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N_φ-type quotient modules on the torus

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Abstract. Structure of the quotient modules in $H^2(\Gamma^2)$ is very complicated. A good understanding of some special examples will shed light on the general picture. This paper studies the so-called N_φ-type quotient modules, namely, quotient modules of the form $H^2(\Gamma^2) \ominus [z - \varphi]$, where $\varphi(w)$ is a function in the classical Hardy space $H^2(\Gamma)$ and $[z-\varphi]$ is the submodule generated by $z - \varphi(w)$. This type of quotient module provides good examples in many studies. A notable fact is its close connections with some classical operators, namely the Jordan block and the Bergman shift. This paper studies spectral properties of the compressions S_z and S_w, compactness of evaluation operators, and essential reductivity of $H^2(\Gamma^2) \ominus [z - \varphi]$.

Contents

1. Introduction

Let $H^2(\Gamma^2)$ be the Hardy space on the two-dimensional torus Γ^2. We denote by z and w the coordinate functions. Shift operators T_z and T_w on $H^2(\Gamma^2)$ are defined by $T_z f = zf$ and $T_w f = wf$ for $f \in H^2(\Gamma^2)$. Clearly, both T_z and T_w have infinite multiplicity. A closed subspace M of $H^2(\Gamma^2)$

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is called a *submodule* (over the algebra $H^\infty(\mathbb{D}^2)$), if it is invariant under multiplications by functions in $H^\infty(\mathbb{D}^2)$. Here $\mathbb{D}$ stands for the open unit disk. Equivalently, M is a submodule if it is invariant for both T_z and T_w. The quotient space $N := H^2(\Gamma^2) \ominus M$ is called a *quotient module*. Clearly $T_z^* N \subset N$ and $T_w^* N \subset N$. And for this reason N is also said to be backward shift invariant. In the study here, it is necessary to distinguish the classical Hardy space in the variable z and that in the variable w, for which we denote by $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$, respectively. $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ are thus different subspaces in $H^2(\Gamma^2)$. We will simply write $H^2(\Gamma)$ when there is no need to tell the difference. In $H^2(\Gamma)$, it is well-known as the Beurling theorem that if $M \subset H^2(\Gamma)$ is invariant for T_z, then $M = qH^2(\Gamma)$ for an inner function $q(z)$. The structure of submodules in $H^2(\Gamma^2)$ is much more complex, and there has been a great amount of work on this subject in recent years. A good reference of this work can be found in [3]. One natural approach to the problem is to find and study some relatively simple submodules, and hope that the study will generate concepts and general techniques that will lead to a better understanding of the general picture. This in fact has become an interesting and encouraging work.

In this paper, we look at submodules of the form $[z - \varphi(w)]$, where φ is a function in $H^2(\Gamma_w)$ with $\varphi \neq 0$ and $[z - \varphi(w)]$ is the closure of $(z - \varphi)H^\infty(\Gamma^2)$ in $H^2(\Gamma^2)$. For simplicity we denote $[z - \varphi(w)]$ by M_φ. One good way of studying M_φ is through the so-called *two variable Jordan block* (S_z, S_w) defined on the quotient module

$$N_\varphi := H^2(\Gamma^2) \ominus M_\varphi.$$

For every quotient module N, the two variable Jordan block (S_z, S_w) is the compression of the pair (T_z, T_w) to N, or more precisely,

$$S_z f = P_N z f, \quad S_w f = P_N w f, \quad f \in N,$$

where $P_N : H^2(\Gamma^2) \to N$ is the orthogonal projection. This paper studies interconnections between the quotient module N_φ, the two variable Jordan block (S_z, S_w) and the function φ. Some related work has been done in [14, 22, 23]. By [14], $N_\varphi \neq \{0\}$ if and only if $\varphi(\mathbb{D}) \cap \mathbb{D} \neq \emptyset$. If $\varphi = 0$, then $M_\varphi = zH^2(\Gamma^2)$ and $N_\varphi = H^2(\Gamma_w)$, so we assume that $\varphi \neq 0$. For convenience, we let

$$\Omega_\varphi = \{w \in \mathbb{D} : |\varphi(w)| < 1\},$$

and assume throughout the paper that $N_\varphi \neq \{0\}$, i.e., $\varphi(\mathbb{D}) \cap \mathbb{D} \neq \emptyset$. The paper is organized as follows.

Section 1 is the introduction.

Section 2 introduces some useful tools and states a few related known results.

Section 3 studies the spectral properties of the operators S_z and S_w. It is interesting to see how these properties depend on the function φ.

A notable phenomenon in many cases is the compactness of the defect operators $I - S_z S_z^*$ and $I - S_z^* S_z$. Section 4 aims to study how the compactness is related to the properties of φ.

The quotient module N_φ has very rich structure. Indeed, when φ is inner, N_φ can be identified with the tensor product of two well-known classical spaces, namely the quotient space $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L_a^2(\mathbb{D})$. Section 5 makes a detailed study of this case.

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2. Preliminaries

For every $\lambda \in \mathbb{D}$, we define a *left evaluation* operator $L(\lambda)$ from $H^2(\Gamma^2)$ to $H^2(\Gamma_w)$ and a *right evaluation* operator $R(\lambda)$ from $H^2(\Gamma^2)$ to $H^2(\Gamma_z)$ by

$$L(\lambda)f(w) = f(\lambda, w), \quad R(\lambda)f(z) = f(z, \lambda), \quad f \in H^2(\Gamma^2).$$

Clearly, $L(\lambda)$ and $R(\lambda)$ are operator-valued analytic functions over $\mathbb{D}$. Restrictions of $L(\lambda)$ and $R(\lambda)$ to quotient spaces N, $M \ominus zM$ and $M \ominus wM$ play key roles in the study here. The following lemma is from [4].

Lemma 2.1. *The restriction of $R(\lambda)$ to $M \ominus wM$ is equivalent to the characteristic operator function for S_w.*

The following spectral relations are thus clear. Details can be found in [4] and [18].

(a) $\lambda \in \sigma(S_w)$ if and only if $R(\lambda) : M \ominus wM \to H^2(\Gamma_z)$ is not invertible.
(b) $\dim\ker(S_w - \lambda I) = \dim\ker(R(\lambda)|_{M\ominus wM})$.
(c) $S_w - \lambda I$ has a closed range if and only if $R(\lambda)(M \ominus wM)$ is closed.
(d) $S_w - \lambda I$ is Fredholm if and only if $R(\lambda)|_{M\ominus wM}$ is Fredholm, and in this case

$$\operatorname{ind}(S_w - \lambda I) = \operatorname{ind}(R(\lambda)|_{M\ominus wM}).$$

Restrictions $T_z^*|_{M\ominus zM}$ and $T_w^*|_{M\ominus wM}$ are also important here, and for simplicity they are denoted by D_z and D_w, respectively. Clearly,

$$D_z f(z, w) = \frac{f(z, w) - f(0, w)}{z}, \quad D_w f(z, w) = \frac{f(z, w) - f(z, 0)}{w},$$

and it is not hard to check that the ranges of D_z and D_w are subspaces of N. The following lemma (cf. [22]) gives a description of the defect operators for S_z, and it will be used often.

Lemma 2.2. *On a quotient module N:*

(i) $S_z^* S_z + D_z D_z^* = I$.
(ii) $S_z S_z^* + (L(0)|_N)^* L(0)|_N = I$.

A parallel version of Lemma 2.2 for S_w will also be used.

The operator D_z is a useful tool in this study. We first note that

$$D_z^* f = P_M zf, \quad f \in N.$$

So if $D_z^* f = 0$, then $zf \in N$. Clearly $zf \in \ker L(0)|_N$. Conversely, if h is in $\ker L(0)|_N$, then we can write $h = zh_0$. One checks easily that $h_0 \in \ker D_z^*$. This observation shows that

$$z \ker D_z^* = \ker L(0)|_N.$$

So on N_φ, since $L(0)|_{N_\varphi}$ is injective (cf. [14]), D_z^* has trivial kernel, i.e., the range $R(D_z)$ is dense in N_φ. The following theorem describes $R(D_z)$ in detail.

Theorem 2.3. *Let N be a quotient module of $H^2(\Gamma^2)$ and $M = H^2(\Gamma^2) \ominus N$. Suppose that $L(0)|_N$ is one to one and $R(D_z)$ is dense in N. Let $f \in N$. Then $f \in R(D_z)$ if and only if there exists a positive constant C_f depending on f such that $|\langle S_z^* h, f\rangle| \le C_f \|L(0)h\|$ for every $h \in N$.*

Proof. Suppose that $f \in R(D_z)$. Let $g \in M \ominus zM$ with $T_z^* g = f$. We have $g = zf + L(0)g$. Then for $h \in N$,

$$\begin{aligned} |\langle S_z^* h, f\rangle| &= |\langle h, zf\rangle| \\ &= |\langle h, g - L(0)g\rangle| \\ &= |\langle h, L(0)g\rangle| \\ &= |\langle L(0)h, L(0)g\rangle| \\ &\le \|L(0)g\|\|L(0)h\|. \end{aligned}$$

To prove the converse, suppose that there exists a positive constant C_f satisfying

$$|\langle S_z^* h, f\rangle| \le C_f \|L(0)h\|$$

for every $h \in N$. Since $L(0)$ on N is one to one, we have a map Λ defined by

$$\Lambda : L(0)N \ni u(w) \to L(0)^{-1}u \to \langle S_z^* L(0)^{-1}u, f\rangle \in \mathbb{C}.$$

Note that $L(0)^{-1}u \in N$. Obviously, Λ is linear and

$$|\Lambda u| = |\langle S_z^* L(0)^{-1}u, f\rangle| \le C_f \|L(0)L(0)^{-1}u\| = C_f \|u\|.$$

Hence by the Hahn–Banach theorem, Λ is extendable to a bounded linear functional on $H^2(\Gamma_w)$ and there exists $v(w) \in H^2(\Gamma_w)$ satisfying $\langle u, v\rangle = \Lambda u$ for every $u \in L(0)N$. We have

$$\langle u, v\rangle = \langle S_z^* L(0)^{-1}u, f\rangle = \langle L(0)^{-1}u, zf\rangle.$$

Since $v(w) \in H^2(\Gamma_w)$, $\langle u, v\rangle = \langle L(0)^{-1}u, v\rangle$. Therefore

$$\langle L(0)^{-1}u, zf - v\rangle = 0$$

for every $u \in L(0)N$. Since $L(0)^{-1}(L(0)N) = N$, we get $zf - v \perp N$. Hence $zf - v \in M$. Since $v(w) \in H^2(\Gamma_w)$, we have $T_z^*(zf - v) = f \in N$. This implies that $zf - v \in M \ominus zM$. Thus we get $f \in R(D_z)$. □

In the case of N_φ, [14] provides a very useful description of the functions in the space. Let $\varphi(w) \in H^2(\Gamma_w)$. For $f(w) \in H^2(\Gamma_w)$, we formally define a function

$$(T_\varphi^* f)(w) = \sum_{n=0}^{\infty} a_n w^n,$$

where

$$a_n = \int_0^{2\pi} \overline{\varphi}(e^{i\theta}) f(e^{i\theta}) e^{-in\theta} d\theta/2\pi = \langle f(w), \varphi(w) w^n \rangle.$$

Generally, $T_\varphi^* f$ may not be in $H^2(\Gamma_w)$. When $T_\varphi^* f \in H^2(\Gamma_w)$, we can define $T_\varphi^{*2} f = T_\varphi^*(T_\varphi^* f)$. Inductively if $T_\varphi^{*n} f \in H^2(\Gamma_w)$, we can define $T_\varphi^{*(n+1)} f = T_\varphi^*(T_\varphi^{*n} f)$. For convenience, we let

$$A_\varphi f(z,w) = \sum_{n=0}^{\infty} z^n T_\varphi^{*n} f(w)$$

be an operator defined at every $f \in H^2(\Gamma_w)$ for which $A_\varphi f \in H^2(\Gamma^2)$. Then it is shown in [14] that $L(0)$ is one-to-one on N_φ and

$$N_\varphi = \left\{ A_\varphi f : f(w) \in H^2(\Gamma_w), \sum_{n=0}^{\infty} \|T_\varphi^{*n} f\|^2 < \infty \right\}. \tag{2.1}$$

It is easy to see that $L(0)A_\varphi f = f$. Moreover by [14, Corollary 2.8], $L(0)N_\varphi$ is dense in $H^2(\Gamma_w)$.

The following two lemmas are needed for the study of $\sigma(S_z)$.

Lemma 2.4. *Let $\varphi(w), g(w) \in H^2(\Gamma_w)$ and $\psi(w) \in H^\infty(\Gamma_w)$. Then*

$$T_\varphi^* T_\psi^* g = T_{\psi\varphi}^* g.$$

Moreover if $T_\varphi^* g \in H^2(\Gamma_w)$, then $T_\psi^* T_\varphi^* g = T_{\psi\varphi}^* g$.

Proof. Let $n \geq 0$. Then by the definitions above,

$$\langle T_\varphi^* T_\psi^* g, z^n \rangle = \langle g, \varphi\psi z^n \rangle = \langle T_{\varphi\psi}^* g, z^n \rangle.$$

Thus $T_\varphi^* T_\psi^* g = T_{\varphi\psi}^* g$. Suppose that $T_\varphi^* g \in H^2(\Gamma_w)$. We have $\overline{\varphi} g - T_\varphi^* g \in \overline{zH^1}$. Hence

$$\begin{aligned} \langle T_\psi^* T_\varphi^* g, z^n \rangle &= \langle T_\varphi^* g, \psi z^n \rangle \\ &= \int_0^{2\pi} \overline{\varphi}(e^{i\theta}) g(e^{i\theta}) \overline{\psi}(e^{i\theta}) e^{-in\theta} d\theta/2\pi \\ &= \langle g, \psi\varphi z^n \rangle. \end{aligned}$$

Thus we get our assertion. □

Let $w_0 \in \Omega_\varphi$. The following lemma follows easily from the calculation

$$T_\varphi^* \frac{1}{1-\overline{w}_0 w} = \frac{\overline{\varphi(w_0)}}{1-\overline{w}_0 w}.$$

Lemma 2.5. *For $w_0 \in \Omega_\varphi$,we have*

$$\frac{1}{(1-\overline{\varphi(w_0)}z)(1-\overline{w}_0 w)} \in N_\varphi.$$

3. The spectra of S_z and S_w

The spectra of S_z and S_w on N_φ is evidently dependent on φ. This section aims to figure out how they are exactly related. Lemma 2.1 and the description in (2.1) are helpful to this end.

Proposition 3.1. *$\overline{\varphi(\mathbb{D}) \cap \mathbb{D}} \subset \sigma(S_z) \subset \overline{\varphi(\mathbb{D})} \cap \overline{\mathbb{D}}$.*

Proof. Let $w_0 \in \varphi(\mathbb{D}) \cap \mathbb{D}$. Then $w_0 = \varphi(w_1)$ for some $w_1 \in \mathbb{D}$ and

$$\begin{aligned} S_z^* \left(\frac{1}{(1-\overline{\varphi(w_1)}z)(1-\overline{w}_1 w)} \right) &= \sum_{n=1}^{\infty} \left(\overline{\varphi(w_1)}^n (1-\overline{w}_1 w)^{-1} \right) z^{n-1} \\ &= \overline{\varphi(w_1)} \left(\frac{1}{(1-\overline{\varphi(w_1)}z)(1-\overline{w}_1 w)} \right). \end{aligned}$$

By Lemma 2.5, $\overline{\varphi(w_1)}$ is a point spectrum of S_z^*. Thus we get $\overline{\varphi(\mathbb{D}) \cap \mathbb{D}} \subset \sigma(S_z)$.

Let $\lambda \notin \overline{\varphi(\mathbb{D})}$. Then $1/(\varphi(w) - \lambda) \in H^\infty(\Gamma_w)$. Let $F \in N_\varphi$. We have

$$\begin{aligned} S_{1/(\varphi-\lambda)}^* F &= S_{1/(\varphi-\lambda)}^* \sum_{n=0}^{\infty} (T_\varphi^{*n} L(0)F) z^n \\ &= \sum_{n=0}^{\infty} (T_\varphi^{*n} T_{1/(\varphi-\lambda)}^* L(0)F) z^n \qquad \text{by Lemma 2.4.} \end{aligned}$$

Hence

$$\begin{aligned} S_{1/(\varphi-\lambda)}^* S_{z-\lambda}^* F &= \sum_{n=0}^{\infty} (T_\varphi^{*n} T_{1/(\varphi-\lambda)}^* L(0) S_{z-\lambda}^* F) z^n \\ &= \sum_{n=0}^{\infty} (T_\varphi^{*n} T_{1/(\varphi-\lambda)}^* T_{\varphi-\lambda}^* L(0)F) z^n \\ &= \sum_{n=0}^{\infty} (T_\varphi^{*n} L(0)F) z^n \qquad \text{by Lemma 2.4} \\ &= F. \end{aligned}$$

Also we have

$$\begin{aligned}
&S^*_{z-\lambda}S^*_{1/(\varphi-\lambda)}F \\
&\quad= \sum_{n=1}^{\infty}(T^{*n}_\varphi T^*_{1/(\varphi-\lambda)}L(0)F)z^{n-1} - \bar\lambda\sum_{n=0}^{\infty}(T^{*n}_\varphi T^*_{1/(\varphi-\lambda)}L(0)F)z^n \\
&\quad= \sum_{n=0}^{\infty}(T^{*n}_\varphi T^*_\varphi T^*_{1/(\varphi-\lambda)}L(0)F)z^n - \bar\lambda\sum_{n=0}^{\infty}(T^{*n}_\varphi T^*_{1/(\varphi-\lambda)}L(0)F)z^n \\
&\quad= \sum_{n=0}^{\infty}(T^{*n}_\varphi T^*_{(\varphi-\lambda)}T^*_{1/(\varphi-\lambda)}L(0)F)z^n \\
&\quad= F.
\end{aligned}$$

Thus $(S_z-\lambda)^{-1} = S_{1/(\varphi-\lambda)}$ and hence $\lambda \notin \sigma(S_z)$.

Since $\|S_z\| \le 1$, we have our assertion. □

For a submodule M in $H^2(\Gamma^2)$, the quotient space $M \ominus zM$ is a wandering subspace for the multiplication by z and we have

$$M = \sum_{n=0}^{\infty} \oplus\, z^n(M \ominus zM).$$

For a fixed $\lambda \in \mathbb{D}$ and every $f \in M$, we write $f = \sum_{j=0}^{\infty} z^j f_j$ for some unique sequence $\{f_j\}$ in $M \ominus zM$. So

$$f = \sum_{j=0}^{\infty}\lambda^j f_j + \sum_{j=0}^{\infty}(z^j - \lambda^j)f_j,$$

which means that $f = h_1 + (z-\lambda)h_2$ for some $h_1 \in M \ominus zM$ and $h_2 \in M$. If $h_1 + (z-\lambda)h_2 = 0$, then $h_1 + zh_2 = \lambda h_2$, and hence $|\lambda|^2\|h_2\|^2 = \|h_1\|^2 + \|h_2\|^2$, which is possible only if $h_1 = h_2 = 0$. This observation shows that M can be expressed as the direct sum

$$M = (M \ominus zM) \dotplus (z-\lambda)M. \tag{3.1}$$

We now look at the spectral properties of S_w.

Proposition 3.2. *On N_φ:*

(i) $\overline{\Omega}_\varphi \subset \sigma(S_w)$.
(ii) $S_w - \alpha I$ *is Fredholm for every* $\alpha \in \Omega_\varphi$ *and* $\mathrm{ind}(S_w - \alpha I) = -1$.

Proof. We use Lemma 2.1 to this end.

(i) It is sufficient to show $\Omega_\varphi \subset \sigma(S_w)$. If $\alpha \in \Omega_\varphi$, then for any function $(z-\varphi)h(z,w)$ in $M_\varphi \ominus wM_\varphi$, $(z-\varphi(\alpha))h(z,\alpha)$ vanishes at $\varphi(\alpha)$, and therefore $R(\alpha)(M_\varphi \ominus wM_\varphi) \subset (z-\varphi(\alpha))H^2(\Gamma_z) \ne H^2(\Gamma_z)$. By Lemma 2.1, $\alpha \in \sigma(S_w)$.

(ii) It is equivalent to show that $R(\alpha)|_{M_\varphi \ominus wM_\varphi}$ is Fredholm with index -1. We first show that $R(\alpha)$ is injective on $M_\varphi \ominus wM_\varphi$ for every $\alpha \in \Omega_\varphi$. Let

$(z-\varphi)h(z,w)$ be in M_φ. Then there is a sequence of polynomials $\{p_n(z,w)\}_n$ such that $(z-\varphi)p_n$ converges to $(z-\varphi)h$ in the norm of $H^2(\Gamma^2)$. Since $R(\alpha)$ is a bounded operator, $(z-\varphi(\alpha))p_n(z,\alpha)$ converges to $(z-\varphi(\alpha))h(z,\alpha)$, which, by the fact $|\varphi(\alpha)|<1$, implies that $p_n(z,\alpha)$ converges to $h(z,\alpha)$ in $H^2(\Gamma_z)$. Since for every $f\in H^2(\Gamma_z)$, we have $\|\varphi f\|=\|\varphi\|\|f\|$ and hence

$$\|(z-\varphi)f\|\le\|zf\|+\|\varphi f\|=(1+\|\varphi\|)\|f\|<\infty, \tag{3.2}$$

so $(z-\varphi)p_n(z,\alpha)$ converges to $(z-\varphi)h(z,\alpha)$ in M_φ. It follows that

$$\lim_{n\to\infty}(z-\varphi)\frac{p_n-p_n(\cdot,\alpha)}{w-\alpha}=(z-\varphi)\frac{h-h(\cdot,\alpha)}{w-\alpha},$$

which implies that $(z-\varphi)\frac{h-h(\cdot,\alpha)}{w-\alpha}\in M_\varphi$. If $(z-\varphi)h(z,w)$ is in $M_\varphi\ominus wM_\varphi$ such that $(z-\varphi(\alpha))h(z,\alpha)=0$, then $h(z,\alpha)=0$, and it follows from the observation above that

$$(z-\varphi)h=(w-\alpha)(z-\varphi)\frac{h}{w-\alpha}\in(w-\alpha)M_\varphi,$$

and hence by (3.1) $(z-\varphi)h(z,w)=0$ which implies that $R(\alpha)$ is injective on $M_\varphi\ominus wM_\varphi$.

In the proof of (i), we showed that $R(\alpha)(M_\varphi\ominus wM_\varphi)\subset(z-\varphi(\alpha))H^2(\Gamma_z)$. On the other hand, for every $g\in H^2(\Gamma_z)$, $(z-\varphi)g$ is in M_φ by (3.2), and by (3.1)

$$(z-\varphi(\alpha))g\in R(\alpha)(M_\varphi)=R(\alpha)(M_\varphi\ominus wM_\varphi).$$

This shows that

$$R(\alpha)(M_\varphi\ominus wM_\varphi)=(z-\varphi(\alpha))H^2(\Gamma_z),$$

i.e., $R(\alpha)|_{M_\varphi\ominus wM_\varphi}$ has a closed range with codimension 1, and this completes the proof in view of Lemma 2.1. □

Corollary 3.3. *If φ is bounded with $\|\varphi\|_\infty\le1$, then $\sigma(S_w)=\overline{\mathbb{D}}$ and $\sigma_e(S_w)=\Gamma$.*

Proof. By Proposition 3.2 and the fact that S_w is a contraction, $\sigma(S_w)=\overline{\mathbb{D}}$ and $\sigma_e(S_w)\subset\Gamma$. Since $\operatorname{ind}(S_w)=-1$, $\sigma_e(S_w)$ is a closed curve, and therefore $\sigma_e(S_w)=\Gamma$. □

We will mention another somewhat deeper consequence of Proposition 3.2 near the end of this section. Here we continue to study the Fredholmness of S_z. Unfortunately, the techniques used for Proposition 3.2(ii) can not be applied directly to the case here and a technical difficulty seems hard to overcome. So instead we use (3.1) in this case. We begin with some simple observations.

Lemma 3.4. *Let $\varphi(w)=b(w)h(w)$ be the inner-outer factorization of $\varphi(w)$. Then $\ker S_z^*=H^2(\Gamma_w)\ominus b(w)H^2(\Gamma_w)$.*

Proof. Since the functions in $H^2(\Gamma_w)\ominus b(w)H^2(\Gamma_w)$ depend only on w, the inclusion

$$H^2(\Gamma_w)\ominus b(w)H^2(\Gamma_w)\subset \ker S_z^*$$

is easy to check.

If f is a function in N_φ such that $S_z^* f = 0$, then $\bar{z}f$ is orthogonal to $H^2(\Gamma^2)$ which means f is independent of the variable z. Since for every nonnegative integer j

$$0=\langle (z-\varphi)w^j, f\rangle = \langle -\varphi w^j, f\rangle,$$

f is in $H^2(\Gamma_w)\ominus b(w)H^2(\Gamma_w)$. □

Theorem 3.5. *Let $\varphi(w)=b(w)h(w)$ be the inner-outer factorization of φ and*

$$\alpha=\inf_{w\in\mathbb{D}}|h(w)|.$$

Then S_z^* has a closed range if and only if $\alpha\neq 0$, and in this case $S_z^* N_\varphi = N_\varphi$.

Proof. Write $K_b = H^2(\Gamma_w)\ominus b(w)H^2(\Gamma_w)$. By Lemma 3.4, $\ker S_z^* = K_b$.

Suppose that $\alpha>0$. Then $h(w)^{-1}\in H^\infty(\Gamma_w)$ and $\|T^*_{h^{-1}}\| = \|h^{-1}\|_\infty = \alpha^{-1}$. Let $F\in N_\varphi\ominus K_b$. We can write $(L(0)F)(w)=b(w)f(w)$. Then by (2.1),

$$\begin{aligned}
\|F\|^2 &= \left\|\sum_{n=0}^\infty z^n T_\varphi^{*n} bf\right\|^2\\
&= \sum_{n=0}^\infty \|T_\varphi^{*n} bf\|^2\\
&\geq \|f\|^2 + \|T_\varphi^* bf\|^2\\
&= \|f\|^2 + \|T_h^* f\|^2\\
&= \|f\|^2 + \alpha^2\alpha^{-2}\|T_h^* f\|^2\\
&= \|f\|^2 + \alpha^2\|T^*_{h^{-1}}\|^2\|T_h^* f\|^2\\
&\geq \|f\|^2 + \alpha^2\|f\|^2 \qquad \text{by Lemma 2.4}\\
&= (1+\alpha^2)\|L(0)F\|^2.
\end{aligned}$$

Since by Lemma 2.2 $\|S_z^*F\|^2 + \|L(0)F\|^2 = \|F\|^2$,

$$\|S_z^*F\|^2 = \|F\|^2 - \|L(0)F\|^2 \geq \Big(1-\frac{1}{1+\alpha^2}\Big)\|F\|^2 = \frac{\alpha^2}{1+\alpha^2}\|F\|^2.$$

This implies that S_z^* is bounded below on $N_\varphi\ominus K_b$, and hence S_z^* has a closed range.

Suppose that $\alpha = 0$. Let $\{w_k\}_k$ be a sequence in $\mathbb{D}$ satisfying $|h(w_k)| < 1$ and $h(w_k) \to 0$ as $k \to \infty$. Let

$$F_k(z,w) = \frac{b(w)}{1-\overline{w}_k w} + \sum_{n=1}^{\infty} z^n \frac{\overline{b(w_k)}^{(n-1)}\ \overline{h(w_k)}^n}{1-\overline{w}_k w}.$$

Then

$$\|F_k\|^2 \geq \left\| \frac{1}{1-\overline{w}_k w} \right\|^2.$$

Using the fact that $T_g^*(1/(1-\overline{w}_k w)) = \overline{g(w_k)}(1/(1-\overline{w_k} w))$ for every $g \in H^2(\Gamma_w)$, we have

$$F_k(z,w) = \sum_{n=0}^{\infty} z^n T_\varphi^{*n} \frac{b(w)}{1-\overline{w}_k w} \in N_\varphi \ominus K_b,$$

and therefore

$$S_z^* F_k = \sum_{n=0}^{\infty} z^n \frac{\overline{b(w_k)}^n\ \overline{h(w_k)}^{(n+1)}}{1-\overline{w}_k w},$$

and

$$\|S_z^* F_k\|^2 \leq \left\| \frac{1}{1-\overline{w}_k w} \right\|^2 \frac{|h(w_k)|^2}{1-|h(w_k)|^2}.$$

It follows

$$\|S_z^* F_k\|^2 \leq \frac{|h(w_k)|^2}{1-|h(w_k)|^2} \|F_k\|^2.$$

This implies that S_z^* is not bounded below on $N_\varphi \ominus K_b$. Since S_z^* is one-to-one on $N_\varphi \ominus K_b$, $S_z^*(N_\varphi \ominus K_b)$ is not a closed subspace. Since $S_z^*(N_\varphi) = S_z^*(N_\varphi \ominus K_q)$, S_z^* does not have a closed range.

Next we shall prove that $S_z^* N_\varphi = N_\varphi$ when $\alpha > 0$. Let $g(w) \in L(0)N_\varphi$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \|T_\varphi^{*n} T_{h^{-1}}^* bg\|^2 &= \|T_{h^{-1}}^* bg\|^2 + \sum_{n=1}^{\infty} \|T_\varphi^{*(n-1)} g\|^2 \\ &\leq \|h^{-1}\|_\infty^2 \|g\|^2 + \|L(0)^{-1} g\|^2 \\ &< \infty.\end{aligned}$$

Hence $T_{h^{-1}}^* bg \in L(0)N_\varphi$, and

$$\begin{aligned}S_z^* L(0)^{-1} T_{h^{-1}}^* bg &= \sum_{n=1}^{\infty} z^{n-1} T_\varphi^{*n} T_{h^{-1}}^* bg \\ &= \sum_{n=1}^{\infty} z^{n-1} T_\varphi^{*(n-1)} g \\ &= L(0)^{-1} g.\end{aligned}$$

This implies that $S_z^* N_\varphi = N_\varphi$. □

Corollary 3.6. *With notations as in Theorem* 3.5*, the following conditions are equivalent.*

(i) $\alpha \neq 0$.
(ii) S_z^* *has a closed range.*
(iii) $S_z^* N_\varphi = N_\varphi$.
(iv) $T_\varphi^* L(0) N_\varphi = L(0) N_\varphi$.

Theorem 3.5 in particular shows that S_z is injective when $\alpha > 0$. This is in fact a general phenomenon on N_φ. The following fact (cf. [5, p. 85]) is needed to this end.

Lemma 3.7. *Let $h(w)$ be an outer function on Γ_w. Then there is a sequence of outer functions $\{h_k\}_k$ in $H^\infty(\Gamma_w)$ such that $\|h_k h\|_\infty \le 1$ and $h_k h \to 1$ a.e. on Γ_w as $k \to \infty$.*

Theorem 3.8. *S_z is injective on N_φ.*

Proof. We show that S_z^* has a dense range. Let $\varphi(w) = b(w)h(w)$ be the inner-outer factorization of φ. By Lemma 3.7, there is a sequence $\{h_k\}_k$ in $H^\infty(\Gamma_w)$ such that

$$\|h_k h\|_\infty \le 1 \ \text{ and } \ h_k h \to 1 \ \text{ a.e. on } \Gamma_w \text{ as } k \to \infty. \tag{3.3}$$

Let $g(w) \in L(0)N_\varphi$. By Lemma 2.4, we have

$$\begin{aligned}
\sum_{n=0}^\infty \|T_\varphi^{*n} T_{h_k}^* bg\|^2 &= \|T_{h_k}^* bg\|^2 + \sum_{n=1}^\infty \|T_{h_k h}^* T_\varphi^{*(n-1)} g\|^2 \\
&\le \|h_k\|_\infty^2 \|g\|^2 + \sum_{n=1}^\infty \|T_\varphi^{*(n-1)} g\|^2 \qquad \text{by (3.3)} \\
&= \|h_k\|_\infty^2 \|g\|^2 + \|L(0)^{-1} g\|^2 \\
&< \infty.
\end{aligned}$$

Hence $T_{h_k}^* bg \in L(0)N_\varphi$, and we have

$$\begin{aligned}
\|S_z^* L(0)^{-1} T_{h_k}^* bg - L(0)^{-1} g\|^2 &= \sum_{n=0}^\infty \|T_\varphi^{*(n+1)} T_{h_k}^* bg - T_\varphi^{*n} g\|^2 \\
&= \sum_{n=0}^\infty \|T_{h_k h - 1}^* T_\varphi^{*n} g\|^2 \\
&\le \sum_{n=0}^\infty \|(\overline{h_k h} - 1) T_\varphi^{*n} g\|^2 \\
&= \int_0^{2\pi} |(hh_k)(e^{i\theta}) - 1|^2 \sum_{n=0}^\infty |(T_\varphi^{*n} g)(e^{i\theta})|^2 \, \frac{d\theta}{2\pi}.
\end{aligned}$$

Since $g \in L(0)N_\varphi$,

$$\sum_{n=0}^{\infty} |T_\varphi^{*n} g|^2 \in L^1(\Gamma_w).$$

Hence by (3.3) and the Lebesgue dominated convergence theorem,

$$\|S_z^* L(0)^{-1} T_{h_k}^* bg - L(0)^{-1} g\|^2 \to 0 \quad \text{as } k \to \infty.$$

This implies that S_z^* has a dense range. □

Corollary 3.9. *Let $\varphi(w) = b(w)h(w)$ be the inner-outer factorization of $\varphi(w)$. Then the following are equivalent.*

(i) *S_z is Fredholm.*
(ii) *$b(w)$ is a finite Blaschke product and $h^{-1}(w) \in H^\infty(\Gamma_w)$.*

In this case, $-\operatorname{ind}(S_z)$ is the number of zeros of $b(w)$ in $\mathbb{D}$ counting multiplicites.

Proof. We let $\alpha = \inf_{w\in\mathbb{D}} |h(w)|$. S_z is Fredholm if and only if S_z^* is Fredholm, and by Lemma 3.4 and Theorem 3.5 this is equivalent to b being a finite Blaschke product and $\alpha > 0$. Clearly, $\alpha > 0$ if and only if $h^{-1}(w) \in H^\infty(\Gamma_w)$. □

A quotient module N is said to be *essentially reductive* if both S_z and S_w are essentially normal, i.e., $[S_z^*, S_z]$ and $[S_w^*, S_w]$ are both compact. Essential reductivity is an important concept and has been studied recently in various contexts. In the context here, it will be interesting to see what type of φ makes N_φ essentially reductive. Proposition 3.2 has a couple of consequences to this end. A general study will be made in a different paper.

Corollary 3.10. *For every $\varphi \in H^2(\Gamma_w)$, $[S_z^*, S_w]$ is Hilbert–Schmidt on N_φ.*

Proof. We let R_z and R_w denote the multiplications by z and w on the submodule M_φ, respectively. It then follows from Proposition 3.2 and Theorem 2.3 in [21] that $[R_z^*, R_z][R_w^*, R_w]$ is Hilbert–Schmidt, and the corollary thus follows from Theorem 2.6 in [21]. □

In the case φ is in the disk algebra $A(\mathbb{D})$, there is a sequence of polynomials $\{p_n\}_n$ satisfying $p_n \to \varphi$ in $A(\mathbb{D})$, and hence $[S_z^*, p_n(S_w)] \to [S_z^*, \varphi(S_w)]$ in operator norm. Since $S_z = \varphi(S_w)$ on N_φ, we easily obtain the following corollary.

Corollary 3.11. *If $\varphi \in A(\mathbb{D})$, then S_z is essentially normal.*

Question 1. *For what $\varphi \in H^2(\Gamma_w)$ is S_w essentially normal on N_φ?*

In the case φ is inner, this question can be settled by direct calculations. We will do it in Section 5.

4. Compactness of $L(0)|_{N_\varphi}$ and D_z

In view of Lemma 2.2, the compactness of $L(0)|_N$ or D_z will give us much information about the operator S_z. So to determine whether $L(0)|_N$ or D_z is compact for a certain quotient module N is of great interest. In the case of N_φ, the compactness is undoubtly dependent on the properties of φ. This section aims to unveil the connection.

We first look at the compactness of $L(0)|_{N\varphi}$. For each fixed $\zeta \in \mathbb{D}$, we denote by $Z_\varphi(\zeta)$ the number of zeros of $\zeta - \varphi(w)$ in $\mathbb{D}$ counting multiplicities. This integer-valued function has an important role to play in this study. As a matter of fact, in [22, Theorem 5.2.2], the second author showed that if $L(0)$ on N_φ is compact, then $Z_\varphi(\zeta)$ is a finite constant on $\mathbb{D}$. The following describes the functions φ for which this is the case.

Lemma 4.1. *Let $\varphi(w) = b(w)h(w)$ be the inner-outer factorization of φ. Then $Z_\varphi(\zeta)$ is a finite constant on $\mathbb{D}$ if and only if b is a finite Blaschke product and $|h(w)| \geq 1$ for every $w \in \mathbb{D}$.*

Proof. It is easy to see that that b is a finite Blaschke product and $|h(w)| \geq 1$ for every $w \in \mathbb{D}$ if and only if

$$\liminf_{|w|\to 1} |\varphi(w)| \geq 1.$$

Suppose that $c = Z_\varphi(\zeta)$ for every $\zeta \in \mathbb{D}$. To prove the necessity by contradiction, we assume that there exists a sequence $\{w_n\}_n$ in $\mathbb{D}$ such that $\sup_n |\varphi(w_n)| < 1$ and $|w_n| \to 1$. We may assume that $\varphi(w_n) \to \zeta_0 \in \mathbb{D}$. Then there exists $r_0, 0 < r_0 < 1$, such that the number of zeros of $\zeta_0 - \varphi(w)$ in $r_0\mathbb{D}$ is equal to c. By the Hurwitz theorem, for a large positive integer n_0, the number of zeros of $\varphi(w_{n_0}) - \varphi(w)$ in $r_0\mathbb{D}$ is equal to c. Further, we may assume that $w_{n_0} \notin r_0\mathbb{D}$. Hence the number of zeros of $\varphi(w_{n_0}) - \varphi(w)$ in $\mathbb{D}$ is greater than c which contradicts the fact that $Z_\varphi(\zeta)$ is a constant.

The sufficiency is an easy consequence of Rouché's theorem in complex analysis. In fact, if $b(w)$ is a finite Blaschke product and $h(w)$ is an outer function with $|h(w)| \geq 1$ on $\mathbb{D}$, then by Rouché's theorem, for each $\zeta \in \mathbb{D}$ the number of zeros of $\zeta - \varphi(w)$ in $\mathbb{D}$ coincides with the number of zeros of $b(w)$ in $\mathbb{D}$. So $Z_\varphi(\zeta)$ is a finite constant. □

Theorem 4.2. *Let $\varphi(w) = b(w)h(w)$ be the inner-outer factorization of φ. Then the following conditions are equivalent.*

(i) *$L(0)$ on N_φ is compact.*
(ii) *b is a finite Blaschke product and $|h(w)| \geq 1$ for every $w \in \mathbb{D}$.*

Proof. (i) $\Rightarrow$ (ii) If $L(0)$ on N_φ is compact, then by Theorem 5.2.2 in [22] $Z_\varphi(\zeta)$ is a finite constant, and (ii) thus follows from Lemma 4.1.

(ii) $\Rightarrow$ (i) Since b is a finite Blaschke product, for any positive integer m, we have $\dim \left(H^2(\Gamma_w) \ominus b^m(w)H^2(\Gamma_w)\right) < \infty$ and $H^2(\Gamma_w) \ominus b^m(w)H^2(\Gamma_w)$

is contained in the disk algebra $A(\mathbb{D})$. One easily sees that

$$T_\varphi^{*j}\big(H^2(\Gamma_w) \ominus b^m(w)H^2(\Gamma_w)\big) = \{0\}, \quad j > m,$$

so that

$$H^2(\Gamma_w) \ominus b^m(w)H^2(\Gamma_w) \subset L(0)N_\varphi.$$

Then

$$L(0)N_\varphi = (H^2(\Gamma_w) \ominus b^m H^2(\Gamma_w)) \oplus (b^m H^2(\Gamma_w) \cap L(0)N_\varphi)$$

and hence

$$N_\varphi = L(0)^{-1}(H^2(\Gamma_w) \ominus b^m H^2(\Gamma_w)) \dotplus L(0)^{-1}(b^m H^2(\Gamma_w) \cap L(0)N_\varphi),$$

which is in fact a direct sum because $L(0)|_{N_\varphi}$ is injective. For simplicity we write this decomposition as

$$N_\varphi = N_{1,m} \dotplus N_{2,m}.$$

Since $\dim(N_{1,m}) < \infty$, to prove that $L(0)$ on N_φ is compact it is sufficient to prove that $\lim_{m\to\infty} \|L(0)|_{N_{2,m}}\| = 0$, i.e.,

$$\sup_{b^m g \in L(0)N_\varphi} \frac{\|b^m g\|^2}{\|L(0)^{-1} b^m g\|^2} \to 0 \quad \text{as } m \to \infty.$$

Let $b^m g \in L(0)N_\varphi$ and $0 \le n \le m$. By Lemma 2.4, $T_h^* b^{m-1} g = T_\varphi^* b^m g \in H^2(\Gamma_w)$, so that

$$T_h^{*2} b^{m-2} g = T_h^* T_h^* T_b^* b^{m-1} g = T_h^* T_b^* T_h^* b^{m-1} g = T_\varphi^{*2} b^m g \in H^2(\Gamma_w).$$

Repeating this, we have

$$T_h^{*n} b^{m-n} g = T_\varphi^{*n} b^m g \in H^2(\Gamma_w). \tag{4.1}$$

Using the fact that $L(0)A_\varphi f = f$, i.e.,

$$L(0)^{-1} f = \sum_{j=0}^{\infty} z^j T_\varphi^{*j} f,$$

and that $\|h^{-1}\|_\infty \le 1$, we calculate that

$$
\begin{aligned}
\sup_{b^m g\in L(0)N_\varphi} \frac{\|b^m g\|^2}{\|L(0)^{-1}b^m g\|^2} &= \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{j=0}^{\infty}\|T_\varphi^{*j}b^m g\|^2} \\
&\le \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{j=0}^{m}\|T_\varphi^{*j}b^m g\|^2} \\
&= \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{j=0}^{m}\|T_h^{*j}b^{m-j} g\|^2} \qquad \text{by (4.1)} \\
&\le \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{j=0}^{m}\|T_{h^{-1}}^{*j}\|^2\|T_h^{*j}b^{m-j} g\|^2} \\
&\le \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{j=0}^{m}\|b^{m-j} g\|^2} \qquad \text{by Lemma 2.4} \\
&= \frac{1}{m+1}.
\end{aligned}
$$

So it follows that $\lim_{m\to\infty}\|L(0)|_{N_{2,m}}\| = 0$ and this completes the proof. □

Corollary 4.3. *If $L(0)$ and $R(0)$ are both compact on N_φ then φ is a finite Blaschke product.*

Proof. If $R(0)$ is compact on N_φ, then by the parallel statement of Theorem 5.2.2 in [22] for $R(0)$, the number of zeros of $z-\varphi(\lambda)$ in $\mathbb{D}$ is a constant with respect to $\lambda\in\mathbb{D}$. Since N_φ is nontrivial, this constant is equal to 1. So $\|\varphi\|_\infty \le 1$, and it follows that $\|h\|_\infty \le 1$. If $L(0)$ is also compact on N_φ, then by Theorem 4.2 h is a constant of modulous 1, hence φ is a finite Blaschke product. □

In fact the converse of Corollary 4.3 is also true and we will see it in Section 5.

Next we study the compactness of D_z. In fact, the compactness of D_z and that of $L(0)|_{N_\varphi}$ are closely related.

Theorem 4.4. *If φ is bounded, then $L(0)|_{N_\varphi}$ is compact if and only if D_z is compact.*

Proof. The fact that the compactness of $L(0)|_{N_\varphi}$ implies the compactness of D_z follows from Theorem 3.8 and [22, Theorem 5.3.1].

To show that the compactness of D_z implies that of $L(0)|_{N_\varphi}$, we first check that S_z is Fredholm in this case. If D_z is compact, then by Lemma 2.2 $S_z^*S_z$ is Fredholm, and hence S_z^* has closed range. Moreover, it follows from Theorem 3.8 that S_z^* is in fact onto. So it remains to show that S_z^* has a finite-dimensional kernel. If we let $\varphi = bh$ be the inner-outer factorization of φ, then by Lemma 3.4 we need to show that $H^2(\Gamma_w)\ominus bH^2(\Gamma_w)$ is a finite-dimensional subspace in N_φ, or equivalently, b is a Blaschke product.

For every $f \in H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$ and integers $i, j \geq 0$, one checks that

$$\langle D_z^* f, (z-\varphi)z^i w^j\rangle = \langle zf, (z-\varphi)z^i w^j\rangle = \langle f, z^i w^j\rangle.$$

So $D_z^* f$ is orthogonal to $(z-\varphi)z^i w^j$ when $i \geq 1$. Therefore,

$$\begin{aligned}\|D_z^* f\| &= \|P_{M_\varphi} zf\| \\ &\geq \sup_{\|(z-\varphi)p\|\leq 1} |\langle zf, (z-\varphi)p\rangle|, \quad p \text{ is polynomial in } H^2(\Gamma_w) \\ &= \sup_{\|(z-\varphi)p\|\leq 1} |\langle f, p\rangle|.\end{aligned}$$

Since

$$\|(z-\varphi)p\|^2 = \|p\|^2 + \|\varphi p\|^2 \leq \|p\|^2(1+\|\varphi\|_\infty^2),$$

we have

$$\|D_z^* f\| \geq \sup_{\|p\|\leq (1+\|\varphi\|_\infty^2)^{-1/2}} |\langle f, p\rangle| = (1+\|\varphi\|_\infty^2)^{-1/2}\|f\|,$$

which means D_z^* is bounded below by a positive constant on $H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$. Since D_z is compact, $H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$ is finite-dimensional, and we conclude that S_z is Fredholm.

Now we show that $L(0)|_{N_\varphi}$ is compact. For this, we recall the equality (cf. Proposition 5.1.1 in [22])

$$S_z D_z + (L(0)|_{N_\varphi})^*(L(0)|_{M_\varphi \ominus zM_\varphi}) = 0.$$

Since D_z is compact, $(L(0)|_{N_\varphi})^*(L(0)|_{M_\varphi \ominus zM_\varphi})$ is compact. Since we have shown that S_z is Fredholm in this case, $L(0)|_{M_\varphi \ominus zM_\varphi}$ is Fredholm by Lemma 2.1, and therefore $L(0)|_{N_\varphi}$ is compact. □

The following example gives a simple illustration for the compactness of $L(0)|_{N_\varphi}$.

Example 1. We consider a function $\varphi(w) = aw$, where $a \in \mathbb{C}$ and $a \neq 0$. Let

$$R_j = \sqrt{1+|a|^2+\cdots+|a|^{2j}}$$

and

$$e_j = \frac{w^j + (\bar{a}z)w^{j-1} + \cdots + (\bar{a}z)^j}{R_j}.$$

Then it is not difficult to check that $\{e_j\}_j$ is an orthonormal basis of N_φ, and one verifies that

$$\|L(0)e_j\|^2 = \left\|\frac{w^j}{R_j}\right\|^2 = R_j^{-2}.$$

So if $|a| < 1$, then $\|L(0)e_j\|^2 \geq 1-|a|^2$ and hence $L(0)$ on N_φ is not compact. If $|a| \geq 1$, then $\lim_{j\to\infty}\|L(0)e_j\| = 0$ which shows that $L(0)$ on N_φ is compact.

It is clear by Corollary 3.11 that S_z is essentially normal in this case. It is easy to give a direct calculation of $[S_z^*, S_z]$. In fact,

$$S_z e_j = \frac{aR_j}{R_{j+1}} e_{j+1}, \quad S_z^* e_j = \frac{\bar{a}R_{j-1}}{R_j} e_{j-1},$$

so

$$\begin{aligned}(S_z^* S_z - S_z S_z^*) e_j &= |a|^2 \left(\frac{R_j^2}{R_{j+1}^2} - \frac{R_{j-1}^2}{R_j^2} \right) e_j \\ &= \left(\frac{|a|^2 + \cdots + |a|^{2(j+1)}}{1 + |a|^2 + \cdots + |a|^{2(j+1)}} - \frac{|a|^2 + \cdots + |a|^{2j}}{1 + |a|^2 + \cdots + |a|^{2j}} \right) e_j \\ &:= c_j e_j.\end{aligned}$$

It is clear that $c_j \to 0$ as $j \to \infty$. One also observes that S_z on N_{aw} is hyponormal.

By [14], we know that $\|S_z\| = \|\varphi\|_\infty$ if $\|\varphi\|_\infty \le 1$, and $\|S_z\| = 1$ for other cases. In the last part of this section, we calculate the norm and the essential norm of $L(0)|_{N_\varphi}$ and S_z. First we recall that the essential norm $\|A\|_e$ is the norm of A in the Calkin algebra.

Since $\|S_z^* F\|^2 + \|L(0)F\|^2 = \|F\|^2$ for every $F \in N_\varphi$, we have

$$\|S_z^*\|^2 = \sup_{F \in N_\varphi, \|F\|=1} \|S_z^* F\|^2 = 1 - \inf_{F \in N_\varphi, \|F\|=1} \|L(0)F\|^2$$

and

$$\inf_{F \in N_\varphi, \|F\|=1} \|S_z^* F\|^2 = 1 - \sup_{F \in N_\varphi, \|F\|=1} \|L(0)F\|^2 = 1 - \|L(0)|_{N_\varphi}\|^2. \tag{4.2}$$

Hence

$$\inf_{F \in N_\varphi, \|F\|=1} \|L(0)F\| = \begin{cases} \sqrt{1 - \|\varphi\|_\infty^2}, & \text{if } \|\varphi\|_\infty \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.5. *Let* $\alpha = \inf_{w \in \mathbb{D}} |\varphi(w)|$. *Then* $\alpha < 1$ *and*

$$\|L(0)|_{N_\varphi}\| = \sqrt{1 - \alpha^2}.$$

Proof. By [14, Corollary 2.7], $\varphi(\mathbb{D}) \cap \mathbb{D} \ne \emptyset$. Hence $\alpha < 1$. Let $w_0 \in \Omega_\varphi$ and

$$F = \frac{2}{(1 - \overline{\varphi(w_0)} z)(1 - \overline{w}_0 w)}.$$

Then by Lemma 2.5, $F \in N_\varphi$ and

$$\frac{\|L(0)F\|^2}{\|F\|^2} = 1 - |\varphi(w_0)|^2.$$

This implies $1 - |\varphi(w_0)|^2 \le \|L(0)|_{N_\varphi}\|^2$. Thus we get

$$\sqrt{1 - \alpha^2} \le \|L(0)\| \le 1. \tag{4.3}$$

If $\alpha = 0$, then $\|L(0)|_{N_\varphi}\| = 1$.

Suppose that $\alpha > 0$. Then $(1/\varphi)(w) \in H^\infty(\Gamma_w)$, and by Lemma 2.4 we have $T^*_{1/\varphi^n} T^{*n}_\varphi = I$ on $L(0)N_\varphi$ for every $n \geq 0$. Let $h \in L(0)N_\varphi$. We have

$$\begin{aligned}\|h\| &= \|T^*_{1/\varphi^n} T^{*n}_\varphi h\| \\ &\leq \|T^*_{1/\varphi^n}\| \|T^{*n}_\varphi h\| \\ &= \|1/\varphi\|^n_\infty \|T^{*n}_\varphi h\| \\ &= \|T^{*n}_\varphi h\|/\alpha^n.\end{aligned}$$

Then $\alpha^n \|h\| \leq \|T^{*n}_\varphi h\|$ for every $h \in L(0)N_\varphi$ and n. Hence

$$\|h\|^2 \frac{1}{1-\alpha^2} \leq \sum_{n=0}^\infty \|T^{*n}_\varphi h\|^2 = \|L(0)^{-1} h\|^2$$

for every $h \in L(0)N_\varphi$, and $\|L(0)F\|^2 \leq (1-\alpha^2)\|F\|$ for every $F \in N_\varphi$. Therefore $\|L(0)|_{N_\varphi}\| \leq \sqrt{1-\alpha^2}$. By (4.3), $\|L(0)|_{N_\varphi}\| = \sqrt{1-\alpha^2}$. □

A combination of (4.2), Propositions 3.1 and 4.5 leads to the following.

Corollary 4.6. *Let $\alpha = \inf_{w \in \mathbb{D}} |\varphi(w)|$. Then S^*_z is invertible if and only if $\alpha > 0$. In this case,*

$$\|S^{*-1}_z\|^{-1} = \inf_{F \in N_\varphi, \|F\|=1} \|S^*_z F\| = \alpha.$$

For $\zeta \in \Omega_\varphi$, let

$$k_\zeta(z,w) = \frac{\sqrt{1-|\varphi(\zeta)|^2}}{1-\overline{\varphi(\zeta)}z} \frac{\sqrt{1-|\zeta|^2}}{1-\bar{\zeta}w}.$$

By Lemma 2.5, $k_\zeta \in N_\varphi$ and $\|k_\zeta\| = 1$.

Theorem 4.7. *Let $\varphi(w) \in H^2(\Gamma_w)$ and $\varphi(w) = b(w)h(w)$ be the outer-inner factorization of φ. Suppose that $L(0)$ on N_φ is not compact. Let $\gamma = \liminf_{|w|\to 1} |\varphi(w)|$. Then $\gamma < 1$ and $\|L(0)|_{N_\varphi}\|_e = \sqrt{1-\gamma^2}$. Moreover $\|L(0)|_{N_\varphi}\|_e \neq \|L(0)|_{N_\varphi}\|$ if and only if $b(w)$ is a nonconstant finite Blaschke product and $1/h(w) \in H^\infty(\Gamma_w)$.*

Proof. By Theorem 4.2, $\gamma < 1$. Take a sequence $\{w_j\}_j$ in Ω_φ such that $|\varphi(w_j)| \to \gamma$ and $|w_j| \to 1$ as $j \to \infty$. We have

$$\begin{aligned}\|L(0)k_{w_j}\| &= \sqrt{1-|w_j|^2}\sqrt{1-|\varphi(w_j)|^2}\Big\|\frac{1}{1-\overline{w}_0 w}\Big\| \\ &= \sqrt{1-|\varphi(w_j)|^2} \\ &\to \sqrt{1-\gamma^2}.\end{aligned}$$

Let K be a compact operator from N_φ to $H^2(\Gamma_w)$. Since $k_{w_j} \to 0$ weakly in N_φ, $\|(L(0)+K)k_{w_j}\| \to \sqrt{1-\gamma^2}$. Hence $\|L(0)|_{N_\varphi}\|_e \geq \sqrt{1-\gamma^2}$.

Suppose that $\gamma = 0$. Then $1 \leq \|L(0)|_{N_\varphi}\|_e \leq \|L(0)|_{N_\varphi}\| \leq 1$. In this case, either b is not a finite Blaschke product or $1/h \notin H^\infty(\Gamma_w)$.

Suppose that $0 < \gamma < 1$. Then b is a finite Blaschke product. By Proposition 4.5, $\|L(0)|_{N_\varphi}\| = \sqrt{1-\alpha^2}$, where $\alpha = \inf_{w\in\mathbb{D}} |\varphi(w)|$. We note that $\alpha \le \gamma$. If $\alpha = \gamma$, then we have $\|L(0)|_{N_\varphi}\| = \|L(0)|_{N_\varphi}\|_e = \sqrt{1-\gamma^2}$. In this case, b is a constant function and $1/h \in H^\infty(\Gamma_w)$.

If $\alpha < \gamma$, then b is a nonconstant finite Blaschke product and $1/h \in H^\infty(\Gamma_w)$. This implies that $\alpha = 0$ and $\|L(0)|_{N_\varphi}\| = 1$. In this case we shall prove that $\|L(0)|_{N_\varphi}\|_e = \sqrt{1-\gamma^2}$. We note that $\|1/h\|_\infty = 1/\gamma$. The idea of the proof is the same as that of Theorem 4.2. We have

$$\begin{aligned}\sup_{b^m g\in L(0)N_\varphi} \frac{\|b^m g\|^2}{\|L^{-1}(0)b^m g\|^2} &\le \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{n=0}^m \|T_h^{*n} b^{m-n} g\|^2}\\ &= \sup_{b^m g\in L(0)N_\varphi} \frac{\|g\|^2}{\sum_{n=0}^m \gamma^{2n}\|T_{1/h}^{*n}\|^2\|T_h^{*n} b^{m-n} g\|^2}\\ &\le \frac{1}{\sum_{n=0}^m \gamma^{2n}}.\end{aligned}$$

Hence $\|L(0)|_{N_\varphi}\|_e \le \sqrt{1-\gamma^2}$, so that we obtain

$$\|L(0)|_{N_\varphi}\|_e = \sqrt{1-\gamma^2} < \sqrt{1-\alpha^2} = \|L(0)|_{N_\varphi}\|. \qquad \square$$

Theorem 4.8. *$\|S_z\|_e = \|S_z\|$ for every N_φ.*

Proof. First, suppose that $0 < \|\varphi\|_\infty \le 1$. Let K be a compact operator on N_φ. Let $\{w_j\}_j$ be a sequence in Ω_φ such that $|\varphi(w_j)| \to \|\varphi\|_\infty$ as $j \to \infty$. Then $Kk_{w_j} \to 0$ as $j \to \infty$. One easily sees that $\|S_z^* k_{w_j}\| = |\varphi(w_j)|$, so that $\|S_z^* k_{w_j}\| \to \|\varphi\|_\infty$ as $j \to \infty$. Hence $\|S_z^* + K\| \ge \|\varphi\|_\infty$. By [14, Proposition 3.5], $\|S_z^*\| = \|\varphi\|_\infty$, so that

$$\|S_z\|_e = \|S_z^*\|_e \ge \|\varphi\|_\infty = \|S_z^*\| = \|S_z\|.$$

Thus we get $\|S_z\|_e = \|S_z\|$.

Next, suppose that $1 < \|\varphi\|_\infty \le \infty$. By [14, Proposition 3.5], $\|S_z\| = 1$. Suppose that $\liminf_{|w|\to 1} |\varphi(w)| \ge 1$. By Theorem 4.2, $L(0)$ is compact on N_φ. Since $S_z S_z^* = I - (L(0)|_{N_\varphi})^* L(0)|_{N_\varphi}$, $\|S_z S_z^*\|_e = 1$, so that $\|S_z\|_e = 1$.

Suppose that $\alpha := \liminf_{|w|\to 1} |\varphi(w)| < 1$. Take a sequence $\{w_j\}_j$ in Ω_φ such that $\liminf_{j\to\infty} |\varphi(w_j)| = \alpha$ and $|w_j| \to 1$ as $j \to \infty$. Let $\alpha_j = \max_{w\in\Gamma} |\varphi(w_j w)|$. Since $\|\varphi\|_\infty > 1$, we may assume that $\alpha_j > 1$ for every j. Since $|\varphi(w_j)| < 1$, $\varphi(w_j\Gamma)$ is a closed curve in $\mathbb{C}$ which interesects with both $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$. Hence there is $\zeta_j \in \Gamma$ satisfying $1 - 1/j < |\varphi(w_j\zeta_j)| < 1$. Note that $w_j\zeta_j \in \Omega_\varphi$. Let K be a compact operator on N_φ. Then $\|(S_z^* + K)k_{w_j\zeta_j}\| = |\varphi(w_j\zeta_j)| \to 1$ as $j \to \infty$, so $\|S_z^* + K\| \ge 1$. Hence

$$\|S_z\|_e = \|S_z^*\|_e \ge 1 \ge \|S_z\| \ge \|S_z\|_e.$$

Thus we get the assertion. $\square$

5. The case when φ is inner

This section gives a detailed study for the case when φ is inner. On the one hand, the fact that φ is inner makes this case very computable, and, as a consequence, many of the earlier results have a clean illustration in this case. On the other hand, the case has a close connection with the two classical spaces, namely the quotient space $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L^2_a(\mathbb{D})$. This fact suggests that the space N_φ indeed has very rich structure.

Some preparations are needed to start the discussion. With every inner function $\theta(w)$ in the Hardy space $H^2(\Gamma_w)$ over the unit circle Γ_w, there is an associated contraction $S(\theta)$ on $H^2(\Gamma_w) \ominus \theta H^2(\Gamma_w)$ defined by

$$S(\theta)f = P_\theta wf, \quad f(w) \in H^2(\Gamma_w) \ominus \theta H^2(\Gamma_w),$$

where P_θ is the projection from $H^2(\Gamma_w)$ onto $H^2(\Gamma_w) \ominus \theta H^2(\Gamma_w)$. The operator $S(\theta)$ is the classical Jordan block, and its properties have been very well studied (cf. [1, 18]). We will state some of the related facts later in the section. Here, we display an orthonormal basis for N_φ.

Lemma 5.1. *Let $\varphi(w)$ be a one variable nonconstant inner function. Let $\{\lambda_k(w)\}_{k=0}^m$ be an orthonormal basis of $H^2(\Gamma_w) \ominus \varphi(w)H^2(\Gamma_w)$, and*

$$e_j = \frac{w^j + w^{j-1}z + \cdots + z^j}{\sqrt{j+1}}$$

for each integer $j \geq 0$. Then

$$\{\lambda_k(w)e_j(z,\varphi(w)) : k = 0,1,2,\dots,m,\ j = 1,2,\dots\}$$

is an othonormal basis for N_φ.

Proof. First of all, we have the facts that

$$N_\varphi = \left\{A_\varphi f : f \in H^2(\Gamma_w), \sum_{n=0}^{\infty} \|T^*_{\varphi^n} f\|^2 < \infty\right\},$$

and

$$H^2(\Gamma_w) = \sum_{j=0}^{\infty} \oplus\, \varphi^j(w)\big(H^2(\Gamma_w) \ominus \varphi(w)H^2(\Gamma_w)\big).$$

Write

$$E_{k,j} = \lambda_k(w)e_j(z,\varphi(w)).$$

Then if $(k,j) \neq (s,t)$ and $j \leq t$,

$$\begin{aligned}\langle E_{k,j}, E_{s,t}\rangle &= \frac{1}{\sqrt{j+1}\sqrt{t+1}} \sum_{l=0}^{j}\sum_{i=0}^{t} \big\langle \lambda_k(w)\varphi^{j-l}(w)z^l, \lambda_s(w)\varphi^{t-i}(w)z^i\big\rangle \\ &= \frac{(j+1)\big\langle \lambda_k(w), \varphi^{t-j}(w)\lambda_s(w)\big\rangle}{\sqrt{j+1}\sqrt{t+1}} \\ &= 0,\end{aligned}$$

and $\|E_{k,j}\| = 1$ for every k, j. Let $f(w) \in H^2(\Gamma_w)$ and write

$$f(w) = \sum_{j=0}^{\infty} \oplus \bigg(\sum_{k=0}^{m} a_{k,j} \lambda_k(w) \bigg) \varphi^j(w), \quad \sum_{j=0}^{\infty} \sum_{k=0}^{m} |a_{k,j}|^2 < \infty.$$

Then

$$\sum_{n=0}^{\infty} \|T^*_{\varphi^n} f(w)\|^2 = \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \sum_{k=0}^{m} |a_{k,j}|^2 = \sum_{j=0}^{\infty} (j+1) \sum_{k=0}^{m} |a_{k,j}|^2.$$

Hence

$$\sum_{n=0}^{\infty} z^n T^*_{\varphi^n} f(w) \in N_\varphi \iff \sum_{j=0}^{\infty} (j+1) \sum_{k=0}^{m} |a_{k,j}|^2 < \infty.$$

In this case, we have

$$\begin{aligned} \sum_{n=0}^{\infty} z^n T^*_{\varphi^n} f(w) &= \sum_{j=0}^{\infty} \bigg(\sum_{k=0}^{m} a_{k,j} \lambda_k(w) \bigg) (\varphi^j(w) + \varphi^{j-1}(w) z + \cdots + z^j) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{m} \sqrt{j+1} a_{k,j} E_{k,j}. \end{aligned}$$

This shows that $\{E_{k,j}\}_{k,j}$ is an othonormal basis of $N_\varphi = H^2(\Gamma^2) \ominus M_\varphi$. $\square$

The operators $L(0)|_{N_\varphi}$, $R(0)|_{N_\varphi}$ and D_z are easy to calculate in this case. In fact, one checks that

$$L(0) E_{k,j} = \frac{\lambda_k(w) \varphi^j(w)}{\sqrt{j+1}},$$

and

$$R(0) E_{k,j} = \frac{\lambda_k(0)(\varphi(0)^j + \varphi(0)^{j-1} z + \cdots + z^j)}{\sqrt{j+1}}.$$

So $L(0)|_{N_\varphi}$ and $R(0)|_{N_\varphi}$ are both compact if $m < \infty$, that is, $\varphi(w)$ is a finite Blaschke product. We summarize this observation and Corollary 4.3 in the following corollary.

Corollary 5.2. *For $\varphi \in H^2(\Gamma_w)$, $L(0)$ and $R(0)$ are both compact on N_φ if and only if φ is a finite Blaschke product.*

The operator D_z is also easy to calculate in this case. One first verifies that

$$X_{k,j} := \frac{\lambda_k(w)}{\sqrt{j+2}} \big(z e_j(z, \varphi(w)) - \sqrt{j+1} \varphi^{j+1}(w) \big), \quad 0 \le k \le m, \ 0 \le j < \infty,$$

is an othonormal basis for $M_\varphi \ominus z M_\varphi$. Then

$$D_z X_{k,j} = \frac{\lambda_k(w) e_j(z, \varphi(w))}{\sqrt{j+2}} = \frac{1}{\sqrt{j+2}} E_{k,j} \tag{5.1}$$

which is also compact if $\varphi(w)$ is a finite Blaschke product.

Two other observations are also worth mentioning. First one calculates that

$$\begin{aligned}\langle zE_{k,j}, E_{s,t}\rangle &= \frac{1}{\sqrt{j+1}\sqrt{t+1}} \sum_{l=0}^{j}\sum_{i=0}^{t} \big\langle z\lambda_k(w)\varphi^{j-l}(w)z^l, \lambda_s(w)\varphi^{t-i}(w)z^i\big\rangle \\ &= \frac{1}{\sqrt{j+1}\sqrt{t+1}} \sum_{l=0}^{j}\sum_{i=0}^{t} \big\langle \lambda_k(w), \lambda_s(w)\varphi^{t+l-i-j}(w)z^{i-l-1}\big\rangle.\end{aligned}$$

Hence

$$\langle zE_{k,j}, E_{s,t}\rangle \neq 0 \iff t = j+1 \text{ and } k = s,$$

and

$$\begin{aligned}S_zE_{k,j} &= \langle S_zE_{k,j}, E_{k,j+1}\rangle E_{k,j+1} \\ &= \frac{1}{\sqrt{j+1}\sqrt{j+2}} \sum_{l=0}^{j} \langle\lambda_k(w), \lambda_k(w)\rangle E_{k,j+1} \\ &= \frac{\sqrt{j+1}}{\sqrt{j+2}} E_{k,j+1}.\end{aligned}$$

This calculation reminds us of the Bergman shift B on the Bergman space $L_a^2(\mathbb{D})$ with the orthonormal basis $\{\sqrt{j+1}\zeta^j\}_j$. In fact, if we define the operator

$$U : N_\varphi \longrightarrow \big(H^2(\Gamma) \ominus \varphi H^2(\Gamma)\big) \otimes L_a^2(\mathbb{D})$$

by

$$U(E_{k,j}) = \lambda_k(w)\sqrt{j+1}\zeta^j, \tag{5.2}$$

then U is clearly a unitary operator, and one checks that

$$US_z = (I \otimes B)U. \tag{5.3}$$

So from this view point N_φ can be identified as $\big(H^2(\Gamma) \ominus \varphi H^2(\Gamma)\big) \otimes L_a^2(\mathbb{D})$. As both $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and $L_a^2(\mathbb{D})$ are classical subjects, this observation indicates that the space N_φ indeed has very rich structure.

The other observation is about the range $R(D_z)$. Let $F \in N_\varphi$. Then by Theorem 2.3,

$$F \in D_z\big(M_\varphi \ominus zM_\varphi\big) \iff \sup_{G\in N_\varphi, \|G\|=1} \frac{|\langle S_z^*G, F\rangle|}{\|L(0)G\|} < \infty.$$

Write

$$\begin{aligned}F &= \sum_{k=0}^{m}\sum_{j=0}^{\infty} a_{k,j}E_{k,j}, \quad \sum_{k=0}^{m}\sum_{j=0}^{\infty} |a_{k,j}|^2 < \infty, \\ G &= \sum_{k=0}^{m}\sum_{j=0}^{\infty} b_{k,j}E_{k,j}, \quad \sum_{k=0}^{m}\sum_{j=0}^{\infty} |b_{k,j}|^2 = 1.\end{aligned}$$

Then

$$\begin{aligned}\frac{|\langle S_z^*G,F\rangle|}{\|L(0)G\|} &= \frac{\left|\left\langle \sum_{k=0}^m\sum_{j=0}^\infty b_{k,j}E_{k,j}, \sum_{k=0}^m\sum_{j=0}^\infty a_{k,j}S_zE_{k,j}\right\rangle\right|}{\left\|\sum_{k=0}^m\sum_{j=0}^\infty b_{k,j}\frac{\lambda_k(w)\varphi^j(w)}{\sqrt{j+1}}\right\|}\\ &= \frac{\left|\sum_{k=0}^m\left\langle \sum_{j=0}^\infty b_{k.j}E_{k,j}, \sum_{j=0}^\infty a_{k,j}S_zE_{k,j}\right\rangle\right|}{\sqrt{\sum_{k=0}^m\sum_{j=0}^\infty\frac{|b_{k,j}|^2}{j+1}}}\\ &= \frac{\left|\sum_{k=0}^m\sum_{j=0}^\infty\frac{\sqrt{j+1}}{\sqrt{j+2}}b_{k,j+1}\bar{a}_{k,j}\right|}{\sqrt{\sum_{k=0}^m\sum_{j=0}^\infty\frac{|b_{k,j}|^2}{j+1}}}\end{aligned}$$

and

$$\sup_{G\in N_\varphi,\|G\|=1}\frac{|\langle S_z^*G,F\rangle|}{\|L(0)G\|} = \sqrt{\sum_{k=0}^m\sum_{j=0}^\infty (j+1)|a_{k,j}|^2}\,.$$

Write $c_{k,j}=\sqrt{j+1}a_{k,j}$, then we have $F\in D_z\big(M_\varphi\ominus zM_\varphi\big)$ if and only if

$$F=\sum_{k=0}^m\sum_{j=0}^\infty\frac{c_{k,j}E_{k,j}}{\sqrt{j+1}},\quad \sum_{k=0}^m\sum_{j=0}^\infty|c_{k,j}|^2<\infty.$$

So

$$U(R(D_z))=\big(H^2(\Gamma)\ominus\varphi H^2(\Gamma)\big)\otimes H^2(\Gamma).$$

The above fact also can be proved using (5.1) and (5.2).

It follows directly from (5.3) that S_z on N_φ is essentially normal if and only if φ is a finite Blaschke product. Now we take a look at the essential normality of S_w. Some facts about the space $H^2(\Gamma)\ominus\varphi H^2(\Gamma)$ need to be mentioned here. We recall that the Jordan block $S(\varphi)$ is defined by

$$S(\varphi)g=P_\varphi wg,\quad g\in H^2(\Gamma)\ominus\varphi H^2(\Gamma),$$

where P_φ is the orthogonal projection from $H^2(\Gamma)$ onto $H^2(\Gamma)\ominus\varphi H^2(\Gamma)$. The two functions $P_\varphi 1$ and $P_\varphi\overline{w}\varphi$ play important roles here, and we let the operator T_0 on $H^2(\Gamma)\ominus\varphi H^2(\Gamma)$ be defined by $T_0g=\langle g,P_\varphi\overline{w}\varphi\rangle P_\varphi 1$. One verifies that

$$T_0^*T_0g=\|P_\varphi 1\|^2\langle g,P_\varphi\overline{w}\varphi\rangle P_\varphi\overline{w}\varphi,\quad T_0T_0^*g=\|P_\varphi\overline{w}\varphi\|^2\langle g,P_\varphi 1\rangle P_\varphi 1,$$

and

$$I-S(\varphi)^*S(\varphi)=\|P_\varphi 1\|^{-2}T_0^*T_0,\quad I-S(\varphi)S(\varphi)^*=\|P_\varphi\overline{w}\varphi\|^{-2}T_0T_0^*. \tag{5.4}$$

For every $g(w)\in H^2(\Gamma_w)\ominus\varphi H^2(\Gamma_w)$, we decompose wg as

$$wg(w)=S(\varphi)g(w)+(I-P_\varphi)wg(w).$$

Using the facts that $(I-P_\varphi)wg = \langle wg,\varphi\rangle\varphi$, $P_\varphi 1 = 1-\overline{\varphi(0)}\varphi$ and $S_\varphi = S_z$, where $S_\varphi g = P_{N_\varphi}\varphi g$, we have

$$\begin{aligned}
&S_w g(w)e_j(z,\varphi(w))\\
&= \sum_{m,n}\langle wg(w)e_j(z,\varphi(w)), E_{m,n}\rangle E_{m,n}\\
&= \sum_{m,n}\left\langle (S(\varphi)g)e_j(z,\varphi(w)) + \langle wg,\varphi\rangle\frac{\varphi P_\varphi 1}{1-\overline{\varphi(0)}\varphi}e_j(z,\varphi(w)), E_{m,n}\right\rangle E_{m,n}\\
&= (S(\varphi)g)e_j(z,\varphi(w)) + \langle wg,\varphi\rangle\sum_{m,n}\left\langle \frac{\varphi P_\varphi 1}{1-\overline{\varphi(0)}\varphi}e_j(z,\varphi(w)), E_{m,n}\right\rangle E_{m,n}\\
&= (S(\varphi)g)e_j(z,\varphi(w)) + \langle g, P_\varphi\overline{w}\varphi\rangle(I-\overline{\varphi(0)}S_z)^{-1}S_z(P_\varphi 1\cdot e_j(z,\varphi(w))).
\end{aligned}$$

So

$$US_wU^* = S(\varphi)\otimes I + T_0\otimes(I-\overline{\varphi(0)}B)^{-1}B. \tag{5.5}$$

For further discussion, we assume φ is not a singular inner function, i.e., φ has a zero in $\mathbb{D}$. We first look at the case when $\varphi(0)=0$. In this case (5.5) reduces to the cleaner expression

$$US_wU^* = S(\varphi)\otimes I + T_0\otimes B. \tag{5.6}$$

Using (5.6) and the fact $S(\varphi)^*T_0 = T_0S(\varphi)^* = 0$, one easily verifies that

$$US_w^*S_wU^* = S(\varphi)^*S(\varphi)\otimes I + T_0^*T_0\otimes B^*B,$$

and

$$US_wS_w^*U^* = S(\varphi)S(\varphi)^*\otimes I + T_0T_0^*\otimes BB^*.$$

Then by (5.4)

$$\begin{aligned}
U[S_w^*,S_w]U^* &= (I-S(\varphi)S(\varphi)^*)\otimes I - (I-S(\varphi)^*S(\varphi))\otimes I\\
&\quad + T_0^*T_0\otimes B^*B - T_0T_0^*\otimes BB^*\\
&= T_0T_0^*\otimes(I-BB^*) - T_0^*T_0\otimes(I-B^*B).
\end{aligned} \tag{5.7}$$

Since T_0 is of rank 1 and it is well-known that $I-BB^*$ and $I-BB^*$ are Hilbert–Schmidt, (5.7) implies that $[S_w^*,S_w]$ is Hilbert–Schmidt. The Hilbert–Schmidt norm of $[S_w^*,S_w]$ can be readily calculated in this case. First of all, $P_{N_\varphi}1 = 1$ and $P_{N_\varphi}\overline{w}\varphi = \overline{w}\varphi$. Let $\lambda_k(w), k=0,1,2,\dots,$ be an orthonormal basis of $H^2(\Gamma_w)\ominus\varphi H^2(\Gamma_w)$ and $\lambda_0(w)=1$. Then by (5.7),

$$\begin{aligned}
&[S_w^*,S_w]\lambda_k(w)e_j(z,\varphi(w))\\
&\qquad = \frac{(T_0T_0^*\lambda_k(w))e_j(z,\varphi(w))}{j+1} - \frac{(T_0^*T_0\lambda_k(w))e_j(z,\varphi(w))}{j+2}\\
&\qquad = \frac{\lambda_k(0)e_j(z,\varphi(w))}{j+1} - \frac{\langle\lambda_k(w),\overline{w}\varphi(w)\rangle\overline{w}\varphi(w)e_j(z,\varphi(w))}{j+2},
\end{aligned}$$

and one calculates that

$$\sum_k \|[S_w^*, S_w]\lambda_k(w)e_j(z,\varphi(w))\|^2 = \frac{1}{(j+1)^2} + \frac{1}{(j+2)^2} - \frac{2|\varphi'(0)|^2}{(j+1)(j+2)},$$

from which it follows that

$$\|[S_w^*, S_w]\|_{H.S}^2 = \frac{\pi^2}{3} - 1 - 2|\varphi'(0)|^2.$$

In the case $\varphi(0) \neq 0$, we need an additional general fact. For $\alpha \in \mathbb{D}$, we let $\tau_\alpha(w) = \frac{\alpha - w}{1-\bar{\alpha}w}$. So if we let operator U_α be defined by

$$U_\alpha(f)(z,w) := \frac{\sqrt{1-|\alpha|^2}}{1-\bar{\alpha}w} f(z,\tau_\alpha(w)), \quad f \in H^2(\mathbb{D}^2),$$

then it is well-known that U_α is a unitary. We let $M' = U_\alpha([z-\varphi]) = [z - \varphi(\tau_\alpha)]$ and $N' = H^2(\mathbb{D}^2) \ominus M'$. The two variable Jordan block on N' is denoted by (S'_z, S'_w). Then by [25],

$$U_\alpha S_z U_\alpha^* = S'_z, \quad U_\alpha S_w U_\alpha^* = \tau_\alpha(S'_w).$$

Since $\tau_\alpha(\tau_\alpha(w)) = w$, we also have

$$U_\alpha \tau_\alpha(S_w) U_\alpha^* = S'_w.$$

So if $\varphi(0) \neq 0$, we pick any zero of φ, say α. Since $\varphi(\tau_a(0)) = \varphi(\alpha) = 0$, $[{S'_w}^*, S'_w]$ is Hilbert–Schmidt by the above calculations, and it then follows that $[S_w^*, S_w]$ is Hilbert–Schmidt (cf. [20, Lemma 1.3]). So in conclusion, when φ is not singular $[S_w^*, S_w]$ is Hilbert–Schmidt on N_φ.

These calculations on S_z and S_w prove the following theorem.

Theorem 5.3. *Let φ be an one variable inner function. Then N_φ is essentially reductive if and only if φ is a finite Blaschke product.*

On N_φ, the commutater $[S_z^*, S_w]$ can also be easily calculated. One sees that

$$\begin{aligned} US_z^*S_wU^* &= (I \otimes B^*)\left(S(\varphi) \otimes I + T_0 \otimes (I - \overline{\varphi(0)}B)^{-1}B\right) \\ &= S(\varphi) \otimes B^* + T_0 \otimes B^*(I - \overline{\varphi(0)}B)^{-1}B, \end{aligned}$$

and

$$\begin{aligned} US_wS_z^*U^* &= \left(S(\varphi) \otimes I + T_0 \otimes (I - \overline{\varphi(0)}B)^{-1}B\right)(I \otimes B^*) \\ &= S(\varphi) \otimes B^* + T_0 \otimes (I - \overline{\varphi(0)}B)^{-1}BB^*. \end{aligned}$$

So

$$U[S_z^*, S_w]U^* = T_0 \otimes [B^*, (I - \overline{\varphi(0)}B)^{-1}B].$$

It was shown in [26] that

$$\operatorname{tr}[f(B)^*, g(B)] = \int_{\mathbb{D}} f'(w)\overline{g'(w)}dA, \tag{5.8}$$

where f and g are analytic functions on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$ and the derivatives f' and g' are in $L_a^2(\mathbb{D})$. Using (5.8), one easily verifies that $[B^*, (1-\overline{\varphi(0)}B)^{-1}B]$ is trace class with $\mathrm{tr}[B^*, (1-\overline{\varphi(0)}B)^{-1}B] = 1$. Therefore, $[S_z^*, S_w]$ is trace class with

$$\begin{aligned}\mathrm{tr}[S_z^*, S_w] &= \mathrm{tr}\, T_0 \cdot \mathrm{tr}[B^*, (I-\overline{\varphi(0)}B)^{-1}B] \\ &= \mathrm{tr}\, T_0 \\ &= \overline{\varphi'(0)}.\end{aligned}$$

Example 2. As we have remarked before that S_z on N_w is equivalent to the Bergman shift B and $S_z = S_w$ in this case, and moreover $\varphi' = 1$. So from the calculations above

$$\mathrm{tr}[B^*, B] = 1, \quad \text{and} \quad \|[B^*, B]\|_{H.S.}^2 = \frac{\pi^2}{3} - 3.$$

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