}\\
r_{19}^0 &= x_3\,y_{13}\\
r_{20}^+ &= y_{26}\,y_{19}^{-1}\,y_{17}\\
r_{21}^- &= x_2^{-1}\,y_{20}^{-1}\,y_{61}\,y_{69}^{-1}\,y_{18}\\
r_{22}^0 &= y_{20}\\
r_{23}^0 &= y_{65}^{-1}\,y_{85}\,y_{21}\,y_{19}^{-1}\\
r_{24}^+ &= y_{23}^{-1}\,y_{21}\\
r_{25}^- &= y_{24}^{-1}\,x_2\,y_{89}\,y_{81}^{-1}\,y_{22}\\
r_{26}^0 &= y_{24}\\
r_{27}^0 &= x_2\,y_{23}^{-1}\\
r_{28}^+ &= y_{28}^{-1}\,y_{26}\\
r_{29}^- &= y_{27}^{-1}\,y_{25}\\
r_{30}^0 &= y_{27}\\
r_{31}^0 &= y_{30}^{-1}\,y_{28}\\
r_{32}^+ &= y_{32}^{-1}\,y_{30}\\
r_{33}^- &= y_{31}^{-1}\,y_{29}\\
r_{34}^0 &= y_{31}
\end{aligned}
$$

*Continued on next page*

Table 1 (*continued*)

$$
\begin{aligned}
r^0_{35} &= y^{-1}_{34}\, y_{32} \\
r^+_{36} &= y^{-1}_{36}\, y_{34} \\
r^-_{37} &= y^{-1}_{35}\, y_{33} \\
r^0_{38} &= y_{35} \\
r^0_{39} &= y^{-1}_{38}\, y_{36} \\
r^+_{40} &= y^{-1}_{40}\, y_{38} \\
r^-_{41} &= y^{-1}_{39}\, y_{37} \\
r^0_{42} &= y_{39} \\
r^0_{43} &= y^{-1}_{42}\, y_{40} \\
r^+_{44} &= y^{-1}_{44}\, y_{42} \\
r^-_{45} &= y^{-1}_{43}\, y_{41} \\
r^0_{46} &= y_{43} \\
r^0_{47} &= y^{-1}_{46}\, y_{44} \\
r^+_{48} &= y^{-1}_{48}\, y_{46} \\
r^-_{49} &= y^{-1}_{47}\, y_{45} \\
r^0_{50} &= y_{47} \\
r^0_{51} &= y^{-1}_{50}\, y_{48} \\
r^+_{52} &= y^{-1}_{52}\, y_{50} \\
r^-_{53} &= y^{-1}_{51}\, y_{49} \\
r^0_{54} &= y_{51} \\
r^0_{55} &= y^{-1}_{54}\, y_{52} \\
r^+_{56} &= y^{-1}_{56}\, y_{54} \\
r^-_{57} &= y^{-1}_{55}\, y_{53} \\
r^0_{58} &= y_{55} \\
r^0_{59} &= y^{-1}_{58}\, y_{56} \\
r^+_{60} &= y^{-1}_{60}\, y_{58} \\
r^-_{61} &= y^{-1}_{59}\, y_{57} \\
r^0_{62} &= y_{59} \\
r^0_{63} &= y^{-1}_{62}\, y_{60} \\
r^+_{64} &= y^{-1}_{64}\, y_{62} \\
r^-_{65} &= y^{-1}_{63}\, y_{61} \\
r^0_{66} &= y_{63} \\
r^0_{67} &= y^{-1}_{66}\, y_{64} \\
r^+_{68} &= y^{-1}_{68}\, y_{66} \\
r^-_{69} &= y^{-1}_{67}\, y_{65} \\
r^0_{70} &= y_{67} \\
r^0_{71} &= y^{-1}_{70}\, y_{68} \\
r^+_{72} &= y^{-1}_{72}\, y_{70} \\
r^-_{73} &= y^{-1}_{71}\, y_{69} \\
r^0_{74} &= y_{71} \\
r^0_{75} &= y^{-1}_{74}\, y_{72}
\end{aligned}
$$

*Continued on next page*

Table 1 (*continued*)

$$
\begin{array}{rcl}
r^{+}_{76} &=& y^{-1}_{76}\, y_{74} \\
r^{-}_{77} &=& y^{-1}_{75}\, y_{73} \\
r^{0}_{78} &=& y_{75} \\
r^{0}_{79} &=& y^{-1}_{78}\, y_{76} \\
r^{+}_{80} &=& y^{-1}_{80}\, y_{78} \\
r^{-}_{81} &=& y^{-1}_{79}\, y_{77} \\
r^{0}_{82} &=& y_{79} \\
r^{0}_{83} &=& y^{-1}_{82}\, y_{80} \\
r^{+}_{84} &=& y^{-1}_{84}\, y_{82} \\
r^{-}_{85} &=& y^{-1}_{83}\, y_{81} \\
r^{0}_{86} &=& y_{83} \\
r^{0}_{87} &=& y^{-1}_{86}\, y_{84} \\
r^{+}_{88} &=& y^{-1}_{88}\, y_{86} \\
r^{-}_{89} &=& y^{-1}_{87}\, y_{85} \\
r^{0}_{90} &=& y_{87} \\
r^{0}_{91} &=& y^{-1}_{90}\, y_{88} \\
r^{+}_{92} &=& y^{-1}_{92}\, y_{90} \\
r^{-}_{93} &=& y^{-1}_{91}\, y_{89} \\
r^{0}_{94} &=& y_{91} \\
r^{0}_{95} &=& x_2\, x^{-1}_1\, x_2\, y_{92}
\end{array}
$$

**Lemma 5.5.** *The presentation $P_{A^1(M_1)}$ presents an algebraically co-2-collapsible complex.*

**Proof.** Given $P_{A^1(M_1)}$ as presented in Lemma 5.4, we claim that the subset $\{r_3, \dots, r_{95}\}$ is algebraically collapsible on $\{x_1, x_2, x_3, y_1, \dots, y_{92}\}$.

To see this, examine Table 2 which presents the relators from Table 1 according to the following convention: The general entry,

$$(i)\, r^{\epsilon_{(i)}}_{(i)} = u_{(i)} x_{[i]} v_{(i)} \qquad [x_{[i]}]$$

corresponds to the permutations $\Gamma \in S_{95}$ and $\Delta \in S_{93}$ so $\Delta(i) = (i)$, $\Gamma(i) = [i]$. In addition, inspection of Table 2 demonstrates that $u_i = u_i(x_{[i+1]}, \dots, x_{[95]})$ and $v_i = v_i(x_{[i+1]}, \dots, x_{[95]})$ for all $i = 1, \dots, 93$. Lemma 2.14 then directly implies that $\{r_1, \dots, r_{95}\}$ forms an algebraically co-2-collapsible set on $\{x_1, x_2, x_3, y_1, \dots, y_{95}\}$. □

**Theorem 5.2.** Given $M_{1\star} \times [-1, 1]$, Lemma 5.4 implies that there exists an admissible system $A^1$ representing $M_{1\star} \times [-1, 1]$ whose presentation is given by $P_{A^1} = \langle x_1, x_2, x_3, y_1, \dots, y_{92} \mid r_1, r_2, \dots, r_{95} \rangle$. By Theorem 4.2, the extended Nielsen genus of $P_{M_1}$ is 2. Therefore, Lemma 5.5 implies that $P_{A^1}$ presents an algebraically co-en$(M_1)$-collapsible presentation.

From Lemma 2.15 there is an automorphism $\sigma : F \to F$ where $\sigma(r_{(i)}) = x_{[i]}$, where $x_{[i]}$ is an element of the ordered collection $(x_1, x_2, x_3, y_1, \dots, y_{92})$

for $1 \leq i \leq 93$, so that

$$\begin{aligned} P_{A^1} &= \langle x_1, x_2, x_3, y_1, \ldots, y_{92} \mid r_1, r_2, \ldots r_{95} \rangle \\ &\overset{\text{en}}{\sim} \langle x_{[1]}, \ldots x_{[95]} \mid \sigma(r_1), \sigma(r_2), \sigma(r_{(1)}), \ldots, \sigma(r_{(93)}) \rangle \\ &\overset{\text{en}}{\sim} \langle x_{[1]}, \ldots x_{[95]} \mid r_1', r_2', x_{[1]}, \ldots, x_{[93]} \rangle. \end{aligned}$$

There exists an admissible system $A^2$ for $M_{1\star} \times [-1, 1]$ and a sequence of admissible systems which take $A^1$ to $A^2$ having a presentation

$$P_{A^2} = \langle x_{[1]}, \ldots x_{[95]} \mid r_1', r_2', x_{[1]}, \ldots, x_{[93]} \rangle.$$

Table 2: Sequence of collapses in order by relator number and generator.

| | | | | |
|---|---|---|---|---|
| (1) | $r^0_{95}$ | = | $x_2\, x_1^{-1}\, x_2\, y_{92}$ | $[x_1]$ |
| (2) | $r^+_{92}$ | = | $y_{92}^{-1}\, y_{90}$ | $[y_{92}]$ |
| (3) | $r^0_{91}$ | = | $y_{90}^{-1}\, y_{88}$ | $[y_{90}]$ |
| (4) | $r^+_{88}$ | = | $y_{88}^{-1}\, y_{86}$ | $[y_{88}]$ |
| (5) | $r^0_{87}$ | = | $y_{86}^{-1}\, y_{84}$ | $[y_{86}]$ |
| (6) | $r^+_{84}$ | = | $y_{84}^{-1}\, y_{82}$ | $[y_{84}]$ |
| (7) | $r^0_{83}$ | = | $y_{82}^{-1}\, y_{80}$ | $[y_{82}]$ |
| (8) | $r^+_{80}$ | = | $y_{80}^{-1}\, y_{78}$ | $[y_{80}]$ |
| (9) | $r^0_{79}$ | = | $y_{78}^{-1}\, y_{76}$ | $[y_{78}]$ |
| (10) | $r^+_{76}$ | = | $y_{76}^{-1}\, y_{74}$ | $[y_{76}]$ |
| (11) | $r^0_{75}$ | = | $y_{74}^{-1}\, y_{72}$ | $[y_{74}]$ |
| (12) | $r^+_{72}$ | = | $y_{72}^{-1}\, y_{70}$ | $[y_{72}]$ |
| (13) | $r^0_{71}$ | = | $y_{70}^{-1}\, y_{68}$ | $[y_{70}]$ |
| (14) | $r^+_{68}$ | = | $y_{68}^{-1}\, y_{66}$ | $[y_{68}]$ |
| (15) | $r^0_{67}$ | = | $y_{66}^{-1}\, y_{64}$ | $[y_{66}]$ |
| (16) | $r^+_{64}$ | = | $y_{64}^{-1}\, y_{62}$ | $[y_{64}]$ |
| (17) | $r^0_{63}$ | = | $y_{62}^{-1}\, y_{60}$ | $[y_{62}]$ |
| (18) | $r^+_{60}$ | = | $y_{60}^{-1}\, y_{58}$ | $[y_{60}]$ |
| (19) | $r^0_{59}$ | = | $y_{58}^{-1}\, y_{56}$ | $[y_{58}]$ |
| (20) | $r^+_{56}$ | = | $y_{56}^{-1}\, y_{54}$ | $[y_{56}]$ |
| (21) | $r^0_{55}$ | = | $y_{54}^{-1}\, y_{52}$ | $[y_{54}]$ |
| (22) | $r^+_{52}$ | = | $y_{52}^{-1}\, y_{50}$ | $[y_{52}]$ |
| (23) | $r^0_{51}$ | = | $y_{50}^{-1}\, y_{48}$ | $[y_{50}]$ |
| (24) | $r^+_{48}$ | = | $y_{48}^{-1}\, y_{46}$ | $[y_{48}]$ |
| (25) | $r^0_{47}$ | = | $y_{46}^{-1}\, y_{44}$ | $[y_{46}]$ |
| (26) | $r^+_{44}$ | = | $y_{44}^{-1}\, y_{42}$ | $[y_{44}]$ |
| (27) | $r^0_{43}$ | = | $y_{42}^{-1}\, y_{40}$ | $[y_{42}]$ |
| (28) | $r^+_{40}$ | = | $y_{40}^{-1}\, y_{38}$ | $[y_{40}]$ |
| (29) | $r^0_{39}$ | = | $y_{38}^{-1}\, y_{36}$ | $[y_{38}]$ |

*Continued on next page*

Table 2 (*continued*)

| | | | | |
|---|---|---|---|---|
| (30) | $r^+_{36}$ | $=$ | $y^{-1}_{36}\, y_{34}$ | $[y_{36}]$ |
| (31) | $r^0_{35}$ | $=$ | $y^{-1}_{34}\, y_{32}$ | $[y_{34}]$ |
| (32) | $r^+_{32}$ | $=$ | $y^{-1}_{32}\, y_{30}$ | $[y_{32}]$ |
| (33) | $r^0_{31}$ | $=$ | $y^{-1}_{30}\, y_{28}$ | $[y_{30}]$ |
| (34) | $r^+_{28}$ | $=$ | $y^{-1}_{28}\, y_{26}$ | $[y_{28}]$ |
| (35) | $r^+_{20}$ | $=$ | $y_{26}\, y^{-1}_{19}\, y_{17}$ | $[y_{26}]$ |
| (36) | $r^0_{3}$ | $=$ | $y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}$ | $[y_{17}]$ |
| (37) | $r^-_{13}$ | $=$ | $y_{77}\, y^{-1}_{53}\, y^{-1}_{12}\, x^{-1}_3\, y_{41}\, y^{-1}_{25}\, y_{10}$ | $[y_{10}]$ |
| (38) | $r^-_{29}$ | $=$ | $y^{-1}_{27}\, y_{25}$ | $[y_{25}]$ |
| (39) | $r^0_{30}$ | $=$ | $y_{27}$ | $[y_{27}]$ |
| (40) | $r^-_{45}$ | $=$ | $y^{-1}_{43}\, y_{41}$ | $[y_{41}]$ |
| (41) | $r^0_{46}$ | $=$ | $y_{43}$ | $[y_{43}]$ |
| (42) | $r^0_{14}$ | $=$ | $y_{12}$ | $[y_{12}]$ |
| (43) | $r^-_{57}$ | $=$ | $y^{-1}_{55}\, y_{53}$ | $[y_{53}]$ |
| (44) | $r^0_{58}$ | $=$ | $y_{55}$ | $[y_{55}]$ |
| (45) | $r^-_{81}$ | $=$ | $y^{-1}_{79}\, y_{77}$ | $[y_{77}]$ |
| (46) | $r^0_{82}$ | $=$ | $y_{79}$ | $[y_{79}]$ |
| (47) | $r^+_{8}$ | $=$ | $y^{-1}_{6}\, y^{-1}_{18}\, y_8$ | $[y_8]$ |
| (48) | $r^-_{21}$ | $=$ | $x^{-1}_2\, y^{-1}_{20}\, y_{61}\, y^{-1}_{69}\, y_{18}$ | $[y_{18}]$ |
| (49) | $r^-_{73}$ | $=$ | $y^{-1}_{71}\, y_{69}$ | $[y_{69}]$ |
| (50) | $r^0_{74}$ | $=$ | $y_{71}$ | $[y_{71}]$ |
| (51) | $r^-_{65}$ | $=$ | $y^{-1}_{63}\, y_{61}$ | $[y_{61}]$ |
| (52) | $r^0_{66}$ | $=$ | $y_{63}$ | $[y_{63}]$ |
| (53) | $r^0_{22}$ | $=$ | $y_{20}$ | $[y_{20}]$ |
| (54) | $r^0_{10}$ | $=$ | $y_6$ | $[y_6]$ |
| (55) | $r^+_{4}$ | $=$ | $y_{29}\, y^{-1}_{37}\, y_3\, y_{57}\, y^{-1}_{73}\, y^{-1}_1$ | $[y_1]$ |
| (56) | $r^-_{33}$ | $=$ | $y^{-1}_{31}\, y_{29}$ | $[y_{29}]$ |
| (57) | $r^0_{34}$ | $=$ | $y_{31}$ | $[y_{31}]$ |
| (58) | $r^-_{41}$ | $=$ | $y^{-1}_{39}\, y_{37}$ | $[y_{37}]$ |
| (59) | $r^0_{42}$ | $=$ | $y_{39}$ | $[y_{39}]$ |
| (60) | $r^0_{7}$ | $=$ | $y^{-1}_{16}\, y^{-1}_3$ | $[y_3]$ |
| (61) | $r^-_{17}$ | $=$ | $y^{-1}_{45}\, x^{-1}_3\, y^{-1}_{14}\, y_{16}$ | $[y_{16}]$ |
| (62) | $r^0_{18}$ | $=$ | $y_{14}$ | $[y_{14}]$ |
| (63) | $r^-_{49}$ | $=$ | $y^{-1}_{47}\, y_{45}$ | $[y_{45}]$ |
| (64) | $r^0_{50}$ | $=$ | $y_{47}$ | $[y_{47}]$ |
| (65) | $r^-_{61}$ | $=$ | $y^{-1}_{59}\, y_{57}$ | $[y_{57}]$ |
| (66) | $r^0_{62}$ | $=$ | $y_{59}$ | $[y_{59}]$ |
| (67) | $r^-_{77}$ | $=$ | $y^{-1}_{75}\, y_{73}$ | $[y_{73}]$ |
| (68) | $r^0_{78}$ | $=$ | $y_{75}$ | $[y_{75}]$ |
| (69) | $r^-_{9}$ | $=$ | $y^{-1}_{22}\, y^{-1}_7\, y_5\, y^{-1}_{33}$ | $[y_5]$ |
| (70) | $r^-_{25}$ | $=$ | $y^{-1}_{24}\, x_2\, y_{89}\, y^{-1}_{81}\, y_{22}$ | $[y_{22}]$ |

*Continued on next page*

Table 2 (*continued*)

| | | | | |
|---|---|---|---|---|
| (71) | $r^0_{26}$ | $=$ | $y_{24}$ | $[y_{24}]$ |
| (72) | $r^-_{93}$ | $=$ | $y_{91}^{-1}\, y_{89}$ | $[y_{89}]$ |
| (73) | $r^0_{94}$ | $=$ | $y_{91}$ | $[y_{91}]$ |
| (74) | $r^-_{85}$ | $=$ | $y_{83}^{-1}\, y_{81}$ | $[y_{81}]$ |
| (75) | $r^0_{86}$ | $=$ | $y_{83}$ | $[y_{83}]$ |
| (76) | $r^0_{11}$ | $=$ | $y_7^{-1}$ | $[y_7]$ |
| (77) | $r^-_{37}$ | $=$ | $y_{35}^{-1}\, y_{33}$ | $[y_{33}]$ |
| (78) | $r^0_{38}$ | $=$ | $y_{35}$ | $[y_{35}]$ |
| (79) | $r^-_5$ | $=$ | $y_4\, y_2^{-1}$ | $[y_4]$ |
| (80) | $r^0_6$ | $=$ | $y_9^{-1}\, y_2$ | $[y_2]$ |
| (81) | $r^+_{12}$ | $=$ | $y_{11}^{-1}\, y_9$ | $[y_9]$ |
| (82) | $r^0_{15}$ | $=$ | $y_{15}^{-1}\, y_{11}^{-1}$ | $[y_{11}]$ |
| (83) | $r^+_{16}$ | $=$ | $y_{13}^{-1}\, y_{15}$ | $[y_{15}]$ |
| (84) | $r^0_{19}$ | $=$ | $x_3\, y_{13}$ | $[y_{13}]$ |
| (85) | $r^0_{23}$ | $=$ | $y_{65}^{-1}\, y_{85}\, y_{21}\, y_{19}^{-1}$ | $[y_{19}]$ |
| (86) | $r^-_{69}$ | $=$ | $y_{67}^{-1}\, y_{65}$ | $[y_{65}]$ |
| (87) | $r^0_{70}$ | $=$ | $y_{67}$ | $[y_{67}]$ |
| (88) | $r^-_{89}$ | $=$ | $y_{87}^{-1}\, y_{85}$ | $[y_{85}]$ |
| (89) | $r^0_{90}$ | $=$ | $y_{87}$ | $[y_{87}]$ |
| (90) | $r^+_{24}$ | $=$ | $y_{23}^{-1}\, y_{21}$ | $[y_{21}]$ |
| (91) | $r^0_{27}$ | $=$ | $x_2\, y_{23}^{-1}$ | $[y_{23}]$ |
| (92) | $r^-_{53}$ | $=$ | $y_{51}^{-1}\, y_{49}$ | $[y_{49}]$ |
| (93) | $r^0_{54}$ | $=$ | $y_{51}$ | $[y_{51}]$ |

Let $\mathcal{H}$ be the handle presentation for $M_{1\star} \times [-1, 1]$ whose associated presentation is given by $P_{A^2}$. Then with $2 = \mathrm{en}(P_{A^1}) = \mathrm{en}(P_{A^2})$, and for $3 \le j \le 95$ where $r_j = r_{(i-2)}$, $r_j$ freely reduces to $x_{[i]}$ after the change of basis, so that $\mathcal{H}$ is algebraically minimal. □

## 6. Derivation of an admissible system for $M_1$

This section details a calculation of a 2-complex spine and corresponding 2-handle presentation for $M_{1\star} \times [-1, 1]$ following the derivation of Montesinos presented in Theorem 4.2. The calculation consists of generating a series of admissible systems to produce a 2-complex whose associated presentation is algebraically co-2-collapsible. Plate B1 shows the 2-spine presentation given by the first equation of Theorem 4.2, and commences by performing a nonsingular slide. The corresponding reading is recorded below it.

The diagrams which follow Plate B1 represent the effect of the projection maps

$$p_+ \;:\; J \times \{+1\} \to J \times \{0\} \quad \text{and}$$
$$p_- \;:\; J \times \{-1\} \to J \times \{0\}$$

and are presented using an admissible representation. A high resolution collection of plates is available at [7] in addition to those presented here.

Each diagram is accompanied by a table at each stage of the calculation which corresponds to the relators of the complex whose presentation is given by

$$P_n = \langle x_1, x_2, x_3, y_1, y_2, \dots, y_n \mid r_1, r_2, \dots r_{n+3}\rangle.$$

where $r_1 = x_3x_1x_3x_1^{-1}$ and $r_2 = x_2x_1x_2x_1^{-3}$are the relators of $P_{M_1}$ as given in Equation (3). The generating symbols corresponding to the 1-handles are taken from the set $\{x_1, x_2, x_3, y_1, y_2, \dots\}$, where the generators $\{x_1, x_2, x_3\}$ correspond to the generators of $\pi_1(M_1)$ and $\{y_1, y_2, \dots\}$ are introduced by repeated stabilizations as in Lemma 5.3.

To convey the information associated with the admissible system at each stage, we adopt the following notational conventions:

(1) If $r_i$ corresponds to a 2-handle attachment in $\dot{J} \times [\frac{1}{2}, 1]$, it will be recorded as $r_i^+$. Similarly, a 2-handle attachment in $\dot{J} \times [-1, -\frac{1}{2}]$ will be recorded as $r_i^-$ and those nonsingular members of the disk system will be denoted as $r_i^0$. In terms of the admissible disk system structure this implies that $r_i^+ = (r_i, +1)$, $r_i^- = (r_i, -1)$, and $r_i^0 = (r_i, 0)$.
(2) The basepoint of each relator curve $r_i$ is denoted as $\star_i$. This symbol is located near the line segment denoting the starting position of the associated reading (the *initial* segment of $r_i$).
(3) For noninitial segments, the $m$th line segment of curve $r_i$ is labeled $i.m$. If $i.k$ denotes the terminal segment of $r_i$, additionally this segment will contain the basepoint. When space is available, the terminal segment may contain the symbols $i.k$ and $i.1$ in addition to the basepoint marker $\star_n$. However, the terminal segment and the segment containing the basepoint are always assumed to be the same.
(4) The admissible slide construction of Lemma 5.3 is used to realize 2-handle slides geometrically in $M_{1\star} \times [-1, 1]$. Segments which correspond to the demonstration of Theorem 4.2 are underlined as they are first encountered in the derivation.

The plates which follow have been carefully checked for accuracy. However, it remains possible that a mislabeled segment or an out of sequence segment numbering has been overlooked. In this case, the reader should proceed with the logical indexing that the particular situation calls for.

**Reading for Plate B1.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = (x_3\, x_2\, x_1^{-1})^3 (x_2\, x_1^{-1})^2$$

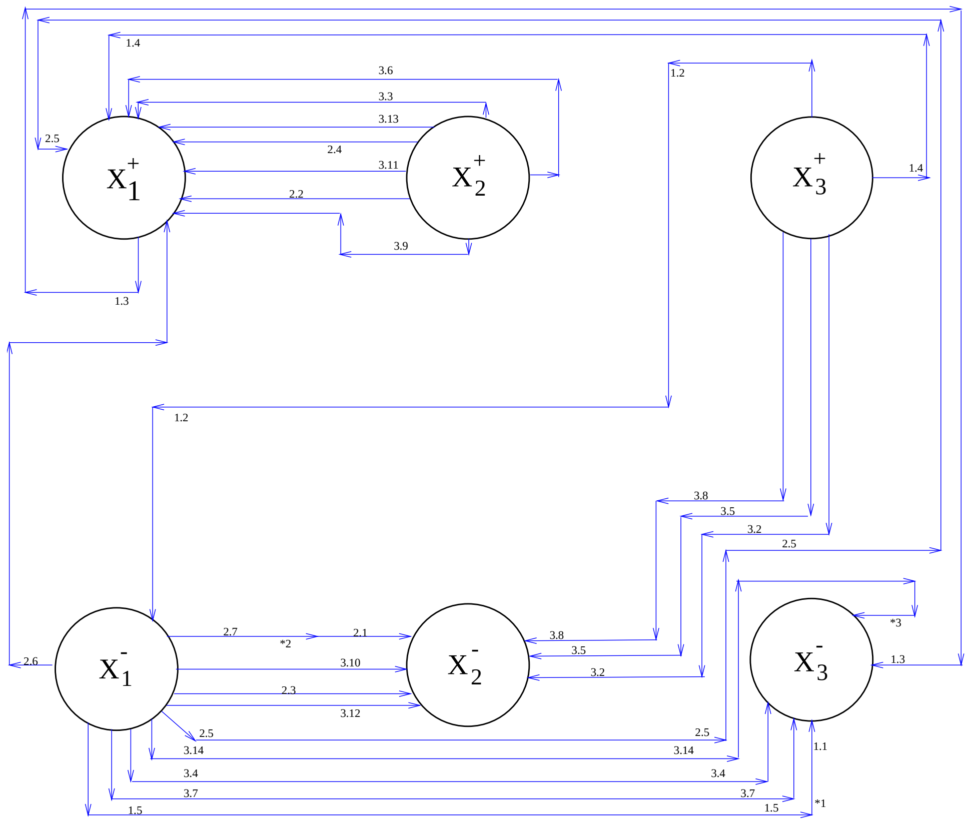


FIGURE 4. Plate B1.

**Reading for Plate B2.**

$$r_1^0 = x_3\,x_1\,x_3\,x_1^{-1}$$
$$r_2^0 = x_2\,x_1^{-1}\,x_2\,x_1^{-3}$$
$$r_3^0 = (x_3\,x_2\,x_1^{-1})^3(x_2\,x_1^{-1})^2\underline{(x_2\,x_1^{-1})^{-2}x_1^2}$$

**Reading for Plate B3.**

$$r_1^0 = x_3\,x_1\,x_3\,x_1^{-1}$$
$$r_2^0 = x_2\,x_1^{-1}\,x_2\,x_1^{-3}$$
$$r_3^0 = (x_3\,x_2\,x_1^{-1})^2x_3\,x_2\,x_1$$

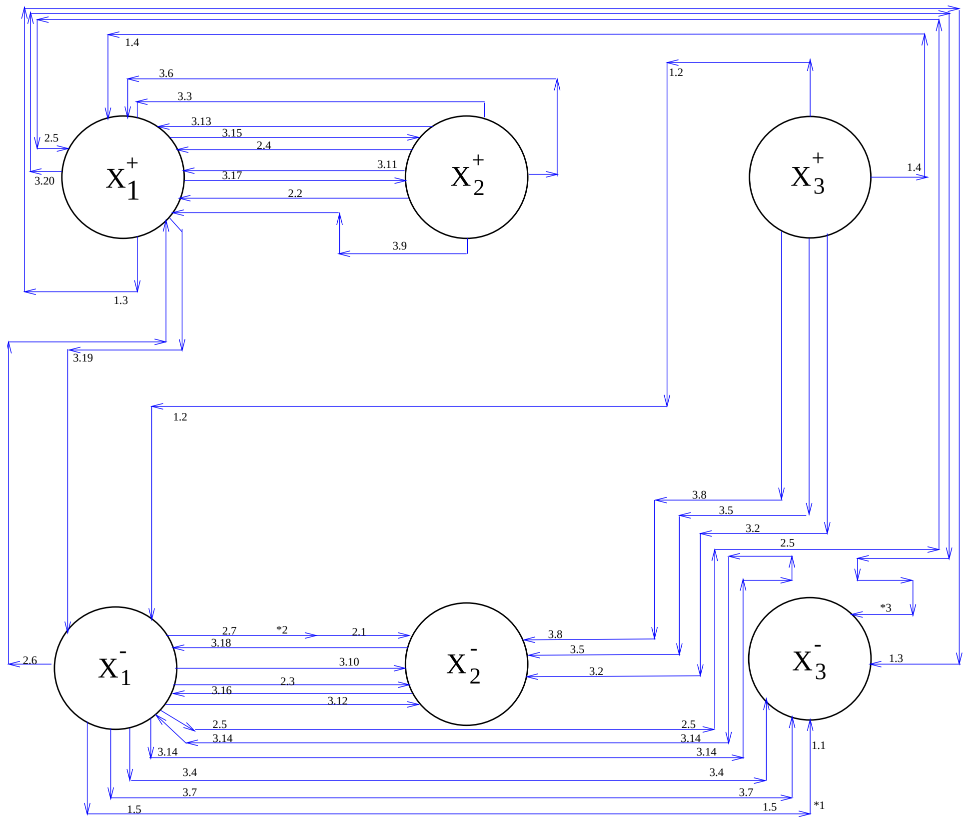


FIGURE 5. Plate B2.

**Reading for Plate B4.**

$$r_1^0 = x_3\,x_1\,x_3\,x_1^{-1}$$
$$r_2^0 = x_2\,x_1^{-1}\,x_2\,x_1^{-3}$$
$$r_3^0 = x_3\,x_2\,x_1^{-1}\,x_3\,y_1\,x_2\,x_1^{-1}\,x_3\,y_4\,x_2\,x_1$$
$$r_4^+ = y_3\,y_1^{-1}$$
$$r_5^- = y_4\,y_2^{-1}$$
$$r_6^0 = y_2$$
$$r_7^0 = \underline{(x_3^{-1}\,x_1\,x_3^{-1}\,x_1^{-1}\,)}y_3^{-1}$$

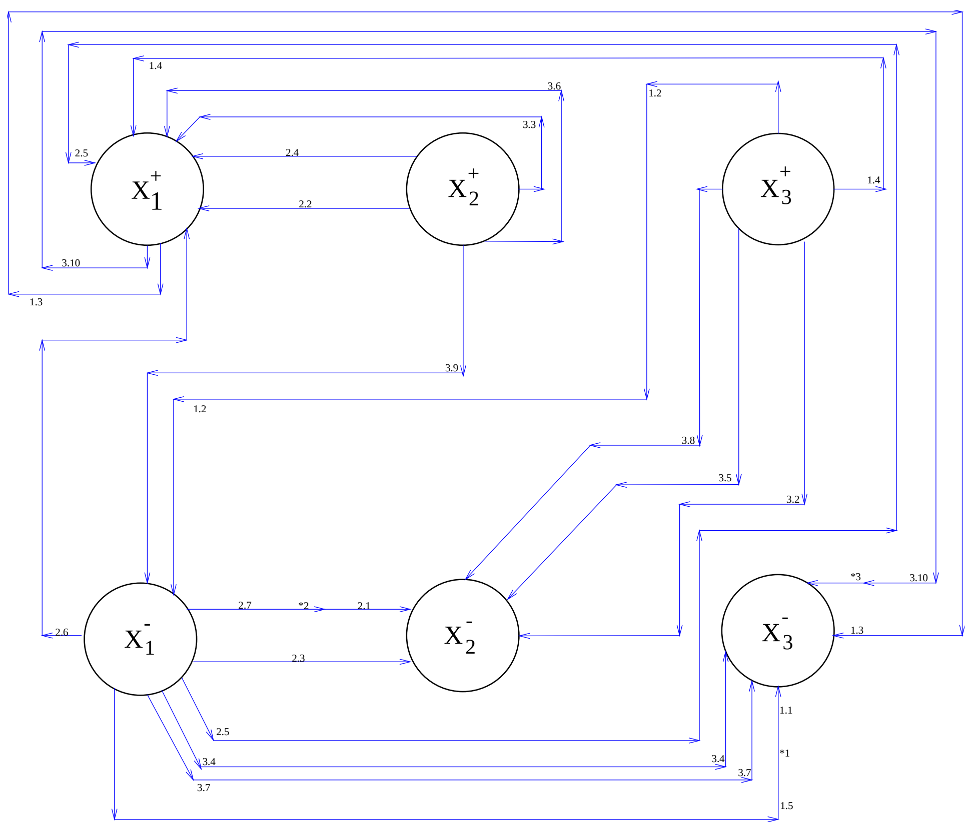


Figure 6. Plate B3.

**Reading for Plate B5.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = x_3\, x_2\, y_1\, x_2\, x_1^{-1}\, x_3\, y_4\, x_2\, x_1$$
$$r_4^+ = x_1^{-1}\, x_3\, x_3^{-1}\, x_1\, y_3\, y_1^{-1}$$
$$r_5^- = y_4\, y_2^{-1}$$
$$r_6^0 = y_2$$
$$r_7^0 = x_3^{-1}\, x_1^{-1}\, y_3^{-1}$$

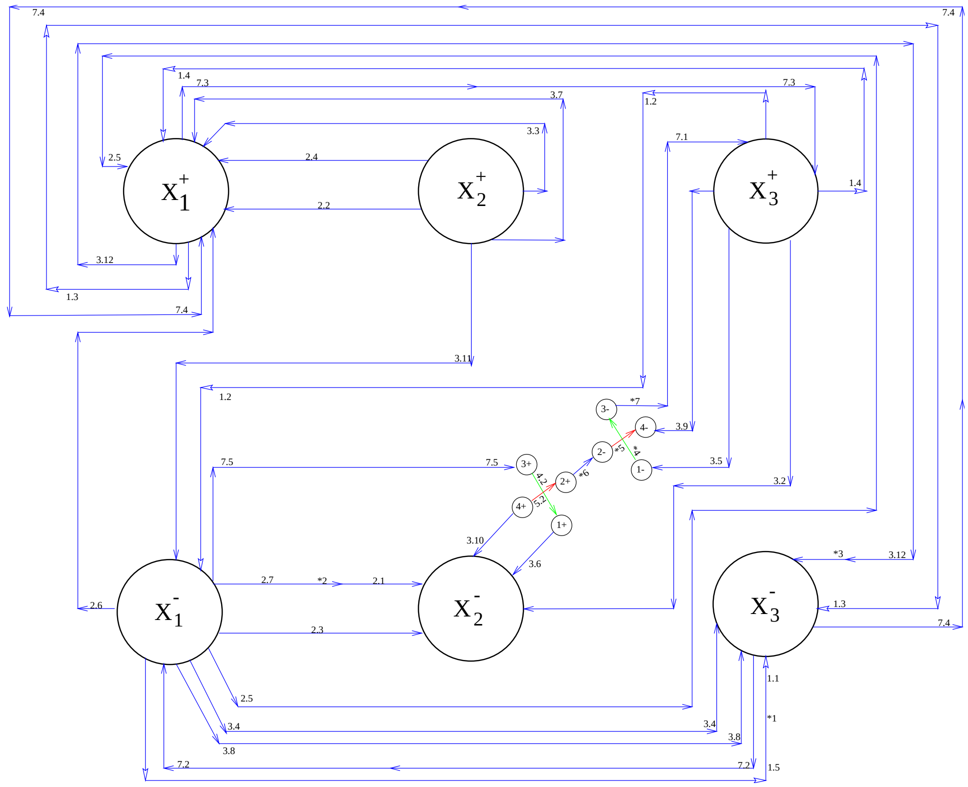


FIGURE 7. Plate B4.

**Reading for Plate B6.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = x_3\, x_2\, y_8\, y_1\, x_2\, y_5\, x_1^{-1}\, x_3\, y_4\, x_2\, x_1$$
$$r_4^+ = x_1^{-1}\, x_3\, x_3^{-1}\, x_1\, y_3\, y_1^{-1}$$
$$r_5^- = y_4\, y_2^{-1}$$
$$r_6^0 = y_2$$
$$r_7^0 = x_3^{-1}\, x_1^{-1}\, y_3^{-1}$$
$$r_8^+ = y_6^{-1}\, y_8$$
$$r_9^- = y_7^{-1}\, y_5$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = \underline{(x_2^{-1}\, x_1\, x_2^{-1}\, x_1^3)}y_7^{-1}$$

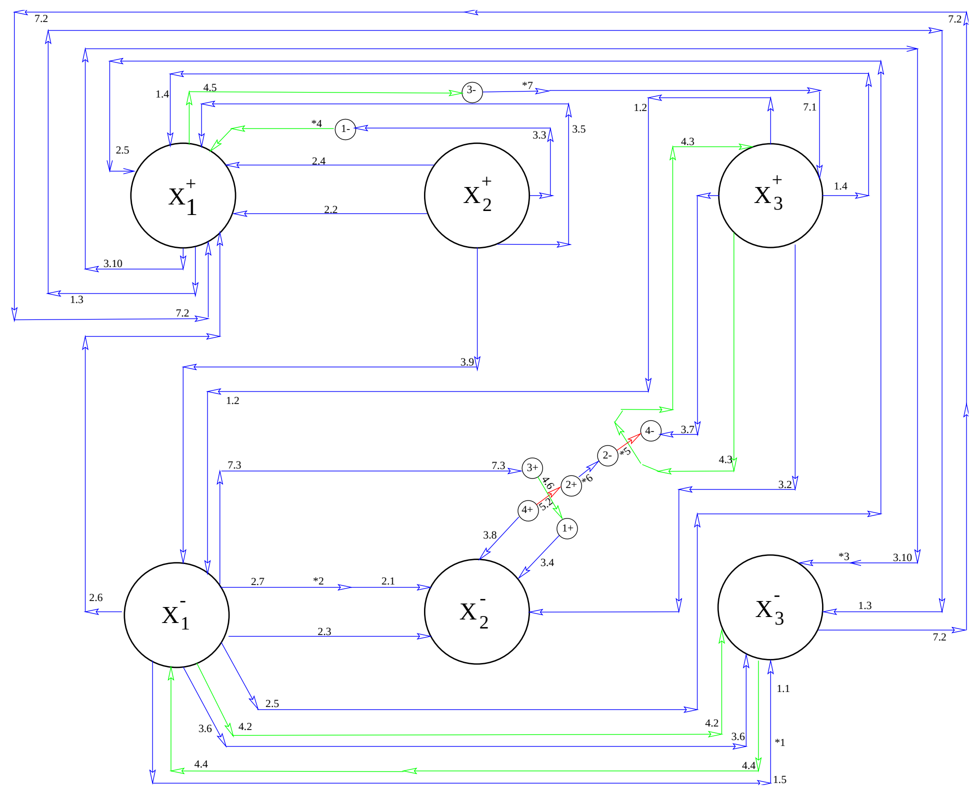

FIGURE 8. Plate B5.

**Reading for Plate B7.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = x_3\, x_2\, y_8\, y_1\, y_5\, x_3\, y_4\, x_2\, x_1$$
$$r_4^+ = x_1^{-1}\, x_3\, x_3^{-1}\, x_1\, y_3\, y_1^{-1}$$
$$r_5^- = y_4\, y_2^{-1}$$
$$r_6^0 = y_2$$
$$r_7^0 = x_3^{-1}\, x_1^{-1}\, y_3^{-1}$$
$$r_8^+ = y_6^{-1}\, y_8$$
$$r_9^- = x_1^{-1}\, y_7^{-1}\, x_2\, x_2^{-1}\, y_5\, x_1$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = x_1\, x_2^{-1}\, x_1^2\, y_7^{-1}$$

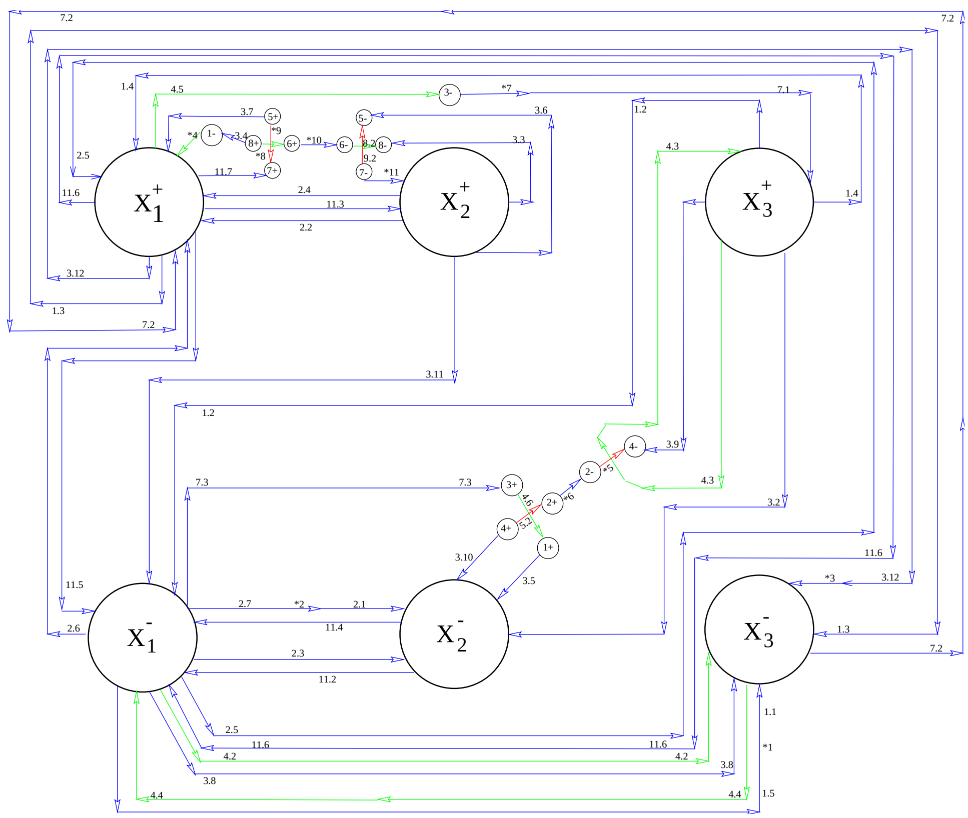


FIGURE 9. Plate B6.

**Reading for Plate B8.**

$$r_1^0 = x_3\,x_1\,x_3\,x_1^{-1}$$
$$r_2^0 = x_2\,x_1^{-1}\,x_2\,x_1^{-3}$$
$$r_3^0 = x_3\,y_{10}\,x_2\,y_8\,y_1\,y_5\,x_3\,y_4\,x_2\,x_1$$
$$r_4^+ = x_1^{-1}\,x_3\,x_3^{-1}\,x_1\,y_3\,y_1^{-1}$$
$$r_5^- = y_4\,y_2^{-1}$$
$$r_6^0 = y_9^{-1}\,y_2$$
$$r_7^0 = x_3^{-1}\,x_1^{-1}\,y_3^{-1}$$
$$r_8^+ = y_6^{-1}\,y_8$$
$$r_9^- = x_1^{-1}\,y_7^{-1}\,x_2\,x_2^{-1}\,y_5\,x_1$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = x_1\,x_2^{-1}\,x_1^2\,y_7^{-1}$$

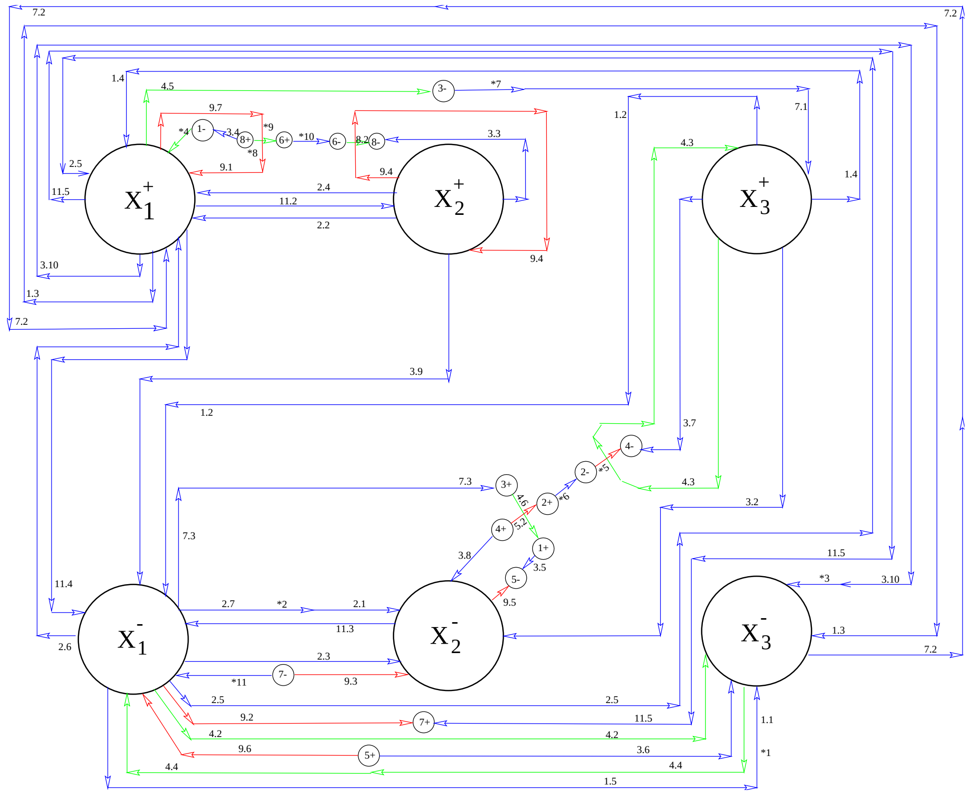

FIGURE 10. Plate B7.

$$r_{12}^{+} = y_{11}^{-1}\, y_9$$
$$r_{13}^{-} = y_{12}^{-1}\, y_{10}$$
$$r_{14}^{0} = y_{12}$$
$$r_{15}^{0} = \underline{(x_3^{-1}\, x_1^{-1}\, x_3^{-1}\, x_1\,)}y_{11}^{-1}$$

**Reading for Plate B9.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = y_{10}\, x_2\, y_8\, y_1\, y_5\, y_4\, x_2\, x_1$$
$$r_4^{+} = x_1^{-2}\, x_1^{2}\, y_3\, y_1^{-1}$$
$$r_5^{-} = y_4\, x_1\, x_1^{-1}\, y_2^{-1}$$
$$r_6^0 = y_9^{-1}\, y_2$$
$$r_7^0 = x_3^{-1}\, y_3^{-1}$$
$$r_8^{+} = y_6^{-1}\, y_8$$

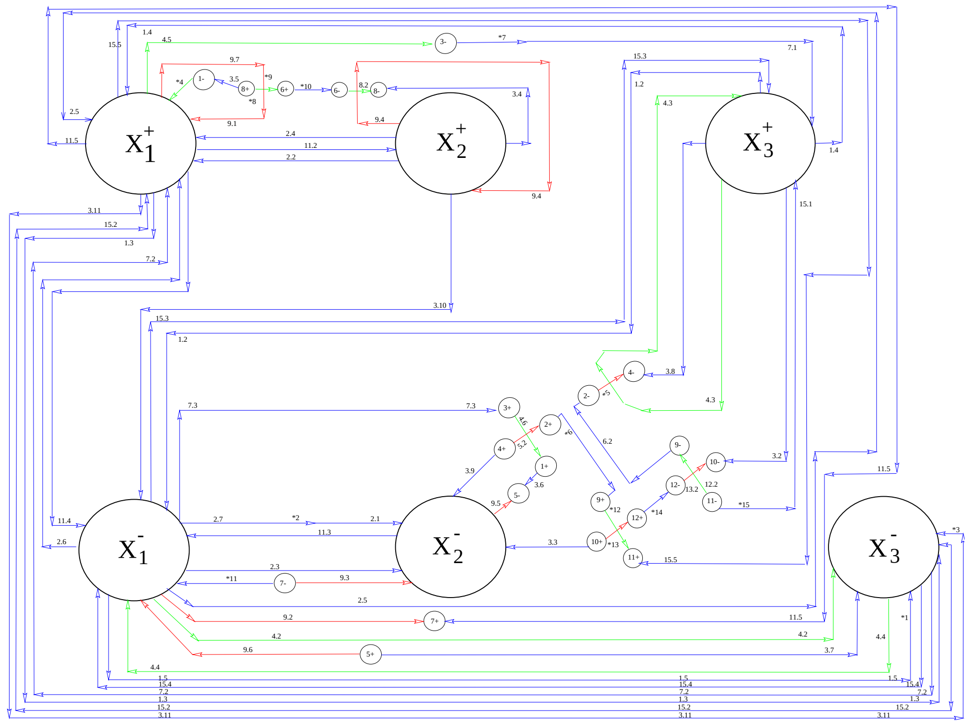


FIGURE 11. Plate B8.

$$r_9^- = x_1^{-2}\, y_7^{-1}\, x_1^{-1}\, x_2\, x_2^{-1}\, x_1\, y_5\, x_1^2$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = x_2^{-1}\, x_1\, y_7^{-1}$$
$$r_{12}^+ = y_{11}^{-1}\, y_9$$
$$r_{13}^- = y_{12}^{-1}\, x_3^{-1}\, x_1^{-1}\, x_1\, y_{10}$$
$$r_{14}^0 = y_{12}$$
$$r_{15}^0 = x_3^{-1}\, x_1\, y_{11}^{-1}$$

**Reading for Plate B10.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = y_{10}\, x_2\, y_8\, y_1\, y_5\, y_4\, x_2\, x_1$$
$$r_4^+ = x_1^{-2}\, x_1^2\, y_3\, y_1^{-1}$$
$$r_5^- = y_4\, x_1\, x_1^{-1}\, y_2^{-1}$$
$$r_6^0 = y_9^{-1}\, y_2$$

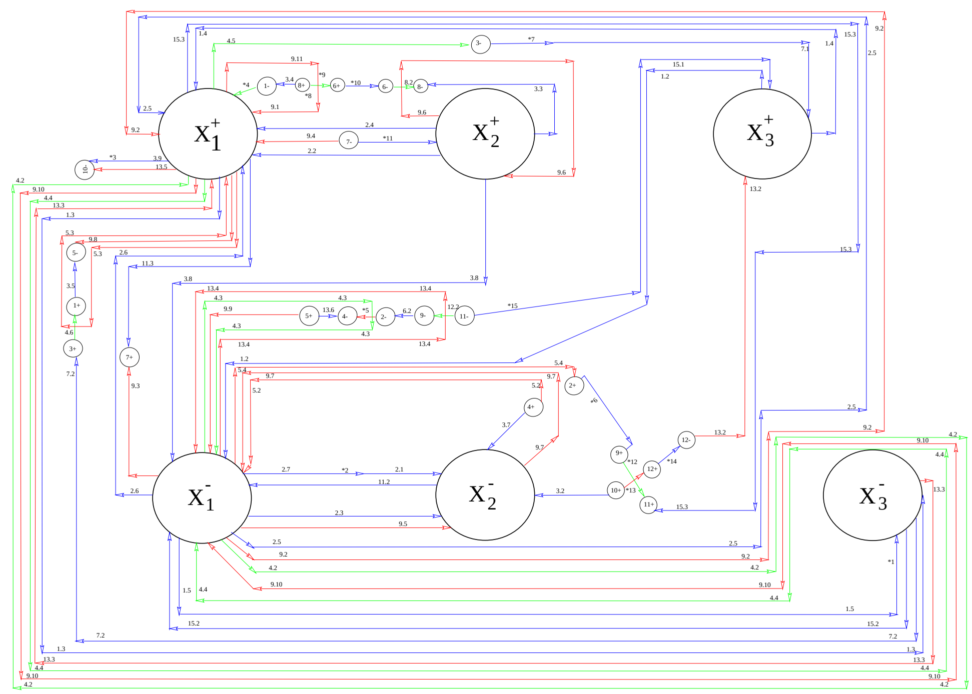

FIGURE 12. Plate B9.

$$r_7^0 = x_3^{-1}\,y_{16}^{-1}\,y_3^{-1}$$
$$r_8^+ = y_6^{-1}\,y_8$$
$$r_9^- = x_1^{-2}\,y_7^{-1}\,x_1^{-1}\,x_2\,x_2^{-1}\,x_1\,y_5\,x_1^2$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = x_2^{-1}\,x_1\,y_7^{-1}$$
$$r_{12}^+ = y_{11}^{-1}\,y_9$$
$$r_{13}^- = y_{12}^{-1}\,x_3^{-1}\,x_1^{-1}\,x_1\,y_{10}$$
$$r_{14}^0 = y_{12}$$
$$r_{15}^0 = x_3^{-1}\,y_{15}^{-1}\,x_1\,y_{11}^{-1}$$
$$r_{16}^+ = y_{13}^{-1}\,y_{15}$$
$$r_{17}^- = y_{14}^{-1}\,y_{16}$$
$$r_{18}^0 = y_{14}$$
$$r_{19}^0 = \underline{(x_3\,x_1^{-1}\,x_3\,x_1\,)}y_{13}$$

**Reading for Plate B11.**

$$r_1^0 = x_3\,x_1\,x_3\,x_1^{-1}$$

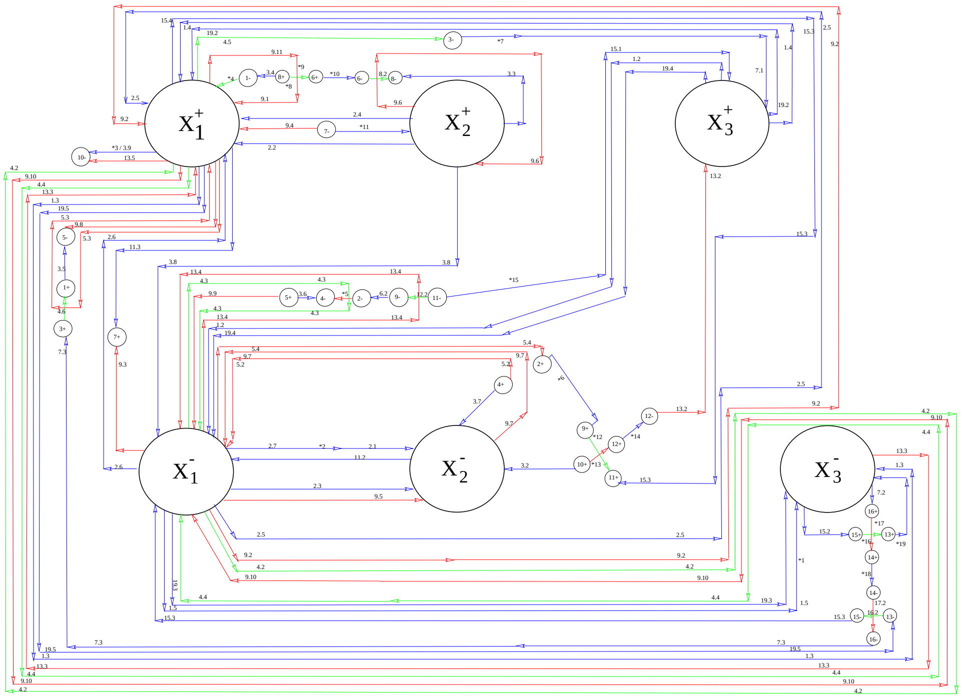


FIGURE 13. Plate B10.

$$r^0_2 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r^0_3 = y_{10}\, y_8\, y_1\, y_5\, y_4\, x_2\, x_1$$
$$r^+_4 = x_1^{-2}\, x_1^2\, y_3\, y_1^{-1}$$
$$r^-_5 = y_4\, x_1\, x_1^{-1}\, y_2^{-1}$$
$$r^0_6 = y_9^{-1}\, y_2$$
$$r^0_7 = x_3^{-1}\, y_{16}^{-1}\, y_3^{-1}$$
$$r^+_8 = y_6^{-1}\, x_2^{-1}\, y_8$$
$$r^-_9 = x_1^{-2}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, x_1^2$$
$$r^0_{10} = y_6$$
$$r^0_{11} = x_2^{-1}\, x_1\, y_7^{-1}$$
$$r^+_{12} = y_{11}^{-1}\, y_9$$
$$r^-_{13} = y_{12}^{-1}\, x_3^{-1}\, x_1^{-1}\, x_1\, y_{10}$$
$$r^0_{14} = y_{12}$$
$$r^0_{15} = y_{15}^{-1}\, x_1\, y_{11}^{-1}$$

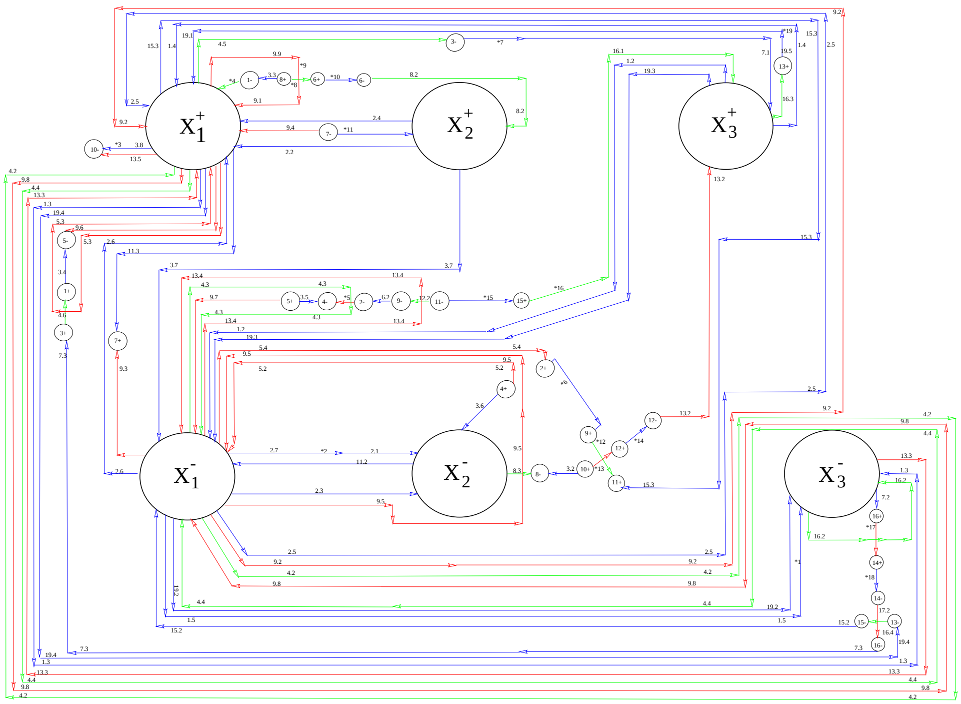


FIGURE 14. Plate B11.

$$r_{16}^{+} = x_3^{-1}\, x_3\, y_{13}^{-1}\, y_{15}$$
$$r_{17}^{-} = y_{14}^{-1}\, y_{16}$$
$$r_{18}^{0} = y_{14}$$
$$r_{19}^{0} = x_1^{-1}\, x_3\, x_1\, y_{13}$$

**Reading for Plate B12.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}\, x_2\, x_1$$
$$r_4^{+} = x_1^{-2}\, x_1^{2}\, y_3\, y_1^{-1}$$
$$r_5^{-} = y_4\, x_1\, x_1^{-1}\, y_2^{-1}$$
$$r_6^0 = y_9^{-1}\, y_2$$
$$r_7^0 = x_3^{-1}\, y_{16}^{-1}\, y_3^{-1}$$
$$r_8^{+} = y_6^{-1}\, x_2^{-1}\, y_{18}^{-1}\, y_8$$
$$r_9^{-} = x_1^{-2}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, x_1^{2}$$

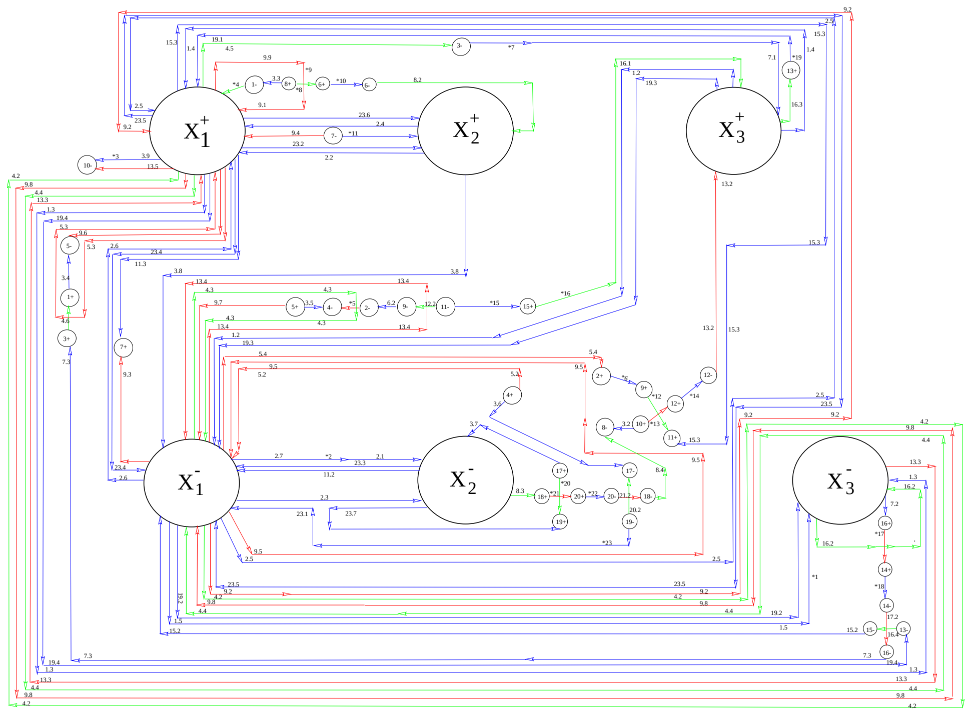


FIGURE 15. Plate B12.

$$r^0_{10} = y_6$$
$$r^0_{11} = x_2^{-1}\,x_1\,y_7^{-1}$$
$$r^+_{12} = y_{11}^{-1}\,y_9$$
$$r^-_{13} = y_{12}^{-1}\,x_3^{-1}\,x_1^{-1}\,x_1\,y_{10}$$
$$r^0_{14} = y_{12}$$
$$r^0_{15} = y_{15}^{-1}\,x_1\,y_{11}^{-1}$$
$$r^+_{16} = x_3^{-1}\,x_3\,y_{13}^{-1}\,y_{15}$$
$$r^-_{17} = y_{14}^{-1}\,y_{16}$$
$$r^0_{18} = y_{14}$$
$$r^0_{19} = x_1^{-1}\,x_3\,x_1\,y_{13}$$
$$r^+_{20} = y_{19}^{-1}\,y_{17}$$
$$r^-_{21} = y_{20}^{-1}\,y_{18}$$
$$r^0_{22} = y_{20}$$
$$r^0_{23} = \underline{(x_1\,x_2^{-1}\,x_1^3\,x_2^{-1})}y_{19}^{-1}$$

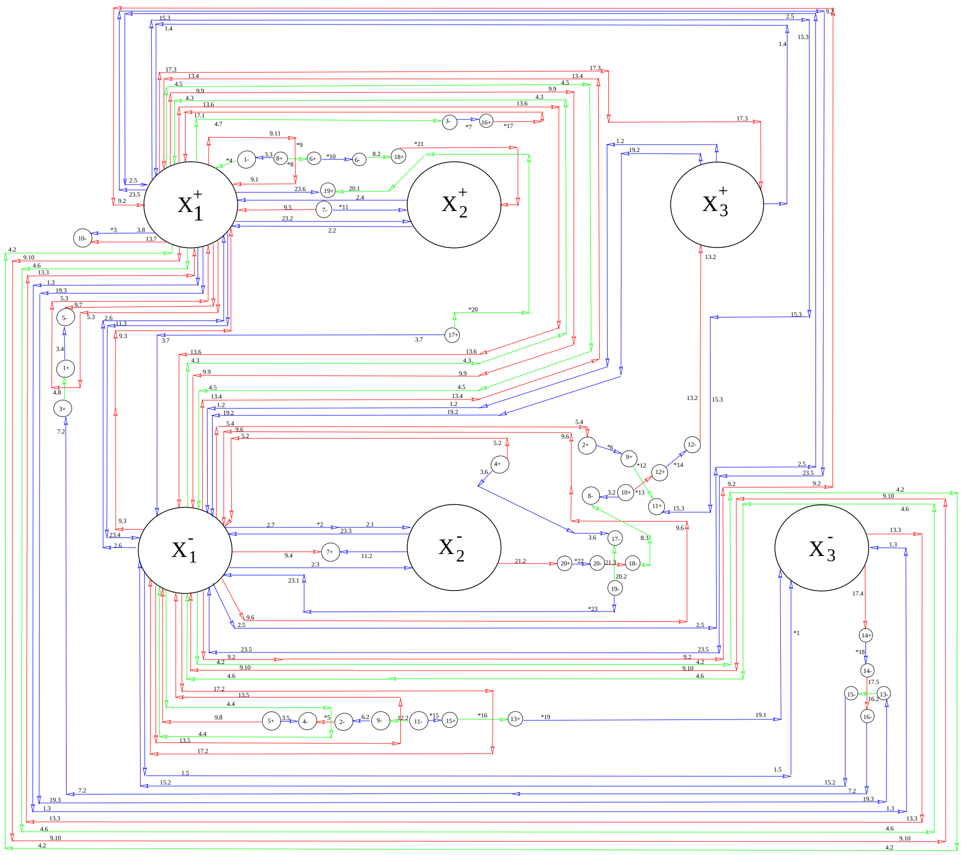


FIGURE 16. Plate B13.

**Reading for Plate B13.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}\, x_1$$
$$r_4^+ = x_1^{-3}\, x_1^3\, y_3\, y_1^{-1}$$
$$r_5^- = y_4\, x_1\, x_1^{-1}\, y_2^{-1}$$
$$r_6^0 = y_9^{-1}\, y_2$$
$$r_7^0 = y_{16}^{-1}\, y_3^{-1}$$
$$r_8^+ = y_6^{-1}\, y_{18}^{-1}\, y_8$$
$$r_9^- = x_1^{-3}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, x_1^3$$
$$r_{10}^0 = y_6$$

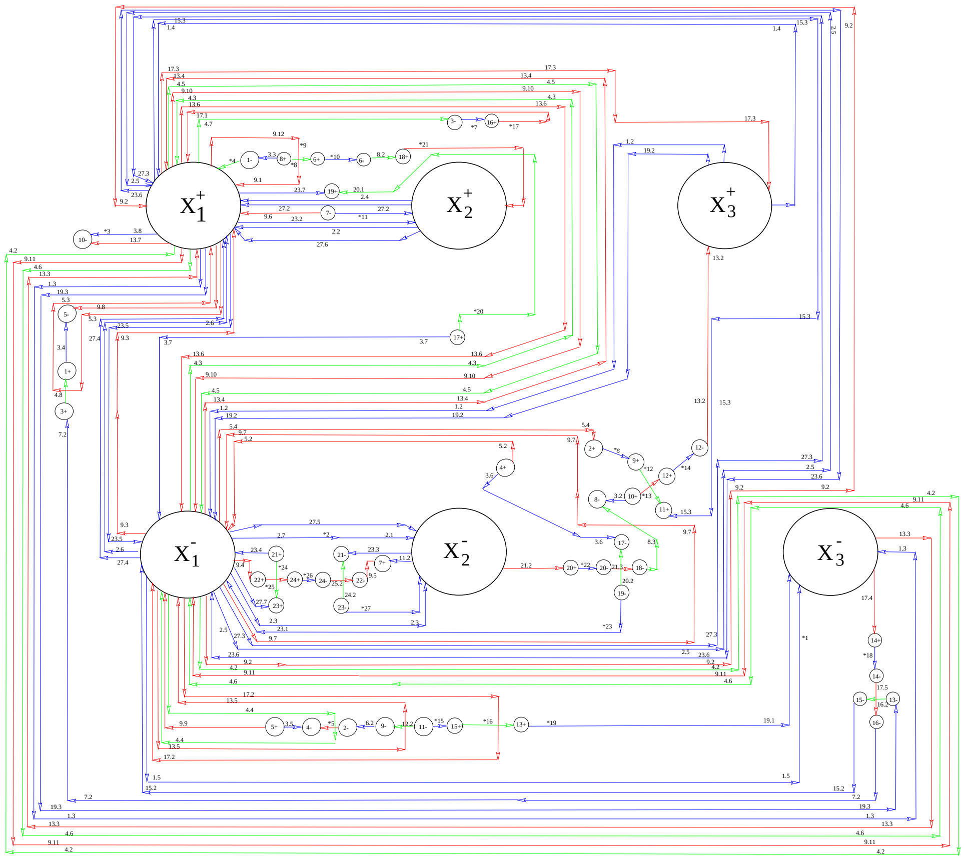

FIGURE 17. Plate B14.

$$r^{0}_{11} = x_2^{-1}\, y_7^{-1}$$
$$r^{+}_{12} = y_{11}^{-1}\, y_9$$
$$r^{-}_{13} = y_{12}^{-1}\, x_3^{-1}\, x_1^{-2}\, x_1^{2}\, y_{10}$$
$$r^{0}_{14} = y_{12}$$
$$r^{0}_{15} = y_{15}^{-1}\, x_1\, y_{11}^{-1}$$
$$r^{+}_{16} = y_{13}^{-1}\, y_{15}$$
$$r^{-}_{17} = x_1^{-1}\, x_1\, x_3^{-1}\, y_{14}^{-1}\, y_{16}$$
$$r^{0}_{18} = y_{14}$$
$$r^{0}_{19} = x_3\, x_1\, y_{13}$$
$$r^{+}_{20} = y_{19}^{-1}\, y_{17}$$
$$r^{-}_{21} = x_2^{-1}\, y_{20}^{-1}\, y_{18}$$

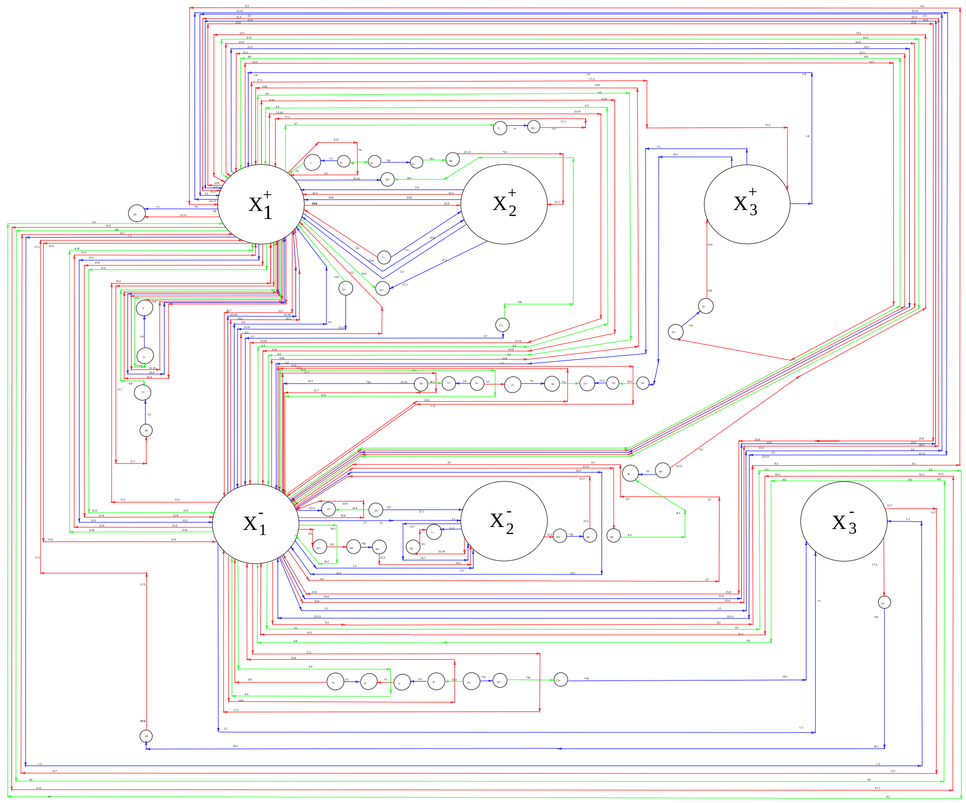


FIGURE 18. Plate B15.

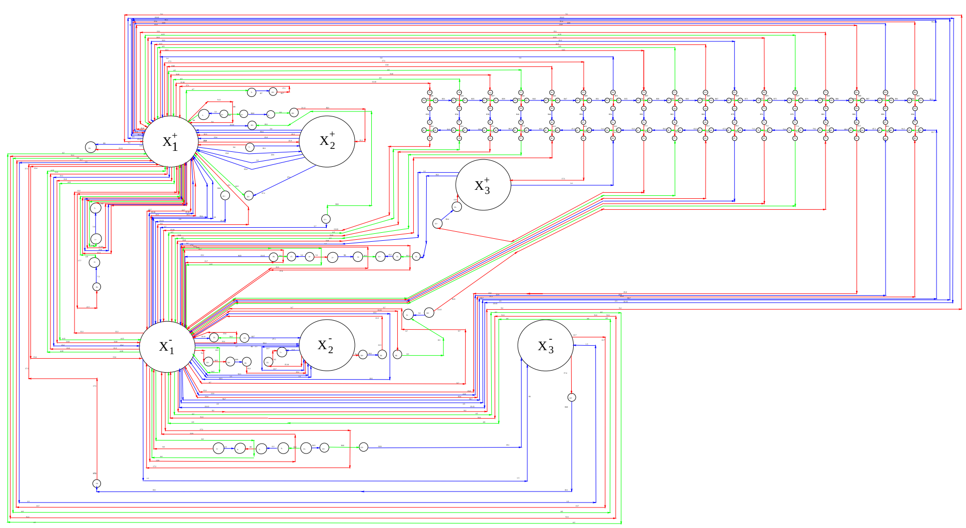


FIGURE 19. Plate B16.

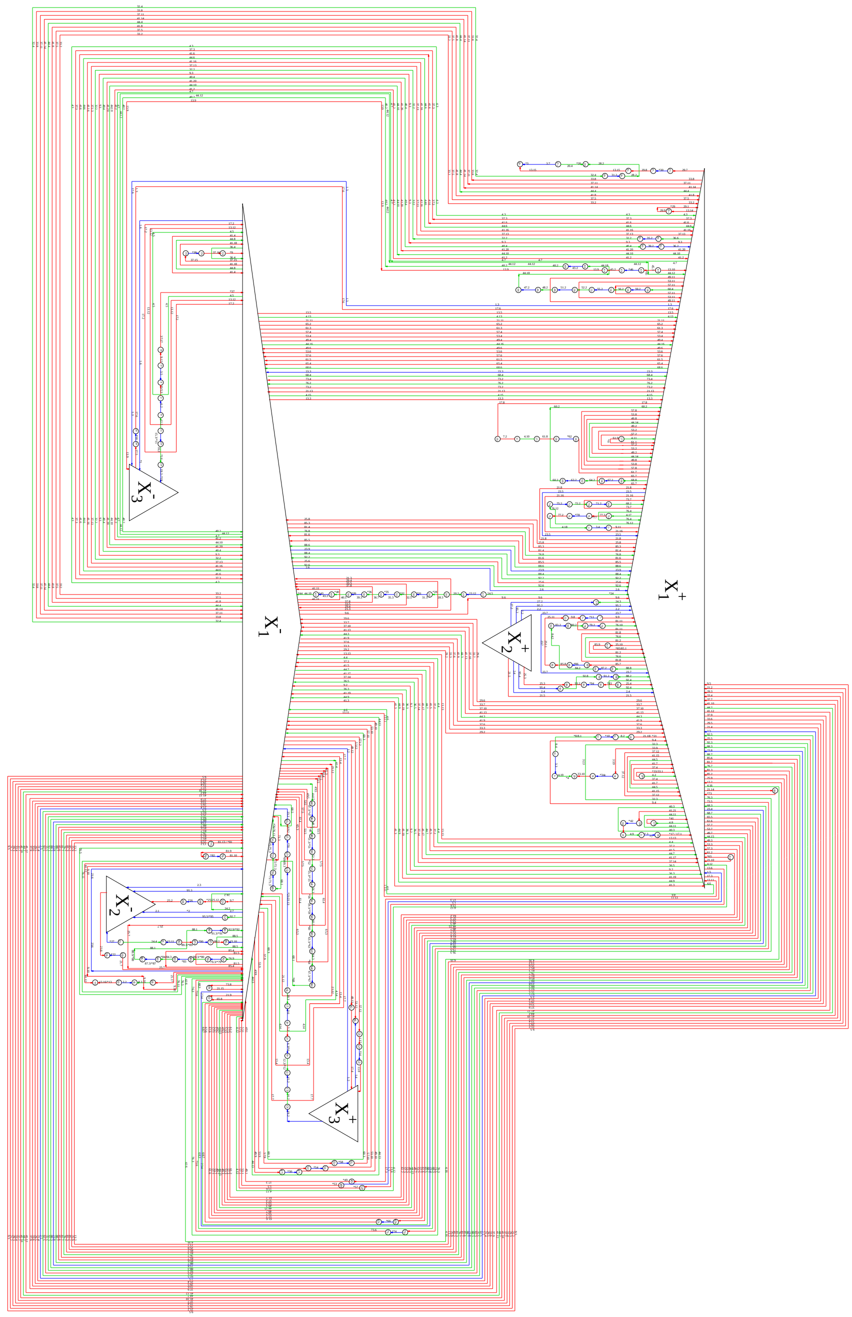


FIGURE 20. Plate B17.

$$r^0_{22} = y_{20}$$
$$r^0_{23} = x_1\, x_2^{-1}\, x_1^3\, y_{19}^{-1}$$

**Reading for Plate B14.**

$$r^0_1 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r^0_2 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r^0_3 = y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}\, x_1$$
$$r^+_4 = x_1^{-3}\, x_1^3\, y_3\, y_1^{-1}$$
$$r^-_5 = y_4\, x_1\, x_1^{-1}\, y_2^{-1}$$
$$r^0_6 = y_9^{-1}\, y_2$$
$$r^0_7 = y_{16}^{-1}\, y_3^{-1}$$
$$r^+_8 = y_6^{-1}\, y_{18}^{-1}\, y_8$$
$$r^-_9 = x_1^{-3}\, y_{22}^{-1}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, x_1^3$$
$$r^0_{10} = y_6$$
$$r^0_{11} = x_2^{-1}\, y_7^{-1}$$
$$r^+_{12} = y_{11}^{-1}\, y_9$$
$$r^-_{13} = y_{12}^{-1}\, x_3^{-1}\, x_1^{-2}\, x_1^2\, y_{10}$$
$$r^0_{14} = y_{12}$$
$$r^0_{15} = y_{15}^{-1}\, x_1\, y_{11}^{-1}$$
$$r^+_{16} = y_{13}^{-1}\, y_{15}$$
$$r^-_{17} = x_1^{-1}\, x_1\, x_3^{-1}\, y_{14}^{-1}\, y_{16}$$
$$r^0_{18} = y_{14}$$
$$r^0_{19} = x_3\, x_1\, y_{13}$$
$$r^+_{20} = y_{19}^{-1}\, y_{17}$$
$$r^-_{21} = x_2^{-1}\, y_{20}^{-1}\, y_{18}$$
$$r^0_{22} = y_{20}$$
$$r^0_{23} = x_1\, x_2^{-1}\, y_{21}\, x_1^3\, y_{19}^{-1}$$
$$r^+_{24} = y_{23}^{-1}\, y_{21}$$
$$r^-_{25} = y_{24}^{-1}\, y_{22}$$
$$r^0_{26} = y_{24}$$
$$r^0_{27} = \underline{(x_2\, x_1^{-3}\, x_2\, x_1^{-1})}y_{23}^{-1}$$

**Reading for Plate B15.**

$$r_1^0 = x_3\, x_1\, x_3\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r_3^0 = y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}\, x_1$$
$$r_4^+ = x_1^{-3}\, x_1^3\, y_3\, x_1^{-3}\, x_1^3\, y_1^{-1}$$
$$r_5^- = y_4\, y_2^{-1}$$
$$r_6^0 = y_9^{-1}\, y_2$$
$$r_7^0 = y_{16}^{-1}\, y_3^{-1}$$
$$r_8^+ = y_6^{-1}\, y_{18}^{-1}\, y_8$$
$$r_9^- = x_1^{-3}\, y_{22}^{-1}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, x_1^3$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = x_2^{-1}\, y_7^{-1}$$
$$r_{12}^+ = y_{11}^{-1}\, y_9$$
$$r_{13}^- = x_1^{-2}\, x_1^2\, y_{12}^{-1}\, x_3^{-1}\, x_1^{-2}\, x_1^2\, y_{10}$$
$$r_{14}^0 = y_{12}$$
$$r_{15}^0 = y_{15}^{-1}\, y_{11}^{-1}$$
$$r_{16}^+ = y_{13}^{-1}\, y_{15}$$
$$r_{17}^- = x_1^{-1}\, x_1\, x_3^{-1}\, y_{14}^{-1}\, x_1^{-1}\, x_1\, y_{16}$$
$$r_{18}^0 = y_{14}$$
$$r_{19}^0 = x_3\, y_{13}$$
$$r_{20}^+ = y_{19}^{-1}\, y_{17}$$
$$r_{21}^- = x_2^{-1}\, y_{20}^{-1}\, x_1\, x_1^{-3}\, x_1^3\, x_1^{-1}\, y_{18}$$
$$r_{22}^0 = y_{20}$$
$$r_{23}^0 = x_1^3\, x_1^{-1}\, x_1\, x_2^{-1}\, x_2\, x_1^{-3}\, y_{21}\, x_1^2\, y_{19}^{-1}$$
$$r_{24}^+ = x_1^{-1}\, x_1\, y_{23}^{-1}\, y_{21}$$
$$r_{25}^- = y_{24}^{-1}\, x_2\, x_1^{-3}\, x_1^3\, x_2^{-1}\, y_{22}$$
$$r_{26}^0 = y_{24}$$
$$r_{27}^0 = x_2\, y_{23}^{-1}$$

**Reading for Plate B16.**

$$r_1^0 = x_3\, x_1\, x_3\, y_{49}\, x_1^{-1}$$
$$r_2^0 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$

$$r_3^0 = y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}\, x_1$$
$$r_4^+ = x_1^{-2}\, y_{29}\, x_1^{-1}\, x_1\, y_{37}^{-1}\, x_1^2\, y_3\, x_1^{-1}\, y_{57}\, x_1^{-2}\, x_1^2\, y_{73}^{-1}\, x_1\, y_1^{-1}$$
$$r_5^- = y_4\, y_2^{-1}$$
$$r_6^0 = y_9^{-1}\, y_2$$
$$r_7^0 = y_{16}^{-1}\, y_3^{-1}$$
$$r_8^+ = y_6^{-1}\, y_{18}^{-1}\, y_8$$
$$r_9^- = x_1^{-3}\, y_{22}^{-1}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, x_1\, y_{33}^{-1}\, x_1^2$$
$$r_{10}^0 = y_6$$
$$r_{11}^0 = x_2^{-1}\, y_7^{-1}$$
$$r_{12}^+ = y_{11}^{-1}\, y_9$$
$$r_{13}^- = y_{77}\, x_1^{-2}\, x_1^2\, y_{53}^{-1}\, y_{12}^{-1}\, x_3^{-1}\, x_1^{-1}\, y_{41}\, x_1^{-1}\, x_1\, y_{25}^{-1}\, x_1\, y_{10}$$
$$r_{14}^0 = y_{12}$$
$$r_{15}^0 = y_{15}^{-1}\, y_{11}^{-1}$$
$$r_{16}^+ = y_{13}^{-1}\, y_{15}$$
$$r_{17}^- = x_1^{-1}\, x_1\, y_{45}^{-1}\, x_3^{-1}\, y_{14}^{-1}\, x_1^{-1}\, x_1\, y_{16}$$
$$r_{18}^0 = y_{14}$$
$$r_{19}^0 = x_3\, y_{13}$$
$$r_{20}^+ = y_{26}\, y_{19}^{-1}\, y_{17}$$
$$r_{21}^- = x_2^{-1}\, y_{20}^{-1}\, x_1\, x_1^{-1}\, y_{61}\, x_1^{-2}\, x_1^2\, y_{69}^{-1}\, x_1\, x_1^{-1}\, y_{18}$$
$$r_{22}^0 = y_{20}$$
$$r_{23}^0 = x_1^2\, y_{65}^{-1}\, x_1\, x_1^{-1}\, x_1\, x_2^{-1}\, x_2\, x_1^{-1}\, y_{85}\, x_1^{-2}\, y_{21}\, x_1^2\, y_{19}^{-1}$$
$$r_{24}^+ = x_1^{-1}\, x_1\, y_{23}^{-1}\, y_{21}$$
$$r_{25}^- = y_{24}^{-1}\, x_2\, x_1^{-1}\, y_{89}\, x_1^{-2}\, x_1^2\, y_{81}^{-1}\, x_1\, x_2^{-1}\, y_{22}$$
$$r_{26}^0 = y_{24}$$
$$r_{27}^0 = x_2\, y_{23}^{-1}$$
$$r_{28}^+ = y_{28}^{-1}\, y_{26}$$
$$r_{29}^- = y_{27}^{-1}\, y_{25}$$
$$r_{30}^0 = y_{27}$$
$$r_{31}^0 = y_{30}^{-1}\, y_{28}$$
$$r_{32}^+ = y_{32}^{-1}\, y_{30}$$
$$r_{33}^- = y_{31}^{-1}\, y_{29}$$

$$r^0_{34} = y_{31}$$
$$r^0_{35} = y^{-1}_{34}\, y_{32}$$
$$r^+_{36} = y^{-1}_{36}\, y_{34}$$
$$r^-_{37} = y^{-1}_{35}\, y_{33}$$
$$r^0_{38} = y_{35}$$
$$r^0_{39} = y^{-1}_{38}\, y_{36}$$
$$r^+_{40} = y^{-1}_{40}\, y_{38}$$
$$r^-_{41} = y^{-1}_{39}\, y_{37}$$
$$r^0_{42} = y_{39}$$
$$r^0_{43} = y^{-1}_{42}\, y_{40}$$
$$r^+_{44} = y^{-1}_{44}\, y_{42}$$
$$r^-_{45} = y^{-1}_{43}\, y_{41}$$
$$r^0_{46} = y_{43}$$
$$r^0_{47} = y^{-1}_{46}\, y_{44}$$
$$r^+_{48} = y^{-1}_{48}\, y_{46}$$
$$r^-_{49} = y^{-1}_{47}\, y_{45}$$
$$r^0_{50} = y_{47}$$
$$r^0_{51} = y^{-1}_{50}\, y_{48}$$
$$r^+_{52} = y^{-1}_{52}\, y_{50}$$
$$r^-_{53} = y^{-1}_{51}\, y_{49}$$
$$r^0_{54} = y_{51}$$
$$r^0_{55} = y^{-1}_{54}\, y_{52}$$
$$r^+_{56} = y^{-1}_{56}\, y_{54}$$
$$r^-_{57} = y^{-1}_{55}\, y_{53}$$
$$r^0_{58} = y_{55}$$
$$r^0_{59} = y^{-1}_{58}\, y_{56}$$
$$r^+_{60} = y^{-1}_{60}\, y_{58}$$
$$r^-_{61} = y^{-1}_{59}\, y_{57}$$
$$r^0_{62} = y_{59}$$
$$r^0_{63} = y^{-1}_{62}\, y_{60}$$
$$r^+_{64} = y^{-1}_{64}\, y_{62}$$

$r_{65}^{-} = y_{63}^{-1}\, y_{61}$
$r_{66}^{0} = y_{63}$
$r_{67}^{0} = y_{66}^{-1}\, y_{64}$
$r_{68}^{+} = y_{68}^{-1}\, y_{66}$
$r_{69}^{-} = y_{67}^{-1}\, y_{65}$
$r_{70}^{0} = y_{67}$
$r_{71}^{0} = y_{70}^{-1}\, y_{68}$
$r_{72}^{+} = y_{72}^{-1}\, y_{70}$
$r_{73}^{-} = y_{71}^{-1}\, y_{69}$
$r_{74}^{0} = y_{71}$
$r_{75}^{0} = y_{74}^{-1}\, y_{72}$
$r_{76}^{+} = y_{76}^{-1}\, y_{74}$
$r_{77}^{-} = y_{75}^{-1}\, y_{73}$
$r_{78}^{0} = y_{75}$
$r_{79}^{0} = y_{78}^{-1}\, y_{76}$
$r_{80}^{+} = y_{80}^{-1}\, y_{78}$
$r_{81}^{-} = y_{79}^{-1}\, y_{77}$
$r_{82}^{0} = y_{79}$
$r_{83}^{0} = y_{82}^{-1}\, y_{80}$
$r_{84}^{+} = y_{84}^{-1}\, y_{82}$
$r_{85}^{-} = y_{83}^{-1}\, y_{81}$
$r_{86}^{0} = y_{83}$
$r_{87}^{0} = y_{86}^{-1}\, y_{84}$
$r_{88}^{+} = y_{88}^{-1}\, y_{86}$
$r_{89}^{-} = y_{87}^{-1}\, y_{85}$
$r_{90}^{0} = y_{87}$
$r_{91}^{0} = y_{90}^{-1}\, y_{88}$
$r_{92}^{+} = y_{92}^{-1}\, y_{90}$
$r_{93}^{-} = y_{91}^{-1}\, y_{89}$
$r_{94}^{0} = y_{91}$
$r_{95}^{0} = \underline{(x_1^{-2}\, x_2\, x_1^{-1}\, x_2\, x_1^{-1})}y_{92}$

**Reading for Plate B17.**

$$r^0_1 = x_3\, x_1\, x_3\, y_{49}\, x_1^{-1}$$
$$r^0_2 = x_2\, x_1^{-1}\, x_2\, x_1^{-3}$$
$$r^0_3 = y_{10}\, y_8\, y_1\, y_5\, y_4\, y_{17}$$
$$r^+_4 = y_{29}\, x_1^{-3}\, x_1^3\, y_{37}^{-1}\, y_3\, y_{57}\, x_1^{-3}\, x_1^3\, y_{73}^{-1}\, y_1^{-1}$$
$$r^-_5 = y_4\, y_2^{-1}$$
$$r^0_6 = y_9^{-1}\, y_2$$
$$r^0_7 = y_{16}^{-1}\, y_3^{-1}$$
$$r^+_8 = y_6^{-1}\, y_{18}^{-1}\, y_8$$
$$r^-_9 = x_1^3\, x_1^{-3}\, y_{22}^{-1}\, y_7^{-1}\, x_1^{-1}\, x_1\, y_5\, y_{33}^{-1}$$
$$r^0_{10} = y_6$$
$$r^0_{11} = y_7^{-1}$$
$$r^+_{12} = y_{11}^{-1}\, y_9$$
$$r^-_{13} = y_{77}\, x_1^{-2}\, x_1^2\, y_{53}^{-1}\, y_{12}^{-1}\, x_3^{-1}\, y_{41}\, x_1^{-2}\, x_1^2\, y_{25}^{-1}\, y_{10}$$
$$r^0_{14} = y_{12}$$
$$r^0_{15} = y_{15}^{-1}\, y_{11}^{-1}$$
$$r^+_{16} = y_{13}^{-1}\, y_{15}$$
$$r^-_{17} = x_1^{-1}\, x_1\, y_{45}^{-1}\, x_3^{-1}\, y_{14}^{-1}\, x_1^{-1}\, x_1\, y_{16}$$
$$r^0_{18} = y_{14}$$
$$r^0_{19} = x_3\, y_{13}$$
$$r^+_{20} = y_{26}\, y_{19}^{-1}\, y_{17}$$
$$r^-_{21} = x_1^{-2}\, x_1^2\, x_2^{-1}\, y_{20}^{-1}\, x_1\, x_1^{-1}\, y_{61}\, x_1^{-2}\, x_1^2\, y_{69}^{-1}\, x_1\, x_1^{-1}\, y_{18}$$
$$r^0_{22} = y_{20}$$
$$r^0_{23} = y_{65}^{-1}\, x_1^3\, x_1^{-1}\, x_1\, x_1^{-3}\, y_{85}\, y_{21}\, y_{19}^{-1}$$
$$r^+_{24} = x_1^{-1}\, x_1\, y_{23}^{-1}\, y_{21}$$
$$r^-_{25} = y_{24}^{-1}\, x_2\, y_{89}\, x_1^{-3}\, x_1^3\, y_{81}^{-1}\, y_{22}$$
$$r^0_{26} = y_{24}$$
$$r^0_{27} = x_2\, y_{23}^{-1}$$
$$r^+_{28} = y_{28}^{-1}\, y_{26}$$
$$r^-_{29} = x_1^{-3}\, x_1^3\, y_{27}^{-1}\, y_{25}$$
$$r^0_{30} = y_{27}$$

$r^0_{31} = y^{-1}_{30}\, y_{28}$
$r^+_{32} = y^{-1}_{32}\, x_1\, x^{-1}_1\, y_{30}$
$r^-_{33} = x^{-4}_1\, x^4_1\, y^{-1}_{31}\, y_{29}$
$r^0_{34} = y_{31}$
$r^0_{35} = y^{-1}_{34}\, y_{32}$
$r^+_{36} = y^{-1}_{36}\, x^{-2}_1\, x^2_1\, y_{34}$
$r^-_{37} = x^3_1\, x^{-3}_1\, x^{-1}_1\, x_1\, x^3_1\, x^{-3}_1\, y^{-1}_{35}\, y_{33}$
$r^0_{38} = y_{35}$
$r^0_{39} = y^{-1}_{38}\, y_{36}$
$r^+_{40} = y^{-1}_{40}\, x_1\, x^{-1}_1\, y_{38}$
$r^-_{41} = x^{-3}_1\, x^3_1\, x^{-4}_1\, x^4_1\, x^{-3}_1\, x^3_1\, y^{-1}_{39}\, y_{37}$
$r^0_{42} = y_{39}$
$r^0_{43} = y^{-1}_{42}\, y_{40}$
$r^+_{44} = x_1\, x^3_1\, x^{-3}_1\, x_1\, x^2_1\, x^{-2}_1\, x_1\, x^{-3}_1\, y^{-1}_{44}\, y_{42}$
$r^-_{45} = y^{-1}_{43}\, y_{41}$
$r^0_{46} = y_{43}$
$r^0_{47} = y^{-1}_{46}\, y_{44}$
$r^+_{48} = y^{-1}_{48}\, y_{46}$
$r^-_{49} = x_1\, x^{-1}_1\, x^{-2}_1\, x^2_1\, x_1\, x^{-1}_1\, y^{-1}_{47}\, x_1\, x^{-1}_1\, y_{45}$
$r^0_{50} = y_{47}$
$r^0_{51} = y^{-1}_{50}\, y_{48}$
$r^+_{52} = y^{-1}_{52}\, y_{50}$
$r^-_{53} = x_1\, x^{-1}_1\, x^{-2}_1\, x^2_1\, x_1\, x^{-1}_1\, y^{-1}_{51}\, x_1\, x^{-1}_1\, y_{49}$
$r^0_{54} = y_{51}$
$r^0_{55} = y^{-1}_{54}\, y_{52}$
$r^+_{56} = y^{-1}_{56}\, y_{54}$
$r^-_{57} = x_1\, x^{-1}_1\, x^{-2}_1\, x^2_1\, x_1\, x^{-1}_1\, y^{-1}_{55}\, x_1\, x^{-1}_1\, y_{53}$
$r^0_{58} = y_{55}$
$r^0_{59} = y^{-1}_{58}\, y_{56}$
$r^+_{60} = y^{-1}_{60}\, x^{-1}_1\, x_1\, y_{58}$
$r^-_{61} = x^{-3}_1\, x^3_1\, y^{-1}_{59}\, y_{57}$

$r^0_{62} = y_{59}$
$r^0_{63} = y^{-1}_{62}\, y_{60}$
$r^+_{64} = y^{-1}_{64}\, y_{62}$
$r^-_{65} = x^{-2}_1\, x^2_1\, y^{-1}_{63}\, x_1\, x^{-1}_1\, y_{61}$
$r^0_{66} = y_{63}$
$r^0_{67} = y^{-1}_{66}\, y_{64}$
$r^+_{68} = y^{-1}_{68}\, x^{-3}_1\, x^3_1\, y_{66}$
$r^-_{69} = y^{-1}_{67}\, y_{65}$
$r^0_{70} = y_{67}$
$r^0_{71} = y^{-1}_{70}\, y_{68}$
$r^+_{72} = y^{-1}_{72}\, y_{70}$
$r^-_{73} = x^{-2}_1\, x^2_1\, y^{-1}_{71}\, x_1\, x^{-1}_1\, y_{69}$
$r^0_{74} = y_{71}$
$r^0_{75} = y^{-1}_{74}\, y_{72}$
$r^+_{76} = x^3_1\, x^{-1}_1\, x_1\, x^{-3}_1\, y^{-1}_{76}\, x^{-1}_1\, x_1\, y_{74}$
$r^-_{77} = y^{-1}_{75}\, y_{73}$
$r^0_{78} = y_{75}$
$r^0_{79} = y^{-1}_{78}\, y_{76}$
$r^+_{80} = y^{-1}_{80}\, y_{78}$
$r^-_{81} = x_1\, x^{-1}_1\, x^{-2}_1\, x^2_1\, x_1\, x^{-1}_1\, y^{-1}_{79}\, x_1\, x^{-1}_1\, y_{77}$
$r^0_{82} = y_{79}$
$r^0_{83} = y^{-1}_{82}\, y_{80}$
$r^+_{84} = y^{-1}_{84}\, y_{82}$
$r^-_{85} = x^{-3}_1\, x^3_1\, y^{-1}_{83}\, y_{81}$
$r^0_{86} = y_{83}$
$r^0_{87} = y^{-1}_{86}\, y_{84}$
$r^+_{88} = y^{-1}_{88}\, x^{-3}_1\, x^3_1\, y_{86}$
$r^-_{89} = y^{-1}_{87}\, y_{85}$
$r^0_{90} = y_{87}$
$r^0_{91} = y^{-1}_{90}\, y_{88}$
$r^+_{92} = x^3_1\, x^{-3}_1\, y^{-1}_{92}\, y_{90}$

$$r^-_{93} = y^{-1}_{91}\, y_{89}$$
$$r^0_{94} = y_{91}$$
$$r^0_{95} = x_2\, x^{-1}_1\, x_2\, y_{92}$$

## References


[1] Brown, Richard A. Generalized group presentation and formal deformations of CW complexes. *Trans. Amer. Math. Soc.* **334** (1992), no. 2, 519–549. MR1153010 (93h:57001), Zbl 0764.57012.

[2] Boileau, M.; Zieschang, Heiner. Heegaard genus of closed orientable Seifert 3-manifolds. *Invent. Math.* **76** (1984), no. 3, 455–468. MR0746538 (86a:57008), Zbl 0538.57004.

[3] Craggs, Robert. On doubled 3-manifolds and minimal handle presentations for 4-manifolds. manuscript, 2000.

[4] Craggs, Robert. Freely reducing group readings for 2-complexes in 4-manifolds. *Topology* **28** (1989), 247–271. MR1003586 (90k:57018), Zbl 0695.57002.

[5] Craggs, Robert. Links in 3-manifolds as obstructions in free reduction problems. *Top. and App.* **49** (1993), 15–53. MR1202875 (93k:57002), Zbl 0782.57003.

[6] Haken, Wolfgang. Some results on surfaces in 3-manifolds. *Studies in Modern Topology*, 39–98. *Math. Assoc. of Amer.*, 1968. MR0224071 (36 #7118), Zbl 0194.24902.

[7] Kapitza, Paul J. Plates for 'On small geometric invariants of 3-manifolds'. New York Journal of Mathematics. `http://nyjm.albany.edu/j/2011/pkplates.zip`.

[8] Kreher, Reinhold; Metzler, Wolfgang. Simpliziale Transformationen von Polyedern und die Zeeman-Vermutung, *Topology* **22** (1983) , no. 1, 19–26. MR0682057 (84e:57022), Zbl 0521.57014.

[9] Montesinos-Amilibia, José M. La discrepancia entre el rango y el genero de Heegaard de una 3-variedad. *Conference on Differential Geometry and Topology* (Italian) (Lecce, 1989). *Note di Matematica* **IX** (1989), suppl., 101-117. MR1154135 (93e:57030), Zbl 0747.57009.

[10] Montesinos, José María. Note on a result of Boileau–Zieschang. *Low-dimensional topology and Kleinian groups* (Coventry/Durham, 1984), 241-252. London Math. Soc. Lecture Note Ser., 112. *Cambridge Univ. Press, Cambridge*, 1986. MR0903868 (89f:57002), Zbl 0619.57003.

[11] Schultens, Jennifer; Weidman, Richard. On the geometric and algebraic rank of graph manifolds. *Pacific J. Math.* **231** (2007), 481–510. MR2346507 (2009a:57030), Zbl 1171.57020.

[12] Waldhausen, Friedhelm. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math.* (2) **87** (1968), 56–88. MR0224099 (36 #7146),Zbl 0157.30603.

[13] Whitehead, J. H. C. On certain sets of elements in a free group. *Proc. London Math. Soc.* (2) **41** (1936) 48–56. Zbl 0013.24801.

[14] Wright, Perrin. Group presentations and formal deformations. *Trans. Amer. Math. Soc.* **208** (1975) 161–169. MR0380813 (52 #1710), Zbl 0318.57010.

[15] Young, S. Contractible 2-complexes. Christ's College, University of Cambridge U.K., unpublished manuscript, preprint, 1976. 161–169.


Berry College
`pkapitza@berry.edu`

This paper is available via `http://nyjm.albany.edu/j/2011/17-18.html`.